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Abstract: I consider a real business cycle model in which agents have private information about the i.i.d. realizations of their value of leisure. For the case of logarithmic preferences I provide an analytical characterization of the solution to the associated mechanism design problem. Moreover, I show a striking irrelevance result: That the stationary behavior of all aggregate variables are exactly the same in the private information economy as in the full information case. Numerical simulations indicate that the irrelevance result approximately holds for more general CRRA preferences.

Keywords: Risk sharing, business cycles, private information, social insurance, optimal contracts, heterogeneous agents.

1 Introduction

At least since the seminal paper by Krusell et al. (1998) there has been a long literature analyzing the effects of exogenous forms of market incompleteness on aggregate fluctuations (e.g. exogenous

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borrowing constraints, exogenous collateral requirements, etc.). This paper takes a first step towards building a more primitive approach: It explores the business cycle effects of restrictions to perfect risk sharing, but when these restrictions arise optimally in response to information frictions. In particular, the paper merges two basic benchmarks in the macroeconomics and private information literatures: A standard real business cycle (RBC) model and a Mirrlees economy. The mechanism design problem for the resulting economy is then solved for and its business cycle fluctuations compared to those of the full information case. The paper is interested in evaluating the effects of private information on aggregate fluctuations, in characterizing the cyclical behavior of the optimal contracts, and in exploring the implications for the optimal amount of consumption and employment inequality over the business cycle. While realism would require considering persistent idiosyncratic shocks, the analysis of this type of shocks is significantly more complicated (e.g. Fernandes and Phelan (2000)). For this reason, I consider the case of i.i.d. idiosyncratic shocks as a first step and leave the analysis of persistent shocks for future research.

The model used in this paper is a simple RBC model with private information. Agents value consumption and leisure and receive idiosyncratic shocks to their value of leisure. These shocks, which are i.i.d. over time and across individuals, are assumed to be private information. The production technology is standard. Output, which can be consumed or invested, is produced using capital and labor. The aggregate production function is subject to aggregate productivity shocks that follow a standard AR(1) process.

A social planner designs dynamic contracts for the agents in this economy. Following the literature, a dynamic contract is given a standard recursive formulation where a promised value to the agent describes its state. Given the current state, the contract specifies current consumption, current hours worked, and next-period state-contingent promised values as a function of the value of leisure reported by the agent. Since the model has a large number of agents and the shocks to the value of leisure are idiosyncratic, the social planner needs to keep track of the whole distribution of individuals across promised values as a state variable. Given this distribution, the aggregate stock of capital, and the aggregate productivity level, the social planner seeks to maximize the present discounted utility of agents subject to incentive compatibility, promise keeping, and aggregate resource feasibility constraints.

For the case in which the utilities of consumption and leisure are both logarithmic (a benchmark case in the RBC literature), I am able to provide a sharp analytical characterization of the solution to the mechanism design problem. In particular, I show that the utility of consumption, utility of leisure, and next-period promised values are all linear, strictly increasing functions of the current promised value. The slopes of these functions are all independent of the reported value of leisure and, while the utilities of consumption and leisure have a common slope less than one, the slope of next-period promised values is equal to one (as a consequence, promised values follow a random walk). Over the business cycle, all of these functions shift vertically while keeping constant the differences across reported values of leisure. In turn, the distributions of promised values and log-consumption levels shift horizontally over the business cycle while maintaining their shapes. While optimal consumption inequality is constant, the optimal dispersion of the distribution of log-hours worked is countercyclical. In terms of aggregate dynamics, I find a striking irrelevance result: The business cycle fluctuations of all macroeconomic variables (i.e., aggregate output, consumption, investment, hours worked, and capital) are exactly the same under private information as under full information. Once the information frictions are dealt with in an optimal way, they have no implications for the aggregate dynamics of the economy.

For more general preferences, analytical results are no longer available and the model must be solved for numerically. However, I obtain the same basic irrelevance result for all the CRRA preferences that I consider: The stationary behavior of all macroeconomic variables in the economy with private information is numerically indistinguishable from the same economy with full information. This is true even though the cross-sectional distributions of promised values, instead of shifting horizontally over time, now changes its shape.

The basic reason why the irrelevance result holds exactly under logarithmic preferences is that the Inverse Euler equations, which characterize the optimality conditions under private information, become linear. When these equations are then integrated across the cross-sectional distribution of agents, the Euler equations of the full-information representative-agent economy are obtained. As a consequence, all aggregate variables become the same under full and private information. With more general preferences, this exact aggregation result does not hold because Jensen's inequality works against it. However, with i.i.d. idiosyncratic shocks, the cross-sectional amount of heterogeneity generated is so small that the adjustments due to Jensen's inequality become negligible, and an approximate aggregation result is obtained.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 builds intuition for the main results in the paper by analyzing a simple static economy. Section 4 describes the results for the full dynamic economy. Finally, Section 5 concludes the paper.

2 Related Literature

This paper is closely related to previous work in the social insurance and dynamic public finance literatures (e.g., Atkeson and Lucas (1992), Green (1987), Golosov et al. (2007), Farhi and Werning (2012)). However, interactions with aggregate fluctuations have been mostly neglected in the literature. Notable exceptions are Phelan (1994), da Costa and Luz (2018), Werning (2007), and Scheuer (2013). Phelan (1994) considered a production economy without capital, hidden actions, i.i.d. aggregate shocks, and unobservable i.i.d. idiosyncratic shocks. Under assumptions of CARA preferences and agents facing a constant probability of dying, he characterized the model analytically and found two main results: that the cross-sectional distribution of consumption levels depends on the entire history of aggregate shocks, and that there is a well defined long-run distribution over cross-sectional consumption distributions. The Mirrlees RBC model in this paper differs from Phelan (1994), not only because it has hidden types instead of hidden actions, but because it has CRRA preferences and a neoclassical production function with capital and persistent aggregate shocks. In terms of results, an apparent similarity between the papers is that even in my model with logarithmic preferences, the cross-sectional distributions of consumption and leisure depends on the entire history of aggregate shocks. However, this is only due to the presence of capital. Without it, the cross-sectional distribution would depend only on the current realization of aggregate productivity.

In fact, this lack of memory in the case of no capital and logarithmic preferences has already been shown by da Costa and Luz (2018). In that paper, da Costa and Luz consider a finite horizon version of Phelan's economy in which agents have CRRA preferences and live as long as the economy. Contrary to Phelan (1994), their cross-sectional distribution of consumption becomes degenerate as the time horizon of the economy becomes large. Interestingly, da Costa and Luz find that when log preferences are used, the cross-sectional distribution of consumption does not depend on the entire history of aggregate shocks but only on the current realization. However, when the elasticity of intertemporal substitution is different from one, the cross-sectional distribution of consumption has memory of the past history. Relative to da Costa and Luz (2018), a major contribution of the analysis of the logarithmic Mirrlees economy in this paper is that, in addition to considering an economy with capital and persistent aggregate shocks, I am able to provide a tight analytical characterization of the optimal contracts and an irrelevance result of the information frictions for aggregate dynamics. Da Costa and Luz focus on the dependence of the cross-sectional distribution on past aggregate shocks and they provide no comparisons of aggregate dynamics under full and private information. For preferences different from the logarithmic case, I am able to compute solutions for infinite horizon economies.

Werning (2007) considered an RBC Mirrlees economy with different permanent types of agents, in which the types are private information. Assuming separable utility functions, he provided a sharp characterization of the optimal savings and labor wedges over the business cycle. In particular, he showed that savings wedges are always zero in the cross-section and over the business cycle. In contrast, labor wedges are positive in the cross-section and, if the distribution of labor productivity is fixed across types, constant over time. The RBC Mirrlees economy in this paper differs from his in that the source of the private information is not permanent types but idiosyncratic i.i.d. shocks that change over time. Consequently, instead of having incentive compatibility constraints only at time zero, here they must hold at every time period and history of idiosyncratic and aggregate shocks. In addition to this difference in environments, Werning focused on characterizing optimal wedges and not on the effects of the private information on aggregate dynamics.

Scheuer (2013) considered a static economy with different types of agents subject to idiosyncratic and aggregate shocks. Individual output levels depend on the realizations of the idiosyncratic and aggregate shocks, probability distributions over idiosyncratic shocks depend on individual effort levels and on the aggregate shock, and preferences depend on consumption and effort levels. All these dependencies differ across agent types. While the agent types are public information, effort levels are hidden. Scheuer shows that in a constrained efficient allocation, the ratios of expected inverse marginal utilities between different aggregate shocks must be equalized across the different types of agents. The rest of the paper is devoted to implementing the efficient allocation as a competitive equilibrium with transfers and taxes on financial markets. In addition to corresponding to a dynamic economy with hidden types instead of a static economy with hidden actions, the optimal allocation of my Mirrlees RBC model is not characterized by Scheuer's intratemporal condition because the underlying economy has ex-ante identical agents instead of heterogeneous types. I don't address the issue of implementability, but focus instead on the consequences of private information for aggregate dynamics and on characterizing the optimal amount of inequality over the business cycle (issues not considered by Scheuer).[2]

The irrelevance result in my RBC Mirrlees economy is related to others in the literature. Krueger and Lustig (2010) considered an incomplete markets endowment economy with idiosyncratic and aggregate shocks. The economy has a Lucas tree that yields a fraction of an aggregate stochastic endowment, and a continuum of agents that receive idiosyncratic shocks to their shares on the non-tree part of the aggregate endowment. Agents cannot insure against their idiosyncratic shocks: They can only trade in a risk-free bond and on the Lucas tree, subject to solvency constraints. Krueger and Lustig show that if preferences are CRRA, the aggregate endowment follows a random walk, and the distribution of idiosyncratic endowment shares is independent of the aggregate endowment shock, then there is no trade in the bonds market and only the stock market operates. Moreover, the cross-sectional distributions of wealth and consumption are independent of the aggregate shocks, and the absence of insurance markets is completely irrelevant for the aggregate risk premium. On the surface, these results are closely related to the irrelevance result for the Mirrlees economy in this paper.[3] However, while Krueger and Lustig consider an endowment economy, I consider a production economy. Thus, while the incomplete markets in Krueger and Lustig (2010) cannot affect aggregate dynamics by assumption, I am able to address the effects of information frictions on aggregate dynamics. Furthermore, the structure of equilibria with incomplete markets is very different from those of constrained-efficient allocations under private information, in which incentive compatibility constraints must be satisfied.

This difference is most clearly seen when comparing this paper with Werning (2015). Most of Werning's paper focuses on the demand side of a deterministic Bewley-Huggett-Aiyagari incomplete markets model with a fixed outside asset, and shows that under certain conditions, aggregate consumption and interest rates are related by the Euler equation of a representative agent. However, this representative agent does not correspond to the one obtained under complete markets (in particular his discount factor depends on the amount of idiosyncratic uncertainty while the complete markets representative agent does not). As a result, aggregate consumption levels dif-

[2]In principle, the implementation with non-linear taxes in Albanesi and Sleet (2006) could be extended to the stochastic optimal allocation of my Mirrlees economy.

[3]However, my irrelevance result does not require aggregate consumption to follow a random walk, only log preferences.

fer under incomplete and complete markets.[4] By comparison, in this paper I provide conditions under which aggregate allocations are identical under private or full information.[5] In the last section of his paper, Werning introduces capital accumulation and aggregate shocks, and provides a full irrelevance result for an RBC economy that is closely related to the one in this paper. In his economy agents value consumption, dislike working, and receive idiosyncratic shocks to their labor productivity. Agents can save in capital but cannot borrow. There are spot markets for labor and capital that are used by firms as inputs to a production function, subject to aggregate productivity shocks. For this economy, Werning shows that if agents value consumption according to log preferences, their disutility of labor supply is isoelastic, the depreciation rate of capital is equal to one, and the production function is Cobb-Douglas, then the aggregate dynamics of capital and labor are identical to their counterparts under complete markets. In this equilibrium, aggregate hours worked are constant over time. Moreover, if the initial distribution of wealth is at an invariant steady state, the cross-sectional distributions of consumption and hours worked are also constant over time. In contrast, the irrelevance result in my paper is obtained under any neoclassical production function and depreciation rate of capital; and it holds even though aggregate hours worked and the cross-sectional distribution of hours worked fluctuate over time. The only requirement is that preferences be logarithmic with respect to consumption and leisure. The sharp differences between the conditions needed to obtain the irrelevance results in Werning (2015) and in this paper point to the fundamentally different structures of equilibria with incomplete markets and of constrained-efficient allocations under private information. Neither irrelevance result reduces to the other.

The work that is most closely related to the irrelevance result in my paper is Farhi and Werning (2012). Fahri and Werning consider a very similar Mirrlees economy, except that it has no aggregate productivity shock, idiosyncratic shocks are persistent, and the social planner is only

[4]However, if the amount of idiosyncratic uncertainty is constant over time, Werning argues that the responses of aggregate consumption to changes in interest rates are the same under incomplete and complete markets. This is a potentially useful result that could greatly simplify the analysis of aggregate dynamics in different contexts.

[5]The aggregate allocations under private and full information coincide with those of a **common** representative agent. If the social planner uses a different social discount factor than the private discount factor, then this representative agent has time varying discount factors (even though the amount of idiosyncratic uncertainty is constant over time). However, this is completely unimportant for the irrelevance result.

allowed to optimize with respect to the consumption allocations (labor allocations are taken to be beyond the planner's control). Starting from the steady state of a Bewley economy, Fahri and Werning perform the dynamic public finance experiment of evaluating the welfare gains associated with moving to an optimal consumption plan. They show that when preferences are logarithmic in consumption, along the transitionary dynamics of the model all aggregate variables behave exactly the same as in the representative agent of the full information case. Thus, my irrelevance result in this paper can be seen as extending Fahri and Werning's result to allow the social planner to optimize with respect to labor as well as consumption and to do so in an environment subject to aggregate uncertainty.[6]

3 A static economy

This section analyzes the optimal provision of social insurance and incentives in a simple static economy. The purpose is to build intuition towards one of the main results in the paper: The irrelevance of private information for aggregate allocations in the case of logarithmic preferences.

The economy is populated by a unit measure of agents with preferences given by

$$E\left\{u\left(c\right)+\alpha\ n\left(1-h\right)\right\}$$

where c is consumption, h is hours worked, α is the idiosyncratic value of leisure and u and n are continuously differentiable, strictly increasing and strictly concave utility functions. The idiosyncratic value of leisure α takes two possible values: α_L and α_H, with $\alpha_L < \alpha_H$. Realizations of α are i.i.d. across individuals and are distributed according to a distribution function $\psi = (\psi_L, \psi_H)$. A key assumption is that α is private information of the individual.

Output is produced according to the following production function:

$$Y = e^z F(H),$$

where Y is aggregate output, H is aggregate hours worked and F is continuously differentiable, strictly increasing, concave and satisfies the Inada conditions.

[6] Contrary to Farhi and Werning (2012), the transitionary dynamics in my Mirrlees RBC economy coincide with those of a representative agent economy only if the agent's preferences shift over time in a particular way. This is due to the overlapping generations structure (introduced to obtain a stationary distribution of agents) and only happens if the social planner discounts the future with a discount rate that is different from the agents'.

A social planner decides utilities of consumption u_s and utilities of leisure n_s as functions of the reported value of leisure α_s.[7] The mechanism design problem is the following:

$$\max \sum_s \left[u_s + \alpha_s n_s\right] \psi_s \tag{3.1}$$

subject to

$$\sum_s u^{-1}\left(u_s\right) \psi_s \leq e^z F(H), \tag{3.2}$$

$$H \leq \sum_s \left[1 - n^{-1}\left(n_s\right)\right] \psi_s, \tag{3.3}$$

$$u_L + \alpha_L n_L \geq u_H + \alpha_L n_H, \tag{3.4}$$

where equation (3.2) is the aggregate feasibility constraint for the consumption good, equation (3.3) is the aggregate feasibility constraint for hours worked and equation (3.4) is the binding incentive compatibility constraint.[8]

The unique solution to this problem satisfies equations (3.2)-(3.4) and the following first order conditions:

$$0 = \psi_L - \lambda \frac{1}{u'\left(c_L\right)} \psi_L + \lambda \xi, \tag{3.5}$$

$$0 = \psi_H - \lambda \frac{1}{u'\left(c_H\right)} \psi_H - \lambda \xi, \tag{3.6}$$

$$0 = \alpha_L \psi_L - \lambda q \frac{1}{n'\left(1 - h_L\right)} \psi_L + \alpha_L \lambda \xi, \tag{3.7}$$

$$0 = \alpha_H \psi_H - \lambda q \frac{1}{n'\left(1 - h_H\right)} \psi_H - \alpha_L \lambda \xi, \tag{3.8}$$

$$q = e^z F'(H), \tag{3.9}$$

where λ, λq, and $\lambda \xi$ are the Lagrange multipliers of equations (3.2), (3.3) and (3.4), respectively, and where $c_s = u^{-1}\left(u_s\right)$ and $1 - h_s = n^{-1}\left(n_s\right)$.

Observe that from equations (3.6) and (3.8) we have

$$q \frac{1}{n'\left(1 - h_H\right)} = \alpha_H \frac{1}{u'\left(c_H\right)} + \frac{\left(\alpha_H - \alpha_L\right) \lambda \xi}{\lambda \psi_H}.$$

[7]I formulate the planning problem in terms of utilities of consumption and leisure (instead of consumption and leisure levels) in order to obtain a convex feasible set, which is crucial for characterizing the solution using first order conditions.

[8]It can be shown that the truth-telling constraint for an agent with the high value of leisure will not be binding under the optimal allocation.

Hence,

$$q > \alpha_H \frac{n'(1-h_H)}{u'(c_H)}. \tag{3.10}$$

Since the marginal rate of substitution of leisure for consumption is less than the shadow wage rate q, it follows that under an optimal plan agents with the high value of leisure are "taxed" their labor supply. On the contrary, from equations (3.5) and (3.7) we have that

$$q = \alpha_L \frac{n'(1-h_L)}{u'(c_L)}. \tag{3.11}$$

That is, the labor supply decision of agents with the low value of leisure is undistorted.

Consider now the social planner problem of this same economy but under full information. This problem is to maximize equation (3.1) subject to equations (3.2) and (3.3). Setting $\xi = 0$ in equations (3.5)-(3.9) we get that the optimal allocation under full information satisfies:

$$C^* = e^z F(H^*), \tag{3.12}$$

$$H^* = h_L^* \psi_L + h_H^* \psi_H, \tag{3.13}$$

$$\lambda^* = u'(C^*), \tag{3.14}$$

$$0 = \alpha_L - \lambda^* q^* \frac{1}{n'(1-h_L^*)}, \tag{3.15}$$

$$0 = \alpha_H - \lambda^* q^* \frac{1}{n'(1-h_H^*)}, \tag{3.16}$$

$$q^* = e^z F'(H^*). \tag{3.17}$$

That is, under full information agents' consumption is fully insured and $h_H^* < h_L^*$.

A crucial question is under what conditions the aggregate allocation of the full information economy (C^*, H^*) is identical to that of the private information economy. To see this, lets try to seek a solution to equations (3.2)-(3.9) that satisfy that $H = H^*$. In fact, in doing so it will be convenient to rewrite those equations as follows:

$$c_L \psi_L + c_H \psi_H = C^* \tag{3.18}$$

$$h_L \psi_L + h_H \psi_H = H^* \tag{3.19}$$

$$u(c_L) + \alpha_L n(1-h_L) = u(c_H) + \alpha_L n(1-h_H) \tag{3.20}$$

$$0 = \psi_L - \lambda \frac{1}{u'(c_L)} \psi_L + \lambda \xi, \tag{3.21}$$

$$1 = \lambda \left[\frac{1}{u'(c_L)} \psi_L + \frac{1}{u'(c_H)} \psi_H \right] \tag{3.22}$$

$$0 = \alpha_L \psi_L - \lambda q^* \frac{1}{n'(1-h_L)} \psi_L + \alpha_L \lambda \xi, \tag{3.23}$$

$$\bar{\alpha} \left[\frac{1}{u'(c_L)} \psi_L + \frac{1}{u'(c_H)} \psi_H \right] = q^* \left[\frac{1}{n'(1-h_L)} \psi_L + \frac{1}{n'(1-h_H)} \psi_H \right] \tag{3.24}$$

where $\bar{\alpha} = \alpha_L \psi_L + \alpha_H \psi_H$ and where, using equations (3.9) and (3.17), q has already been substituted by q^*. Observe that equation (3.22) is obtained by adding equations (3.5) and (3.6), and that equation (3.24) is obtained by adding equations (3.7) and (3.8) and using (3.22).

Equations (3.18)-(3.24) form a system of 7 equations in 6 unknowns: c_L, c_H, h_L, h_H, λ and ξ. As a consequence, a solution will generally not exist. In particular, suppose that we have a solution $(c_L, c_H, h_L, h_H, \lambda, \xi)$ to equations (3.18)-(3.23). Then, generally equation (3.24) will not be satisfied. However, there is an exception: when $1/u'$ is a linear function of c and $1/n'$ is a linear function of $1-h$. Observe that in this case equation (3.24) reduces to

$$\bar{\alpha} \frac{1}{u'(c_L \psi_L + c_H \psi_H)} = q^* \frac{1}{n'(1 - h_L \psi_L - h_H \psi_H)} \tag{3.25}$$

and, using equations (3.18) and (3.19), to the following:

$$\bar{\alpha} \frac{1}{u'(C^*)} = q^* \frac{1}{n'(1-H^*)}. \tag{3.26}$$

But this equation is guaranteed to hold since C^* and H^* correspond to a solution of the full information planning problem. To see this, multiply equation (3.15) by ψ_L and equation (3.16) by ψ_H, add them and use equation (3.14) to get:

$$\bar{\alpha} \frac{1}{u'(C^*)} = q^* \left[\frac{1}{n'(1-h_L^*)} \psi_L + \frac{1}{n'(1-h_H^*)} \psi_H \right]. \tag{3.27}$$

Equation (3.26) now follows from equation (3.27) and the linearity of $1/n'$.

This argument has established that logarithmic functional forms for both u and n are generally needed to get identical aggregate allocations under private and full information. Moreover, equation (3.26) indicates that under logarithmic preferences the aggregate allocation of the private information economy coincides with the aggregate allocation of a representative agent model with preferences given by

$$\ln(C) + \bar{\alpha} \ln(1-H).$$

Furthermore, from equations (3.12), (3.17) and (3.26) it can be verified that aggregate hours worked H^* are independent of aggregate productivity z (a standard result under separable and log of consumption preferences). In addition, equations (3.2)-(3.9) imply that h_L and h_H are independent of z while c_L and c_H vary proportionately with it. It follows that the cross sectional variances of log-hours worked and of log-consumption levels are independent of z.

It is also useful to observe from equations (3.14), (3.18) and (3.22) that under logarithmic preferences $\lambda = \lambda^*$. From equations (3.5), (3.6), (3.14) and the concavity of u we then see that $c_H < C^* < c_L$. From equations (3.7), (3.8), (3.15), (3.16), the concavity of n and the fact that $q = q^*$ we also see that $1 - h_L < 1 - h_L^*$ and that $1 - h_H^* < 1 - h_H$. Since, $1 - h_L^* < 1 - h_H^*$ if follows that under private information agents not only receive less insurance in terms of consumption levels but also in terms of leisure. Thus, while under logarithmic preferences aggregate allocations are identical in the private and full information cases, there are key differences in terms of individual allocations.

4 The dynamic economy

The previous section showed that when preferences are logarithmic (both in consumption and in leisure), that the presence of private information becomes irrelevant for the optimal aggregate allocation of a static economy. In what follows I explore if this irrelevance result can be extended to a dynamic setting. There are three reasons for doing this. First, a static economy with logarithmic preferences is quite uninteresting from a macroeconomic point of view since, as was previously mentioned, aggregate hours are not affected by the realization of aggregate productivity. Second, in a dynamic setting the social planner uses intertemporal rewards and punishments to induce truthful revelation in addition to the intratemporal elements already present in a static environment. It is unclear whether logarithmic preferences will be able to jointly aggregate these intertemporal and intratemporal margins into those of a representative agent economy with full information. Third, even if private information under logarithmic preferences plays no role for aggregate allocations it seems important to characterize the cyclical behavior of the optimal amount of cross-sectional inequality in consumption and hours worked within the realm of a realistic business cycle model. The reason is that the presence of private information may be able to shed light on certain cross-sectional cyclical observations that a representative agent model is not able to address. For these reasons, this section incorporates the private information structure of the previous section into a standard real business cycle model and characterizes its optimal allocation.

The economy is populated by a unit measure of agents subject to stochastic lifetimes. Whenever an agent dies he is immediately replaced by a newborn, leaving the aggregate population level constant.[9] The preferences of an individual born at date T are given by

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[u\left(c_t\right) + \alpha_t n\left(1-h_t\right)\right] \right\}, \tag{4.1}$$

where σ is the survival probability, $0 < \beta < 1$ is the discount factor, $\alpha_t \in \{\alpha_1, ..., \alpha_S\}$ is the idiosyncratic value of leisure, and u and n are continuously differentiable, strictly increasing and strictly concave utility functions. Realizations of α_t are assumed to be i.i.d. both across individuals and across time, and private information. The probability that $\alpha_t = \alpha_s$ is given by ψ_s.

Output, which can be consumed or invested, is produced with the following production function:

$$Y_t = e^{z_t} F(K_{t-1}, H_t),$$

where Y_t is output, z_t is aggregate productivity, K_{t-1} is capital, H_t is hours worked, and F is a neoclassical production function. The aggregate productivity level z_t follows a standard AR(1) process given by:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1},$$

where $0 < \rho < 1$ and ε_{t+1} is normally distributed with mean zero and standard deviation σ_ε.

Capital is accumulated using a standard linear technology given by

$$K_t = (1-\delta) K_{t-1} + I_t,$$

where I_t is gross investment and $0 < \delta < 1$. The initial values of z_0 and K_{-1} are given.

4.1 Recursive mechanism design problem

This section provides a recursive formulation to the problem of a social planner that seeks to maximize utility subject to incentive compatibility, promise keeping and resource feasibility constraints. In order to do this it will be important to distinguish between two types of agents: young

[9] As in Phelan (1994), the stochastic lifetime guarantees that there will be a stationary distribution of agents across individual states.

and old. A young agent is one that has been born at the beginning of the current period. An old agent is one that has been born in some previous period. The social planner must decide recursive plans for both types of agents. The state of a recursive plan is the value (i.e., discounted expected utility) that the agent is entitled to at the beginning of the period. Given this promised value, the recursive plan specifies the current utility of consumption, the current utility of leisure, and next-period promised values as functions of the value of leisure currently reported by the agent. The social planner is fully committed to the recursive plans they choose and agents have no outside opportunities available.

A key difference between the young and the old is in terms of promised values. Since during the previous period the social planner has already decided on some recursive plan for a currently old agent, the planner is restricted to delivering the corresponding promised value during the current period. In contrast, the social planner is free to deliver any value to a currently young agent since this is the first period they are alive. Reflecting this difference, I will specify the individual state of an old agent to be their promised value v and their current value of leisure s (similarly to Section 3, I will refer to the value of leisure α_s by its subindex s). At date t, their current utility of consumption, utility of leisure, and next-period promised value are denoted by $u_{ost}(v)$, $n_{ost}(v)$ and $w_{os,t+1}(v)$, respectively, where $w_{os,t+1}(v)$ is a random variable contingent on the realization of z_{t+1}.[10] In turn, the individual state of a young agent is solely given by their current value of leisure s. At date t, the agent's current utility of consumption, utility of leisure, and next-period promised value are denoted by u_{yst}, n_{yst} and $w_{ys,t+1}$ respectively, where $w_{ys,t+1}$ is also contingent on the realization of z_{t+1}.

The aggregate state of the economy is given by the triplet (z_t, K_{t-1}, μ_t), where z_t is the aggregate productivity level, K_{t-1} is the stock of capital, and μ_t is a measure describing the number of old agents across individual promised values v. The social planner seeks to maximize the weighted sum of welfare levels of current and future generations of young agents (the welfare levels of old agents are predetermined by their promised values at the beginning of the period). In recursive

[10] I follow the convention that a variable is dated t if it becomes known at date t.

form, the planner's problem at $t \geq 1$ is given by[11]

$$V_t(z_t, K_{t-1}, \mu_t) = \max \left\{ (1-\sigma) \sum_s [u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t(w_{ys,t+1})] \psi_s + \theta E_t [V_{t+1}(z_{t+1}, K_t, \mu_{t+1})] \right\} \tag{4.2}$$

subject to:

$$(1-\sigma) \sum_s c(u_{yst}) \psi_s + \int \sum_s c(u_{ost}(v)) \psi_s d\mu_t + K_t - (1-\delta) K_{t-1} \leq e^{z_t} F(K_{t-1}, H_t), \tag{4.3}$$

$$H_t \leq (1-\sigma) \sum_s h(n_{yst}) \psi_s + \int \sum_s h(n_{ost}(v)) \psi_s d\mu_t, \tag{4.4}$$

$$u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t [w_{ys,t+1}] \geq u_{yjt} + \alpha_s n_{yjt} + \beta\sigma E_t [w_{yj,t+1}], \tag{4.5}$$

$$u_{ost}(v) + \alpha_s n_{ost}(v) + \beta\sigma E_t [w_{os,t+1}(v)] \geq u_{ojt}(v) + \alpha_s n_{ojt}(v) + \beta\sigma E_t [w_{oj,t+1}(v)], \tag{4.6}$$

$$v = \sum_s \{u_{ost}(v) + \alpha_s n_{ost}(v) + \beta\sigma E_t [w_{os,t+1}(v)]\} \psi_s, \tag{4.7}$$

$$\mu_{t+1}(B) = \sigma \sum_s \int_{\{v:\ w_{os,t+1}(v) \in B\}} \psi_s d\mu_t + (1-\sigma) \sigma \sum_{s:\ w_{ys,t+1} \in B} \psi_s, \tag{4.8}$$

where E_t denotes expectation conditional on z_t and $\beta\sigma < \theta < 1$ is the welfare weight of the next-period generation relative to the current-period generation. Equation (4.3) describes the aggregate feasibility constraint for the consumption good. It states that the total consumption of young and old agents, plus aggregate investment cannot exceed aggregate output.[12] Equation (4.4) is the aggregate labor feasibility constraint. It states that the input of hours into the production function cannot exceed the total hours worked by young and old agents. Equations (4.5) and (4.6) are the incentive compatibility constraints of young and old agents, respectively, and must hold for every (s, j). Equation (4.7) is the promise keeping constraint. It states that the recursive plan for an old agent with promised value v must provide him an expected utility equal to that promised value. Finally, equation (4.8) is the law of motion for the measure of old agents across promised values. It states that the number of old agents that at the beginning of the following period will have a promised value in the Borel set B is given by the sum of two terms. The first term sums

[11] Actually, the planning problem described here has a recursive structure and, therefore, the time subscripts of all variables could be removed. The advantages of leaving them explicitly will become apparent below.

[12] Observe that, given the constant probability of dying $1-\sigma$ and the immediate replacement with newborns, the number of young agents in the economy is always equal to $1-\sigma$.

all currently old agents that receive a next-period promised value in the set B and do not die. The second term does the same for all currently young agents. Observe that since next-period promised values $w_{os,t+1}(v)$ and $w_{ys,t+1}$ are contingent on the realization of next-period aggregate productivity z_{t+1}, that the same is true for the measure μ_{t+1}.

Since the objective function in equation (4.2) is linear and increasing and equations (4.3)-(4.8) define a convex feasible set, the solution to the social planning problem starting from any initial (z_1, K_0, μ_1) is unique. This solution satisfies equations (4.3)-(4.8) and the following first order conditions:

$$0 = \frac{1}{\lambda_t}\psi_s - \frac{1}{u'(c_{yst})}\psi_s + \sum_j \xi_{ysjt} - \sum_j \xi_{yjst}, \tag{4.9}$$

$$0 = \frac{\alpha_s}{\lambda_t}\psi_s - q_t\frac{1}{n'(1-h_{yst})}\psi_s + \sum_j \alpha_s\xi_{ysjt} - \sum_j \alpha_j\xi_{yjst}, \tag{4.10}$$

$$0 = \frac{\beta\sigma}{\lambda_t}\psi_s - \theta\sigma\frac{\lambda_{t+1}}{\lambda_t}\eta_{o,t+1}(w_{ys,t+1})\psi_s + \beta\sigma\sum_j \xi_{ysjt} - \beta\sigma\sum_j \xi_{yjst}, \tag{4.11}$$

$$0 = -\frac{1}{u'[c_{ost}(v)]}\psi_s + \sum_j \xi_{osjt}(v) - \sum_j \xi_{ojst}(v) + \eta_{ot}(v)\psi_s, \tag{4.12}$$

$$0 = -q_t\frac{1}{n'[1-h_{ost}(v)]}\psi_s + \sum_j \alpha_s\xi_{osjt}(v) - \sum_j \alpha_j\xi_{ojst}(v) + \eta_{ot}(v)\alpha_s\psi_s, \tag{4.13}$$

$$0 = -\theta\sigma\frac{\lambda_{t+1}}{\lambda_t}\eta_{o,t+1}[w_{os,t+1}(v)]\psi_s + \beta\sigma\sum_j \xi_{osjt}(v) - \beta\sigma\sum_j \xi_{ojst}(v) + \beta\sigma\eta_{ot}(v)\psi_s, \tag{4.14}$$

$$0 = e^{z_t}F_H(K_{t-1}, H_t) - q_t, \tag{4.15}$$

$$0 = -1 + \theta E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\left[e^{z_{t+1}}F_K(K_t, H_{t+1}) + 1 - \delta\right]\right\}, \tag{4.16}$$

where λ_t, $\lambda_t q_t$, $\lambda_t\xi_{ysjt}$, $\lambda_t\xi_{osjt}(v)$ and $\lambda_t\eta_t(v)$ are the Lagrange multipliers of equations (4.3)-(4.7), respectively.

From equations (4.12)-(4.13) we have, for every s, that

$$q_t\frac{1}{n'[1-h_{ost}(v)]} = \alpha_s\frac{1}{u'[c_{ost}(v)]} + \frac{1}{\psi_s}\sum_j(\alpha_s - \alpha_j)\xi_{ojst}(v). \tag{4.17}$$

Since the last term of this equation is generally different from zero and q_t satisfies equation (4.15), it follows that the labor supply decision of old agents is generally distorted (i.e. there is a wedge between the marginal product of labor and the marginal rate of substitution of old agents).

From equations (4.12), (4.14), and (4.16) we get, for every s, that

$$u'\left[c_{ost}\left(v\right)\right]=\beta E_{t}\left\{\left[e^{z_{t+1}}F_{K}\left(K_{t},H_{t+1}\right)+1-\delta\right]\left[\sum_{s}\frac{1}{u'\left[c_{os,t+1}\left(v\right)\right]}\psi_{s}\right]^{-1}\right\}, \tag{4.18}$$

a relation known in the Dynamic Public Finance literature as the Inverse Euler equation. Applying Jensen's inequality to equation (4.18) we get that

$$u'\left[c_{ost}\left(v\right)\right]\leq\beta E_{t}\left\{\left(e^{z_{t+1}}F_{K}\left(K_{t},H_{t+1}\right)+1-\delta\right)\sum_{s}u'\left[c_{os,t+1}\left(v\right)\right]\psi_{s}\right\}, \tag{4.19}$$

with strict inequality whenever $c_{os,t+1}\left(v\right)$ varies across s. That is, in general there is a wedge in the intertemporal Euler equations of old agents. Using equations (4.9)-(4.11) we derive similar relations to equations (4.17) and (4.18) but for young agents. We conclude that, irrespective of being young or old, under an optimal allocation agents have their intratemporal and intertemporal margins distorted. Therefore, individual allocations differ from the full information case.

4.2 An irrelevance result under logarithmic preferences

The previous section described the mechanism design problem in recursive form. However, date 0 is special because it has no ongoing recursive plans in place on which promised values must be delivered. As a consequence, all agents at date 0 must be treated as young. The date-0 mechanism design problem is thus given by

$$\max\left\{\sum_{s}\left[u_{ys0}+\alpha_{s}n_{ys0}+\beta\sigma E_{t}\left(w_{ys1}\right)\right]\psi_{s}+\theta E_{0}V_{1}\left(z_{1},K_{0},\mu_{1}\right)\right\} \tag{4.20}$$

subject to

$$\sum_{s}c\left(u_{ys0}\right)\psi_{s}+K_{0}-\left(1-\delta\right)K_{-1}\leq e^{z_{0}}F\left[K_{-1},H_{0}\right], \tag{4.21}$$

$$H_{0}\leq\sum_{s}h\left(n_{ys0}\right)\psi_{s}, \tag{4.22}$$

$$u_{ys0}+\alpha_{s}n_{ys0}+\beta\sigma E_{0}\left[w_{ys,1}\right]\geq u_{yj0}+\alpha_{s}n_{yj0}+\beta\sigma E_{0}\left[w_{yj,1}\right],\ \text{for every }\left(s,j\right), \tag{4.23}$$

$$\mu_{1}\left(B\right)=\sigma\sum_{s:\ w_{ys1}\in B}\psi_{s}, \tag{4.24}$$

where V_1 is the value function in equation (4.2) at $t=1$, and (z_0, K_{-1}) are taken as given. The solution to this problem satisfies equations (4.3)-(4.16) for $t\geq 1$, equations (4.21)-(4.24), and equations (4.9)-(4.11) and (4.15)-(4.16) for $t=0$.

By contrast, consider the following non-stationary representative agent planning problem:

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \phi_t \theta^t \left[u\left(C_t\right) + \bar{\alpha} n \left(1 - H_t\right) \right] \right\} \tag{4.25}$$

subject to:

$$C_t + K_t - (1-\delta) K_{t-1} \leq e^{z_t} F\left(K_{t-1}, H_t\right). \tag{4.26}$$

where $\phi_t > 0$ is a deterministic preference shifter with positive limit and (z_0, K_{-1}) is taken as given. Its solution is characterized by equation (4.26) and the following first order conditions:

$$\bar{\alpha} n'\left(1 - H_t\right) = u'\left(C_t\right) e^{z_t} F_H\left(K_{t-1}, H_t\right), \text{ for } t \geq 0, \tag{4.27}$$

$$1 = \theta \frac{\phi_{t+1}}{\phi_t} E_t \left\{ \frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)} \left[e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta \right] \right\}, \text{ for } t \geq 0. \tag{4.28}$$

In what follows I show that when u and n are both logarithmic, the optimal aggregate allocation of the economy with private information coincides with the solution to the representative agent planning problem (4.25) under a particular sequence $\{\phi_t\}_{t=0}^{\infty}$.

Adding equation (4.9) across all s (which cancels all the Lagrange multipliers ξ_{ysjt}), adding equation (4.10) across all s (which cancels all terms $\alpha_s \xi_{ysjt}$) and equating gives the following intratemporal condition:

$$q_t \sum \frac{1}{n'\left(1 - h_{yst}\right)} \psi_s = \bar{\alpha} \sum_s \frac{1}{u'\left(c_{yst}\right)} \psi_s \tag{4.29}$$

Also, adding equations (4.12) across all s (which cancels all the Lagrange multipliers $\xi_{osjt}(v)$), adding equation (4.13) across all s (which cancels all the terms $\alpha_s \xi_{osjt}(v)$) and equating gives the following intratemporal condition:

$$q_t \sum_s \frac{1}{n'\left[1 - h_{ost}(v)\right]} \psi_s = \bar{\alpha} \sum_s \frac{1}{u'\left[c_{ost}(v)\right]} \psi_s \tag{4.30}$$

Under logarithmic u and n, the intratemporal condition (4.29) becomes linear and we have for $t \geq 0$ that

$$q_t \left(1 - H_t^y\right) = \bar{\alpha} C_t^y, \tag{4.31}$$

where $H_t^y = \sum_s h_{yst} \psi_s$ is the average hours worked of young agents and $C_t^y = \sum_s c_{yst} \psi_s$ is the average consumption of young agents. Also, under logarithmic u and n, the intratemporal condition (4.30) becomes linear. Integrating across old agents we have for $t \geq 1$ that

$$q_t \left(1 - H_t^o\right) = \bar{\alpha} C_t^o \tag{4.32}$$

where $H_t^o = \frac{1}{\sigma} \int \sum_s h_{ost}(v) \psi_s d\mu_t$ is the average hours of old agents and $C_t^o = \frac{1}{\sigma} \int \sum_s c_{ost}(v) \psi_s d\mu_t$ is the average consumption of old agents.

Since aggregate consumption is given by

$$C_t = \begin{cases} (1-\sigma) C_t^y + \sigma C_t^o, \text{ for } t \geq 1, \\ C_0^y, \text{ for } t = 0, \end{cases}$$

and aggregate hours worked are given by

$$H_t = \begin{cases} (1-\sigma) H_t^y + \sigma H_t^o, \text{ for } t \geq 1, \\ H_0^y, \text{ for } t = 0, \end{cases}$$

from equations (4.31) and (4.32) we have that

$$q_t (1 - H_t) = \bar{\alpha} C_t \tag{4.33}$$

which is the representative agent intratemporal condition (4.27) under logarithmic preferences, once equation (4.15) is used.

Deriving the representative agent intertemporal Euler equation (4.28) is somewhat more involved. First, observe that adding equations (4.9) and (4.11) across all s gives

$$\frac{1}{\lambda_t} = \sum_s \frac{1}{u'(c_{yst})} \psi_s \tag{4.34}$$

$$\frac{\beta\sigma}{\lambda_t} = \theta\sigma \frac{\lambda_{t+1}}{\lambda_t} \sum_s \eta_{o,t+1}(w_{ys,t+1}) \psi_s \tag{4.35}$$

and that adding equations (4.12) and (4.14) across all s gives

$$\eta_{ot}(v) = \sum_s \frac{1}{u'[c_{ost}(v)]} \psi_s \tag{4.36}$$

$$\beta\sigma\eta_{ot}(v) = \theta\sigma \frac{\lambda_{t+1}}{\lambda_t} \sum_s \eta_{o,t+1}[w_{os,t+1}(v)] \psi_s \tag{4.37}$$

From equations (4.8), (4.35) and (4.37) we then have that

$$\begin{aligned} \int \eta_{o,t+1}(v) d\mu_{t+1} &= \sigma \int \sum_s \eta_{o,t+1}[w_{os,t+1}(v)] \psi_s d\mu_t + (1-\sigma)\sigma \sum_s \eta_{o,t+1}(w_{ys,t+1}) \psi_s \\ &= \frac{\lambda_t \beta\sigma}{\theta\lambda_{t+1}} \int \eta_{ot}(v) d\mu_t + (1-\sigma) \frac{\beta\sigma}{\theta\lambda_{t+1}}, \text{ for } t \geq 1 \end{aligned} \tag{4.38}$$

and from equations (4.24) and (4.35) we have that

$$\int \eta_{o1}(v)\, d\mu_1 = \sigma \sum_s \eta_{o1}(w_{ys,1})\, \psi_s = \frac{\beta\sigma}{\theta\lambda_1}. \tag{4.39}$$

Let $\{\rho_t\}_{t=1}^{\infty}$ be defined as follows:

$$\begin{aligned} \rho_1 &= \frac{\beta\sigma}{\theta}, \\ \rho_{t+1} &= \frac{\beta\sigma}{\theta}\rho_t + (1-\sigma)\frac{\beta\sigma}{\theta}, \text{ for } t \geq 1. \end{aligned} \tag{4.40}$$

From equations (4.38) and (4.39) it follows that

$$\rho_t = \lambda_t \int \eta_{ot}(v)\, d\mu_t, \text{ for } t \geq 1.$$

Under logarithmic u, from equations (4.34) and (4.36) we have

$$\begin{aligned} C_t^y &= \frac{1}{\lambda_t}, \text{ for } t \geq 0, \\ C_t^o &= \frac{1}{\sigma}\int \eta_t(v)\, d\mu_t, \text{ for } t \geq 1. \end{aligned}$$

Then, the ratio of the average consumption of old agents to the average consumption of young agents is given by

$$\frac{C_t^o}{C_t^y} = \frac{\frac{1}{\sigma}\int \eta_t(v)\, d\mu_t}{\frac{1}{\lambda_t}} = \frac{1}{\sigma}\rho_t, \text{ for } t \geq 1,$$

and, consequently,

$$\frac{C_t}{C_{t+1}} = \frac{(1-\sigma)C_t^y + \sigma C_t^o}{(1-\sigma)C_{t+1}^y + \sigma C_{t+1}^o} = \frac{(1-\sigma)+\rho_t}{(1-\sigma)+\rho_{t+1}}\frac{C_t^y}{C_{t+1}^y} = \frac{(1-\sigma)+\rho_t}{(1-\sigma)+\rho_{t+1}}\frac{\lambda_{t+1}}{\lambda_t}, \text{ for } t \geq 1. \tag{4.41}$$

Also, observe that

$$\frac{C_0}{C_1} = \frac{C_0^y}{(1-\sigma)C_1^y + \sigma C_1^o} = \frac{1}{(1-\sigma)C_1^y + \rho_1 C_1^y}\frac{1}{\lambda_0} = \frac{1}{(1-\sigma)+\rho_1}\frac{\lambda_1}{\lambda_0}. \tag{4.42}$$

Defining $\{\phi_t\}_{t=0}^{\infty}$ as

$$\phi_t = \begin{cases} 1, \text{ for } t = 0 \\ (1-\sigma)+\rho_t, \text{ for } t \geq 1, \end{cases} \tag{4.43}$$

equations (4.41) and (4.42) can then be written as

$$\frac{C_t}{C_{t+1}}\frac{\phi_{t+1}}{\phi_t} = \frac{\lambda_{t+1}}{\lambda_t}, \text{ for } t \geq 0. \tag{4.44}$$

From equations (4.16) and (4.44) it follows that, when u is logarithmic, the intertemporal Euler equation of the representative agent (4.28) holds for $t \geq 0$. Since equations (4.3) and (4.21) imply equation (4.26), I have thus established the following Lemma:

Lemma 1 *Suppose that u and n are logarithmic. Define* $\phi = \{\phi_t\}_{t=0}^{\infty}$ *as in equations (4.43) and (4.40). Then, the optimal aggregate allocation of the economy with* ***private information*** *is identical to the optimal allocation of the representative agent economy with preference shifters* ϕ.

Observe that the optimal allocation of the full information economy can be obtained by dropping the incentive compatibility constraints (4.5), (4.6) and (4.23), and setting ξ_{ysjt} and $\xi_{osjt}(v)$ to zero in all first order conditions. Also observe that none of those incentive compatibility constraints or positive values for ξ_{ysjt} or $\xi_{osjt}(v)$ were used in the derivations of equations (4.33) and (4.44). I thus have a second important result:

Lemma 2 *Suppose that u and n are logarithmic. Define* $\phi = \{\phi_t\}_{t=0}^{\infty}$ *as in equations (4.43) and (4.40). Then, the optimal aggregate allocation of the economy with* ***full information*** *is identical to the optimal allocation of the representative agent economy with preference shifters* ϕ.

In addition, since the optimal aggregate allocations of the economy with private information and the economy with full information are equal to the same object, I have the following Corollary.

Corollary 3 *Suppose that u and n are logarithmic. Then, the optimal aggregate allocation of the economy with private information is identical to the optimal aggregate allocation of the economy with full information.*

This Corollary provides a strong irrelevance result: Under logarithmic preferences and i.i.d. shocks, the information frictions play no role for aggregate dynamics. The information frictions affect individual allocations since agents are not fully insured and suffer from the lack of insurance. However, this has no effect whatsoever on aggregate variables.

The reason why the allocations of the private information and full information economy do not aggregate to a representative agent economy with stationary preferences (and preference shifters are generally needed) is because the social planner is allowed to discount the welfare of future generations at a different rate than private agents discount future utility. In fact, if we set the relative Pareto weight θ to the private discount factor β we see from equation (4.40) that $\rho_t = \sigma$ for all $t \geq 1$ and from equation (4.43) that $\phi_t = 1$ for all $t \geq 0$. That is, in this case the representative agent economy has standard stationary preferences.

Observe that independently of the value of θ, from equation (4.40) we verify that ρ_t converges to a positive value and, therefore, that ϕ_t converges to a positive value as well.[13] Since it is well known that the solution to the representative agent economy with stationary preferences (constant ϕ_t) converges to a stationary stochastic process, I can say the same about the aggregate optimal allocations of the economies with private and full information (using Lemmas 1 and 2). Thus, I have the following result.

Corollary 4 *Suppose that u and n are logarithmic. Then, the aggregate optimal allocations of the economies with private and full information converge to a stationary stochastic process. Moreover, this stationary process is the one associated with a representative agent economy with stationary preferences (zero preference shifters).*

4.3 Cross-sectional heterogeneity under log preferences

The previous section showed that private information is irrelevant for aggregate business cycle fluctuations when preferences are logarithmic. However, even in this case the lack of perfect insurance generates endogenous heterogeneity across individual agents that may help understand cross-sectional features of the business cycle that economies with full information cannot address. This is particularly interesting since logarithmic preferences represent a benchmark case in the RBC literature (e.g. Cooley and Prescott (1995)).

In order to study cross-sectional properties of the business cycle we need a sharper characterization of the private information optimal stationary allocation. The next Lemma provides such characterization.

Lemma 5 *Suppose that u and n are logarithmic. Define $\Delta \ln \lambda_t = \ln \lambda_t - \ln \lambda^*$ and $\Delta \ln q_t = \ln q_t - \ln q^*$, where λ^* and q^* are the deterministic steady state values of λ_t and q_t, respectively. Then, the stationary solution to the private information planning problem satisfies, for every s,*

[13] Recall that θ was assumed to be greater than $\beta\sigma$.

that

$$u_{yst} = u^*_{ys} - \Delta \ln \lambda_t \quad (4.45)$$

$$n_{yst} = n^*_{ys} - \Delta \ln \lambda_t - \Delta \ln q_t \quad (4.46)$$

$$w_{ys,t+1} = w^*_{ys} - \frac{1}{b}\left(\Delta \ln \lambda_{t+1} + \Delta \pi_{t+1}\right) \quad (4.47)$$

$$u_{ost}(v) = u^*_{os} + bv + \Delta \pi_t \quad (4.48)$$

$$n_{ost}(v) = n^*_{os} + bv + \Delta \pi_t - \Delta \ln q_t \quad (4.49)$$

$$w_{os,t+1}(v) = w^*_{os} + v - \frac{1}{b}\left(\Delta \ln \lambda_{t+1} + \Delta \pi_{t+1} - \Delta \ln \lambda_t - \Delta \pi_t\right) \quad (4.50)$$

where $0 < b = \frac{1-\beta\sigma}{1+\bar{\alpha}} < 1$ *and* $\Delta \pi_t$ *is given by*

$$\Delta \pi_t = -\beta\sigma\Delta \ln \lambda_t + (1-\beta\sigma)\sum_{k=1}^{\infty}(\beta\sigma)^k E_t\left[\Delta \ln \lambda_{t+k}\right] + b\bar{\alpha}\sum_{k=0}^{\infty}(\beta\sigma)^k E_t\left[\Delta \ln q_{t+k}\right].$$

Proof: *These functional forms satisfy all constraints and first-order conditions.*■[14]

Equations (4.45)-(4.47) indicate that for young agents the utility of consumption, the utility of leisure, and next-period promised values shift over the business cycle by amounts that are independent of the reported type. Equation (4.48) states that all $u_{ost}(v)$ are linear parallel functions that shift vertically over the business cycle by amounts that are independent of the reported type. While equations (4.49) and (4.50) show that the same is true for the utility of leisure and next-period promised values, the slopes of all $w_{os,t+1}(v)$ are equal to one. Thus, promised values follow a random walk process with innovations that depend on the realization of the idiosyncratic and aggregate shocks.[15]

I now turn to characterize the behavior of the cross-sectional distributions of promised values, consumption levels and hours worked implied by the optimal allocation rules described in Lemma 5. Observe, from equations (4.8) and (4.50) that for every interval (a_1, a_2) the steady state

[14] See the Technical Appendix for a complete proof.

[15] Even with no aggregate fluctuations, promised values follow a random walk. However, contrary to Atkeson and Lucas (1992), an immizerizing result is not obtained because of the stochastic lifetimes. As people die and are replaced by young agents, there is enough "reversion to the mean" in promised values that an invariant distribution is obtained (see Phelan 1994). The immizerizing result actually applies within each cohort of agents: Within each cohort the distribution of promised values spreads out over time.

distribution μ^* satisfies that:

$$\mu^* [(a_1, a_2)] = \sigma \sum_s \psi_s \mu^* [(a_1 - w_{os}^*, a_2 - w_{os}^*)] + (1 - \sigma)\, \sigma \sum_{s:\ w_{ys}^* \in (a_1, a_2)} \psi_s. \tag{4.51}$$

Define

$$\Delta_t = \frac{\Delta \ln \lambda_t + \Delta \pi_t}{b}. \tag{4.52}$$

From equations (4.8), (4.47) and (4.50) we have that for every interval $(a_1 - \Delta_{t+1}, a_2 - \Delta_{t+1})$:

$$\begin{aligned} \mu_{t+1} [(a_1 - \Delta_{t+1}, a_2 - \Delta_{t+1})] &= \sigma \sum_s \psi_s \mu_t [(a_1 - \Delta_t - w_{os}^*, a_2 - \Delta_t - w_{os}^*)] \\ &\quad + (1 - \sigma)\, \sigma \sum_{s:\ w_{ys}^* \in (a_1, a_2)} \psi_s. \end{aligned} \tag{4.53}$$

From equations (4.51) and (4.53) it then follows that for every interval (a_1, a_2):

$$\mu_t [(a_1 - \Delta_t, a_2 - \Delta_t)] = \mu^* [(a_1, a_2)]\,. \tag{4.54}$$

That is, μ_t is merely a Δ_t horizontal translation of the steady state distribution μ^*. In particular, since promised values increase during a boom, μ_t shifts to the right during such an episode. We thus have the following Lemma.

Lemma 6 *The dispersion of the cross-sectional distribution of promised values is constant over the business cycle.*

Now let's consider the associated behavior of the cross-sectional distribution φ_t of utilities of consumption u_t. From equations (4.45) and (4.48) we have that φ_t satisfies that for every Borel set B,

$$\varphi_t (B) = \sum_s \int_{\{v:\ u_{os}^* + bv + \Delta \pi_t\ \in\ B\}} \psi_s d\mu_t + \sum_{s:\ u_{ys}^* - \Delta \ln \lambda_t\ \in\ B} \psi_s.$$

It follows that for every interval (a_1, a_2),

$$\varphi^* [(a_1, a_2)] = \sum_s \psi_s \mu^* \left[\left(\frac{a_1 - u_{os}^*}{b}, \frac{a_2 - u_{os}^*}{b} \right) \right] + \sum_{s:\ u_{ys}^*\ \in\ (a_1, a_2)} \psi_s$$

and

$$\begin{aligned} &\varphi_t [(a_1 - \Delta \ln \lambda_t, a_2 - \Delta \ln \lambda_t)] \\ = &\sum_s \psi_s \mu_t \left[\left(\frac{a_1 - \Delta \ln \lambda_t - u_{os}^* - \Delta \pi_t}{b}, \frac{a_2 - \Delta \ln \lambda_t - u_{os}^* - \Delta \pi_t}{b} \right) \right] + \sum_{s:\ u_{ys}^*\ \in\ (a_1, a_2)} \psi_s. \end{aligned}$$

From equations (4.52) and (4.54) we then have that

$$\varphi_t \left[(a_1 - \Delta \ln \lambda_t, a_2 - \Delta \ln \lambda_t) \right] = \varphi^* \left[(a_1, a_2) \right]. \tag{4.55}$$

Thus, φ_t is also a $\Delta \ln \lambda_t$ horizontal translation of the steady state distribution φ^*. Since u is logarithmic, we then have the following Lemma.

Lemma 7 *The dispersion of the cross-sectional distribution of log-consumption levels is constant over the business cycle.*

Finally, let's turn to characterizing the behavior of the cross-sectional distribution ζ_t of utilities of leisure n_t. From equations (4.45) and (4.46) we have that cyclical shifts in n_{yst} differ from the cyclical shifts in u_{yst} by the amount $-\Delta \ln q_t$. From equations (4.48) and (4.49) we also see that $n_{ost}(v)$ is parallel to $u_{ost}(v)$ and that its vertical shifts differ from those in $u_{ost}(v)$ by the amount $-\Delta \ln q_t$. Following the same steps as those used to derive equation (4.55) we thus have that,

$$\zeta_t \left[(a_1 - \Delta \ln \lambda_t - \Delta \ln q_t, a_2 - \Delta \ln \lambda_t - \Delta \ln q_t) \right] = \zeta^* \left[(a_1, a_2) \right].$$

That is, ζ_t is a $\Delta \ln \lambda_t + \Delta \ln q_t$ horizontal translation of the steady state distribution ζ^*. Since the utilities of leisure decrease during a boom, it follows that ζ_t shifts to the left during such an episode.

Observe that the log of hours worked are related to utilities of leisure according to $\ln(h) = \ln(1 - e^n)$. Since this is a strictly decreasing and strictly concave function of n it follows that when the distribution of utilities of leisure shifts to the left, that the dispersion of the distribution of log hours decreases. Thus, we have our last Lemma.

Lemma 8 *The dispersion of the cross-sectional distribution of log-hours worked is countercyclical.*

There is a considerable empirical literature analyzing the behavior of consumption and labor income inequality over time. While most of the literature has focused on trends a few studies have considered business cycle frequencies as well. Heathcote et al. (2010) is a recent example. A key finding in that paper is that U.S. labor earnings inequality widens sharply in recessions and that this is driven by an increase in labor supply inequality (since the cross-sectional distribution of wages is not much affected). Krueger et al. (2010) reported similar findings for eight other countries considered in their study. This empirical evidence is broadly in line with the theoretical results obtained in this section. In particular, Lemma 8 indicates that the model's labor supply inequality increases during recessions and, since all agents earn the same wage rate q, that this translates into an increase in labor income inequality.[16] The empirical evidence on the cyclical behavior of consumption inequality is less clear. Summarizing the international evidence, Krueger et al. (2010) reported that most recessions are accompanied by an uptick in consumption inequality that is much smaller than the associated increase in earnings inequality. Focusing on the U.S. Great Recession of 2007-2009, Krueger et al. (2016) found that consumption inequality increased during that recession as well. However, using a structural factor model Giorgi and Gambetti (2017) found that TFP shocks generate pro-cyclical movement in U.S. consumption inequality. Given these opposing results it seems that the acyclical consumption inequality described in Lemma 7 represents a rough compromise between the different empirical studies.

4.4 General CRRA preferences

So far I have been able to provide an analytical characterization of the solution to the mechanism design problem by assuming that preferences are logarithmic. However, when preferences differ from this case such characterization is no longer possible and the model must be solved for numerically. This is a challenging task: Not only one of the state variables to the social planner problem, μ_t, is infinite dimensional but it is state-contingent (since next-period promised values are contingent on the realization of the aggregate shock). This non-standard feature of the social planner problem makes previous computational methods unsuitable for the task. Fortunately, in Veracierto (2019) I introduce a general computational method that can handle this case without difficulty, making it particularly useful for solving aggregate fluctuations of economies with private information.[17] It turns out that when this method is applied to the Mirrlees economy with

[16]Discussing labor income and wages actually requires specifying a decentralization. To fix concepts it may be useful to consider a simple decentralization in which households have a continuum of members and all the dynamic contracting is done within the family. Output is produced by competitive firms and the market structure consists of spot markets for labor and capital, and a complete set of Arrow securities.

[17]Aside for being able to solve equilibria of economies with private information, three other features that make this computational method particularly attractive relative to other alternatives are the following: 1) it keeps track

logarithmic preferences analyzed above, it reproduces all the analytical results exactly, providing considerable confidence about its accuracy. In what follows, I use it to solve the case of more general CRRA preferences. In particular, I consider preferences of the following form:

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[\frac{c_t^{1-\kappa} - 1}{1-\kappa} + \alpha_t \frac{(1-h_t)^{1-\pi} - 1}{1-\pi} \right] \right\},$$

where $\kappa \neq 1$ and $\pi \neq 1$. In terms of the production function, I assume a Cobb-Douglas specification

$$F(K_{t-1}, H_t) = K_{t-1}^{\gamma} H_t^{1-\gamma},$$

where $0 < \gamma < 1$.

I calibrate the model under different values of κ and π. In order to simplify computations, I select the model time period to be one year and consider only two values for the idiosyncratic shocks: α_L and α_H, with $\alpha_L < \alpha_H$. Following the RBC literature, I select a labor share $1-\gamma$ of 0.64, a depreciation rate δ of 0.10, a private discount factor β of 0.96, a persistence of aggregate productivity ρ of 0.95, and a variance of the innovations to aggregate productivity σ_{ε}^2 equal to 4×0.007^2. The social discount factor θ is chosen to be the same as the private discount factor β. The values of leisure α_L and α_H are chosen to satisfy two criteria: Aggregate hours worked H equal 0.31 (a standard target in the RBC literature) and that the hours worked by old agents with the high value of leisure and the highest possible promised value be a small but positive number.[18] The rationale for this second criterion is that I want to maximize the relevance of the information frictions while keeping an internal solution for hours worked.[19] The probability of drawing a high value of leisure ψ_H is chosen to maximize the standard deviation of the invariant distribution of promised values. It turns out that a value of $\psi_H = 0.50$ achieves this. In terms of the life-cycle structure, I choose $\sigma = 0.975$ to generate an expected lifespan of 40 years.

Before turning to the business cycle results, I describe different steady state features of interest. In order to streamline the presentation I focus on the $\kappa = 2$ and $\pi = 2$ case as an illustration throughout most of the discussion, but present results for other cases when needed.

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of the full distribution of agents across individual states as a state variable (no summary statistics), 2) it handles irregular shapes for this distribution, and 3) it incorporates the distribution's exact law of motion.

[18]The computational method requires working with promised values that lie in a closed interval.

[19]Table 1 describes the values for α_L and α_H corresponding to the different combinations of (κ, π) considered in this section.

In computations I represent the allocation rules u_{ost}, n_{ost} and $w_{os,t+1}$ as spline approximations defined over 20 grid points for the promised value v. Figure 1 shows the invariant distribution of promised values across the 19 intervals defined by the corresponding consecutive grid points. While it is hard to see at this coarseness level, the distribution is approximately symmetrical. More importantly, we see that the invariant distribution puts very little mass at extreme values. In consequence, in what follows I report allocation rules only between the 6th and 16th grid points. The reason is not only that there are too few agents at the tails of the distribution for them to matter, but also that being close to the extremes greatly distorts the shape of the allocation rules.

While not apparent in Figure 1, the invariant distribution of promised values generates too little heterogeneity. The standard deviations of the cross-sectional distribution of log-consumption levels and log-hours worked are 0.04 and 0.41, respectively. This compares with values of 0.50 and 0.82 reported by Heathcote et al. (2010) for 1981 (the year of lowest consumption heterogeneity in their sample).[20] The reason for the small amount of heterogeneity is that there is no persistence in the idiosyncratic shocks: The only way that the model can generate large deviations from the mean is through long streams of repeated bad shocks or good shocks, and these are unlikely to happen. Unsurprisingly, an unrealistic idiosyncratic shock process generates an unrealistic amount of cross-sectional heterogeneity.[21]

Figure 2.A reports the utilities of consumption for old agents $u_{oL}(v)$ and $u_{oH}(v)$ across promised values v, as well as those of young agents u_{yL} and u_{yH} (which are independent of v). Figures 2.B and 2.C do the same for the utilities of leisure and next-period promised values, respectively. The basic economics behind these Figures is quite intuitive: When an agent (young or old) reports a high value of leisure, the planner allows them to enjoy more leisure but, in compensation, they receive less consumption and are promised worse treatment in the future. What I am particularly interested here is in functional forms. We see that, similarly to the logarithmic preferences case, the allocation rules are approximately linear over the relevant range of promised values.[22] However, contrary to the logarithmic case, the allocation rules are not parallel across

[20]See their Figures 10 and 13.

[21]For this reason, there is no point in reporting other features of the cross section, such as optimal labor and capital wedges.

[22]Although extending the domain of these functions to much lower values of v happens to make the existence of

reported types. This is shown in Figure 2.D, which depicts the vertical differences across reported types. We see that only for the utilities of consumption the differences $u_{oH}(v) - u_{oL}(v)$ are constant: The differences $n_{oH}(v) - n_{oL}(v)$ are decreasing in v while the differences $w_{oH}(v) - w_{oL}(v)$ are increasing in v. Thus, the allocation rules do not have the same properties as in the logarithmic case.

Although the allocation rules have different properties from the logarithmic case, the irrelevance result of private information for steady state dynamics is still obtained. This is shown in Table 2, which reports for different values of (κ, π) the deterministic steady state values of all macroeconomic variables for the economies with private information and full information. We see that in each parametrization, all variables are nearly identical in both information scenarios.

The discussion of business cycle dynamics that follows centers on the analysis of the impulse responses of different variables to a one standard deviation increase in aggregate productivity. For simplicity, I once again focus on the $\kappa = 2$ and $\pi = 2$ case.[23] Before turning to the behavior of macroeconomic variables, I establish that the cyclical behavior of the cross-sectional allocation rules is significantly different from the logarithmic case. For parsimony, I only do this for the utilities of leisure (however, the utilities of consumption and next-period promised values are also affected). Figure 3.A shows the impulse responses of the utility of leisure of young agents n_{yL} and n_{yH}. Far from being identical as in the logarithmic case, we see that the response of n_{yH} is much smaller than that of n_{yL}. Figure 3.B shows the impulse response of the utility of leisure of old agents with a low value of leisure $n_{oL}(v)$, at each of the eleven grid points $(v_j)_{j=6}^{16}$. While in the logarithmic case these impulse responses overlap perfectly, we see that in the $\kappa = 2$ and $\pi = 2$ case the response of $n_{oL}(v)$ is larger at low values of v. As a consequence, $n_{oL}(v)$ increases its slope as it shifts down on impact, and then slowly reverts to its steady-state shape. Figure 3.B shows the same for $n_{oH}(v)$. However, differently from the logarithmic case, we see that the response of $n_{oH}(v)$ is smaller than that of $n_{oL}(v)$.

Since the cyclical behavior of the cross-sectional allocation rules are significantly different from the logarithmic case, it is not surprising that the cyclical behavior of the cross-sectional amount of inequality is also quite different. This is shown in Figure 4.A. Similarly to the logarithmic case, we

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non-linearities quite evident.

[23]Similar results are obtained in all the other cases.

see that the amount of consumption inequality is constant. However, contrary to the logarithmic case, the amount of hours inequality now increases with the aggregate shock and the dispersion of the cross-sectional distribution of promised values decreases. Despite this, the information frictions remain irrelevant for aggregate dynamics. Figure 4.B reports the impulse responses of all macroeconomic variables in the economy with private information while Figure 6.C does the same for the economy with full information. We see that both sets of impulse responses are identical. Thus, similar to the log-log case, the stationary behavior of the aggregate variables of the economy is not affected by the presence of information frictions.

The basic reason why the irrelevance result holds under more general CRRA preferences is the small amount of heterogeneity that the model with i.i.d. shocks is able to generate. To see this recall the proof of Lemmas 1 and 2 in Section 4.2. The proof showed that, under logarithmic preferences, the inverse Euler equations that characterize the optimal allocation under private information become linear. This allowed me to integrate the inverse Euler equations across all individuals and obtain a relation between aggregate variables that coincides with the direct Euler equation of the representative agent. When preference are CRRA, Jensen's inequality breaks the integration of the inverse Euler equations into an elemental relation between aggregate variables. However, under i.i.d. shocks, the amount of cross-sectional heterogeneity is so small that the correction from Jensen's inequality becomes negligible.

5 Conclusions

The paper analyzed the effects of restrictions to risk sharing on macroeconomic dynamics when these restrictions are not exogenously imposed but arise endogenously as the optimal response to the presence of private information. For this purpose, the paper brought together two benchmark models in the macroeconomics and information economics literatures, respectively: A real business cycle model and a Mirrlees economy. In particular, the paper considered a RBC model in which agents are subject to i.i.d. idiosyncratic shocks to their value of leisure and these shocks are private information. In this framework the paper analyzed the mechanism design problem of maximizing utility subject to incentive compatibility, promise keeping and aggregate feasibility constraints.

For the case of log-log preferences, which is a standard case in the RBC literature, the paper obtained a sharp analytical characterization. In particular, the utility of consumption, the utility of leisure and next-period promised values are all linear functions of current promised values. Over the business cycle these functions shift vertically in such a way that the distributions of promised values and log-consumption shift horizontally while maintaining their shapes. However, consistent with empirical evidence, the cross-sectional dispersion of log-hours worked is countercyclical. A striking result of the paper is that under logarithmic preferences the business cycle fluctuations of all macroeconomic variables are exactly the same under private information as under full information.

For preferences other than the log-log case analytical results are no longer available. However, numerical simulations indicate that the irrelevance result for aggregate dynamics still holds under more general CRRA preferences: While the cross-sectional distribution of promised values now changes its shape, the business cycle fluctuations of all macroeconomic variables are still unaffected by the presence of private information.

The paper opens wide possibilities for future research. The analysis was done under i.i.d. shocks, which admittedly are highly unrealistic. Considering the case of persistent shocks would be essential for giving richer empirical content to the analysis. From a theoretical point of view it would also be extremely interesting to see if the irrelevance result found in this paper extends to that case.

Also, the paper compared the constrained-efficient aggregate fluctuations under private information with those of the full information counterpart. This isolated the importance of the private information for aggregate dynamics. However, it would be interesting to compare them with versions of the model with realistic financial markets and public policy in order to see how far from their socially optimum fluctuations actual economies may be. One could also compare them to the aggregate fluctuations obtained under optimal policy instruments restricted to belong to a certain class, to see how close to achieving constrained-efficient outcomes those policy instruments may be.

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Table 1

Idiosyncratic shock values

(κ, π)	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
α_L	1.643	1.394	1.471	1.051
α_H	2.177	2.426	2.349	2.770

Table 2

Steady state macroeconomic variables

(κ, π)	Information	Y	C	I	H	K
$(1, 1)$	Private	0.52381	0.39070	0.13311	0.30999	1.3311
	Full	0.52381	0.39070	0.13311	0.31074	1.3311
$(1, 2)$	Private	0.42990	0.32065	0.10925	0.25441	1.0924
	Full	0.42995	0.32069	0.10926	0.25444	1.0926
$(2, 1)$	Private	0.75151	0.56054	0.19097	0.44474	1.9097
	Full	0.75168	0.56066	0.19101	0.44483	1.9101
$(2, 2)$	Private	0.64229	0.47907	0.16322	0.38010	1.6322
	Full	0.64261	0.47931	0.16330	0.38029	1.6330

Figure 1: Invariant distribution of promised values

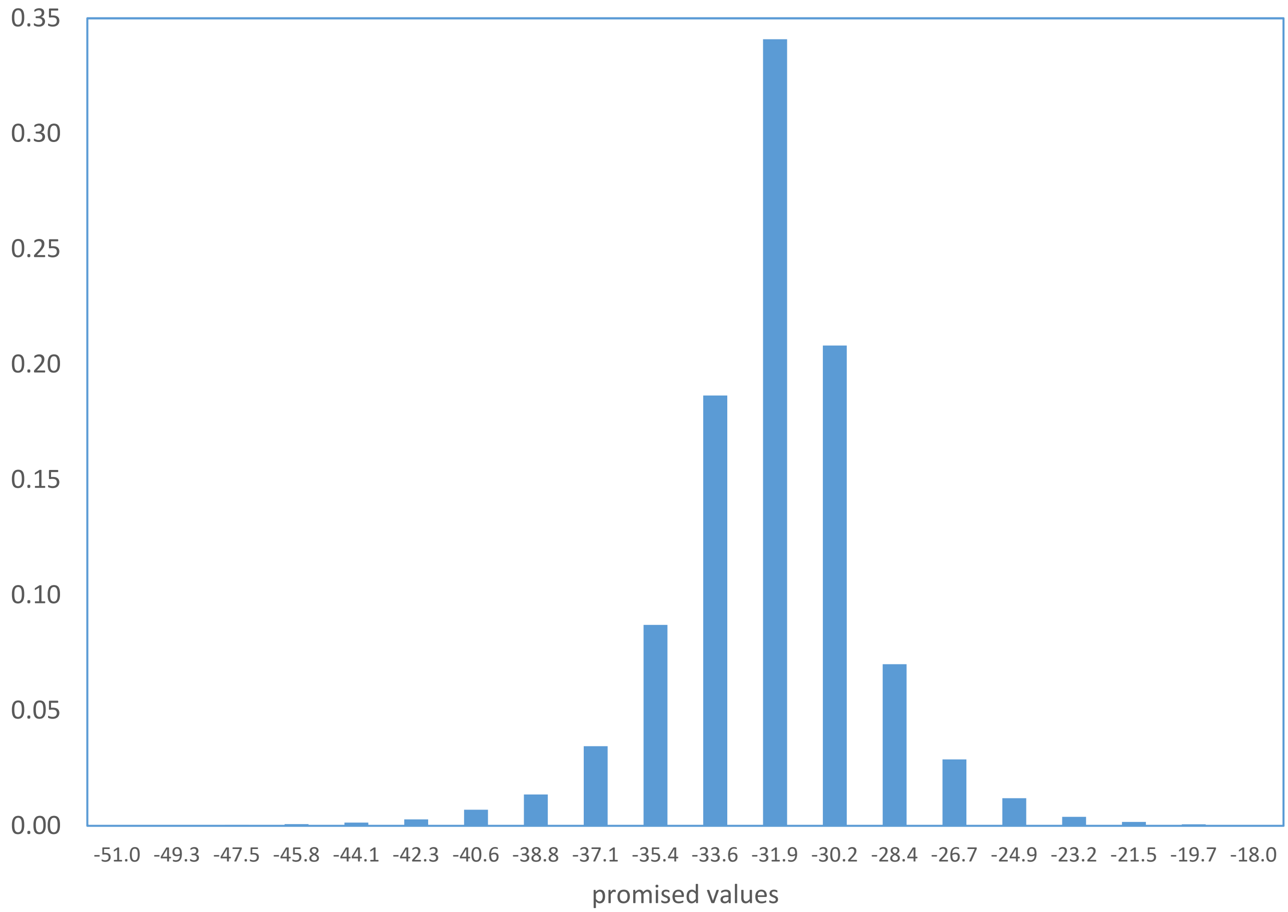


Figure 2: Steady state allocation rules

Figure 3: Impulse responses for leisure utilities

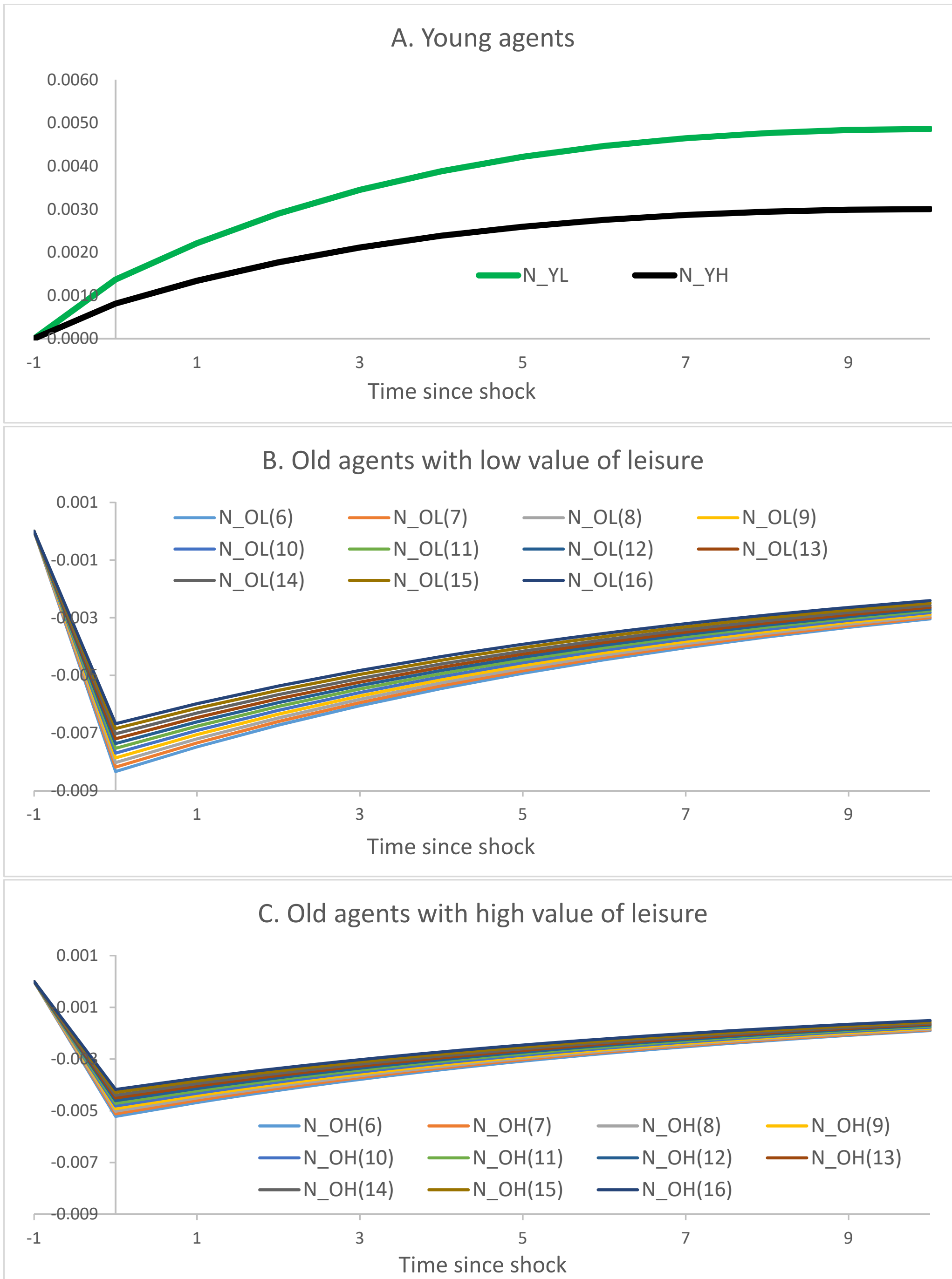
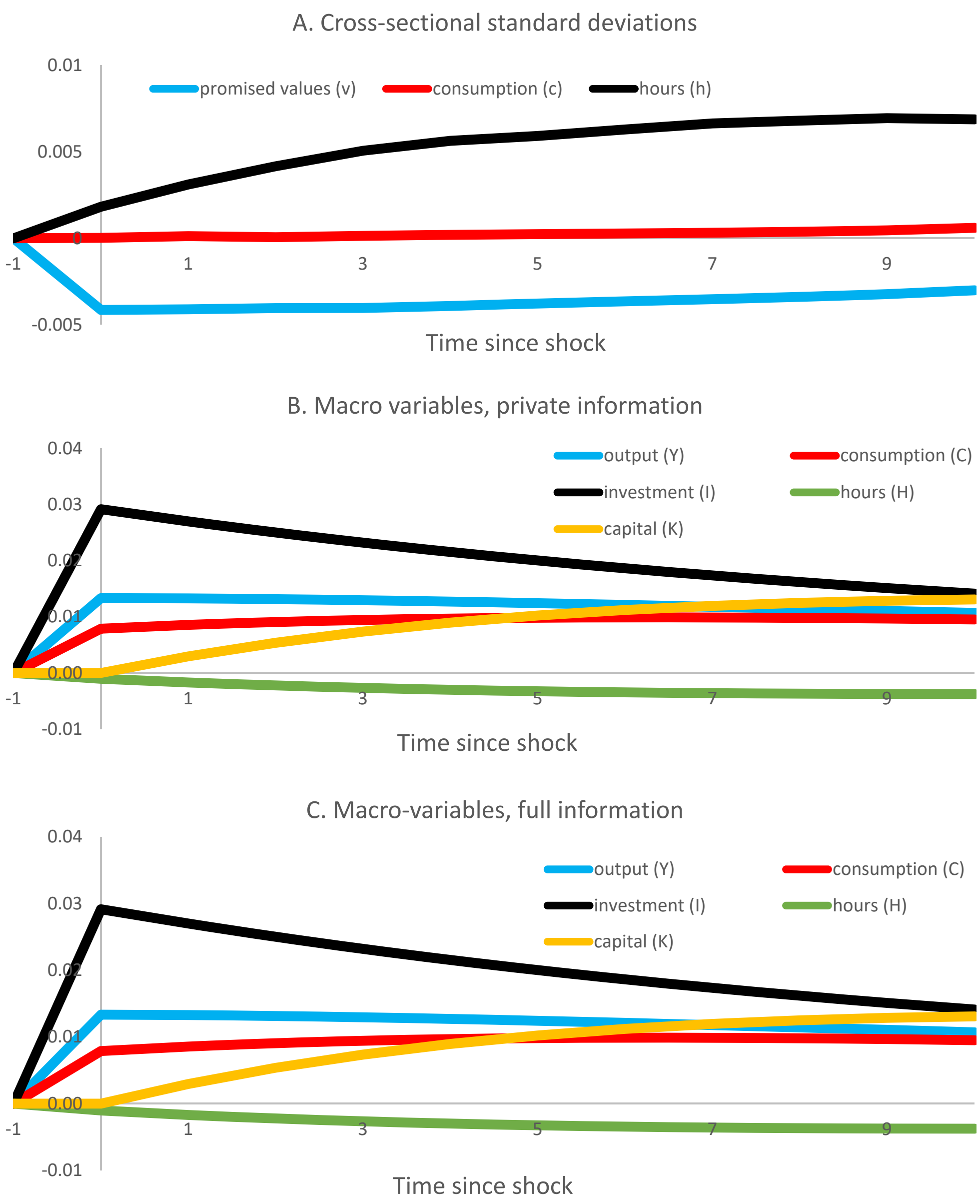


Figure 4: Cross-sectional distributions and macro variables



Technical Appendix for

Business Cycle Fluctuations in Mirrlees Economies: The case of i.i.d. shocks

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6 Proof of Lemma 5

6.1 Representative agent economy with stationary preferences

For future reference it will be useful to characterize the solution to the representative agent planning problem given by equations (4.25)-(4.26), with $\phi_t = 1$ and log preferences. In terms of our redefined variables, this problem is the following:

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \theta^t \left[u_t + \bar{\alpha} n_t \right] \right\}$$

subject to

$$e^{u_t} + I_t \leq e^{z_t} F\left(K_{t-1}, H_t\right)$$

$$H_t \leq 1 - e^{n_t}$$

$$K_t \leq (1-\delta) K_{t-1} + I_t$$

with (z_0, K_{-1}) given.

The first order conditions are:

$$q_t = e^{z_t} F_H\left(K_{t-1}, H_t\right) \tag{6.1}$$

$$\lambda_t = \theta E_t \left\{ \lambda_{t+1} \left[e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta \right] \right\} \tag{6.2}$$

$$1 = \lambda_t e^{u_t} \tag{6.3}$$

$$\bar{\alpha} = \lambda_t q_t e^{n_t} \tag{6.4}$$

For any variable x_t define $\Delta x_t = x_t - x$, where x is its deterministic steady state value. The solution to the representative agent planning problem can then be characterized as follows:

$$q_t = e^{z_t} F_H\left(K_{t-1}, H_t\right) \tag{6.5}$$

$$e^{\Delta \ln \lambda_t} = \theta E_t \left\{ e^{\Delta \ln \lambda_{t+1}} \left[e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta \right] \right\} \tag{6.6}$$

$$K_t = (1-\delta) K_{t-1} + I_t \tag{6.7}$$

$$e^{-\Delta \ln \lambda_t} e^{u} + I_t = e^{z_t} F\left(K_{t-1}, H_t\right) \tag{6.8}$$

$$H_t \leq 1 - e^{-\Delta \ln \lambda_t - \Delta \ln q_t} e^{n} \tag{6.9}$$

where u and n are deterministic steady state values for u_t and n_t, respectively.

6.2 First-order conditions for the private information economy

Under logarithmic preferences, the first order conditions become the following:

$$u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t\left[w_{ys,t+1}\right] \geq u_{yjt} + \alpha_s n_{yjt} + \beta\sigma E_t\left[w_{yj,t+1}\right], \tag{6.10}$$

$$0 = \xi_{ysjt}\left[u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t\left[w_{ys,t+1}\right] - u_{yjt} - \alpha_s n_{yjt} - \beta\sigma E_t\left[w_{yj,t+1}\right]\right] \tag{6.11}$$

$$0 = \psi_s - \lambda_t e^{u_{yst}} + \sum_j \lambda_t \xi_{ysjt} - \sum_j \lambda_t \xi_{yjst} \tag{6.12}$$

$$0 = \alpha_s \psi_s - \lambda_t q_t e^{n_{yst}} + \sum_j \alpha_s \lambda_t \xi_{ysjt} - \sum_j \alpha_j \lambda_t \xi_{yjst} \tag{6.13}$$

$$0 = \beta\sigma\psi_s - \theta\sigma\lambda_{t+1}\eta_{o,t+1}\left(w_{ys,t+1}\right) + \beta\sigma\sum_j \lambda_t \xi_{ysjt} - \beta\sigma\sum_j \lambda_t \xi_{yjst} \tag{6.14}$$

$$u_{ost}(v) + \alpha_s n_{ost}(v) + \beta\sigma E_t\left[w_{os,t+1}(v)\right] \geq u_{ojt}(v) + \alpha_s n_{ojt}(v) + \beta\sigma E_t\left[w_{oj,t+1}(v)\right], \tag{6.15}$$

$$0 = \xi_{osjt}(v)\left[u_{ost}(v) + \alpha_s n_{ost}(v) + \beta\sigma E_t\left[w_{os,t+1}(v)\right] - u_{ojt}(v) - \alpha_s n_{ojt}(v) - \beta\sigma E_t\left[w_{oj,t+1}(v)\right]\right] \tag{6.16}$$

$$v = \sum_s \left\{u_{ost}(v) + \alpha_s n_{ost}(v) + \beta\sigma E_t\left[w_{os,t+1}(v)\right]\right\}\psi_s, \tag{6.17}$$

$$0 = -e^{u_{ost}(v)}\psi_s + \sum_j \xi_{osjt}(v) - \sum_j \xi_{ojst}(v) + \eta_{ot}(v)\psi_s \tag{6.18}$$

$$0 = -q_t e^{n_{ost}(v)}\psi_s + \sum_j \alpha_s \xi_{osjt}(v) - \sum_j \alpha_j \xi_{ojst}(v) + \eta_{ot}(v)\alpha_s\psi_s \tag{6.19}$$

$$0 = -\theta\sigma\lambda_{t+1}\eta_{o,t+1}\left[w_{os,t+1}(v)\right]\psi_s + \beta\sigma\sum_j \lambda_t \xi_{osjt}(v) - \beta\sigma\sum_j \lambda_t \xi_{ojst}(v) + \beta\sigma\lambda_t\eta_{ot}(v)\psi_s \tag{6.20}$$

$$0 = e^{z_t} F_H\left(K_{t-1}, H_t\right) - q_t \tag{6.21}$$

$$0 = \theta E_t\left[\lambda_{t+1}\left(e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta\right)\right] - \lambda_t \tag{6.22}$$

$$K_t = (1-\delta) K_{t-1} + I_t, \tag{6.23}$$

$$(1-\sigma)\sum_s e^{u_{yst}}\psi_s + \int \sum_s e^{u_{ost}(v)}\psi_s d\mu_t + I_t = e^{z_t} F\left(K_{t-1}, H_t\right), \tag{6.24}$$

$$H_t = (1-\sigma)\sum_s \left(1 - e^{n_{yst}}\right)\psi_s + \int \sum_s \left(1 - e^{n_{ost}(v)}\right)\psi_s d\mu_t, \tag{6.25}$$

$$\mu_{t+1}(B) = \sigma\sum_s \int_{\{v:\ w_{os,t+1}(v)\in B\}} \psi_s d\mu_t + (1-\sigma)\sigma \sum_{s:\ w_{ys,t+1}\in B} \psi_s, \tag{6.26}$$

where, from Corollary 4, we know that the stationary stochastic process for the aggregate variables K_{t-1}, H_t and I_t is the stationary solution to the social planner's problem of the representative agent economy in Section 6.1.

6.3 Linear allocation rules

In what follows, I will find it convenient to write ξ_{ysjt} as

$$\xi_{ysjt} = \begin{cases} e^{g_{ysjt}}, \text{ if } I^C_{ysjt} = 0 \\ 0, \text{ otherwise} \end{cases}$$

where

$$I^C_{ysjt} = u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t\left[w_{ys,t+1}\right] - u_{yjt} - \alpha_s n_{yjt} - \beta\sigma E_t\left[w_{yj,t+1}\right].$$

Guess that the allocation rules for old agents have the following functional forms:

$$\begin{aligned} u_{ost}(v) &= u_{ost} + bv \\ n_{ost}(v) &= n_{ost} + bv \\ w_{os,t+1}(v) &= w_{os,t+1} + v \\ \eta_{ot}(v) &= e^{\pi_t + bv} \\ \xi_{osjt}(v) &= \begin{cases} e^{g_{osjt}+bv}, \text{ if } I^C_{osjt} = 0 \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

where

$$0 < b = \frac{1-\beta\sigma}{1+\bar{\alpha}} < 1, \tag{6.27}$$

and

$$I^C_{osjt} = u_{ost} + \alpha_s n_{ost} + \beta\sigma E_t\left[w_{os,t+1}\right] - u_{ojt} - \alpha_s n_{ojt} - \beta\sigma E_t\left[w_{oj,t+1}\right].$$

It is straightforward to verify that this guess satisfies equations (6.10)-(6.26) and that these equations take the following simplified form:

$$u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t\left[w_{ys,t+1}\right] \geq u_{yjt} + \alpha_s n_{yjt} + \beta\sigma E_t\left[w_{yj,t+1}\right], \tag{6.28}$$

$$0 = \psi_s - \lambda_t e^{u_{yst}} + \sum_{j:I^C_{ysjt}=0} \lambda_t e^{g_{ysjt}} - \sum_{j:I^C_{yjst}=0} \lambda_t e^{g_{yjst}} \tag{6.29}$$

$$0 = \alpha_s\psi_s - \lambda_t q_t e^{n_{yst}} + \sum_{j:I^C_{ysjt}=0} \alpha_s\lambda_t e^{g_{ysjt}} - \sum_{j:I^C_{yjst}=0} \alpha_j\lambda_t e^{g_{yjst}} \tag{6.30}$$

$$0 = \beta\sigma\psi_s - \theta\sigma\lambda_{t+1}e^{\pi_{t+1}+bw_{ys,t+1}} + \beta\sigma \sum_{j:I^C_{ysjt}=0} \lambda_t e^{g_{ysjt}} - \beta\sigma \sum_{j:I^C_{yjst}=0} \lambda_t e^{g_{yjst}} \tag{6.31}$$

$$u_{ost} + \alpha_s n_{ost} + \beta\sigma E_t\left[w_{os,t+1}\right] \geq u_{ojt} + \alpha_s n_{ojt} + \beta\sigma E_t\left[w_{oj,t+1}\right] \tag{6.32}$$

$$0 = \sum_s \left\{u_{ost} + \alpha_s n_{ost} + \beta\sigma E_t\left[w_{os,t+1}\right]\right\}\psi_s, \tag{6.33}$$

$$0 = -e^{u_{ost}}\psi_s + \sum_{j:I^C_{osjt}=0} e^{g_{osjt}} - \sum_{j:I^C_{ojst}=0} e^{g_{ojst}} + e^{\pi_t}\psi_s \tag{6.34}$$

$$0 = -q_t e^{n_{ost}}\psi_s + \sum_{j:I^C_{osjt}=0} \alpha_s e^{g_{osjt}} - \sum_{j:I^C_{ojst}=0} \alpha_j e^{g_{ojst}} + e^{\pi_t}\alpha_s\psi_s \tag{6.35}$$

$$0 = -\theta\sigma\lambda_{t+1}e^{\pi_{t+1}+bw_{os,t+1}}\psi_s + \beta\sigma \sum_{j:I^C_{osjt}=0} \lambda_t e^{g_{osjt}} - \beta\sigma \sum_{j:I^C_{ojst}=0} \lambda_t e^{g_{ojst}} + \beta\sigma\lambda_t e^{\pi_t}\psi_s \tag{6.36}$$

$$0 = e^{z_t}F_H\left(K_{t-1}, H_t\right) - q_t \tag{6.37}$$

$$0 = \theta E_t\left[\lambda_{t+1}\left(e^{z_{t+1}}F_K\left(K_t, H_{t+1}\right) + 1 - \delta\right)\right] - \lambda_t \tag{6.38}$$

$$K_t = \left(1-\delta\right)K_{t-1} + I_t, \tag{6.39}$$

$$\left(1-\sigma\right)\sum_s e^{u_{yst}}\psi_s + V_t\sum_s e^{u_{ost}}\psi_s + I_t = e^{z_t}F\left(K_{t-1}, H_t\right), \tag{6.40}$$

$$H_t = \left(1-\sigma\right)\sum_s\left(1-e^{n_{yst}}\right)\psi_s + \sigma - V_t\sum_s e^{n_{ost}}\psi_s, \tag{6.41}$$

$$V_{t+1} = \sigma V_t\sum_s e^{bw_{os,t+1}}\psi_s + \left(1-\sigma\right)\sigma\sum_s e^{bw_{ys,t+1}}\psi_s \tag{6.42}$$

where

$$V_t = \int e^{bv}d\mu_t \tag{6.43}$$

and where equation (6.42) has been derived as follows:

$$\begin{aligned} V_{t+1} &= \int e^{bv}d\mu_{t+1} \\ &= \sigma\sum_s\int e^{bw_{os,t+1}(v)}\psi_s d\mu_t + \left(1-\sigma\right)\sigma\sum_s e^{bw_{ys,t+1}}\psi_s \\ &= \sigma\sum_s\int e^{bw_{os,t+1}+bv}\psi_s d\mu_t + \left(1-\sigma\right)\sigma\sum_s e^{bw_{ys,t+1}}\psi_s \\ &= \sigma\sum_s e^{bw_{os,t+1}}\psi_s\int e^{bv}d\mu_t + \left(1-\sigma\right)\sigma\sum_s e^{bw_{ys,t+1}}\psi_s \\ &= \sigma V_t\sum_s e^{bw_{os,t+1}}\psi_s + \left(1-\sigma\right)\sigma\sum_s e^{bw_{ys,t+1}}\psi_s \end{aligned}$$

6.4 Deterministic steady state

From equations (6.28)-(6.42) we have that a deterministic steady state (with $z_t \equiv 0$) is characterized by the following conditions:

$$u_{ys} + \alpha_s n_{ys} + \beta\sigma w_{ys} \geq u_{yj} + \alpha_s n_{yj} + \beta\sigma w_{yj} \tag{6.44}$$

$$0 = \psi_s - \lambda e^{u_{ys}} + \sum_{j:I^C_{ysj}=0} \lambda e^{g_{ysj}} - \sum_{j:I^C_{yjs}=0} \lambda e^{g_{yjs}} \tag{6.45}$$

$$0 = \alpha_s \psi_s - \lambda q e^{n_{ys}} + \sum_{j:I^C_{ysj}=0} \alpha_s \lambda e^{g_{ysj}} - \sum_{j:I^C_{yjs}=0} \alpha_j \lambda e^{g_{yjs}} \tag{6.46}$$

$$0 = \beta\sigma\psi_s - \theta\sigma\lambda e^{\pi + b w_{ys}} + \beta\sigma \sum_{j:I^C_{ysj}=0} \lambda e^{g_{ysj}} - \beta\sigma \sum_{j:I^C_{yjs}=0} \lambda e^{g_{yjs}} \tag{6.47}$$

$$u_{os} + \alpha_s n_{os} + \beta\sigma w_{os} \geq u_{oj} + \alpha_s n_{oj} + \beta\sigma w_{oj} \tag{6.48}$$

$$0 = \sum_s \left\{ u_{os} + \alpha_s n_{os} + \beta\sigma w_{os} \right\} \psi_s, \tag{6.49}$$

$$0 = -e^{u_{os}} \psi_s + \sum_{j:I^C_{osj}=0} e^{g_{osj}} - \sum_{j:I^C_{ojs}=0} e^{g_{ojs}} + e^{\pi} \psi_s \tag{6.50}$$

$$0 = -q e^{n_{os}} \psi_s + \sum_{j:I^C_{osj}=0} \alpha_s e^{g_{osj}} - \sum_{j:I^C_{ojs}=0} \alpha_j e^{g_{ojs}} + e^{\pi} \alpha_s \psi_s \tag{6.51}$$

$$0 = -\theta\sigma\lambda e^{\pi + b w_{os}} \psi_s + \beta\sigma \sum_{j:I^C_{osj}=0} \lambda e^{g_{osj}} - \beta\sigma \sum_{j:I^C_{ojs}=0} \lambda_t e^{g_{ojs}} + \beta\sigma\lambda e^{\pi} \psi_s \tag{6.52}$$

$$0 = F_H (K, H) - q \tag{6.53}$$

$$0 = \theta \left[F_K (K, H) + 1 - \delta \right] - 1 \tag{6.54}$$

$$0 = \delta K - I \tag{6.55}$$

$$(1 - \sigma) \sum_s e^{u_{ys}} \psi_s + V \sum_s e^{u_{os}} \psi_s + I = F (K, H) \tag{6.56}$$

$$H = (1 - \sigma) \sum_s \left(1 - e^{n_{ys}}\right) \psi_s + \sigma - V \sum_s e^{n_{os}} \psi_s, \tag{6.57}$$

$$V = \sigma V \sum_s e^{b w_{os}} \psi_s + (1 - \sigma) \sigma \sum_s e^{b w_{ys}} \psi_s \tag{6.58}$$

6.5 Fluctuations of optimal decision rules

We will guess and verify the following: 1) that $\Delta \ln \lambda_t$ and $\Delta \ln q_t$ are exactly the same as in the representative agent economy, 2) that for every s

$$\Delta u_{yst} = -\Delta \ln \lambda_t \tag{6.59}$$

$$\Delta n_{yst} = -\Delta \ln \lambda_t - \Delta \ln q_t \tag{6.60}$$

$$\Delta w_{ys,t+1} = -\frac{1}{b}\Delta \ln \lambda_{t+1} - \frac{1}{b}\Delta \pi_{t+1} \tag{6.61}$$

$$\Delta g_{ysjt} = -\Delta \ln \lambda_t \tag{6.62}$$

$$\Delta u_{ost} = \Delta \pi_t \tag{6.63}$$

$$\Delta n_{ost} = \Delta \pi_t - \Delta \ln q_t \tag{6.64}$$

$$\Delta w_{os,t+1} = -\frac{1}{b}\Delta \ln \lambda_{t+1} - \frac{1}{b}\Delta \pi_{t+1} + \frac{1}{b}\Delta \ln \lambda_t + \frac{1}{b}\Delta \pi_t \tag{6.65}$$

$$\Delta g_{osjt} = \Delta \pi_t \tag{6.66}$$

and 3) that

$$\Delta \ln V_t = -\Delta \ln \lambda_t - \Delta \pi_t, \tag{6.67}$$

where $\Delta \pi_t$ is given by:

$$\Delta \pi_t = -\beta\sigma \Delta \ln \lambda_t + (1-\beta\sigma) \sum_{k=1}^{\infty} (\beta\sigma)^k E_t [\Delta \ln \lambda_{t+k}] + b\bar{\alpha} \sum_{k=0}^{\infty} (\beta\sigma)^k E_t [\Delta \ln q_{t+k}] . \tag{6.68}$$

Lemma 9 *Under this guess*

$$\Delta u_{ost} + \bar{\alpha}\Delta n_{ost} + \beta\sigma E_t [\Delta w_{os,t+1}] = 0, \text{ for every } s,$$

Proof: From equations (6.63), (6.64) and (6.65),

$$\begin{aligned}
& \Delta u_{ost} + \bar{\alpha}\Delta n_{ost} + \beta\sigma E_t [\Delta w_{os,t+1}] \\
= \; & \Delta \pi_t + \bar{\alpha}\Delta \pi_t - \bar{\alpha}\Delta \ln q_t + \beta\sigma E_t \left[-\frac{1}{b}\Delta \ln \lambda_{t+1} - \frac{1}{b}\Delta \pi_{t+1} + \frac{1}{b}\Delta \ln \lambda_t + \frac{1}{b}\Delta \pi_t \right] \\
= \; & \left(1 + \frac{\beta\sigma}{b} + \bar{\alpha}\right) \Delta \pi_t - \beta\sigma E_t \left[\frac{1}{b}\Delta \pi_{t+1}\right] - \bar{\alpha}\Delta \ln q_t + \beta\sigma E_t \left[-\frac{1}{b}\Delta \ln \lambda_{t+1} + \frac{1}{b}\Delta \ln \lambda_t \right]
\end{aligned}$$

Using equation (6.27) we have that

$$\begin{aligned}1+\frac{\beta\sigma}{b}+\bar{\alpha} &= 1+\beta\sigma\frac{1+\bar{\alpha}}{1-\beta\sigma}+\bar{\alpha} \\ &= \frac{1-\beta\sigma+\beta\sigma+\beta\sigma\bar{\alpha}+\bar{\alpha}-\bar{\alpha}\beta\sigma}{1-\beta\sigma} \\ &= \frac{1+\bar{\alpha}}{1-\beta\sigma} \\ &= \frac{1}{b}\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta u_{ost}+\bar{\alpha}\Delta n_{ost}+\beta\sigma E_t\left[\Delta w_{os,t+1}\right] \\ = &\frac{1}{b}\Delta\pi_t-\beta\sigma E_t\left[\frac{1}{b}\Delta\pi_{t+1}\right]-\bar{\alpha}\Delta\ln q_t+\beta\sigma E_t\left[-\frac{1}{b}\Delta\ln\lambda_{t+1}+\frac{1}{b}\Delta\ln\lambda_t\right]\end{aligned}$$

Using equation (6.68) we have that

$$\begin{aligned}&\Delta\pi_t-\beta\sigma E_t\left[\Delta\pi_{t+1}\right] \\ = &-\beta\sigma\Delta\ln\lambda_t+(1-\beta\sigma)\sum_{k=1}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln\lambda_{t+k}\right]+b\bar{\alpha}\sum_{k=0}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln q_{t+k}\right] \\ &-\beta\sigma E_t\left[-\beta\sigma\Delta\ln\lambda_{t+1}+(1-\beta\sigma)\sum_{k=1}^{\infty}(\beta\sigma)^k\Delta\ln\lambda_{t+1+k}+b\bar{\alpha}\sum_{k=0}^{\infty}(\beta\sigma)^k\Delta\ln q_{t+1+k}\right] \\ = &-\beta\sigma\Delta\ln\lambda_t+(1-\beta\sigma)\sum_{k=1}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln\lambda_{t+k}\right]+b\bar{\alpha}\sum_{k=0}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln q_{t+k}\right] \\ &-\beta\sigma E_t\left[-\beta\sigma\Delta\ln\lambda_{t+1}+(1-\beta\sigma)\sum_{k=2}^{\infty}(\beta\sigma)^{k-1}\Delta\ln\lambda_{t+k}+b\bar{\alpha}\sum_{k=1}^{\infty}(\beta\sigma)^{k-1}\Delta\ln q_{t+k}\right] \\ = &-\beta\sigma\Delta\ln\lambda_t+\left[(1-\beta\sigma)\beta\sigma+(\beta\sigma)^2\right]E_t\left[\Delta\ln\lambda_{t+1}\right]+b\bar{\alpha}\Delta\ln q_t \\ = &-\beta\sigma\Delta\ln\lambda_t+\beta\sigma E_t\left[\Delta\ln\lambda_{t+1}\right]+b\bar{\alpha}\Delta\ln q_t\end{aligned}$$

Therefore,

$$\begin{aligned}&\Delta u_{ost}+\bar{\alpha}\Delta n_{ost}+\beta\sigma E_t\left[\Delta w_{os,t+1}\right] \\ = &\frac{1}{b}\left\{-\beta\sigma\Delta\ln\lambda_t+\beta\sigma E_t\left[\Delta\ln\lambda_{t+1}\right]+b\bar{\alpha}\Delta\ln q_t\right\} \\ &-\bar{\alpha}\Delta\ln q_t+\beta\sigma E_t\left[-\frac{1}{b}\Delta\ln\lambda_{t+1}+\frac{1}{b}\Delta\ln\lambda_t\right] \\ = &-\beta\sigma\frac{1}{b}\Delta\ln\lambda_t+\beta\sigma\frac{1}{b}E_t\left[\Delta\ln\lambda_{t+1}\right]+\bar{\alpha}\Delta\ln q_t \\ &-\bar{\alpha}\Delta\ln q_t-\beta\sigma E_t\left[\frac{1}{b}\Delta\ln\lambda_{t+1}\right]+\beta\sigma E_t\left[\frac{1}{b}\Delta\ln\lambda_t\right] \\ = &\ 0\ \blacksquare\end{aligned}$$

Under the guess given by equations (6.59)-(6.68), equations (6.28)-(6.42) become the following:

$$\begin{aligned} & u_{ys} - \Delta \ln \lambda_t + \alpha_s \left(n_{ys} - \Delta \ln \lambda_t - \Delta \ln q_t \right) + \beta \sigma E_t \left[w_{ys} - \frac{1}{b} \Delta \ln \lambda_{t+1} - \frac{1}{b} \Delta \pi_{t+1} \right] \\ \geq \ & u_{yj} - \Delta \ln \lambda_t + \alpha_s \left(n_{yj} - \Delta \ln \lambda_t - \Delta \ln q_t \right) + \beta \sigma E_t \left[w_{yj} - \frac{1}{b} \Delta \ln \lambda_{t+1} - \frac{1}{b} \Delta \pi_{t+1} \right] \end{aligned} \quad (6.69)$$

$$\begin{aligned} 0 \ = \ & \psi_s - e^{\ln \lambda + \Delta \ln \lambda_t} e^{u_{ys} - \Delta \ln \lambda_t} + \sum_{j: I^C_{ysjt} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{ysj} - \Delta \ln \lambda_t} \\ & - \sum_{j: I^C_{yjst} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{yjs} - \Delta \ln \lambda_t} \end{aligned} \quad (6.70)$$

$$\begin{aligned} 0 \ = \ & \alpha_s \psi_s - e^{\ln \lambda + \Delta \ln \lambda_t} e^{\ln q + \Delta \ln q_t} e^{n_{ys} - \Delta \ln \lambda_t - \Delta \ln q_t} \\ & + \sum_{j: I^C_{ysjt} = 0} \alpha_s e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{ysj} - \Delta \ln \lambda_t} - \sum_{j: I^C_{yjst} = 0} \alpha_j e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{yjs} - \Delta \ln \lambda_t} \end{aligned} \quad (6.71)$$

$$\begin{aligned} 0 \ = \ & \beta \sigma \psi_s - \theta \sigma e^{\ln \lambda + \Delta \ln \lambda_{t+1}} e^{\pi + \Delta \pi_{t+1} + b \left(w_{ys} - \frac{1}{b} \Delta \ln \lambda_{t+1} - \frac{1}{b} \Delta \pi_{t+1} \right)} \\ & + \beta \sigma \sum_{j: I^C_{ysjt} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{ysj} - \Delta \ln \lambda_t} - \beta \sigma \sum_{j: I^C_{yjst} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{yjs} - \Delta \ln \lambda_t} \end{aligned} \quad (6.72)$$

$$\begin{aligned} & u_{os} + \Delta \pi_t + \alpha_s \left(n_{os} + \Delta \pi_t - \Delta \ln q_t \right) + \beta \sigma E_t \left[w_{os} + \frac{1}{b} \left(-\Delta \ln \lambda_{t+1} - \Delta \pi_{t+1} + \Delta \ln \lambda_t + \Delta \pi_t \right) \right] \\ & \hspace{30em} (6.73) \\ \geq \ & u_{oj} + \Delta \pi_t + \alpha_s \left(n_{oj} + \Delta \pi_t - \Delta \ln q_t \right) + \beta \sigma E_t \left[w_{oj} + \frac{1}{b} \left(-\Delta \ln \lambda_{t+1} - \Delta \pi_{t+1} + \Delta \ln \lambda_t + \Delta \pi_t \right) \right] \end{aligned}$$

$$\begin{aligned} 0 \ = \ & \sum_s \{ u_{os} + \Delta \pi_t + \alpha_s \left(n_{os} + \Delta \pi_t - \Delta \ln q_t \right) \\ & + \beta \sigma E_t \left[w_{os} + \frac{1}{b} \left(-\Delta \ln \lambda_{t+1} - \Delta \pi_{t+1} + \Delta \ln \lambda_t + \Delta \pi_t \right) \right] \Big\} \psi_s, \end{aligned} \quad (6.74)$$

$$0 = -e^{u_{os} + \Delta \pi_t} \psi_s + \sum_{j: I^C_{osjt} = 0} e^{g_{osj} + \Delta \pi_t} - \sum_{j: I^C_{ojst} = 0} e^{g_{ojs} + \Delta \pi_t} + e^{\pi + \Delta \pi_t} \psi_s \quad (6.75)$$

$$0 = -e^{\ln q + \Delta \ln q_t} e^{n_{os} + \Delta \pi_t - \Delta \ln q_t} \psi_s + \sum_{j: I^C_{osjt} = 0} \alpha_s e^{g_{osj} + \Delta \pi_t} - \sum_{j: I^C_{ojst} = 0} \alpha_j e^{g_{ojs} + \Delta \pi_t} + e^{\pi + \Delta \pi_t} \alpha_s \psi_s \quad (6.76)$$

$$\begin{aligned} 0 \ = \ & -\theta \sigma e^{\ln \lambda + \Delta \ln \lambda_{t+1}} e^{\pi + \Delta \pi_{t+1} + b \left(w_{os} - \frac{1}{b} \Delta \ln \lambda_{t+1} - \frac{1}{b} \Delta \pi_{t+1} + \frac{1}{b} \Delta \ln \lambda_t + \frac{1}{b} \Delta \pi_t \right)} \psi_s \quad (6.77) \\ & + \beta \sigma \sum_{j: I^C_{osjt} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{osj} + \Delta \pi_t} - \beta \sigma \sum_{j: I^C_{ojst} = 0} e^{\ln \lambda + \Delta \ln \lambda_t} e^{g_{ojs} + \Delta \pi_t} + \beta \sigma e^{\ln \lambda + \Delta \ln \lambda_t} e^{\pi + \Delta \pi_t} \psi_s \end{aligned}$$

$$0 = e^{z_t} F_H\left(K_{t-1}, H_t\right) - q_t \tag{6.78}$$

$$0 = \theta E_t\left[e^{\Delta \ln \lambda_{t+1}}\left(e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta\right)\right] - e^{\Delta \ln \lambda_t} \tag{6.79}$$

$$K_t = (1-\delta) K_{t-1} + I_t, \tag{6.80}$$

$$(1-\sigma)\sum_s e^{u_{ys} - \Delta \ln \lambda_t}\psi_s + e^{\ln V - \Delta \ln \lambda_t - \Delta \pi_t}\sum_s e^{u_{os} + \Delta \pi_t}\psi_s + I_t = e^{z_t} F\left(K_{t-1}, H_t\right), \tag{6.81}$$

$$H_t = (1-\sigma)\sum_s\left(1 - e^{n_{ys} - \Delta \ln \lambda_t - \Delta \ln q_t}\right)\psi_s + \sigma - e^{\ln V - \Delta \ln \lambda_t - \Delta \pi_t}\sum_s e^{n_{os} + \Delta \pi_t - \Delta \ln q_t}\psi_s, \tag{6.82}$$

$$\begin{aligned} e^{\ln V - \Delta \ln \lambda_{t+1} - \Delta \pi_{t+1}} &= \sigma e^{\ln V - \Delta \ln \lambda_t - \Delta \pi_t}\sum_s e^{b\left(w_{os} - \frac{1}{b}\Delta \ln \lambda_{t+1} - \frac{1}{b}\Delta \pi_{t+1} + \frac{1}{b}\Delta \ln \lambda_t + \frac{1}{b}\Delta \pi_t\right)}\psi_s \\ &+ (1-\sigma)\,\sigma\sum_s e^{b\left(w_{ys} - \frac{1}{b}\Delta \ln \lambda_{t+1} - \frac{1}{b}\Delta \pi_{t+1}\right)}\psi_s \end{aligned} \tag{6.83}$$

From equations (6.69) and (6.73) we can readily verify that $I^C_{ysjt} = 0 \Leftrightarrow I^C_{ysj} = 0$ and that $I^C_{osjt} = 0 \Leftrightarrow I^C_{osj}$. Using Lemma 9, equations (6.69)-(6.83) thus become the following:

$$u_{ys} + \alpha_s n_{ys} + \beta\sigma E_t\left[w_{ys}\right] \geq u_{yj} + \alpha_s n_{yj} + \beta\sigma E_t\left[w_{yj}\right] \tag{6.84}$$

$$0 = \psi_s - e^{\ln \lambda} e^{u_{ys}} + \sum_{j: I^C_{ysj}=0} e^{\ln \lambda} e^{g_{ysj}} - \sum_{j: I^C_{yjs}=0} e^{\ln \lambda} e^{g_{yjs}} \tag{6.85}$$

$$0 = \alpha_s\psi_s - e^{\ln \lambda} e^{\ln q} e^{n_{ys}} + \sum_{j: I^C_{ysj}=0} \alpha_s e^{\ln \lambda} e^{g_{ysj}} - \sum_{j: I^C_{yjs}=0} \alpha_j e^{\ln \lambda} e^{g_{yjs}} \tag{6.86}$$

$$0 = \beta\sigma\psi_s - \theta\sigma e^{\ln \lambda} e^{\pi + b w_{ys}} + \beta\sigma\sum_{j: I^C_{ysj}=0} e^{\ln \lambda} e^{g_{ysj}} - \beta\sigma\sum_{j: I^C_{yjs}=0} e^{\ln \lambda} e^{g_{yjs}} \tag{6.87}$$

$$u_{os} + \alpha_s n_{os} + \beta\sigma w_{os} \geq u_{oj} + \alpha_s n_{oj} + \beta\sigma w_{oj} \tag{6.88}$$

$$0 = \sum_s\left\{u_{os} + \alpha_s n_{os} + \beta\sigma w_{os}\right\}\psi_s, \tag{6.89}$$

$$0 = -e^{u_{os}}\psi_s + \sum_{j: I^C_{osj}=0} e^{g_{osj}} - \sum_{j: I^C_{ojs}=0} e^{g_{ojs}} + e^{\pi}\psi_s \tag{6.90}$$

$$0 = -e^{\ln q} e^{n_{os}}\psi_s + \sum_{j: I^C_{osj}=0} \alpha_s e^{g_{osj}} - \sum_{j: I^C_{ojs}=0} \alpha_j e^{g_{ojs}} + e^{\pi}\alpha_s\psi_s \tag{6.91}$$

$$0 = -\theta\sigma e^{\ln \lambda} e^{\pi + b w_{os}}\psi_s + \beta\sigma\sum_{j: I^C_{osj}=0} e^{\ln \lambda} e^{g_{osj}} - \beta\sigma\sum_{j: I^C_{ojs}=0} e^{\ln \lambda} e^{g_{ojs}} + \beta\sigma e^{\ln \lambda} e^{\pi}\psi_s \tag{6.92}$$

$$0 = e^{z_t} F_H\left(K_{t-1}, H_t\right) - q_t \tag{6.93}$$

$$0 = \theta E_t\left[e^{\Delta \ln \lambda_{t+1}}\left(e^{z_{t+1}} F_K\left(K_t, H_{t+1}\right) + 1 - \delta\right)\right] - e^{\Delta \ln \lambda_t} \tag{6.94}$$

$$K_t = (1-\delta) K_{t-1} + I_t, \tag{6.95}$$

$$e^{-\Delta \ln \lambda_t} \left\{ (1-\sigma) \sum_s e^{u_{ys}} \psi_s + e^{\ln V} \sum_s e^{u_{os}} \psi_s \right\} + I_t = e^{z_t} F(K_{t-1}, H_t), \tag{6.96}$$

$$H_t = 1 - e^{-\Delta \ln \lambda_t - \Delta \ln q_t} \left\{ (1-\sigma) \sum_s e^{n_{ys}} \psi_s + e^{\ln V} \sum_s e^{n_{os}} \psi_s \right\}, \tag{6.97}$$

$$e^{\ln V} = \sigma e^{\ln V} \sum_s e^{bw_{os}} \psi_s + (1-\sigma) \sigma \sum_s e^{bw_{ys}} \psi_s \tag{6.98}$$

Observe that equations (6.84)-(6.92) are identical to equations (6.44)-(6.52) and that equation (6.98) is identical to equation (6.58). As a consequence those equations hold by the definition of a deterministic steady state. Only equations (6.93)-(6.97) need to be verified to hold. However, by Corollary 4 we know that at a stationary equilibrium all aggregate variables coincide with those of the representative agent economy of Section 6.1. Thus, the term within brackets in equation (6.96) equals the deterministic steady state aggregate consumption e^u in the representative agent economy and the term within brackets in equation (6.97) equals the deterministic steady state aggregate leisure e^n in the representative agent economy. As a consequence, equations (6.93)-(6.97) are identical to equations (6.5)-(6.9) and they do hold since they hold in the representative agent economy.

This establishes that the guess given by equations (6.59)-(6.68) is correct.