Federal Reserve Bank of Chicago

Establishments Dynamics, Vacancies and Unemployment: A Neoclassical Synthesis

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Abstract: This paper develops a Walrasian equilibrium theory of establishment dynamics and matching frictions and uses it to analyze business cycle fluctuations. Two scenarios are considered: one in which the matching process is subject to congestion externalities and another in which it is not. The paper finds that the scenario with congestion externalities replicates U.S. business cycle dynamics much better than the scenario with efficient matching. Reallocation shocks improve the empirical behavior of the model in terms of microeconomic adjustments but have little consequences for aggregate dynamics.

*This paper originated in a conversation with Randall Wright and was heavily influenced by it. I have also benefited from the comments of Bjoern Bruegemann, Mario Cozzi, Nicolas Petrosky-Nadeau, Michael Elsby and numerous seminar participants. The views express here do not necessarily reflect the position of the Federal Reserve Bank of Chicago or the Federal Reserve System. Address: Federal Reserve Bank of Chicago, Research Department, 230 South LaSalle Street, Chicago, IL 60604. E-mail: mveracierto@frbchi.org. Phone: (312) 322-5695.

1 Introduction

The purpose of this paper is to evaluate if standard neoclassical theory can be used to explain the observed behavior of establishment dynamics, vacancies and unemployment both at growth and business cycle frequencies. To this end, the paper constructs a real business cycle model that blends three important strands in the literature: 1) the Hopenhayn and Rogerson (15) model of establishment dynamics, 2) the Mortensen and Pissarides (18) matching model, and 3) the Lucas and Prescott (16) islands model. A key feature of the model is that it fully relies on classical price theory: All prices, including that of labor, are determined in Walrasian markets.

The economy is populated by a representative household that values consumption and leisure. Output, which can be consumed or invested, is produced by a large number of spatially separated establishments that are subject to aggregate and idiosyncratic productivity shocks. The amount of hiring that an establishment can undertake is constrained by the number of recruitment opportunities that it has available. Unemployed workers can become employed only if they gain employment opportunities. Recruitment opportunities for establishments and employment opportunities for workers are jointly produced by a neoclassical recruitment technology that uses unemployed workers and the consumption good as inputs of production. Following the matching literature, the recruitment technology is allowed to be subject to production externalities: The total number of unemployed workers in the economy and the aggregate amount of recruitment expenditures affect its productivity. Any of the workers with employment opportunities can be hired by any of the establishments with recruitment opportunities.

The paper defines and fully characterizes a recursive competitive equilibrium for this economy. It shows that an equilibrium can be constructed by solving a social planning problem with side conditions. The social planner solves a standard utility maximization problem subject to feasibility constraints, except that it takes as given the total number of unemployed workers and the aggregate amount of recruitment expenditures that enter the recruitment technology as external effects. At equilibrium, these variables must be generated by the social planner's optimal decision rules. The recruitment opportunities and employment decision rules of the establishments are also characterized. In particular, they are shown to be of the (S,s) variety. This, together with the assumptions that the idiosyncratic productivity shocks take a finite number of values and that the aggregate productivity shocks are sufficiently small, implies that the distribution over establishment types has a finite support. As a consequence, the social planner's problem can be formulated in terms of a finite number of state and decision variables. This is an important result: Despite the model's complexity, simple linear-quadratic methods can be used for computing a recursive competitive equilibrium.

The paper then evaluates how well the model is able to explain the data. Two versions are considered: A version without external effects in the recruitment technology and a version with external effects. Both versions are calibrated to identical U.S. long-run observations. Some parameter values are closely related to the neoclassical growth model and are calibrated to reproduce similar observations (e.g. the capital/output ratio, the investment/output ratio, etc.). The rest of the parameters are chosen to reproduce observations on establishment dynamics (e.g. the size distribution of establishments, job creation and destruction rates, etc.), worker flows (e.g. the separation rates, the hazard rate from unemployment, etc.), and vacancies (e.g. the vacancy rate, recruitment costs, etc.). When an aggregate productivity shock of empirically relevant magnitude is introduced, the paper finds that the version without external effects in the recruitment technology fails to reproduce the data: The aggregate fluctuations that it generates are too small. On the contrary, the version with external effects generates aggregate fluctuations of reasonable magnitude. Thus the paper indicates that, when looked through the eyes of neoclassical theory, there is empirical support for the hypothesis of congestion externalities in the matching process.

While the paper has a strong empirical focus it also makes a theoretical contribution to the literature on equilibrium unemployment. This literature has been dominated by two main strands: the Mortensen-Pissarides (18) matching model and the Lucas-Prescott (16) islands model. The Mortensen-Pissarides model is extremely useful for analyzing vacancies and unemployment and has been extended to incorporate business cycle fluctuations (e.g. Andolfatto (4), Merz (17), Shimer (20), Hall (13), Hagedorn and Manovskii (9), etc.) and, more recently, establishment dynamics (e.g. Acemoglu and Hawkins (1), Cooper et. al (6), etc.). However, the model has a significant drawback: It introduces free parameters in the wage determination process. Even in the simplest version of the model it is unclear what value to use for the Nash bargaining parameter. In versions with aggregate fluctuations and establishment dynamics, the degrees of freedom multiply since it is possible for the Nash bargaining parameter to vary systematically with the state of the economy or of an individual establishment. The Lucas-Prescott model does not suffer from these difficulties since wages are determined in Walrasian markets.[1] However, there is no notion of vacancies in that model: Firms behave as if they could hire any number of workers at the island specific competitive wage rate. That is, firms do not need to undertake any type of active recruitment effort in order to fill their job openings. This paper avoids these limitations: By blending together the Mortensen-Pissarides model and the Lucas-Prescott model, it delivers a framework for analyzing vacancies and unemployment in which all prices are fully determined by preferences and technology. Incorporating the Hopenhayn-Rogerson model is also important since establishments dynamics are the counterpart to worker flows and vacancies. The result is a comprehensive theory of labor market dynamics.

The paper is organized as follows. Section 2 describes the economy. Section 3 describes a recursive competitive equilibrium. Section 4 characterizes a recursive competitive equilibrium and describes how to compute it. Section 5 calibrates the two versions of the model. Finally, Section 6 presents the results. An appendix provides proofs to the most important claims made in the paper.

2 The economy

The economy is endowed with a measure one of workers. A worker is a capital good that does not depreciate and can not be produced. During any period of time a worker can be in either of two states: *employed* or *unemployed*. Employed workers produce the consumption good while unemployed workers produce home goods. Employed workers can be freely transformed into unemployed workers. However, unemployed workers can only be transformed into employed workers using a costly technology. All workers are subject to an idiosyncratic productivity shock called a *quit* shock, that makes them temporarily unproductive as employed workers. A worker that quits needs to spend a full period of time unemployed before regaining his productive capacity. The probability that a worker quits at the beginning of the following period depends on his current employment status: It is equal to π_n if the worker is currently employed and it is equal to π_u if the worker is

[1] The Lucas-Prescott model has been used, among other things, to study the effects of labor market policies (e.g. Alvarez and Veracierto (3)), business cycle dynamics (e.g. Veracierto (23)), occupational mobility (e.g. Kambourov and Manovskii (11)), and rest unemployment (e.g. Alvarez and Shimer (2)).

currently unemployed.

The economy is populated by a representative household with preferences given by

$$\mathcal{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \varphi U_t \right] \right\}, \tag{1}$$

where C_t is consumption, U_t is the total number of unemployed workers, $\varphi > 0$, $\sigma > 0$ and $0 < \beta < 1$.

The consumption good is produced by a large number of establishments. Each establishment has a production function given by

$$y_t = e^{z_t} s_t F(n_t, k_t),$$

where z_t is an aggregate productivity shock, s_t is an idiosyncratic productivity shock, n_t is the number of employed workers, k_t is physical capital, and F is a continuously differentiable, strictly increasing, strictly concave and decreasing returns to scale production function that satisfies the Inada conditions. The idiosyncratic productivity shock s_t takes values in a finite set S and follows a Markov process with monotone transition matrix Q. Realizations of s_t are independent across establishments and $s_t = 0$ is an absorbing state. Since there are no fixed costs of operation, exit takes place only when the idiosyncratic productivity level becomes zero. In every period of time a measure ϱ of new establishments is exogenously born. Their distribution over initial productivity shocks is given by ψ. The aggregate productivity shock follows an AR(1) process given by

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}, \tag{2}$$

where $0 \leq \rho < 1$, and ε_{t+1} is i.i.d., normally distributed, with variance σ_ε^2 and zero mean.

The number of employed workers n_t at an establishment is given by

$$n_t = n_{t-1} + h_t - f_t,$$

where h_t are the gross employment increases (i.e. *hirings*) and f_t are the gross employment reductions (i.e. *firings*). All the workers that are fired become unemployed. Because of the exogenous quit of employed workers, f_t is effectively constrained as follows

$$\pi_n n_{t-1} \leq f_t \leq n_{t-1}.$$

The number of new hires h_t is limited by the number of *recruiting opportunities* j_t that the establishment has at the beginning of the period, i.e.

$$h_t \leq j_t. \tag{3}$$

Unemployed workers can become employed only if they are transformed into workers with *employment opportunities*. Workers with employment opportunities e_{t+1} and recruiting opportunities j_{t+1} are jointly produced using the following *recruitment technology*:

$$e_{t+1} = G(a_t, u_t, A_t, U_t), \tag{4}$$

$$j_{t+1} = H(a_t, u_t, A_t, U_t), \tag{5}$$

where a_t are recruitment expenditures (in the consumption good), u_t are unemployed workers, A_t is the aggregate amount of recruitment expenditures in the economy, and U_t is the total number of unemployed workers in the economy.[2]

The recruitment technology satisfies the following assumptions: 1) G and H are continuously differentiable, 2) G and H are increasing in (a_t, u_t), 3) G and H are homogenous of degree one with respect to (a_t, u_t) and homogeneous of degree zero with respect to (A_t, U_t), 4) G and H are concave in (a_t, u_t), 5) and G satisfies that

$$G\left(a_t, u_t, A_t, U_t\right) \leq u_t, \text{ for every } \left(a_t, u_t, A_t, U_t\right). \tag{6}$$

Observe from equation (6) that not all unemployed workers that enter the recruitment technology are transformed into workers with employment opportunities:

$$x_{t+1} = u_t - G\left(a_t, u_t, A_t, U_t\right),$$

is the number of *unsuccessful candidates* that the recruitment technology generates.

3 Recursive competitive equilibrium

The state of the economy is given by the quintuple (z, K, E, X, μ), where z is the aggregate productivity level, K is the aggregate stock of capital, E is the aggregate number of workers with employment opportunities, X is the aggregate number of unsuccessful candidates, $\mu\left(s, l \times j\right)$ is a measure of establishments over individual states (s, l, j), and (E, X, μ) satisfies that[3]

$$\int l \; \mu\left(s, dl \times dj\right) + E + X = 1. \tag{7}$$

There are three competitive sectors in the economy: a households sector, an establishments sector, and a recruitment industry.

Households earn income from renting capital to the establishments and from the aggregate profits made by the establishments sector.[4] They spend their income on consumption, on investment and on renting unemployed workers. The individual state of a household is the amount of capital that it owns κ. The household's problem is described by the following Bellman equation:

$$B(\kappa, z, K, E, X, \mu) = \max_{\{c,i,m\}} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \varphi m + \beta \mathcal{E}\left[B\left(\kappa', z', K', E', X', \mu'\right) \mid z\right] \right\} \tag{8}$$

subject to:

$$c + i + r^u\left(z, K, E, X, \mu\right) m \leq r^k\left(z, K, E, X, \mu\right) \kappa + \Pi\left(z, K, E, X, \mu\right), \tag{9}$$

$$\kappa' = \left(1 - \delta\right) \kappa + i \tag{10}$$

$$\left(K', E', X', \mu'\right) = L\left(z, K, E, X, \mu\right). \tag{11}$$

[2] Observe that when G_A, G_U, H_A or H_U are strictly positive, the recruitment technology is subject to production externalities.

[3] Equation (7) implies that either E or X could be removed from the aggregate state vector. However, this would complicate the definition of a recursive competitive equilibrium.

[4] Each household is assumed to own one share of each establishment in the economy.

where r^u is the rental rate of an unemployed worker, r^k is the rental rate of capital, Π are the aggregate profits made by the establishments sector, i is investment, m is the number of unemployed workers that the household rents, and L is the law of motion for the endogenous state of the economy. Equation (9) is the budget constraint of the household, and equation (10) is the law of motion for its stock of capital. The household's optimal decisions are $c = c(\kappa, z, K, E, X, \mu)$, $i = i(\kappa, z, K, E, X, \mu)$, and $m = m(\kappa, z, K, E, X, \mu)$ for consumption, investment and unemployed workers, respectively.

The establishments rent capital, purchase workers with employment opportunities (up to the number of recruitment opportunities that they have at the beginning of the period), sell unemployed workers (up to their previous-period employment level), and purchase next-period recruitment opportunities. The individual state of an establishment is given by a triple (s, l, j), where s is its current idiosyncratic productivity level, l is its previous-period employment level and j is its recruitment opportunities at the beginning of the period. The establishment's problem is described by the following Bellman equation:

$$
\begin{aligned}
W(s,l,j,z,K,E,X,\mu) = \max_{\{f,h,k,n,v\}} \quad & \{e^z sF(n,k) + p^u(z,K,E,X,\mu) f - p^e(z,K,E,X,\mu) h \\
& -r^k(z,K,E,X,\mu) k - p^v(z,K,E,X,\mu) v \\
& +\mathcal{E}\left[\sum_{s'} q(z,K,E,X,\mu,z') W(s',l',j',z',K',E',X',\mu') Q(s,s') \mid z\right]\}
\end{aligned}
\tag{12}
$$

subject to

$$n = l + h - f \tag{13}$$
$$\pi_n l \leq f \tag{14}$$
$$f \leq l \tag{15}$$
$$h \leq j \tag{16}$$
$$l' = n \tag{17}$$
$$j' = v \tag{18}$$
$$(K', E', X', \mu') = L(z, K, E, X, \mu). \tag{19}$$

where p^u is the price of an unemployed worker, p^e is the price of a worker with employment opportunities, p^v is the price of a next-period recruitment opportunity, $q(\cdot, z')$ is the price of an Arrow security that delivers one unit of the consumption good if the next-period aggregate productivity level is equal to z', n is the number of employed workers, k is the capital level, f are the firings, h are the hirings, and v are the purchases of next-period recruitment opportunities. The constraints (13)-(16) have been described in the previous section. The establishment's optimal decisions are $n = n(s,l,j,z,K,E,X,\mu)$, $k = k(s,l,j,z,K,E,X,\mu)$, $f = f(s,l,j,z,K,E,X,\mu)$, $h = h(s,l,j,z,K,E,X,\mu)$, and $v = v(s,l,j,z,K,E,X,\mu)$, for employed workers, capital, firings, hirings and next-period recruitment opportunities, respectively.

The recruitment companies sell workers with employment opportunities and next-period recruitment opportunities. They also buy and sell unemployed workers and rent them to the households sector. The individual state of a recruitment company is a pair (e, x), where e is its number of workers with employment opportunities at the beginning of the period, and x is its number of unsuccessful candidates. The problem of a recruitment company is given as follows:

$$\begin{aligned} R(e,x,z,K,E,X,\mu) = \max_{\{a,b,d,u\}} \quad & \{p^{e}(z,K,E,X,\mu)\,d + p^{v}(z,K,E,X,\mu)\,b \\ & + p^{u}(z,K,E,X,\mu)\,[x+e-d-u] + r^{u}(z,K,E,X,\mu)\,u - a \\ & + \mathcal{E}\left[q(z,K,E,X,\mu,z')\,R(e',x',z',K',E',X',\mu') \mid z\right]\} \end{aligned} \tag{20}$$

subject to

$$\begin{aligned} d &\leq (1-\pi_u)\,e \\ b &= H(a,u,A,U) \\ e' &= G(a,u,A,U) \\ x' &= u - G(a,u,A,U) \\ A &= A(z,K,E,X,\mu) \\ U &= U(z,K,E,X,\mu) \\ (K',E',X',\mu') &= L(z,K,E,X,\mu)\,. \end{aligned}$$

where d is the number of workers with employment opportunities that the recruitment company sells, b is the number of next-period recruitment opportunities that the recruitment company sells, u is the number of unemployed workers that the recruitment company owns, a are the expenditures that the recruitment company makes, A are the aggregate recruitment expenditures in the economy, and U is the aggregate number of unemployed workers. Observe that, since unemployed workers quit at the rate π_u, d cannot exceed $(1-\pi_u)\,e$. Also observe that the recruitment company can sell as unemployed workers all of its unsuccessful candidates x and any of its unsold workers with employment opportunities $e-d$. The recruitment company's optimal decisions are $a = a(e,x,z,K,E,X,\mu)$, $b = b(e,x,z,K,E,X,\mu)$, $d = d(e,x,z,K,E,X,\mu)$, and $u = u(e,x,z,K,E,X,\mu)$, for recruitment expenditures, next-period recruitment opportunities, sales of workers with employment opportunities, and unemployed workers, respectively.[5]

A recursive competitive equilibrium can now be defined.

Definition 1 *A recursive competitive equilibrium (RCE) is a set of value functions* $B(\kappa,z,K,E,X,\mu)$, $W(s,l,j,z,K,E,X,\mu)$, $R(e,x,z,K,E,X,\mu)$, *a set of individual decision rules* $c(\kappa,z,K,E,X,\mu)$, $i(\kappa,z,K,E,X,\mu)$, $m(\kappa,z,K,E,X,\mu)$, $n(s,l,j,z,K,E,X,\mu)$, $k(s,l,j,z,K,E,X,\mu)$, $f(s,l,j,z,K,E,X,\mu)$, $h(s,l,j,z,K,E,X,\mu)$, $v(s,l,j,z,K,E,X,\mu)$, $a(e,x,z,K,E,X,\mu)$, $b(e,x,z,K,E,X,\mu)$, $d(e,x,z,K,E,X,\mu)$, $u(e,x,z,K,E,X,\mu)$, *a pair of aggregate decision rules* $A(z,K,E,X,\mu)$, $U(z,K,E,X,\mu)$, *an aggregate law of motion* $L(z,K,E,X,\mu)$, *an aggregate profits function* $\Pi(z,K,E,X,\mu)$, *and a set of price functions* $r^{k}(z,K,E,X,\mu)$, $r^{u}(z,K,E,X,\mu)$, $p^{u}(z,K,E,X,\mu)$, $p^{e}(z,K,E,X,\mu)$, $p^{v}(z,K,E,X,\mu)$, $q(z,K,E,X,\mu,z')$, *such that:*

(i) the value function $B(\kappa,z,K,E,X,\mu)$ *solves the households' Bellman equation and* $c(\kappa,z,K,E,X,\mu)$, $i(\kappa,z,K,E,X,\mu)$, *and* $m(\kappa,z,K,E,X,\mu)$ *are the associated decision rules,*

[5] Sections 1.1-1.3 in the Technical Appendix provide first-order and envelope conditions for the household, establishment and recruitment company's decision problems, respectively.

(ii) the value function $W(s,l,j,z,K,E,X,\mu)$ solves the establishments' Bellman equation and $n(s,l,j,z,K,E,X,\mu)$, $k(s,l,j,z,K,E,X,\mu)$, $f(s,l,j,z,K,E,X,\mu)$, $h(s,l,j,z,K,E,X,\mu)$, and $v(s, l,j,z,K,E,X,\mu)$ are the associated decision rules,

(iii) the value function $R(e,x,z,K,E,X,\mu)$ solves the Bellman equation of the recruitment companies and $a(e,x,z,K,E,X,\mu)$, $b(e,x,z,K,E,X,\mu)$, $d(e,x,z,K,E,X,\mu)$, and $u(e,x,z,K,E,X,\mu)$ are the associated decision rules,

(iv) the prices of the Arrow securities satisfy that

$$q(z,K,E,X,\mu,z') = \beta \frac{c\left(K,z,K,E,X,\mu\right)^{\sigma}}{c\left(K',z',K',E',X',\mu'\right)^{\sigma}},$$

where $(K',E',X',\mu') = L\left(z,K,E,X,\mu\right)$,

(v) the capital rental market clears, i.e.

$$K = \sum_s \int k(s,l,j,z,K,E,X,\mu)\mu\left(s,dl\times dj\right)$$

(vi) the rental market for unemployed workers clears, i.e.

$$u(E,X,z,K,E,X,\mu) = m\left(K,z,K,E,X,\mu\right)$$

(vii) the ownership market for unemployed workers clears, i.e.

$$\begin{aligned} u(E,X,z,K,E,X,\mu) &= X + E - d(E,X,z,K,E,X,\mu) \\ &\quad + \sum_s \int f(s,l,j,z,K,E,X,\mu)\mu\left(s,dl\times dj\right) \end{aligned}$$

(viii) the market for workers with employment opportunities clears, i.e.

$$d(E,X,z,K,E,X,\mu) = \sum_s \int h(s,l,j,z,K,E,X,\mu)\mu\left(s,dl\times dj\right)$$

(ix) the market for next-period recruitment opportunities clears, i.e.

$$b(E,X,z,K,E,X,\mu) = \sum_s \int v(s,l,j,z,K,E,X,\mu)\mu\left(s,dl\times dj\right)$$

(x) the market for the consumption good clears, i.e.

$$\begin{aligned} &c\left(K,z,K,E,X,\mu\right) + i\left(K,z,K,E,X,\mu\right) + a(E,X,z,K,E,X,\mu) \\ = &\sum_s \int e^{z} sF\left[n(s,l,j,z,K,E,X,\mu),k(s,l,j,z,K,E,X,\mu)\right]\mu\left(s,dl\times dj\right) \end{aligned}$$

(xi) the aggregate decision rules are generated by the optimal individual decisions, i.e.

$$A\left(z,K,E,X,\mu\right) = a(E,X,z,K,E,X,\mu)$$

$$U\left(z,K,E,X,\mu\right) = u(E,X,z,K,E,X,\mu)$$

(xii) the aggregate law of motion is generated by the optimal individual decisions, i.e.

$$(K',E',X',\mu') = L\left(z,K,E,X,\mu\right)$$

is given as follows:

$$K' = (1-\delta) K + i (K, z, K, E, X, \mu)$$

$$E' = G [a(E, X, z, K, E, X, \mu), u(E, X, z, K, E, X, \mu), A (z, K, E, X, \mu), U (z, K, E, X, \mu)]$$

$$X' = u(E, X, z, K, E, X, \mu) - E'$$

$$\mu' (s', \mathcal{L} \times \mathcal{J}) = \sum_s \int_{\mathcal{B}(s, \mathcal{L}\times\mathcal{J})} Q (s, s') \mu (s, dl \times dj) + \varrho\psi (s') \mathcal{I} (\mathcal{L} \times \mathcal{J})$$

where

$$\mathcal{B} (s, \mathcal{L} \times \mathcal{J}) = \{(l, j) : n(s, l, j, z, K, E, X, \mu) \in \mathcal{L} \text{ and } v(s, l, j, z, K, E, X, \mu) \in \mathcal{J}\}$$

and where $\mathcal{I} (\mathcal{L} \times \mathcal{J})$ *is an indicator function which takes a value of one if* $(0, 0) \in \mathcal{L} \times \mathcal{J}$*, and a value of zero otherwise.*[6]

4 Characterization and computation of a RCE

Due to the external effects in the recruitment technology a RCE is generally inefficient and must be solved for directly. The high dimensionality of the sate space makes this a daunting task. However, it can be simplified considerably. This section provides a solution method that can be easily implemented in actual computations. The method relies on two key properties of a RCE. First, that it can be characterized as the solution to a dynamic programming problem with side conditions.[7] Second, that in a neighborhood of the deterministic steady state, the dynamic programming problem can be represented as having a finite number of state and decision variables. The following subsections explain these properties in detail.

4.1 The myopic social planner's problem

Consider the problem of a social planner that seeks to maximize utility subject to the economy's feasibility constraints. However, the social planner is *myopic* in the sense that he does not fully internalize the effects of his decisions on the output produced by the recruitment technology. In particular, the myopic social planner takes the recruitment technology as being the following:

$$E' = G \left(A, U, \hat{A}, \hat{U}\right)$$

$$J' = H \left(A, U, \hat{A}, \hat{U}\right)$$

where E' are next-period workers with employment opportunities, J' are next-period recruitment opportunities, A are recruitment expenditures, U are unemployed workers, and $\hat{A}$ and $\hat{U}$ are exogenous productivity shocks. The shocks $\hat{A}$ and $\hat{U}$ evolve according to the following stochastic process:

$$\hat{A} = \hat{A} \left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$$

[6] It is straightforward to verify that if (E, X, μ) satisfy equation (7), then (E', X', μ') also satisfy equation (7).

[7] The dynamic optimization problem depends on exogenous parameters, wich in turn depend on the solution to the dynamic optimization problem. Finding a RCE is then reduced to solving a fixed point problem on those parameters. This basic strategy for solving for a competitive equilibrium in an economy with externalities is already familiar to the literature, though in much simpler contexts (e.g. Kehoe, Levine and Romer (12), Jones and Manuelli (10), etc.).

$$\hat{U} = \hat{U}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$$

$$\left(\hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) = \hat{L}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$$

where z is the aggregate productivity level, and $\left(\hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$ are variables that lie in the same space as (K, E, X, μ).

The state of the myopic social planner is then given by the state of the economy (z, K, E, X, μ) and by the variables $\left(\hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, which are sufficient statistics for predicting the future behavior of $\hat{A}$ and $\hat{U}$. The problem of the myopic social planner facing a stochastic process $\left(\hat{A}, \hat{U}, \hat{L}\right)$ is described by the following Bellman equation:

$$V(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{\frac{C^{1-\sigma} - 1}{1-\sigma} + \varphi U + \beta \mathcal{E}\left[V\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) \mid z\right]\right\} \tag{21}$$

subject to

$$C + I + A \leq \sum_s \int e^z sF\left[n(s,l,j), k(s,l,j)\right] \mu\left(s, dl \times dj\right) \tag{22}$$

$$\sum_s \int k(s,l,j)\mu\left(s, dl \times dj\right) \leq K \tag{23}$$

$$\sum_s \int v(s,l,j)\mu\left(s, dl \times dj\right) \leq H\left(A, U, \hat{A}, \hat{U}\right) \tag{24}$$

$$U \leq X + E - \sum_s \int h(s,l,j)\mu\left(s, dl \times dj\right) + \sum_s \int f(s,l,j)\mu\left(s, dl \times dj\right) \tag{25}$$

$$\sum_s \int h(s,l,j)\mu\left(s, dl \times dj\right) \leq \left(1 - \pi_u\right) E \tag{26}$$

$$n\left(s,l,j\right) = l + h\left(s,l,j\right) - f\left(s,l,j\right) \tag{27}$$

$$h\left(s,l,j\right) \leq j \tag{28}$$

$$\pi_n l \leq f\left(s,l,j\right) \tag{29}$$

$$f\left(s,l,j\right) \leq l \tag{30}$$

$$K' = \left(1 - \delta\right) K + I \tag{31}$$

$$E' = G\left(A, U, \hat{A}, \hat{U}\right) \tag{32}$$

$$X' = U - G\left(A, U, \hat{A}, \hat{U}\right) \tag{33}$$

$$\mu'\left(s', \mathcal{L} \times \mathcal{J}\right) = \sum_s \int_{\{(l,j):\ n(s,l,j) \in \mathcal{L} \text{ and } v(s,l,j) \in \mathcal{J}\}} Q\left(s, s'\right) \mu\left(s, dl \times dj\right) + \varrho \psi\left(s'\right) \mathcal{I}\left(\mathcal{L} \times \mathcal{J}\right) \tag{34}$$

$$\hat{A} = \hat{A}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) \tag{35}$$

$$\hat{U} = \hat{U}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) \tag{36}$$

$$\left(\hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) = \hat{L}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right). \tag{37}$$

where equations (22)-(34) are feasibility constraints and equations (35)-(37) describe the stochastic process that $\hat{A}$ and $\hat{U}$ follow over time.[8] The myopic social planner's decision rules are $C = C^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $I = I^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $n = n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $k = k^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $f = f^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $h = h^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $v = v^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $U = U^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $A = A^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, for consumption, investment, establishment employment, establishment capital, establishment firings, establishment hirings, establishment recruitment opportunities, unemployment and recruitment expenditures, respectively.

The following proposition provides a characterization of the decision rules to the myopic social planner's problem.

Proposition 2 *Let $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ be the solution to the MSP's with exogenous stochastic process $\left(\hat{A}, \hat{U}, \hat{L}\right)$. Then, there exist thresholds $\underline{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $\bar{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ and $\bar{v}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ and a shadow capital price function $r^k\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$ such that, for every $s > 0$ and $l + j > 0$:*

$$n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{\begin{array}{c} \min\left\{(1-\pi_n)\, l + j, \underline{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right\}, \\ \min\left\{(1-\pi_n)\, l, \bar{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right\} \end{array}\right\},$$

$$v^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{\bar{v}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) - (1-\pi_n) n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), 0\right\},$$

$$h^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) - l, 0\right\}$$

$$f^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{l - n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), 0\right\}$$

$$e^z s F_k\left[n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), k^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right] = r^k\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right),$$

Proof. *In the economy in which $\left(\hat{A}, \hat{U}, \hat{L}\right)$ truly represent exogenous productivity shocks to the recruitment technology, the Welfare Theorems apply. In this case the problem described by equation (21) is the social planner's problem and its solution can be decentralized as a recursive competitive equilibrium in which prices are functions of the state $(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$. The claim then follows from characterizing the optimal decision rules to the associated establishments' problem given by equation (12).*[9] ■

This proposition is important because it can be used to reduce the dimensionality of the decision variables in the myopic social planner's problem: Instead of choosing functions n^m, v^m, h^m, f^m and k^m defined over the infinite number of triples (s, l, j), the myopic social planner can be restricted to choose thresholds $\underline{n}^m$, $\bar{n}^m$ and $\bar{v}^m$ defined over the finite number of singletons s.

The next proposition states that if the solution to a myopic planner's problem satisfies certain side conditions, then it is a RCE.

[8] Observe that if equations (35)-(37) were substituted by $\hat{A} = A$ and $\hat{U} = U$, the solution to this planning problem would be the Pareto optimal allocation.

[9] For details see Sections 2 in the Technical Appendix.

Proposition 3 *Let* $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ *be the solution to the myopic social planner's problem with exogenous stochastic process* $\left(\hat{A}, \hat{U}, \hat{L}\right)$.

Suppose that,

$$\hat{A}(z, K, E, X, \mu) = A^m(z, K, E, X, \mu, K, E, X, \mu),$$

$$\hat{U}(z, K, E, X, \mu) = U^m(z, K, E, X, \mu, K, E, X, \mu).$$

In addition, suppose that

$$\left(\hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) = \hat{L}(z, K, E, X, \mu).$$

satisfies that

$$\hat{K}' = (1-\delta) K + I^m(z, K, E, X, \mu, K, E, X, \mu),$$

$$\begin{aligned} \hat{E}' = G \quad & [A^m(z, K, E, X, \mu, K, E, X, \mu), U^m(z, K, E, X, \mu, K, E, X, \mu), \\ & A^m(z, K, E, X, \mu, K, E, X, \mu), U^m(z, K, E, X, \mu, K, E, X, \mu)], \end{aligned}$$

$$\hat{X}' = U^m(z, K, E, X, \mu, K, E, X, \mu) - \hat{E}',$$

$$\hat{\mu}'(s', \mathcal{L} \times \mathcal{J}) = \sum_s \int_{\mathcal{B}(s, \mathcal{L} \times \mathcal{J})} Q(s, s') \mu(s, dl \times dj) + \varrho \psi(s') \mathcal{I}(\mathcal{L} \times \mathcal{J}),$$

where

$$\mathcal{B}(s, \mathcal{L} \times \mathcal{J}) = \{(l, j) : n^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \in \mathcal{L} \text{ and } v^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \in \mathcal{J}\}.$$

Then, there exists a RCE $\{B,\ W,\ R,\ c,\ i,\ m,\ n,\ k,\ f,\ h,\ v,\ a,\ b,\ d,\ u,\ A,\ U,\ L,\ \Pi,\ r^k,\ r^u,\ p^u,\ p^e,\ p^v,\ q\}$ *such that*

$$\begin{aligned} c(K, z, K, E, X, \mu) &= C^m(z, K, E, X, \mu, K, E, X, \mu) \\ i(K, z, K, E, X, \mu) &= I^m(z, K, E, X, \mu, K, E, X, \mu) \\ n(s, l, j, z, K, E, X, \mu) &= n^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\ f(s, l, j, z, K, E, X, \mu) &= f^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\ h(s, l, j, z, K, E, X, \mu) &= h^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\ v(s, l, j, z, K, E, X, \mu) &= v^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\ A(z, K, E, X, \mu) &= A^m(z, K, E, X, \mu, K, E, X, \mu) \\ U(z, K, E, X, \mu) &= U^m(z, K, E, X, \mu, K, E, X, \mu). \end{aligned}$$

Proof. It follows from comparing the necessary and sufficient conditions for a RCE with the necessary and sufficient conditions to the myopic social planner's problem.[10] ■

[10] For the details, see Section 3.4 in the Technical Appendix.

4.2 State space characterization

When there are no aggregate productivity shocks to the economy (i.e. when z is identical to zero), a deterministic steady state can be defined. In particular, a *myopic steady state* is given by an aggregate state $(K^*, E^*, X^*, \mu^*, \hat{K}^*, \hat{E}^*, \hat{X}^*, \hat{\mu}^*)$ that replicates itself under the myopic planner's optimal decision rules.[11] Characterizing the invariant distribution μ^* of a myopic steady state will turn to be crucial for characterizing the state-space when the economy is subject to small aggregate productivity shocks.

From equation (34) and Proposition 2, observe that the invariant distribution μ^* must satisfy the following equation:

$$\mu^*(s', \mathcal{L} \times \mathcal{J}) = \sum_s \int_{\{(l,j): n^*(s,l,j) \in \mathcal{L} \text{ and } v^*(s,l,j) \in \mathcal{J}\}} Q(s, s') \mu^*(s, dl \times dj) + \varrho \psi(s') \mathcal{I}(\mathcal{L} \times \mathcal{J}),$$

where

$$n^*(s,l,j) = \max \left\{ \begin{array}{c} \min\{(1-\pi_n) l + j, \underline{n}^*(s)\}, \\ \min\{(1-\pi_n) l, \bar{n}^*(s)\} \end{array} \right\}, \tag{38}$$

$$v^*(s,l,j) = \max\{\bar{v}^*(s) - (1-\pi_n) n^*(s,l,j), 0\}. \tag{39}$$

The following proposition characterizes a support to the invariant distribution μ^* in terms of the finite number of steady state thresholds $\underline{n}^*$, $\bar{n}^*$ and $\bar{v}^*$.[12]

Proposition 4 *Let M be a natural number satisfying that*

$$(1-\pi_n)^M \max\{\bar{n}^*(s_{\max}), \bar{v}^*(s_{\max})\} < \min\{\underline{n}^*(s_{\min}), \bar{v}^*(s_{\min})\}. \tag{40}$$

Define the set $\mathcal{N}^*$ as follows:

$$\mathcal{N}^* = \left\{ \bigcup_{s \in S} \bigcup_{k=0}^{M-1} \left\{ (1-\pi_n)^k \underline{n}^*(s), (1-\pi_n)^k \bar{n}^*(s), (1-\pi_n)^k \bar{v}^*(s) \right\} \right\} \cup \{0\}.$$

Then, the set

$$\mathcal{P}^* = \left\{ (s,l,j) : s \in S,\ l \in \mathcal{N}^*,\ \text{and } j \in \bigcup_{s' \in S} \{\max[\bar{v}^*(s') - (1-\pi_n) l, 0]\} \right\} \cup \left\{ \bigcup_{s \in S} \{(s,0,0)\} \right\}$$

is a support of the invariant distribution μ^*.

Proof. *See Appendix A.* ■

Observe that Proposition 4 not only constructs a support $\mathcal{P}^*$ for the invariant distribution μ^*, but determines that it is a finite set.

In order to analyze off-steady state dynamics it will be useful to define $\underline{n}_t$, $\bar{n}_t$, and $\bar{v}_t$, as the threshold functions chosen at date t. In addition, it will be useful to define the following minimum distance:

$$\varepsilon = \min \ |a - b| \tag{41}$$

[11] From Proposition 3 we know that if $(K^*, E^*, X^*, \mu^*) = (\hat{K}^*, \hat{E}^*, \hat{X}^*, \hat{\mu}^*)$, this myopic steady-state constitutes a steady-state equilibrium. See Sections 4.1 and 4.3 in the Technical Appendix for explicit steady state equillibrium conditions and a computational algorithm.

[12] In the statement of the proposition $s_{\max}$ and $s_{\min}$ denote the largest and smallest positive values for s, respectively.

subject to

$$a, b \in \mathcal{D}^* \text{ and } a \neq b,$$

where

$$\mathcal{D}^* = \mathcal{N}^* \cup \left\{ \underset{s \in S}{\cup} \left\{ (1-\pi_n)^M \underline{n}^*(s), (1-\pi_n)^M \bar{n}^*(s), (1-\pi_n)^M \bar{v}^*(s) \right\} \right\}.$$

The following proposition characterizes the distribution μ_{t+1} under the assumptions that μ_t and the finite history of thresholds $\{\underline{n}_{t-k}, \bar{n}_{t-k}, \bar{v}_{t-k}\}_{k=0}^{M+1}$ are sufficiently close to their steady-state counterparts.

Proposition 5 *Let M be defined by equation (40) and ε by equation (41).*

Suppose that

$$\left|\underline{n}_{t-k}(s) - \underline{n}^*(s)\right| < \varepsilon/2, \tag{42}$$

$$\left|\bar{n}_{t-k}(s) - \bar{n}^*(s)\right| < \varepsilon/2, \tag{43}$$

$$\left|\bar{v}_{t-k}(s) - \bar{v}^*(s)\right| < \varepsilon/2, \tag{44}$$

for every s and every $0 \leq k \leq M+1$.

Suppose that the distribution μ_t has a finite support $\mathcal{P}_t$ given by

$$\mathcal{P}_t = \left\{ (s,l,j) : s \in S,\ l \in \mathcal{N}_t,\ \text{and } j \in \underset{s' \in S}{\cup} \{\max[\bar{v}_{t-1}(s') - (1-\pi_n) l, 0]\} \right\} \cup \left\{ \underset{s \in S}{\cup} \{(s,0,0)\} \right\} \tag{45}$$

where

$$\mathcal{N}_t = \left\{ \underset{s \in S}{\cup} \overset{M-1}{\underset{k=0}{\cup}} \left\{ (1-\pi_n)^k \underline{n}_{t-k-1}(s), (1-\pi_n)^k \bar{n}_{t-k-1}(s), (1-\pi_n)^k \bar{v}_{t-k-2}(s) \right\} \right\} \cup \{0\}. \tag{46}$$

In addition, suppose that for every $(s,l,j) \in \mathcal{P}_t$:

$$\mu_t(s,l,j) = \mu^*(s,l^*,j^*), \tag{47}$$

where (s,l^*,j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l-l^*| < \varepsilon/2$ and $|j-j^*| < \varepsilon/2 + (1-\pi)\varepsilon/2$.

Then, the distribution μ_{t+1} has a finite support $\mathcal{P}_{t+1}$ given by

$$\mathcal{P}_{t+1} = \left\{ (s,l,j) : s \in S,\ l \in \mathcal{N}_{t+1},\ \text{and } j \in \underset{s' \in S}{\cup} \{\max[\bar{v}_t(s') - (1-\pi_n) l, 0]\} \right\} \cup \left\{ \underset{s \in S}{\cup} \{(s,0,0)\} \right\}$$

where

$$\mathcal{N}_{t+1} = \left\{ \underset{s \in S}{\cup} \overset{M-1}{\underset{k=0}{\cup}} \left\{ (1-\pi_n)^k \underline{n}_{t-k}(s), (1-\pi_n)^k \bar{n}_{t-k}(s), (1-\pi_n)^k \bar{v}_{t-k-1}(s) \right\} \right\} \cup \{0\}.$$

Moreover, for every $(s,l,j) \in \mathcal{P}_{t+1}$:

$$\mu_{t+1}(s,l,j) = \mu^*(s,l^*,j^*)$$

where (s,l^*,j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l-l^*| < \varepsilon/2$ and $|j-j^*| < \varepsilon/2 + (1-\pi)\varepsilon/2$.

Proof. *See Appendix A.* ■

Proposition 5 plays a crucial role in the solution method to be described below. It implies that if the economy starts at the deterministic steady-state at $t=0$ and the thresholds $\underline{n}_t$, $\bar{n}_t$ and $\bar{v}_t$ thereafter fluctuate within a sufficiently small neighborhood of their steady state values, then the distribution μ_t will always have a finite support $\mathcal{P}_t$ determined by the finite history of thresholds $\left\{\underline{n}_{t-k}, \bar{n}_{t-k}, \bar{v}_{t-k}\right\}_{k=1}^{M+1}$ (equations 45 and 46) and its mass at each point in $\mathcal{P}_t$ will be given by the mass of the invariant distribution μ^* at the corresponding point in $\mathcal{P}^*$ (equation 47). As a result the state to the myopic planner problem can be defined in terms of the finite history of thresholds $\left\{\underline{n}_{t-k}, \bar{n}_{t-k}, \bar{v}_{t-k}\right\}_{k=1}^{M+1}$ instead of the distribution μ_t.

4.3 Solution method

This section redefines the myopic social planner's problem so that standard solution methods can be applied. For this purpose, it will be convenient to return to a recursive formulation and define $\underline{n}_k$, $\bar{n}_k$ and $\bar{v}_k$ as the thresholds that were chosen k periods ago (relative to the current period).

Recall from Section 4.2 that the finite history $\{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}$ can be used to construct the current distribution μ (as long as fluctuations are sufficiently small). Moreover, Proposition 2 states that the current thresholds $(\underline{n}_0, \bar{n}_0, \bar{v}_0)$ fully describe the employment rule n, the vacancies rule v, the hiring rule h and the firing rule f. In turn, the employment decision rule n and the aggregate stock of capital K are sufficient for determining the capital allocation rule k.[13] This suggests that the state vector $(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ in the myopic planner's problem can be replaced by the vector $(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}, \hat{K}, \hat{E}, \{\widehat{\underline{n}}_k, \widehat{\bar{n}}_k, \widehat{\bar{v}}_k\}_{k=1}^{M+1})$ and that the decision variables (k, n, v, h, f) can be replaced by $(\underline{n}_0, \bar{n}_0, \bar{v}_0)$.[14]

Also, observe from equation (32) that A can be written as $A = g_1\left(E', U, \hat{A}, \hat{U}\right)$ for some differentiable function g_1 and, since at equilibrium $A = \widehat{A}$ and $U = \widehat{U}$, that $\hat{A}$ can be written as $\hat{A} = g_2\left(\widehat{E}', \widehat{U}\right)$ for some differentiable function g_2. Moreover, at equilibrium we have that $U = 1 - \int n \, d\mu$ and that $\widehat{U} = 1 - \int \widehat{n} \, d\widehat{\mu}$. Substituting these expressions and equations (22)-(33) into the return function in equation (21), the myopic planner's problem can then be written as follows:[15]

$$
\begin{aligned}
&V\left(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}, \hat{K}, \hat{E}, \{\widehat{\underline{n}}_k, \widehat{\bar{n}}_k, \widehat{\bar{v}}_k\}_{k=1}^{M+1}\right) \qquad (48)\\
&\quad = \max\left\{R\left(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}, \hat{K}, \hat{E}, \{\widehat{\underline{n}}_k, \widehat{\bar{n}}_k, \widehat{\bar{v}}_k\}_{k=1}^{M+1}, K', E', \underline{n}_0, \bar{n}_0, \bar{v}_0, \widehat{E}', \widehat{\underline{n}}_0, \widehat{\bar{n}}_0, \widehat{\bar{v}}_0\right)\right.\\
&\qquad \left. + \beta\mathcal{E}\left[V\left(z', K', E', \{\underline{n}'_k, \bar{n}'_k, \bar{v}'_k\}_{k=1}^{M+1}, \hat{K}', \hat{E}', \left\{\widehat{\underline{n}}'_k, \widehat{\bar{n}}'_k, \widehat{\bar{v}}'_k\right\}_{k=1}^{M+1}\right) \mid z\right]\right\}
\end{aligned}
$$

subject to

$$\underline{n}'_k = \underline{n}_{k-1}, \text{ for } k = 1, ..., M+1 \qquad (49)$$

$$\bar{n}'_k = \bar{n}_{k-1}, \text{ for } k = 1, ..., M+1 \qquad (50)$$

$$\bar{v}'_k = \bar{v}_{k-1}, \text{ for } k = 1, ..., M+1 \qquad (51)$$

[13] Since capital is freely movable, the myopic social planner allocates the aggregate stock of capital K to equate the marginal producitivity of capital across all types of islands $(s, l.j)$, subject to the feasibility constraint (23).

[14] The variables X and $\widehat{X}$ can be removed from the state vector because they are actually redundant (see equation 7).

[15] Observe that equation (24) must be used to remove some vacancy threshold $\bar{v}$ (e.g. $\bar{v}(s_{\min})$) from the formulation of the problem, since it always hold with equality. Similarly, when the deterministic steady state is such that equation (26) holds with equality, it must be used to remove some lower employment thresshold $\underline{n}$ (e.g. $\underline{n}(s_{\min})$) from the formulation of the problem. Otherwise, equation (26) must be ignored.

$$\left(\hat{K}', \hat{E}', \left\{\underline{\widehat{n}}'_k, \widehat{\bar{n}}'_k, \widehat{\bar{v}}'_k\right\}_{k=1}^{M+1}\right) = \hat{L}\left(z, \hat{K}, \hat{E}, \left\{\underline{\widehat{n}}_k, \widehat{\bar{n}}_k, \widehat{\bar{v}}_k\right\}_{k=1}^{M+1}\right), \tag{52}$$

where the vector of decision variables is $\left(K', E', \{\underline{n}'_k, \bar{n}'_k, \bar{v}'_k\}_{k=1}^{M+1}\right)$.

Let the optimal decision rule to the above problem be given by

$$\left(K', E', \{\underline{n}'_k, \bar{n}'_k, \bar{v}'_k\}_{k=1}^{M+1}\right) = D\left(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}, \hat{K}, \hat{E}, \left\{\underline{\widehat{n}}_k, \widehat{\bar{n}}_k, \widehat{\bar{v}}_k\right\}_{k=1}^{M+1}\right).$$

The condition for a RCE in Proposition 3 then becomes:

$$\hat{L}\left(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}\right) = D\left(z, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}, K, E, \{\underline{n}_k, \bar{n}_k, \bar{v}_k\}_{k=1}^{M+1}\right). \tag{53}$$

Observe that there are a finite number of arguments to the return function in equation (48) and that all their values are strictly positive at the deterministic steady state (except for the aggregate productivity level z). Since R is differentiable, a Taylor expansion at the deterministic steady state can then be performed to obtain a quadratic objective function. Since the constraints in equations (49)-(51) are linear, this delivers a standard linear-quadratic RCE structure that can be solved using standard methods (e.g. Hansen and Prescott (14)).[16] The linear decision rule D thus obtained is a good local approximation and, as long as fluctuations in the aggregate productivity shock z are small, it can be used to simulate and analyze equilibrium business cycle fluctuations.

5 Calibration

Throughout the rest of the paper the recruitment technology will be given a matching function interpretation in which employment and recruitment opportunities are produced in pairs at the aggregate level. In particular, the recruitment technology will be restricted to satisfy that

$$G(A, U, A, U) = H(A, U, A, U), \tag{54}$$

for every (A, U).

Two version of the model economy are considered: One where the matching technology is subject to congestion externalities and another where it isn't. The matching technology with congestion externalities is given by

$$G(a, u, A, U) = u\frac{A}{[U^\phi + A^\phi]^{\frac{1}{\phi}}}, \tag{55}$$

$$H(a, u, A, U) = a\frac{U}{[U^\phi + A^\phi]^{\frac{1}{\phi}}}, \tag{56}$$

and the matching technology with no externalities is given by

$$G(a, u, A, U) = H(a, u, A, U) = \frac{u.a}{[u^\phi + a^\phi]^{\frac{1}{\phi}}}. \tag{57}$$

[16]Strictly speaking, the linear-quadratic structure is obtained only when the aggregate law of motion $\hat{L}$ in equation (52) is linear. However, this will be true in equilibrium. In fact, Hansen and Prescott (14) update the linear law of motion $\hat{L}$ at each value function iteration by imposing the RCE condition (53) on the linear decision rule D obtained from the iteration.

Both matching technologies satisfy equation (54) and all the assumptions made in Section 2. Moreover, they both aggregate into a standard den Haan-Ramey-Watson (8) matching function

$$G(A,U,A,U) = \frac{U.A}{[U^{\phi}+A^{\phi}]^{\frac{1}{\phi}}}. \tag{58}$$

Also, observe that the matching technology with congestion externalities given by equations (55) and (56) captures a standard assumption in the matching literature: That the aggregate market-tightness ratio A/U determines the rate at which individual unemployed workers find employment opportunities and the rate at which individual help-wanted ads find recruitment opportunities.

The rest of this section calibrates the steady states of both versions of the model economy to identical long-run U.S. observations. Before proceeding it will be necessary to select a model time-period that is both convenient and consistent with observations.

The Job Openings and Labor Turnover Survey (JOLTS) conducted by the Bureau of Labor Statistics is an important source of information for two key features of the model: the creation of recruitment opportunities (i.e. job openings) and the worker turnover process. JOLTS, which is a monthly survey of continuing nonagricultural establishments, defines job openings as positions for which there is work available, for which a job could start within 30 days, and for which there is an active recruitment effort taking place (such as advertisement in newspapers, radio and television, posting "help wanted signs", interviewing candidates, etc.). Job openings are measured on the last business day of the month. On the contrary, hirings, which are defined as all additions to the establishments' payrolls, are measured over the entire month. The *vacancy yield rate* defined as the average monthly ratio of hirings to job openings over the entire period 2000-2005 is equal to 1.3 (see Davis et. al, (7)).

Since hirings cannot exceed recruitment opportunities in the model economy (see equation 3), a vacancy yield rate greater than one can only be obtained through time aggregation. This suggests calibrating to a short time period. However, computational convenience requires making the time period as large as possible. The largest time period consistent with the above observation is 3 weeks. The reason is simple: if total hirings turned out to be approximately equal to total recruitment opportunities, a monthly vacancy yield rate close to 1.3 would be obtained from the simple fact that a month contains 4/3 three-weeks periods. Observe that, since equation (54) implies that recruitment opportunities are equal to E, equation (26) indicates that a small π_u is a necessary condition for total hirings to be close to total recruitment opportunities. In what follows the time period will thus be tentatively selected to be 3 weeks and π_u will be set to zero.[17] Moreover, it will be assumed that total hirings are approximately equal to total recruitment opportunities. Later on it will be verified that this assumption is correct and that the monthly vacancy yield rate obtained is indeed consistent with the JOLTS measurement.

The next issue that must be addressed is what actual measure of capital should the model capital correspond to. Since the focus is on establishment level dynamics, it seems natural to abstract from capital components such as land, residential structures, and consumer durables. The empirical counterpart for capital is then identified with plant, equipment, and

[17] Observe that, since establishments invest in recruitment opportunities one period in advance and some of them end-up transiting to lower idiosyncratic productivity levels (or even exiting), not all existing recruitment opportunities end-up being exercised. Thus, when π_u is equal to zero equation (26) holds with strict inequality. This in turn implies that p^e must be equal to p^u, since owners of employable workers must be made indifferent between selling them or keeping them as unemployed workers.

inventories. As a result, investment is associated in the National Income and Product Accounts (NIPA) with nonresidential investment plus changes in business inventories. The empirical counterpart for consumption is identified with personal consumption expenditures in nondurable goods and services. Output is then defined as the sum of these investment and consumption measures. The quarterly capital-output ratio and the investment-output ratio corresponding to these measures are 6.8 and 0.15, respectively. Since at steady state $I/Y = \delta(K/Y)$, these ratios require that $\delta = 0.005515$.

The production function is assumed to have the following functional form:

$$y = sn^{\gamma}k^{\theta},$$

where $0 < \gamma + \theta < 1$. Calibrating to an annual interest rate of 4 percent, which is a standard value in the macro literature, requires a time discount factor β equal to 0.99755. Given this value for β, the above value for δ, and given that the capital share satisfies that

$$\theta = \frac{(1/\beta + \delta)\, K}{Y},$$

matching the U.S. capital-output ratio requires choosing a value of θ equal to 0.2168. Similarly, $\gamma = 0.64$ is selected to reproduce the share of labor in National Income.[18] Observe that, since workers are capital goods, the "wage rate" used in calculating the labor share is given by the user cost $(1-\beta)\, p^{e}$. In what follows, the value of p^{e} will be normalized to an arbitrary value and the utility of leisure parameter φ will be selected to generate that value.

The values for the idiosyncratic productivity levels s, the distribution over initial productivity levels ψ and the transition matrix Q are key determinants of the job-flows generated by the model. As a consequence I choose them to reproduce observations from the Business Employment Dynamics (BED) data set, which is a virtual census of establishments level dynamics. Since BED data across establishment sizes can be found for the nine employment ranges shown in the first column of Table 1, I restrict the idiosyncratic productivity levels s to take nine positive values ($s_1, s_2, ..., s_9$, with $s_i < s_j$ for every $i < j$) and choose them so that all establishments with a same idiosyncratic productivity level choose employment levels in the same range.

The average size of new entrants can be obtained by dividing the total gross job gains at opening establishments by the total number of opening establishments. Using data between 1992:Q3 and 2005:Q4, I find that the average size of new entrants is equal to 5.3 employees. Since this is a small number, I restrict the distribution over initial productivity levels ψ to put positive mass on only the two lowest values of s and choose $\psi(s_1)$ to reproduce that average size.

Similarly, the average size at exit can be obtained by dividing the total gross job losses at closing establishments by the total number of closing establishments. Using data for the same time period as above, I find that the average size at exit is equal to 5.2 employees. Since this is also a small number, I restrict the probabilities of transiting to a zero productivity level $Q(s,0)$ to take positive values only at the three lowest values of s. The values for $Q(s_1,0)$, $Q(s_2,0)$ and $Q(s_3,0)$ are then chosen to reproduce three observations: 1) the average size at exit, 2) the average quarterly rate of gross job losses due to closing establishments (JLD), which is equal to 1.6%, and 3) the average quarterly exit rate of establishments, which is

[18] In the model, γ is not strictly the same as the share of labor in National Income. However, under $\gamma = 0.64$ the labor share turns out to be 0.6367.

equal to 5.2%.[19]

The rest of the transition matrix Q is parameterized with enough flexibility to reproduce other important establishment level observations. The only restriction that I impose is that $Q(s_i, s_j) > 0$ only if $j = i - 1$, $j = i$ or $j = i + 1$. Since the rows of Q add to one this introduces 16 parameters (2 parameters each, for $i = 2, ..., 8$, and 1 parameter each, for $i = 1, 9$). Eight of these parameters are chosen to reproduce the shares in total employment across size classes (which provide eight independent observations). The other eight parameters are chosen to reproduce the shares in total gross job gains across size classes (which also provide eight independent observations). I must point out that the BED does not tabulate statistics across size classes in its regular reports. However, these statistics can be found in Okolie (19) (Tables 1 and 3) for the first two quarters of 2000. These statistics together with the corresponding model statistics are shown in the first panel of Table 1. The second panel reports the average sizes at entry and exit both in the model and the data. We see that the model does a good job at reproducing these observations. As a test of the model, Table 1 also includes the shares in total gross job losses across size classes for the first two quarters of 2000 in Okolie (19), and the average quarterly rates of gross job gains due to expanding establishments (JGE), gross job gains due to opening establishments (JGB), and gross job losses due to contracting establishments (JLC) reported by the BED for the period 1992:Q3-2005:Q4. Although the fit is not perfect, we see that the model also does a good job at reproducing these addional statistics.

The exogenous separation rate π_n and the number of establishments created every period ϱ are important determinants of the worker flows in and out of unemployment, so I calibrate them to reproduce this type of observations. In particular, I target an average monthly separation rate from employment equal to 3.5% and an average monthly hazard rate from unemployment equal to 46%, which were estimated by Shimer (21) using CPS data between 1948 and 2004. Since the separation rate of 3.5% is significantly larger than the rate of job losses experienced by establishments, I select a positive value of π_n to reproduce the excess worker reallocation.[20] Also, observe that the separation and hazard rates estimated by Shimer (21) imply a steady state unemployment rate equal to 7.1%. The average size of establishments implied by the distribution reported in Table 1 thus determine the entry rate of establishments ϱ that is needed to generate an aggregate employment level N equal to 0.929.

Based on evidence in Barron et. al (5) and Silva and Toledo (22), Hagedorn and Manovskii (9) determined that the costs of hiring a worker are equivalent to 4.5% of quarterly wages.[21] Since total hirings are assumed to be approximately equal to total recruitment opportunities, the cost of hiring a worker is approximately equal to the price of a next-period recruitment opportunity p^v. This suggest calibrating parameter values to reproduce the following relation:

$$p^v = 0.045 \times 4 \times (1 - \beta)\, p^e, \tag{59}$$

where $(1 - \beta)\, p^e$ represents 3-weeks wages and a factor of 4 is needed to convert them into quarterly wages. Recall that the price of a worker p^e was normalized to an arbitrary value. Thus, equation (59) solely imposes a restriction on p^v.

[19] Since the model time period is three weeks, quarterly statistics are constructed following establishments over four consecutive time periods.

[20] Not surprisingly, my calibrated value of π_n is smaller than the quit rate of workers measured by JOLTS (equal to 2% a month), since many of those separations entail job-to-job transitions that the model abstracts from.

[21] Since in their model capital is iddle while a job is open, Hagedorn and. Manovskii (9) add an imputed opportunity cost of capital to the total costs of hiring a worker. I do not make such adjustment because there is no iddle capital in my model.

The steady-state price of a recruitment opportunity p^v depends on A , U and the matching function curvature parameter ϕ.[22] Also, since (by assumption) total hirings are approximately equal to total recruitment opportunities, we have from equation (58) that

$$\frac{Hirings}{U} \approx \frac{A}{[U^{\phi} + A^{\phi}]^{\frac{1}{\phi}}}, \tag{60}$$

which must be equal to the hazard rate of unemployment that we are calibrating to (i.e. the monthly hazard rate of 46%, estimated by Shimer (21)). Since we are calibrating to a known value of U (equal to 0.071), equations (59) and (60) can be used to solve for A and ϕ. The values thus obtained are quite reasonable. In particular, the implied elasticity of the hazard rate from unemployment $G(A, U, A, U)/U$ to the unemployment-help-wanted-ads ratio U/A turn out to be 0.52 in the case of congestion externalities and 0.64 in the case of efficient matching. This elasticities are within the range estimated by previous studies (e.g. Shimer (20), Hall (13), etc.).

As a test of the model Table 2 reports a set of basic monthly statistics both for JOLTS and the model economy that were not used as calibration targets (except for the vacancy yield rate).[23]. We see that the model does a reasonable job at reproducing not only the vacancy yield rate, but the hiring and separation rates for continuing establishments, the fraction of vacancies with zero hirings and the fraction of hires with zero vacancies. The low rate of exogenous separations π_n explains the model's success in reproducing the fraction of vacancies with zero hirings. The reason is that a significant number of establishments reach the lower thresholds $\underline{n}$ and start hiring just enough workers to replenish the exogenous separation of workers. Since the monthly rate of exogenous separation is less than 1%, following Davis et al. (7), I classify these establishments (and their corresponding vacancies) as having zero hirings.

Observe that the model's ability at reproducing the JOLTS vacancy yield rate confirms that the strategy of calibrating to a three weeks time-period and setting π_u to zero was justified. In fact, the assumption that total hirings are approximately equal to total recruitment opportunities is verified: Total hirings turn out to be 96% as large as total recruitment opportunities.

Finally, the parameters ρ_z and σ_ε^2 governing the aggregate productivity shock process are selected to reproduce the empirical behavior of measured Solow residuals in the U.S. economy.[24] Defining output and capital as above and using civilian employment as the labor input, I find that measured Solow residuals are highly persistent and that their quarterly proportionate changes have a standard deviations equal to 0.0064 over the period 1951:1-2004:4.[25] It turns out that values of $\rho_z = 0.95$ and $\sigma_\varepsilon = 0.0041$ are needed to reproduce these observations using the artificial data generated by both versions of the model economy.[26]

All calibrated parameter values are summarized in Table 3.

[22] For the details, see Section 4.1 in the Technical Appendix.

[23] JOLTS statistics are from Davis et al. (7).

[24] Let γ^e denote the empirical labor share implicit in the National Income and Product Accounts. Proportionate changes in measured Solow residuals are then defined as the proportionate change in aggregate output minus the sum of the proportionate change in labor times γ^e, minus the sum of the proportionate change in capital times $(1 - \gamma^e)$.

[25] Solow residuals are constructed using an empirical labor share γ^e equal to 0.64.

[26] In both model economies quarterly Solow residuals are measured with the same empirical labor share γ^e used to measure Solow residuals in

6 Business cycles

This section uses both versions of the model economy calibrated in the previous section to address three important questions: 1) Can the neoclassical theory developed so far account for U.S. business cycle observations?, 2) Which scenario for the matching process is empirically more plausible: The efficient matching or the congestion externalities scenarios?, and 3) Is the model consistent with microeconomic adjustments at the establishment level?

Before turning to these questions it will be useful to describe salient features of U.S. business cycle fluctuations. Table 4 reports business cycle statistics for a number of U.S. time series corresponding to the period 1951:1 to 2004:4. All time series were logged and detrended using the Hoddrick-Prescott filter with smoothing parameter 1,600 before computing any statistics.

The upper panel of Table 4 reports standard deviations and correlations with output for GDP (Y), consumption (C), investment (I), capital (K), civilian employment (N), and labor productivity (Y/N). These statistics are standard in the RBC literature. They show that consumption, employment and labor productivity fluctuate roughly 61% as much as output, that capital fluctuates only 43% as much as output, that investment fluctuates 3.3 times more than output, and that all variables are strongly procyclical except for capital, which is acyclical.

The lower panel of Table 4 reports standard deviations and the cross-correlation matrix for GDP (Y), employment (N), unemployment (U), help-wanted ads (A), market tightness (A/U), job finding probability (H/U), employment exit probability (F/N), job creation rate (JC) and job destruction rate (JD). The job finding and employment exit probabilities, H/U and F/N, are from Shimer (21). The job creation and job destruction rates, JC and JD, are from Davis, Faberman and Haltiwanger (7) (based on BED data) and correspond to the subperiod 1990:2-2004:4.[27] The statistics in the lower panel of Table 4 have been emphasized in the labor literature (e.g. Hagedorn and Manovskii (9), Shimer (20), Davis and Haltiwanger, etc.). They show that unemployment and help wanted ads fluctuate about 7.3 times more than output, that the employment exit probability fluctuates 2.5 times more than output and that the job finding probability is about 40% more variable than the employment exit probability, and that the job creation rate fluctuates 1.4 times more than output and that job destruction rate is about 65% more variable than the job creation rate. Unemployment is strongly countercyclical, help wanted ads and the job finding rate are strongly procyclical, the employment exit probability is countercyclical, and the job creation and job destruction rates show weak cyclical patterns. Also observe that a "Beveridge curve" is obtained: help-wanted ads and unemployment are strongly negatively correlated (their correlation is -0.92).

6.1 Efficient matching vs. congestion externalities

Table 5 reports business cycle statistics for the model economy with efficient matching. Time series of length equal to 864 time periods were computed for 100 simulations and then aggregated into a quarterly frequency to obtain 216 quarters of

the U.S. economy.

[27] Job creation (JC) corresponds to the sum of gross job gains due to expanding establishments (JGE) and gross job gains due to opening establishments (JGB). Job destruction (JD) corresponds to the sum of gross job losses due to contracting establishments (JLC) and job losses due to closing establishments (JLD).

data (the same length as the U.S. series). The reported statistics are averages across these simulations. With regard to standard RBC statistics, we see from the upper-panels of Tables 4 and 5 that the model with efficient matching reproduces the comovements with output quite well: Except for capital, which is acyclical, all other variables are procyclical. The model's performance is not as good in terms of standard deviations, though. We see that investment fluctuations are as large as in the data but the rest of the variables are much smoother. The largest difference is with consumption, which fluctuates only 29% as much as in the data. However, this is a standard problem with RBC models. The most disappointing performance is with employment, which fluctuates only 37% as much as in the data. This smoothness is in turn inherited by output, which fluctuates only 69% as much as in the U.S. The failure of the model with efficient matching to account for labor market dynamics is evident in the lower panel. We see that unemployment, help-wanted ads, the job finding and employment exit probabilities and the job creation and job destruction rates fluctuate too little compared with the data. Moreover, the model fails to generate a strong "Beveridge curve": the correlation of unemployment with help-wanted ads is only -0.61.

We now turn to the model with congestion externalities in the matching process. Table 6 shows the results. We see that in terms of standard RBC statistics that this version of the model replicates U.S. business cycle observations at least as well as the economy with efficient matching. Comovements with output are still very similar with the data: consumption, investment, employment and labor productivity are all procyclical while capital is acyclical. The dimension in which the economy with congestion externalities outperforms the economy with efficient matching is in standard deviations: Employment, capital and output go from being 37%, 62% and 69% as volatile as the data to being 93%, 82%, 91% as volatile, respectively.[28] The model with congestion externalities also outperforms the model with efficient matching in terms of labor market statistics. In the lower panel of Table 6 we see that unemployment, help-wanted ads, the job-finding probability, the employment-exit probability, the job creation rate and the job destruction rate become much more variable than in the model with efficient matching. Also the economy now generates a more noticeable Beveridge curve: The correlation between unemployment and help-wanted ads is -0.76. Although these are improvements over the model with efficient matching, the empirical performance of the model is far from perfect: 1) help-wanted ads fluctuate 45% more than in the data, 2) the job-finding probability is 9 times more volatile than the employment-exit probability, while in the data they are only 40% more volatile, and 3) job destruction is as variable as job creation, while in the data job destruction is 65% more variable. Before turning to these limitations the rest of this section explores the reasons for the improved performance of the model with matching externalities.

Observe that there are two differences between the economy with efficient matching and the economy with congestion externalities. First, as Table 3 indicates, the economies have different parameter values (in particular, the curvature of the matching function ϕ and the utility of leisure φ are different). Second, although their aggregate matching functions have identical functional forms (see equation 58), their individual matching technologies are different (compare equations 55-56 with equation 57). In order to determine which of these differences drives the result that the economy with congestion externalities outperforms the economy with efficient matching, Table 7 reports the business cycle statistics for the Pareto optimal allocation of the economy with congestion externalities. Since the social planner fully internalizes the effects of the

[28] The only drawback is with labor productivity, which goes from being 76% as volatile as the data to being only 58% as volatile.

congestion externalities, any differences between these statistics and those of the economy with efficient matching reported in Table 5 can be solely attributed to differences in parameter values. Since Table 7 is very similar to Table 5, we conclude that the bulk of the differences in business cycle fluctuations between the economy with efficient matching and the economy with congestion externalities is not due to different parameter values but to the different individual matching technologies.[29]

In order to determine what feature of the matching technology with congestion externalities is essential for generating relatively large business cycle fluctuations, it will be useful to consider the following linear matching technology:

$$G(a,u,A^*,U^*) = u\frac{A^*}{\left[(U^*)^{\phi}+(A^*)^{\phi}\right]^{\frac{1}{\phi}}}, \tag{61}$$

$$H(a,u,A^*,U^*) = a\frac{U^*}{\left[(U^*)^{\phi}+(A^*)^{\phi}\right]^{\frac{1}{\phi}}}, \tag{62}$$

where A^* and U^* are positive parameters. Observe that if an economy had identical parameter values as the economy with congestion externalities but its matching technology was described by equations (61) and (62), with A^* and U^* given by the steady-state values of help-wanted ads and unemployment in the economy with congestion externalities, respectively, its steady state would be identical to the steady state of the economy with congestion externalities. However, its business cycles would be different. The reason is that the linear matching technology in equations (61) and (62) has a constant productivity while the linear matching technology faced by the myopic social planner in the economy with congestion externalities is subject to "productivity shocks" given by the realizations of the market tightness ratio A/U. Table 8 reports the business cycle statistics for this economy. We see that its business cycles are in fact much larger than in the economy with congestion externalities: Except for labor productivity and help-wanted ads, all variables are significantly more volatile than in Table 6. This indicates that the crucial feature generating the relatively large business cycles in the economy with congestion externalities is the linearity of the individual matching technology given by equations (55) and (56): The external effects from endogenous variations in the market tightness ratio A/U only serve to dampen the aggregate fluctuations generated by the economy. This is not surprising. Since equation (26) holds with strict inequality, the technology for creating workers with employment opportunities in equation (55) is largely irrelevant for business cycles. On the contrary, the technology for creating recruitment opportunities in equation (56) binds the amount of hiring that the economy can undertake. Since aggregate market tightness A/U is strongly procyclical in Table 6, the productivity of the technology for creating recruitment opportunities turns out to be countercyclical. This reduces the response of aggregate employment to aggregate productivity shocks, leading to lower employment fluctuations in Table 6 than in Table 8. This also explains why help-wanted ads are more volatile in Table 6 than in Table 8: Help-wanted ads need to respond more to aggregate productivity shocks to partially compensate for the countercyclical productivity of the technology for creating recrutiment opportunities.

[29] From Tables 6 and 7 we also conclude that introducing policies that achieve the first-best allocation would significantly reduce aggregate fluctuations in the economy with congestion externalities. See the working paper version (Veracierto xx) for an analysis of such policies.

6.2 Reallocation shocks

Section 6.1 showed that the economy with congestion externalities generates much more realistic business cycle fluctuations than the economy with efficient matching. However, it had several limitations: 1) help-wanted ads fluctuated much more than in the data, 2) the job-finding probability was several times more volatile than the employment-exit probability (while in the data it is only slightly more volatile), and 3) the job destruction rate was as variable as the job creation rate (while in the data it is considerably more volatile).

The purpose of this section is to explore to what extent the model's performance could be improved by introducing reallocation shocks that affect the idiosyncratic productivity process. This is a natural starting point since reallocation shocks directly influence the behavior of job creation and job destruction. Since the job creation and destruction process have strong implications for help-wanted ads, the job-finding probability and the employment-exit probability, reallocation shocks have the potential of affecting the behavior of these other variables as well. In what follows, reallocation shocks will be introduced to make volatility of job destruction (relative to the volatility of job creation) as large as in the data. A key question will be if these reallocation shocks help improve the model's performance in other dimensions.

A wide variety of reallocation shocks may be analyzed using the computational approach developed in this paper. For instance, the reallocation shock considered could affect the dispersion of the idiosyncratic productivity levels s while leaving the transition matrix Q unchanged. Another possible reallocation shock could leave the idiosyncratic productivity levels s unchanged while affecting the persistence $Q(s,s)$ of the different idiosyncratic productivity levels s. It turns out that these types of reallocations shocks do not improve the model's performance. The reason is that they synchronize the fluctuations in job creation and job destruction and thus fail to generate their asymmetric volatilities.

In order to break this synchronization the following reallocation shock will be considered. Let S^* be the set of idiosyncratic productivity levels and Q^* the transition matrix that were calibrated in Section 5. The reallocation shock r_t analyzed leaves the set of values for the idiosyncratic productivity levels unchanged at S^* but affects the transitions matrix Q_t as follows. For every s and s' in S^*,

$$Q_t(s,s') = \begin{cases} Q^*\left(s,s'\right), \text{ if } s' > s, \\ e^{r_t} Q^*\left(s,s'\right), \text{ if } s' < s, \\ 1 - \sum_{s''<s} e^{r_t} Q^*\left(s,s''\right) - \sum_{s''>s} Q^*\left(s,s''\right), \text{ if } s' = s. \end{cases} \tag{63}$$

Observe that this reallocation shock r_t affects the probabilities of transiting to lower productivity levels but not the probabilities of transiting to higher productivity levels. Any variations in the probabilities of transiting to lower productivity levels are absorbed by the probabilities of no-change.[30]

The reallocation shock r_t and the aggregate productivity shock z_t follow a joint autoregressive process given by

$$\begin{bmatrix} z_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_{zz} & \rho_{zr} \\ \rho_{rz} & \rho_{rr} \end{bmatrix} \begin{bmatrix} z_t \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma_{zz} & \sigma_{zr} \\ \sigma_{rz} & \sigma_{rr} \end{bmatrix} \begin{bmatrix} \varepsilon^z_{t+1} \\ \varepsilon^r_{t+1} \end{bmatrix}, \tag{64}$$

where ε^z_{t+1} and ε^r_{t+1} are normally distributed with zero mean and unit standard deviation.

[30] Restrictions to ensure that the probabilities of no-change $Q(s,s)$ remain positive for every possible realization of the reallocation shock r_t are ignored in equation (63) since these restrictions turn out to be non-binding in the experiments.

Allowing the reallocation shock r_t to be negatively correlated with the aggregate productivity shock z_t is crucial for generating asymmetries in the job creation and job destruction process. To see this, suppose that the economy is hit by a negative aggregate productivity shock that is accompanied by higher transition probabilities to lower idiosyncratic productivity levels. Since these higher transition probabilities generate job destruction, the response of job destruction to the negative aggregate productivity shock will thus be amplified. In addition, if the larger transition probabilities are short-lived (i.e. if ρ_{rr} is close to zero), the response of job creation to the negative aggregate productivity shock will be dampened. The reason, is that after its initial fall, the distribution of idiosyncratic productivity levels will be reverting towards the invariant distribution generated by the transition matrix Q^*, creating job creation over time. Both effects work in the same direction: making job destruction relatively more volatile than job creation.[31]

Given the above discussion, the reallocation shocks will be restricted to be short-lived and perfectly negatively correlated with innovations in aggregate productivity. In turn, the aggregate productivity shock will be allowed to have the same persistence level as in the benchmark case. Under these assumptions the general process in equation (64) reduces to the following specification:

$$\begin{bmatrix} z_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma_{zz} & 0 \\ \sigma_{rz} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^z_{t+1} \\ \varepsilon^r_{t+1} \end{bmatrix}. \tag{65}$$

The parameters σ_{zz} and σ_{rz} in equation (65) are selected to reproduce two important observations. First, that the standard deviation of measured Solow residuals is equal to 0.0064 (the same observation that was used in the benchmark case). Second, that the standard deviation of job destruction is 65% larger than the standard deviation of job creation. The parameter values consistent with these observations turn out to be $\sigma_{zz} = 0.00385$ and $\sigma_{rz} = -0.07$.

Table 9 reports the business cycle statistics for this economy. We see that in terms of standard RBC statistics, that the economy with reallocation shocks is virtually identical to the benchmark economy with congestion externalities (see the upper pannel of Table 6). In fact, despite of the fact that the aggregate productivity shock is 6% less variable and that the reallocation shock is i.i.d., the economy with reallocation shocks is slightly more volatile than than the benchmark case. There are significant differences in terms of labor market statistics, though. The most obvious is that (by construction) job destruction is now much more volatile than job creation, bringing the job creation and destruction process closer to the data. There are significant improvements on other variables as well. In particular, the volatility of help-wanted ads A and market tightness A/U are more in line with the data and a clearer Beveridge curve is now obtained (the correlation between unemployment and help wanted ads is -0.81). We see that the reallocation shocks also help reduce the volatility of the job-finding probability relative to the volatility of the employment-exit probability: the job-finding probability is now 3 times more variable than the employment-exit probability, while it was 9 times larger in the benchmark case. However, this asymmetry is still too large compared to the data. The reason why it is so hard for the model economy to generate large fluctuations in the employment-exit probability is that a large component of it is constant over the cycle: It is given by the exogenous quit rate of workers π_n. Endogenizing this margin may improve the performance of the model in this

[31] The assymmetric volatilities may not be obtained if ρ_{rr} is close to one. If after a negative aggregate productivity shock hits the economy establishments expect good idiosyncratic productivity levels to be much more transient than before, they will have fewer incentives to invest in recruitment opportunities after a high idiosyncratic productivity level is realized. As a consequence, the drop in job creation after the negative aggregate productivity shock hits the economy may actually be amplified.

particular dimension.

We have seen that the reallocation shock introduced improve the empirical performance of the model in terms of its job creation and destruction rates, job-finding and employment-exit probabilities, and help wanted ads. While there is ample evidence that idiosyncratic uncertainty increases during recessions (e.g. Bloom, 2009; Gilchrist et al, 2009; Bachmann and Bayer, 2009), it would be desirable to determine the precise empirical nature of reallocation shocks before jumping to any strong conclusions. The results presented in this section indicates that this may be a fruitful area of research. Having said this, the reallocation shocks introduced had very minor effects on macroeconomic variables such as output, employment, consumption and investment. Thus, reproducing the cyclical microeconomic adjustments observed in the data may turn up unimportant for aggregate business cycle dynamics.

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Appendix A

Proof of Proposition 4: From equations (38) and (39) we know that an establishment of type (s,l,j) transits to a next-period type (s',l',j'), with s' randomly determined,

$$l' = n^*(s,l,j), \tag{66}$$

and

$$j' = \max\{\bar{v}^*(s) - (1-\pi_n)l', 0\}. \tag{67}$$

Define

$$\mathcal{P}^{(0)} = \underset{s\in S}{\cup}\{(s,0,0)\}.$$

Since establishments are created with $(l,j)=(0,0)$, $\mathcal{P}^{(0)}$ describes the set of all possible types (s,l,j) of establishments of zero age.

Define

$$\mathcal{N}^{(0)} = \{0\}.$$

Since $n^*(s,l,j)=0$ whenever $(l,j)=(0,0)$, $N^{(0)}$ describes the set of all possible employment levels of establishments of zero age.

Starting from $\mathcal{N}^{(0)}$, define recursively a sequence of sets $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$ as follows:

$$\mathcal{P}^{(m)} = \left\{(s,l,j)\colon\ s\in S,\ l\in\mathcal{N}^{(m-1)},\text{ and } j\in \underset{s_{-1}\in S}{\cup}\{\max[\bar{v}^*(s_{-1}) - (1-\pi_n)\,l, 0]\}\right\}$$

$$\mathcal{N}^{(m)} = \left\{\underset{s\in S}{\cup}\{\underline{n}^*(s), \bar{n}^*(s), \bar{v}^*(s)\}\right\} \cup \left\{n\colon\ n=(1-\pi_n)\,n_{m-1} \text{ for some } n_{m-1}\in N^{(m-1)}\right\},$$

for $m = 1, 2, ..., \infty$.

From equations (38), (39), (66) and (67) we know that $\mathcal{P}^{(m)}$ contains the set of all possible types (s,l,j) of establishments of age m, and that $\mathcal{N}^{(m)}$ contains the set of all possible employment levels of establishments of age m.[32]

By induction, it can be shown that:

$$\mathcal{N}^{(m)} = \left\{\underset{s\in S}{\cup}\ \overset{m-1}{\underset{k=0}{\cup}}\left\{(1-\pi_n)^k\,\underline{n}^*(s), (1-\pi_n)^k\,\bar{n}^*(s), (1-\pi_n)^k\,\bar{v}^*(s)\right\}\right\}\cup\{0\}, \tag{68}$$

for $m = 1, 2, ..., \infty$.

A direct consequence of equation (68) is that $\mathcal{N}^{(m-1)}\subset\mathcal{N}^{(m)}$, for every $m\geq 1$. Thus, the set $\mathcal{N}^{(m)}$ in fact contains all the possible employment levels of establishments of age m or younger and the set $\mathcal{P}^{(m)}\cup\mathcal{P}^{(0)}$ contains all the possible types of establishments of age m or younger. Moreover,

$$\mathcal{N}^{(m)}/\mathcal{N}^{(m-1)} = \underset{s\in S}{\cup}\left\{(1-\pi_n)^{m-1}\,\underline{n}^*(s), (1-\pi_n)^{m-1}\,\bar{n}^*(s), (1-\pi_n)^{m-1}\,\bar{v}^*(s)\right\}, \tag{69}$$

[32] Observe that the "max" and "min" operators in equation (38) have been disregarded in the construction of the sets $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$. Thus, the set of actual types of establishments of age m and the set of actual employment levels of establishments of age m are smaller than $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$, respectively.

for $m = 1, 2, ..., \infty$, where "/" denotes set difference.

In what follows it will be shown that there exists a $M < \infty$ such that $\mathcal{N}^{(M)}$ contains the set of all possible employment levels of establishments of all ages $m = 0, 1, ..., \infty$. To prove this it suffices to show that there exists a $M < \infty$ such that no establishment of age $M + 1$ will choose an employment level in the set $\mathcal{N}^{(M+1)}/\mathcal{N}^{(M)}$, i.e. all establishments of age $M + 1$ will choose an employment level in the set $\mathcal{N}^{(M)}$.[33]

Let M satisfy equation (40). Since $0 < \pi_n < 1$, such a M exists.

Let $(s, l, j) \in \mathcal{P}^{(M+1)}$.

Suppose that $n^*(s, l, j) \in \mathcal{N}^{(M+1)}/\mathcal{N}^{(M)}$. Since $\mathcal{N}^{(M+1)}/\mathcal{N}^{(M)}$ satisfies equation (69), and M satisfies equation (40), it follows that

$$n^*(s, l, j) \leq (1 - \pi_n)^M \max \{\bar{n}^*(s_{\max}), \bar{v}^*(s_{\max})\} < \min \{\underline{n}^*(s_{\min}), \bar{v}^*(s_{\min})\}. \tag{70}$$

Also, since $n^*(s, l, j)$ satisfies equation (38) and $(s, l, j) \in \mathcal{P}^{(M+1)}$, we have that

$$n^*(s, l, j) \in \{\underline{n}^*(s), \bar{n}^*(s), (1 - \pi_n) l\} \cup \left\{ \bigcup_{s_{-1} \in S} \bar{v}^*(s_{-1}) \right\}. \tag{71}$$

From equation (71) and the last inequality in equation (70), we then have that

$$n^*(s, l, j) = (1 - \pi_n) l.$$

Suppose, first, that $j = 0$.

Suppose that some establishment of age M transits to (s, l, j). From equation (67), this can be the case only if

$$0 = \max \{\bar{v}^*(s_{-1}) - (1 - \pi_n) l, 0\},$$

for some $s_{-1} \in S$.

But, from equation (70)

$$n^*(s, l, j) = (1 - \pi_n) l < \bar{v}^*(s_{-1}),$$

for all $s_{-1} \in S$. A contradiction.

Hence, $(s, l, j) \in \mathcal{P}^{(M+1)}$ does not correspond to an establishment of age $M + 1$.

Suppose now that $j > 0$.

Let s_{-1} be such that $(1 - \pi_n) l + j = \bar{v}^*(s_{-1})$ (since $(s, l, j) \in \mathcal{P}^{(M+1)}$, such an s_{-1} exists).

Then, from equation (38) we have that

$$n^*(s, l, j) = \max \left\{ \begin{array}{l} \min \{\bar{v}^*(s_{-1}), \underline{n}^*(s)\}, \\ \min \{(1 - \pi_n) l, \bar{n}^*(s)\} \end{array} \right\},$$

and, therefore, that

$$n^*(s, l, j) = (1 - \pi_n) l \leq \bar{n}^*(s) \text{ and } n^*(s, l, j) = (1 - \pi_n) l \geq \min \{\bar{v}^*(s_{-1}), \underline{n}^*(s)\}. \tag{72}$$

[33] This condition is sufficient because whenever an establishment reaches age $M + 1$, its age can be reset to M without consequence. This procedure can be repeated an infinite number of times.

The second inequality in equation (72) contradicts equation (70).

We conclude that no establishment of age $M+1$ chooses an employment level in the set $\mathcal{N}^{(M+1)}/\mathcal{N}^{(M)}$. It follows that the set $\mathcal{P}^* = \mathcal{P}^{(M+1)} \cup \mathcal{P}^{(0)}$ is a support of the invariant distribution μ^*.■

Proof of Proposition 5: Observe that the optimal decision rules at period $t-k$ are given by

$$n_{t-k}(s,l,j) = \max \left\{ \begin{array}{c} \min \left\{ (1-\pi_n \right\} l + j, \underline{n}_{t-k}(s), \\ \min \left\{ (1-\pi_n) l, \bar{n}_{t-k}(s) \right\} \end{array} \right\}, \tag{73}$$

and

$$v_{t-k}(s,l,j) = \max \left\{ \bar{v}_{t-k}(s) - (1-\pi_n) n_{t-k}(s,l,j), 0 \right\}, \tag{74}$$

for $k = 0, 1, ..., M+1$.

a) We will first show that $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1}.

Define the sets $\mathcal{A}_t$ and $\mathcal{B}_{t+1}$ as follows:

$$\mathcal{A}_t = \underset{s \in S}{\cup} \left\{ (1-\pi_n)^{M-1} \underline{n}_{t-M}(s), (1-\pi_n)^{M-1} \bar{n}_{t-M}(s), (1-\pi_n)^{M-1} \bar{v}_{t-M-1}(s) \right\},$$

$$\mathcal{B}_{t+1} = \left\{ l' : l' = (1-\pi_n) l, \text{ for some } l \in \mathcal{N}_t / \mathcal{A}_t \right\}. \tag{75}$$

Observe that

$$\mathcal{N}_{t+1} = \mathcal{B}_{t+1} \cup \left\{ \underset{s \in S}{\cup} \left\{ \underline{n}_t(s), \bar{n}_t(s), \bar{v}_{t-1}(s) \right\} \right\}. \tag{76}$$

To show that $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1} it suffices to show that

$$(s,l,j) \in \mathcal{P}_t \Longrightarrow n_t(s,l,j) \in \mathcal{N}_{t+1} \text{ and } v_t(s,l,j) \in \underset{s' \in S}{\cup} \left\{ \max \left[\bar{v}_t(s') - (1-\pi_n) n_t(s,l,j), 0 \right] \right\}. \tag{77}$$

Let $(s,l,j) \in \mathcal{P}_t$.

Suppose, first, that $(l,j) = (0,0)$.

From equation (73) we then have that $n_t(s,l,j) = 0$ and, from equation (74), that $v_t(s,l,j) = \bar{v}_t(s)$. Therefore, equation (77) is satisfied.

Suppose, now, that $(l,j) \neq (0,0)$.

Then,

$$s \in S,\ l \in \mathcal{N}_t \text{ and } j \in \underset{s' \in S}{\cup} \left\{ \max \left[\bar{v}_{t-1}(s') - (1-\pi_n) l, 0 \right] \right\}. \tag{78}$$

From equations (73) and (78), we then have that

$$n_t(s,l,j) = \max \left\{ \begin{array}{c} \min \left\{ \max \left[\bar{v}_{t-1}(s'), (1-\pi_n) l \right], \underline{n}_t(s) \right\}, \\ \min \left\{ (1-\pi_n) l, \bar{n}_t(s) \right\} \end{array} \right\}, \tag{79}$$

for some $s' \in S$.

As a consequence,

$$n_t(s,l,j) \in \left\{ (1-\pi_n) l, \bar{n}_t(s), \underline{n}_t(s) \right\} \cup \left\{ \underset{s' \in S}{\cup} \left\{ \bar{v}_{t-1}(s') \right\} \right\}. \tag{80}$$

From equations (75), (76) and (80) we have that

$$l \in \mathcal{N}_t/\mathcal{A}_t \Rightarrow n_t(s,l,j) \in \mathcal{N}_{t+1}.$$

Suppose that $l \in A_t$. Without loss of generality assume that

$$l = (1-\pi_n)^{M-1}\underline{n}_{t-M}(\widehat{s})$$

for some $\widehat{s} \in S$ (the cases $l = (1-\pi_n)^{M-1}\bar{n}_{t-M}(\widehat{s})$ and $l = (1-\pi_n)^{M-1}\bar{v}_{t-M-1}(\widehat{s})$ can be handled in exactly the same way).

Then, equation (79) becomes

$$n_t(s,l,j) = \max\left\{\begin{array}{c}\min\left\{\max\left[\bar{v}_{t-1}(s'),(1-\pi_n)^M\underline{n}_{t-M}(\widehat{s})\right],\underline{n}_t(s)\right\}, \\ \min\left\{(1-\pi_n)^M\underline{n}_{t-M}(\widehat{s}),\bar{n}_t(s)\right\}\end{array}\right\}. \tag{81}$$

for some $s' \in S$.

But from equation (40) and equations (42)-(44), we have that

$$(1-\pi_n)^M\underline{n}_{t-M}(\widehat{s}) < (1-\pi_n)^M\bar{n}_{t-M}(\widehat{s}) \leq (1-\pi_n)^M\bar{n}_{t-M}(s_{\max}) < \bar{v}_{t-1}(s_{\min}) \leq \bar{v}_{t-1}(s'),$$

and that

$$(1-\pi_n)^M\underline{n}_{t-M}(\widehat{s}) < (1-\pi_n)^M\bar{n}_{t-M}(\widehat{s}) \leq (1-\pi_n)^M\bar{n}_{t-M}(s_{\max}) < \underline{n}_t(s_{\min}) \leq \underline{n}_t(s) < \bar{n}_t(s).$$

Therefore equation (81) becomes

$$\begin{aligned} n_t(s,l,j) &= \max\left\{\begin{array}{c}\min\left\{\bar{v}_{t-1}(s'),\underline{n}_t(s)\right\}, \\ (1-\pi_n)^M\underline{n}_{t-M}(\widehat{s})\end{array}\right\} \\ &= \min\left\{\bar{v}_{t-1}(s'),\underline{n}_t(s)\right\}. \end{aligned}$$

Thus, from equations (76), $n_t(s,l,j) \in \mathcal{N}_{t+1}$.

From equation (74), observe that

$$v_t(s,l,j) = \max\left\{\bar{v}_t(s) - (1-\pi_n)\,n_t(s,l,j), 0\right\}$$

Thus,

$$v_t(s,l,j) \in \bigcup_{s' \in S}\left\{\max\left[\bar{v}_t(s') - (1-\pi_n)\,n_t(s,l,j), 0\right]\right\}.$$

Therefore, $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1}.

b) To prove the second part of the Proposition it will be convenient to define the following (one-to-one and onto) functions.

For every $(s,l,j) \in \mathcal{P}_t$:

$$\begin{aligned} l_t^*(l,j) &= l^* \\ j_t^*(l,j) &= j^* \end{aligned}$$

where (s, l^*, j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l - l^*| < \varepsilon/2$ and $|[(1-\pi_n)\, l + j] - [(1-\pi_n)\, l^* + j^*]| < \varepsilon/2$.

Similarly, for every $(s', l', j') \in \mathcal{P}_{t+1}$:

$$\begin{aligned} l^*_{t+1}(l', j') &= l^* \\ j^*_{t+1}(l', j') &= j^* \end{aligned}$$

where (s', l^*, j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l' - l^*| < \varepsilon/2$ and $|[(1-\pi_n)\, l' + j'] - [(1-\pi_n)\, l^* + j^*]| < \varepsilon/2$.

Observe that, by assumption, we have that for every $(s, l, j) \in \mathcal{P}_t$:

$$\mu_t\,(s, l, j) = \mu^*\,(s, l^*_t(l, j), j^*_t(l, j))\,. \tag{82}$$

We need to show that for every $(s', l', j') \in \mathcal{P}_{t+1}$:

$$\mu_{t+1}\,(s', l', j') = \mu^*\left(s', l^*_{t+1}(l', j'), j^*_{t+1}(l', j')\right). \tag{83}$$

Let $(s', l', j') \in \mathcal{P}_{t+1}$.

Using equation (82), we have that

$$\mu_{t+1}\,(s', l', j') = \sum_{(s,l,j)\ \in\ \mathcal{G}_t(l',j')} Q\,(s, s')\,\mu^*\,(s, l^*_t(l, j), j^*_t(l, j)) + \varrho\psi\,(s')\,\mathcal{I}\,(l', j')\,,$$

where

$$\mathcal{G}_t(l', j') = \{(s, l, j) \in \mathcal{P}_t\colon\ n_t\,(s, l, j) = l' \text{ and } v_t\,(s, l, j) = j'\}\,.$$

Also observe that

$$\mu^*\left(s', l^*_{t+1}(l', j'), j^*_{t+1}(l', j')\right) = \sum_{(s,l^*,j^*)\ \in\ \mathcal{G}^*(l^*_{t+1}(l',j'), j^*_{t+1}(l',j'))} Q\,(s, s')\,\mu^*\,(s, l^*, j^*) + \varrho\psi\,(s')\,\mathcal{I}\left(l^*_{t+1}(l', j'), j^*_{t+1}(l', j')\right),$$

where

$$\mathcal{G}^*(l^*_{t+1}(l', j'), j^*_{t+1}(l', j')) = \left\{(s, l^*, j^*) \in \mathcal{P}^*\colon\ n^*(s, l^*, j^*) = l^*_{t+1}(l', j') \text{ and } v^*(s, l^*, j^*) = j^*_{t+1}(l', j')\right\}.$$

To show that equation (83) holds, it then suffices to show that

$$(l', j') = (0, 0) \Leftrightarrow \left(l^*_{t+1}(l', j'), j^*_{t+1}(l', j')\right) = (0, 0)\,, \tag{84}$$

$$(s, l, j) \in \mathcal{G}_t(l', j') \Longrightarrow (s, l^*_t\,(l, j)\,, j^*_t\,(l, j)) \in \mathcal{G}^*(l^*_{t+1}(l', j'), j^*_{t+1}(l', j')), \tag{85}$$

$$(s, l^*, j^*) \in \mathcal{G}^*(l^*_{t+1}(l', j'), j^*_{t+1}(l', j')) \Longrightarrow \left(s, [l^*_t]^{-1}\,(l^*, j^*), [j^*_t]^{-1}\,(l^*, j^*)\right) \in \mathcal{G}_t(l', j'). \tag{86}$$

where $\left([l^*_t]^{-1}, [j^*_t]^{-1}\right)$ is the inverse function of (l^*_t, j^*_t).

b.1) Proof of equation (84).

It is a direct consequence of how l^*_{t+1} and j^*_{t+1} were defined and equations (42)-(44).

b.2) Proof of equation (85).

Let $(s, l, j) \in \mathcal{G}_t(l', j')$. Then, $(s, l, j) \in \mathcal{P}_t$,

$$l' = \max\left\{\begin{array}{c} \min\left\{(1-\pi_n)\, l + j, \underline{n}_t\,(s)\right\} \\ \min\left\{(1-\pi_n)\, l, \bar{n}_t\,(s)\right\} \end{array}\right\},$$

and

$$(1-\pi_n)\, l' + j' = \max\left\{\bar{v}_t\,(s)\,, (1-\pi_n)\, l'\right\}.$$

Observe that $(s, l_t^*\,(l,j)\,, j_t^*\,(l,j)) \in \mathcal{P}^*$,

$$n^*(s, l_t^*(l,j), j_t^*(l,j)) = \max\left\{\begin{array}{c}\min\left\{(1-\pi_n)\, l_t^*(l,j) + j_t^*(l,j), \underline{n}^*\,(s)\right\}\\ \min\left\{(1-\pi_n)\, l_t^*(l,j), \bar{n}^*\,(s)\right\}\end{array}\right\},$$

and

$$\begin{aligned}&(1-\pi_n)\, n^*(s, l_t^*(l,j), j_t^*(l,j)) + v^*(s, l_t^*(l,j), j_t^*(l,j))\\ =\;&\max\left\{\bar{v}^*\,(s)\,, (1-\pi_n)\, n^*(s, l_t^*(l,j), j_t^*(l,j))\right\}.\end{aligned}$$

Since

$$|(1-\pi_n)\, l - (1-\pi_n)\, l_t^*(l,j)| < \varepsilon/2,$$

$$|[(1-\pi_n)\, l + j] - [(1-\pi_n)\, l_t^*(l,j) + j_t^*(l,j)]| < \varepsilon/2,$$

$$|\underline{n}_t\,(s) - \underline{n}^*\,(s)| < \varepsilon/2,$$

and

$$|\bar{n}_t\,(s) - \bar{n}^*\,(s)| < \varepsilon/2,$$

it follows that

$$|n^*(s, l_t^*(l,j), j_t^*(l,j)) - l'| < \varepsilon/2, \tag{87}$$

and, therefore, that

$$|[(1-\pi_n)\, n^*(s, l_t^*(l,j), j_t^*(l,j)) + v^*(s, l_t^*(l,j), j_t^*(l,j))] - [(1-\pi_n)\, l' + j']| < \varepsilon/2. \tag{88}$$

Since $(s', l', j') \in \mathcal{P}_{t+1}$ and $[s', n^*(s, l_t^*(l,j), j_t^*(l,j)), v^*(s, l_t^*(l,j), j_t^*(l,j))] \in \mathcal{P}^*$, equations (87) and (88) imply that

$$l_{t+1}^*(l', j') = n^*(s, l_t^*(l,j), j_t^*(l,j)),$$

$$j_{t+1}^*(l', j') = v^*(s, l_t^*(l,j), j_t^*(l,j)).$$

Since $(s, l_t^*(l,j), j_t^*(l,j)) \in \mathcal{P}^*$ it follows that

$$(s, l_t^*(l,j), j_t^*(l,j)) \in \mathcal{G}^*(l_{t+1}^*(l', j'), j_{t+1}^*(l', j')).$$

b.3) Proof of equation (86).

Let $(s, l^*, j^*) \in \mathcal{G}^*(l_{t+1}^*(l', j'), j_{t+1}^*(l', j'))$. Then, $(s, l^*, j^*) \in \mathcal{P}^*$,

$$\begin{aligned} l_{t+1}^*(l', j') &= n^*\,(s, l^*, j^*) && (89)\\ &= \max\left\{\begin{array}{c}\min\left\{(1-\pi_n)\, l^* + j^*, \underline{n}^*\,(s)\right\}\\ \min\left\{(1-\pi_n)\, l^*, \bar{n}^*\,(s)\right\}\end{array}\right\}, && (90)\end{aligned}$$

and

$$(1-\pi_n)\, l^*_{t+1}(l',j') + j^*_{t+1}(l',j') = (1-\pi_n)\, n^*(s,l^*,j^*) + v^*(s,l^*,j^*) \tag{91}$$

$$= \max\left\{\bar{v}^*(s), (1-\pi_n)\, l^*_{t+1}(l',j')\right\}. \tag{92}$$

Observe that $\left(s, [l^*_t]^{-1}(l^*,j^*), [j^*_t]^{-1}(l^*,j^*)\right) \in \mathcal{P}_t$,

$$n_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)) = \max\left\{\begin{array}{c} \min\left\{(1-\pi_n)[l^*_t]^{-1}(l^*,j^*) + [j^*_t]^{-1}(l^*,j^*), \underline{n}_t(s)\right\} \\ \min\left\{(1-\pi_n)[l^*_t]^{-1}(l^*,j^*), \bar{n}_t(s)\right\}\end{array}\right\},$$

and

$$\begin{aligned} & (1-\pi_n)\, n_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)) + v_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)) \\ = \; & \max\left\{\bar{v}_t(s), (1-\pi_n)\, n_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*))\right\}. \end{aligned}$$

Also, from equation (77), we have that

$$\left[s', n_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right], v_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right]\right] \in \mathcal{P}_{t+1}$$

for every s'.

Moreover,

$$\begin{aligned} & l^*_{t+1}(n_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right], v_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right]) \\ = \; & n^*(s,l^*,j^*) \end{aligned}$$

and

$$\begin{aligned} & j^*_{t+1}(n_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right], v_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right]) \\ = \; & v^*(s,l^*,j^*) \end{aligned}$$

Hence, from equations (89) and (91), we have that

$$\begin{aligned} & l^*_{t+1}(n_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right], v_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right]) \\ = \; & l^*_{t+1}(l',j') \end{aligned}$$

and

$$\begin{aligned} & j^*_{t+1}(n_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right], v_t\left[s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right]) \\ = \; & j^*_{t+1}(l',j') \end{aligned}$$

It follows that

$$l' = n_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)),$$

$$j' = v_t(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)).$$

Since $\left(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right) \in \mathcal{P}_t$ it follows that

$$\left(s,[l^*_t]^{-1}(l^*,j^*),[j^*_t]^{-1}(l^*,j^*)\right) \in \mathcal{G}_t(l',j') \blacksquare$$

Table 1

Quarterly observations

Panel A: BED data, March 2000 to June 2000						
Size	Data			Model		
Classes* (employees)	Shares in Employment	Shares in Job Gains	Shares in Job Losses	Shares in Employment	Shares in Job Gains	Shares in Job Losses
$[1, 5)$	6.4%	16.9%	9.7%	6.2%	16.2%	8.6%
$[5, 10)$	8.1%	13.1%	11.6%	9.0%	13.1%	10.7%
$[10, 20)$	10.7%	14.9%	13.7%	12.3%	14.1%	10.8%
$[20, 50)$	16.6%	18.3%	18.2%	16.4%	18.4%	14.7%
$[50, 100)$	13.1%	11.6%	12.6%	12.0%	12.2%	16.2%
$[100, 250)$	16.5%	11.9%	14.6%	16.4%	13.1%	13.9%
$[250, 500)$	9.8%	5.9%	8.5%	9.3%	4.9%	12.4%
$[500, 1000)$	7.3%	3.5%	5.4%	7.1%	8.1%	5.3%
$[1000, \infty)$	11.6%	4.2%	5.9%	11.3%	0.0%	7.5%

Panel B: BED data, 1992:3-2005:4		
	Data	Model
size at entry	5.3	4.4
size at exit	5.2	5.8
JGB	1.7%	1.2%
JGE	6.2%	6.2%
JLD	1.6%	1.6%
JLC	6.0%	5.8%
Exit Rate	5.2%	6.2%

Table 2

Monthly observations

Panel A: CPS data, 1948-2004		
	Data	Model
Separation rate	3.5%	3.3%
Hazard rate	46.0%	46.9%

Panel B: JOLTS data, 2000-2005		
	Data	Model
Vacancy rate	2.2%	2.2%
Hiring rate	3.2%	2.9%
Separation rate	3.1%	2.9%
Vacancies yield rate	1.3	1.3
% Vacancies with zero hiring	18.7%	21.9%
% Hiring with zero vacancies	42.3%	39.3%
% Establishments with zero hiring	81.6%	95.8%
% Establishments with zero vacancies	87.6%	97.9%

Table 3

Calibrated parameter values

Parameter	Description	Value
ϱ	entry of establishments	0.0007038
δ	capital depreciation rate	0.0055147
β	discount factor	0.9975517
θ	capital share	0.216757
γ	labor share	0.64
π_u	quit rate, unemployed workers	0
π_n	quit rate, employed workers	0.00675
ϕ	curvature, matching function	1.0161441 (efficient matching)
		0.734344 (congestion externalities)
φ	utility of leisure	0.78439 (efficient matching)
		0.805099 (congestion externalities)
ρ_z	persistence aggregate shocks	0.95
σ_ε	standard deviation aggregate shocks	0.0041

Table 3 (Continued)

Calibrated idiosyncratic process

Idiosyncratic Productivity levels:

$$s_0 = 0.00 \quad s_1 = 6.3 \quad s_2 = 6.7 \quad s_3 = 7.7 \quad s_4 = 8.6$$
$$s_5 = 9.4 \quad s_6 = 10.9 \quad s_7 = 12.1 \quad s_8 = 13.1 \quad s_9 = 14.3$$

Initial distribution:

$$\psi_0 = 0.00 \quad \psi_1 = 0.82 \quad \psi_2 = 0.18 \quad \psi_3 = 0.00 \quad \psi_4 = 0.00$$
$$\psi_5 = 0.00 \quad \psi_6 = 0.00 \quad \psi_7 = 0.00 \quad \psi_8 = 0.00 \quad \psi_9 = 0.00$$

Transition matrix:

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.026 & 0.9467 & 0.0273 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.019 & 0.0147 & 0.9555 & 0.0108 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.007 & 0 & 0.0149 & 0.9582 & 0.0199 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0330 & 0.9309 & 0.0361 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0777 & 0.9119 & 0.0104 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0245 & 0.9571 & 0.0184 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0612 & 0.9262 & 0.0126 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0325 & 0.9335 & 0.0340 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0356 & 0.9644 \end{pmatrix}$$

Table 4

Business Cycle Statistics: U.S. economy (1951:1-2004:4)

A. Macroeconomic variables

Standard deviations

	Y	C	I	K	N	Y/N
	1.58	0.90	6.76	0.68	1.00	0.99

Correlations with output

	Y	C	I	K	N	Y/N
	1.00	0.80	0.91	0.05	0.80	0.79

B. Labor market variables

Standard deviations

	Y	N	U	A	A/U	H/U	F/N	JC^*	JD^*
	1.58	1.00	12.32	13.89	25.66	7.72	5.46	3.70	6.15

Correlations matrix

	Y	N	U	A	A/U	H/U	F/N	JC^*	JD^*
Y	1.00	0.80	-0.84	0.90	0.89	0.82	-0.57	0.49	-0.30
N		1.00	-0.87	0.88	0.89	0.87	-0.38	0.21	-0.02
U			1.00	-0.92	-0.98	-0.92	0.54	-0.43	0.03
A				1.00	0.98	0.91	-0.56	0.47	-0.26
A/U					1.00	0.93	-0.56	0.46	-0.16
H/U						1.00	-0.38	0.45	0.06
F/N							1.00	-0.13	0.47
JC^*								1.00	-0.08

Table 5

Business Cycle Statistics: Efficient Matching

A. Macroeconomic variables

Standard deviations

	Y	C	I	K	N	Y/N
	1.09	0.26	6.23	0.42	0.37	0.75

Correlations with output

	Y	C	I	K	N	Y/N
	1.00	0.75	0.99	0.18	0.96	0.99

B. Labor market variables

Standard deviations

	Y	N	U	A	A/U	H/U	F/N	JC	JD
	1.09	0.37	5.20	4.58	8.78	5.52	0.50	1.83	1.97

Correlations matrix

	Y	N	U	A	A/U	H/U	F/N	JC	JD
Y	1.00	0.96	-0.96	0.76	0.96	0.98	-0.82	0.12	-0.40
N		1.00	-1.00	0.60	0.91	0.97	-0.72	-0.07	-0.22
U			1.00	-0.61	-0.91	-0.97	0.72	0.07	0.23
A				1.00	0.88	0.76	-0.96	0.62	-0.81
A/U					1.00	0.97	-0.93	0.28	-0.56
H/U						1.00	-0.84	0.16	-0.44
F/N							1.00	-0.54	0.76
JC								1.00	-0.70

Table 6

Business Cycle Statistics: Matching externalities

A. Macroeconomic variables						
Standard deviations						
	Y	C	I	K	N	Y/N
	1.44	0.31	8.26	0.56	0.93	0.58
Correlations with output						
	Y	C	I	K	N	Y/N
	1.00	0.66	0.99	0.16	0.97	0.93

B. Labor market variables									
Standard deviations									
	Y	N	U	A	A/U	H/U	F/N	JC	JD
	1.44	0.93	13.68	20.14	31.84	14.57	1.42	4.55	4.77
Correlations matrix									
	Y	N	U	A	A/U	H/U	F/N	JC	JD
Y	1.00	0.97	-0.96	0.79	0.92	0.97	-0.62	0.08	-0.31
N		1.00	-0.99	0.73	0.89	0.97	-0.54	-0.03	-0.20
U			1.00	-0.76	-0.91	-0.98	0.54	0.03	0.20
A				1.00	0.96	0.85	-0.85	0.48	-0.67
A/U					1.00	0.96	-0.77	0.29	-0.51
H/U						1.00	-0.67	0.17	-0.38
F/N							1.00	-0.60	0.84
JC								1.00	-0.72

Table 7

Business Cycle Statistics: Efficient allocation for economy with matching externalities

A. Macroeconomic variables

Standard deviations

	Y	C	I	K	N	Y/N
	1.15	0.28	6.60	0.45	0.48	0.71

Correlations with output

	Y	C	I	K	N	Y/N
	1.00	0.75	0.99	0.18	0.95	0.98

B. Labor market variables

Standard deviations

	Y	N	U	A	A/U	H/U	F/N	JC	JD
	1.15	0.48	4.92	5.39	9.07	5.58	0.60	2.21	2.40

Correlations matrix

	Y	N	U	A	A/U	H/U	F/N	JC	JD
Y	1.00	0.95	-0.95	0.72	0.95	0.97	-0.74	0.16	-0.39
N		1.00	-1.00	0.54	0.86	0.94	-0.60	-0.04	-0.19
U			1.00	-0.54	-0.87	-0.94	0.60	0.04	0.19
A				1.00	0.89	0.77	-0.98	0.69	-0.84
A/U					1.00	0.97	-0.91	0.39	-0.60
H/U						1.00	-0.81	0.28	-0.50
F/N							1.00	-0.66	0.82
JC								1.00	-0.73

Table 8

Business Cycle Statistics: Efficient allocation for economy with linear matching technology

A. Macroeconomic variables						
Standard deviations						
	Y	C	I	K	N	Y/N
	1.82	0.39	10.92	0.71	1.50	0.43
Correlations with output						
	Y	C	I	K	N	Y/N
	1.00	0.71	0.98	0.16	0.99	0.80

B. Labor market variables									
Standard deviations									
	Y	N	U	A	A/U	H/U	F/N	JC	JD
	1.82	1.50	24.80	11.39	30.55	28.34	4.16	9.07	8.37
Correlations matrix									
	Y	N	U	A	A/U	H/U	F/N	JC	JD
Y	1.00	0.99	-0.95	0.34	0.90	0.94	0.03	0.18	-0.06
N		1.00	-0.96	0.31	0.90	0.95	0.04	0.16	-0.04
U			1.00	-0.31	-0.93	-0.98	-0.04	-0.16	0.04
A				1.00	0.63	0.41	-0.39	0.72	-0.63
A/U					1.00	0.95	-0.12	0.40	-0.27
H/U						1.00	0.03	0.28	-0.14
F/N							1.00	-0.58	0.76
JC								1.00	-0.70

Table 9

Business Cycle Statistics: Reallocation shocks

A. Macroeconomic variables

Standard deviations

Y	C	I	K	N	Y/N
1.48	0.34	8.42	0.56	0.96	0.57

Correlations with output

Y	C	I	K	N	Y/N
1.00	0.72	0.99	0.17	0.98	0.94

B. Labor market variables

Standard deviations

Y	N	U	A	A/U	H/U	F/N	JC	JD
1.48	0.96	12.78	14.53	25.96	13.19	3.11	3.54	5.80

Correlations matrix

	Y	N	U	A	A/U	H/U	F/N	JC	JD
Y	1.00	0.98	-0.97	0.80	0.93	0.97	-0.40	-0.06	-0.27
N		1.00	-0.99	0.78	0.92	0.98	-0.34	-0.12	-0.20
U			1.00	-0.81	-0.94	-0.99	0.34	0.12	0.20
A				1.00	0.96	0.86	-0.75	0.38	-0.68
A/U					1.00	0.97	-0.59	0.15	-0.48
H/U						1.00	-0.40	0.02	-0.30
F/N							1.00	-0.63	0.93
JC								1.00	-0.74

Technical Appendix for:
"Establishments Dynamics, Vacancies and Unemployment: A Neoclassical Synthesis"*

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November 25, 2009

Abstract: This is the Technical Appendix for my paper "Establishments Dynamics, Vacancies and Unemployment: A Neoclassical Synthesis".

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1 Equilibrium conditions

1.1 Households' problem

The household's Bellman equation is:

$$B(\kappa, z, K, E, X, \mu) = \max_{\{c,i,m\}} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \varphi m + \beta \mathcal{E} \left[B\left(\kappa', z', K', E', X', \mu'\right) \mid z \right] \right\}$$

subject to:

$$\begin{aligned} c + i + r^u\left(z, K, E, X, \mu\right) m &\leq r^k\left(z, K, E, X, \mu\right) \kappa + \Pi\left(z, K, E, X, \mu\right), & (1) \\ \kappa' &= (1-\delta)\kappa + i & (2) \\ \left(K', E', X', \mu'\right) &= L\left(z, K, E, X, \mu\right). & (3) \end{aligned}$$

Let $\lambda(\kappa, z, K, E, X, \mu)$ be the Lagrange multiplier for equation (1). The first order conditions and envelope conditions are then the following:

$$\begin{aligned} c\left(\kappa, z, K, E, X, \mu\right)^{-\sigma} &= \lambda\left(\kappa, z, K, E, X, \mu\right) \\ \varphi &= \lambda\left(\kappa, z, K, E, X, \mu\right) r^u\left(z, K, E, X, \mu\right) \\ \lambda\left(\kappa, z, K, E, X, \mu\right) &= \beta \mathcal{E}\left[B_\kappa\left(\kappa', z', K', E', X', \mu'\right) \mid z\right] \\ B_\kappa(\kappa, z, K, E, X, \mu) &= \lambda\left(\kappa, z, K, E, X, \mu\right)\left[1 - \delta + r^k\left(z, K, E, X, \mu\right)\right] \end{aligned}$$

1.2 Establishment's problem

The establishments' Bellman equation is the following

$$\begin{aligned} W(s, l, j, z, K, E, X, \mu) = \max_{\{f,h,k,n,v\}} &\left\{ e^z s F\left(n, k\right) + p^u\left(z, K, E, X, \mu\right) f - p^e\left(z, K, E, X, \mu\right) h \right. \\ &- r^k\left(z, K, E, X, \mu\right) k - p^v\left(z, K, E, X, \mu\right) v \\ &\left. + \mathcal{E}\left[\sum_{s'} q\left(z, K, E, X, \mu, z'\right) W\left(s', l', j', z', K', E', X', \mu'\right) Q\left(s, s'\right) \mid z \right] \right\} \end{aligned}$$

subject to

$$\begin{aligned} n &= l + h - f & (4) \\ \pi_n l &\leq f & (5) \\ f &\leq l & (6) \\ h &\leq j & (7) \\ l' &= n & (8) \\ j' &= v & (9) \\ \left(K', E', X', \mu'\right) &= L\left(z, K, E, X, \mu\right). & (10) \end{aligned}$$

Let $\xi(s,l,j,z,K,E,X,\mu)$, $\alpha(s,l,j,z,K,E,X,\mu)$, $\chi(s,l,j,z,K,E,X,\mu)$, and $\eta(s,l,j,z,K,E,X,\mu)$ be the Lagrange multipliers for constraints (4)-(7), respectively. The first order conditions and envelope conditions are then the following:

$$e^z sF_k\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] \leq r^k\left(z,K,E,X,\mu\right), \ (= \text{ if } k(s,l,j,z,K,E,X,\mu) > 0)$$

$$\begin{aligned} & p^u\left(z,K,E,X,\mu\right) - \xi(s,l,j,z,K,E,X,\mu) + \alpha(s,l,j,z,K,E,X,\mu) - \chi(s,l,j,z,K,E,X,\mu) \\ \leq \ & 0, \ (= \text{ if } f(s,l,j,z,K,E,X,\mu) > 0) \end{aligned}$$

$$-p^e\left(z,K,E,X,\mu\right) + \xi(s,l,j,z,K,E,X,\mu) - \eta(s,l,j,z,K,E,X,\mu) \leq 0, \ (= \text{ if } h(s,l,j,z,K,E,X,\mu) > 0)$$

$$\begin{aligned} & e^z sF_n\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] - \xi(s,l,j,z,K,E,X,\mu) \\ & + \mathcal{E}\left[\sum_{s'} q\left(z,K,E,X,\mu,z'\right) W_l\left(s',l',j',z',K',E',X',\mu'\right) Q\left(s,s'\right) \mid z\right] \\ \leq \ & 0, \ (= \text{ if } n(s,l,j,z,K,E,X,\mu) > 0) \end{aligned}$$

$$\begin{aligned} & -p^v\left(z,K,E,X,\mu\right) + \mathcal{E}\left[\sum_{s'} q\left(z,K,E,X,\mu,z'\right) W_j\left(s',l',j',z',K',E',X',\mu'\right) Q\left(s,s'\right) \mid z\right] \\ \leq \ & 0, \ (= \text{ if } v(s,l,j,z,K,E,X,\mu) > 0) \end{aligned}$$

$$\alpha(s,l,j,z,K,E,X,\mu)\left[f(s,l,j,z,K,E,X,\mu) - \pi_n l\right] = 0$$

$$\chi(s,l,j,z,K,E,X,\mu)\left[l - f(s,l,j,z,K,E,X,\mu)\right] = 0$$

$$\eta(s,l,j,z,K,E,X,\mu)\left[j - h(s,l,j,z,K,E,X,\mu)\right] = 0$$

$$W_l(s,l,j,z,K,E,X,\mu) = \xi(s,l,j,z,K,E,X,\mu) - \pi_n\alpha(s,l,j,z,K,E,X,\mu) + \chi(s,l,j,z,K,E,X,\mu)$$

$$W_j(s,l,j,z,K,E,X,\mu) = \eta(s,l,j,z,K,E,X,\mu)$$

1.3 Recruitment company's problem

The Bellman equation of the recruitment company is:

$$\begin{aligned} R\left(e,x,z,K,E,X,\mu\right) \ = \ & \max_{\{a,b,d,u\}} \{p^e\left(z,K,E,X,\mu\right) d + p^v\left(z,K,E,X,\mu\right) b \\ & + p^u\left(z,K,E,X,\mu\right)\left[x + e - d - u\right] + r^u\left(z,K,E,X,\mu\right) u - a \\ & + \mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right) R\left(e',x',z',K',E',X',\mu'\right) \mid z\right]\} \end{aligned}$$

subject to

$$
\begin{aligned}
d &\leq (1-\pi_u)\,e &\quad (11)\\
b &= H(a,u,A,U)\\
e' &= G(a,u,A,U)\\
x' &= u - G(a,u,A,U)\\
A &= A\,(z,K,E,X,\mu)\\
U &= U\,(z,K,E,X,\mu)\\
(K',E',X',\mu') &= L\,(z,K,E,X,\mu)\,.
\end{aligned}
$$

Let $\varsigma\,(e,x,z,K,E,X,\mu)$ be the Lagrange multiplier for constraint (11). The first order conditions and envelope conditions are then the following:

$$
\begin{aligned}
1 &= p^v\,(z,K,E,X,\mu)\,H_a(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A,U)\\
&+\mathcal{E}\,[q\,(z,K,E,X,\mu,z')\,R_e\,(e',x',z',K',E',X',\mu')\,G_a(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A,U) \mid z]\\
&-\mathcal{E}\,[q\,(z,K,E,X,\mu,z')\,R_x\,(e',x',z',K',E',X',\mu')\,G_a(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A,U) \mid z]
\end{aligned}
$$

$$
\begin{aligned}
0 &= p^v\,(z,K,E,X,\mu)\,H_u(a(e,x,z,K,E,X,\mu) - p^u\,(z,K,E,X,\mu) + r^u\,(z,K,E,X,\mu)\\
&+\mathcal{E}\,[q\,(z,K,E,X,\mu,z')\,R_e\,(e',x',z',K',E',X',\mu')\,G_u(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A,U) \mid z]\\
&+\mathcal{E}\,[q\,(z,K,E,X,\mu,z')\,R_x\,(e',x',z',K',E',X',\mu') \mid z]\\
&-\mathcal{E}\,[q\,(z,K,E,X,\mu,z')\,R_x\,(e',x',z',K',E',X',\mu')\,G_u(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A,U) \mid z]
\end{aligned}
$$

$$p^e\,(z,K,E,X,\mu) - p^u\,(z,K,E,X,\mu) - \varsigma\,(e,x,z,K,E,X,\mu) = 0$$

$$\varsigma\,(e,x,z,K,E,X,\mu)\,[(1-\pi_u)\,e - d(e,x,z,K,E,X,\mu)] = 0$$

$$R_e\,(e,x,z,K,E,X,\mu) = p^u\,(z,K,E,X,\mu) + (1-\pi_u)\,\varsigma\,(e,x,z,K,E,X,\mu)$$

$$R_x\,(e,x,z,K,E,X,\mu) = p^u\,(z,K,E,X,\mu)$$

1.4 Conditions for a recursive competitive equilibrium (RCE)

The necessary and sufficient conditions for $\{B,\ W,\ R,\ c,\ i,\ m,\ n,\ k,\ f,\ h,\ v,\ a,\ b,\ d,\ u,\ A,\ U,\ L,\ \Pi,\ r^k,\ r^u,\ p^u,\ p^e,\ p^v,\ q\}$ to be a RCE is that there exist Lagrange multipliers $\lambda\,(\kappa,z,K,E,X,\mu)$, $\xi(s,l,j,z,K,E,X,\mu)$, $\alpha(s,l,j,z,K,E,X,\mu)$, $\chi(s,l,j,z,K,E,X,\mu)$, and $\eta(s,l,j,z,K,E,X,\mu)$ such that equations (12)-(40) hold (equations 41 through 50 are merely definitional).

$$c\,(\kappa,z,K,E,X,\mu)^{-\sigma} = \lambda\,(\kappa,z,K,E,X,\mu) \tag{12}$$

$$1 = \mathcal{E}\left[\beta\frac{\lambda\,[(1-\delta)\,\kappa + i\,(\kappa,z,K,E,X,\mu)\,,z',L\,(z,K,E,X,\mu)]}{\lambda\,(\kappa,z,K,E,X,\mu)}\left[1-\delta+r^k\,(z',L\,(z,K,E,X,\mu))\right] \mid z\right] \tag{13}$$

$$e^z sF_k\,[n(s,l,j,z,K,E,X,\mu),k(s,l,j,z,K,E,X,\mu)] \leq r^k\,(z,K,E,X,\mu)\,,\ (=\ \text{if } k(s,l,j,z,K,E,X,\mu)>0) \tag{14}$$

$$
\begin{aligned}
0 \;=\;& r^{u}\left(z,K,E,X,\mu\right)-p^{u}(z,K,E,X,\mu)\\
&+p^{v}\left(z,K,E,X,\mu\right)H_{u}(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\\
&+\left(1-\pi_{u}\right)G_{u}(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\times\\
&\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)\left[p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)-p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\right]\mid z\right]\\
&+\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\mid z\right] \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
1 \;=\;& p^{v}\left(z,K,E,X,\mu\right)H_{a}(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\\
&+\left(1-\pi_{u}\right)G_{a}(a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\times\\
&\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)\left[p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)-p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\right]\mid z\right] \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}\left[\sum_{s'}q\left(z,K,E,X,\mu,z'\right)\eta\left(s',n(s,l,j,z,K,E,X,\mu),v(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\
\leq\;& p^{v}\left(z,K,E,X,\mu\right),\; (=\text{ if } v(s,l,j,z,K,E,X,\mu)>0) \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&e^{z}sF_{n}\left[n(s,l,j,z,K,E,X,\mu),k(s,l,j,z,K,E,X,\mu)\right]\\
&+\mathcal{E}\left[\sum_{s'}q\left(z,K,E,X,\mu,z'\right)\xi\left(s',n(s,l,j,z,K,E,X,\mu),v(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\
&-\pi_{n}\mathcal{E}\left[\sum_{s'}q\left(z,K,E,X,\mu,z'\right)\alpha\left(s',n(s,l,j,z,K,E,X,\mu),v(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\
&+\mathcal{E}\left[\sum_{s'}q\left(z,K,E,X,\mu,z'\right)\chi\left(s',n(s,l,j,z,K,E,X,\mu),v(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\
\leq\;& \xi(s,l,j,z,K,E,X,\mu),\; (=\text{ if } n(s,l,j,z,K,E,X,\mu)>0) \qquad (18)
\end{aligned}
$$

$$
-p^{e}\left(z,K,E,X,\mu\right)+\xi(s,l,j,z,K,E,X,\mu)\leq\eta(s,l,j,z,K,E,X,\mu),\; (=\text{ if } h(s,l,j,z,K,E,X,\mu)>0) \qquad (19)
$$

$$
\begin{aligned}
&p^{u}\left(z,K,E,X,\mu\right)-\xi(s,l,j,z,K,E,X,\mu)+\alpha(s,l,j,z,K,E,X,\mu)-\chi(s,l,j,z,K,E,X,\mu)\\
\leq\;& 0,\; (=\text{ if } f(s,l,j,z,K,E,X,\mu)>0) \qquad (20)
\end{aligned}
$$

$$
n(s,l,j,z,K,E,X,\mu)=l+h(s,l,j,z,K,E,X,\mu)-f(s,l,j,z,K,E,X,\mu) \qquad (21)
$$

$$
h(s,l,j,z,K,E,X,\mu)\leq j \qquad (22)
$$

$$
\pi_{n}l\leq f(s,l,j,z,K,E,X,\mu) \qquad (23)
$$

$$
f(s,l,j,z,K,E,X,\mu)\leq l \qquad (24)
$$

$$
\eta(s,l,j,z,K,E,X,\mu)\left[j-h(s,l,j,z,K,E,X,\mu)\right]=0 \qquad (25)
$$

$$
\alpha(s,l,j,z,K,E,X,\mu)\left[f(s,l,j,z,K,E,X,\mu)-\pi_{n}l\right]=0 \qquad (26)
$$

$$
\chi(s,l,j,z,K,E,X,\mu)\left[l-f(s,l,j,z,K,E,X,\mu)\right]=0 \qquad (27)
$$

$$
b(e,x,z,K,E,X,\mu)=H\left[a(e,x,z,K,E,X,\mu),u(e,x,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right)\right] \qquad (28)
$$

$$d(e,x,z,K,E,X,\mu) \leq (1-\pi_u)\, e \tag{29}$$

$$0 = \left[p^e(z,K,E,X,\mu) - p^u(z,K,E,X,\mu)\right]\left[(1-\pi_u)\, e - d(e,x,z,K,E,X,\mu)\right] \tag{30}$$

$$r^u(z,K,E,X,\mu) = \varphi\lambda(\kappa,z,K,E,X,\mu)^{-1} \tag{31}$$

$$q(z,K,E,X,\mu,z') = \beta \frac{c(K,z,K,E,X,\mu)^{\sigma}}{c\left((1-\delta)K + i(K,z,K,E,X,\mu), z', L(z,K,E,X,\mu)\right)^{\sigma}} \tag{32}$$

$$\begin{aligned} & c(K,z,K,E,X,\mu) + i(K,z,K,E,X,\mu) + a(E,X,z,K,E,X,\mu) \\ = \quad & \sum_s \int e^z sF\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right]\mu(s, dl \times dj) \end{aligned} \tag{33}$$

$$\sum_s \int k(s,l,j,z,K,E,X,\mu)\mu(s,dl\times dj) = K \tag{34}$$

$$\sum_s \int v(s,l,j,z,K,E,X,\mu)\mu(s,dl\times dj) = b(E,X,z,K,E,X,\mu) \tag{35}$$

$$u(E,X,z,K,E,X,\mu) = X + E - d(E,X,z,K,E,X,\mu) + \sum_s \int f(s,l,j,z,K,E,X,\mu)\mu(s,dl\times dj) \tag{36}$$

$$d(E,X,z,K,E,X,\mu) = \sum_s \int h(s,l,j,z,K,E,X,\mu)\mu(s,dl\times dj) \tag{37}$$

$$m(K,z,K,E,X,\mu) = u(E,X,z,K,E,X,\mu) \tag{38}$$

$$A(z,K,E,X,\mu) = a(E,X,z,K,E,X,\mu) \tag{39}$$

$$U(z,K,E,X,\mu) = u(E,X,z,K,E,X,\mu) \tag{40}$$

$$\begin{aligned} B(\kappa,z,K,E,X,\mu) \quad = \quad & \frac{c(\kappa,z,K,E,X,\mu)^{1-\sigma} - 1}{1-\sigma} + \varphi m(\kappa,z,K,E,X,\mu) && (41) \\ & + \beta\mathcal{E}\left\{B\left[(1-\delta)\kappa + i(\kappa,z,K,E,X,\mu), z', L(z,K,E,X,\mu)\right] \mid z\right\} && (42) \end{aligned}$$

$$\begin{aligned} W(s,l,j,z,K,E,X,\mu) \quad = \quad & e^z sF\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] \\ & + p^u(z,K,E,X,\mu)\, f(s,l,j,z,K,E,X,\mu) - p^e(z,K,E,X,\mu)\, h(s,l,j,z,K,E,X,\mu) \\ & - r^k(z,K,E,X,\mu)\, k(s,l,j,z,K,E,X,\mu) - p^v(z,K,E,X,\mu)\, v(s,l,j,z,K,E,X,\mu) \end{aligned}$$

$$+\mathcal{E}\left[\sum_{s'} q(z,K,E,X,\mu,z')\, W\left(s', n(s,l,j,z,K,E,X,\mu), v(s,l,j,z,K,E,X,\mu), z', L(z,K,E,X,\mu)\right) Q(s,s') \mid z\right] \tag{43}$$

$$\begin{aligned} R(e,x,z,K,E,X,\mu) \quad = \quad & p^e(z,K,E,X,\mu)\, d(e,x,z,K,E,X,\mu) + p^v(z,K,E,X,\mu)\, b(e,x,z,K,E,X,\mu) \\ & + p^u(z,K,E,X,\mu)\left[x + e - d(e,x,z,K,E,X,\mu) - u(e,x,z,K,E,X,\mu)\right] \\ & + r^u(z,K,E,X,\mu)\, u(e,x,z,K,E,X,\mu) - a(e,x,z,K,E,X,\mu) \end{aligned}$$

$$+\mathcal{E}\left[q(z,K,E,X,\mu,z')\, R\left(\begin{array}{c} G\left[\begin{array}{c} a(e,x,z,K,E,X,\mu), u(e,x,z,K,E,X,\mu), \\ A(z,K,E,X,\mu), U(z,K,E,X,\mu) \end{array}\right], \\ u(e,x,z,K,E,X,\mu) - G\left[\begin{array}{c} a(e,x,z,K,E,X,\mu), u(e,x,z,K,E,X,\mu), \\ A(z,K,E,X,\mu), U(z,K,E,X,\mu) \end{array}\right], \\ z', L(z,K,E,X,\mu) \end{array}\right) \mid z \right] \tag{44}$$

$$\left(K',E',X',\mu'\right)=L\left(z,K,E,X,\mu\right) \tag{45}$$

is given by:

$$K'=\left(1-\delta\right)K+i\left(K,z,K,E,X,\mu\right) \tag{46}$$

$$E'=G\left[a(E,X,z,K,E,X,\mu),u(E,X,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right)\right] \tag{47}$$

$$X'=u(E,X,z,K,E,X,\mu)-G\left[a(E,X,z,K,E,X,\mu),u(E,X,z,K,E,X,\mu),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right)\right] \tag{48}$$

$$\mu'\left(s',\mathcal{L}\times\mathcal{J}\right)=\sum_{s}\int_{\mathcal{B}(s,\mathcal{L}\times\mathcal{J})}Q\left(s,s'\right)\mu\left(s,dl\times dj\right)+\varrho\psi\left(s'\right)\mathcal{I}\left(\mathcal{L}\times\mathcal{J}\right) \tag{49}$$

where

$$\mathcal{B}\left(s,\mathcal{L}\times\mathcal{J}\right)=\left\{(l,j):n(s,l,j,z,K,E,X,\mu)\in\mathcal{L}\text{ and }v(s,l,j,z,K,E,X,\mu)\in\mathcal{J}\right\} \tag{50}$$

1.5 Equilibrium allocations and prices

Evaluate equations (12)-(40) at $(\kappa,e,X)=(K,E,X)$ and eliminate m, a, b, d and u to get:

$$r^{u}\left(z,K,E,X,\mu\right)=\varphi c\left(K,z,K,E,X,\mu\right)^{\sigma} \tag{51}$$

$$1=\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)\left[1-\delta+r^{k}\left(z',L\left(z,K,E,X,\mu\right)\right)\right]\mid z\right] \tag{52}$$

$$\begin{aligned}p^{u}(z,K,E,X,\mu) &= r^{u}\left(z,K,E,X,\mu\right)\\ &+p^{v}\left(z,K,E,X,\mu\right)H_{u}(A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\\ &+\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\mid z\right]\\ &+\left(1-\pi_{u}\right)G_{u}(A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\times\\ &\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)\left[p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)-p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\right]\mid z\right]\end{aligned} \tag{53}$$

$$\begin{aligned}1 &= p^{v}\left(z,K,E,X,\mu\right)H_{a}(A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\\ &+\left(1-\pi_{u}\right)G_{a}(A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right),A\left(z,K,E,X,\mu\right),U\left(z,K,E,X,\mu\right))\times\\ &\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)\left[p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)-p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\right]\mid z\right]\end{aligned} \tag{54}$$

$$e^{z}sF_{k}\left[n(s,l,j,z,K,E,X,\mu),k(s,l,j,z,K,E,X,\mu)\right]\leq r^{k}\left(z,K,E,X,\mu\right),\ (=\text{ if }k(s,l,j,z,K,E,X,\mu)>0) \tag{55}$$

$$\begin{aligned}&\mathcal{E}\left[\sum_{s'}q\left(z,K,E,X,\mu,z'\right)\eta\left(s',n(s,l,j,z,K,E,X,\mu),v(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\ \leq\ &p^{v}\left(z,K,E,X,\mu\right),\ (=\text{ if }v(s,l,j,z,K,E,X,\mu)>0)\end{aligned} \tag{56}$$

$$
\begin{aligned}
& e^z sF_n \left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] \\
& +\mathcal{E}\left[\sum_{s'} q\left(z,K,E,X,\mu,z'\right)\xi\left(s', n(s,l,j,z,K,E,X,\mu), v(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right] \\
& -\pi_n\mathcal{E}\left[\sum_{s'} q\left(z,K,E,X,\mu,z'\right)\alpha\left(s', n(s,l,j,z,K,E,X,\mu), v(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right] \\
& +\mathcal{E}\left[\sum_{s'} q\left(z,K,E,X,\mu,z'\right)\chi\left(s', n(s,l,j,z,K,E,X,\mu), v(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right] \\
\leq \quad & \xi(s,l,j,z,K,E,X,\mu), \;\; (= \text{ if } n(s,l,j,z,K,E,X,\mu) > 0) \qquad (57)
\end{aligned}
$$

$$
-p^e\left(z,K,E,X,\mu\right) + \xi(s,l,j,z,K,E,X,\mu) \leq \eta(s,l,j,z,K,E,X,\mu), \;\; (= \text{ if } h(s,l,j,z,K,E,X,\mu) > 0) \qquad (58)
$$

$$
\begin{aligned}
& p^u\left(z,K,E,X,\mu\right) - \xi(s,l,j,z,K,E,X,\mu) + \alpha(s,l,j,z,K,E,X,\mu) - \chi(s,l,j,z,K,E,X,\mu) \\
\leq \quad & 0, \;\; (= \text{ if } f(s,l,j,z,K,E,X,\mu) > 0) \qquad (59)
\end{aligned}
$$

$$
n(s,l,j,z,K,E,X,\mu) = l + h(s,l,j,z,K,E,X,\mu) - f(s,l,j,z,K,E,X,\mu) \qquad (60)
$$

$$
h(s,l,j,z,K,E,X,\mu) \leq j \qquad (61)
$$

$$
\pi_n l \leq f(s,l,j,z,K,E,X,\mu) \qquad (62)
$$

$$
f(s,l,j,z,K,E,X,\mu) \leq l \qquad (63)
$$

$$
\eta(s,l,j,z,K,E,X,\mu)\left[j - h(s,l,j,z,K,E,X,\mu)\right] = 0 \qquad (64)
$$

$$
\alpha(s,l,j,z,K,E,X,\mu)\left[f(s,l,j,z,K,E,X,\mu) - \pi_n l\right] = 0 \qquad (65)
$$

$$
\chi(s,l,j,z,K,E,X,\mu)\left[l - f(s,l,j,z,K,E,X,\mu)\right] = 0 \qquad (66)
$$

$$
\sum_s \int h(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right) \leq \left(1 - \pi_u\right) E \qquad (67)
$$

$$
0 = \left[p^e\left(z,K,E,X,\mu\right) - p^u\left(z,K,E,X,\mu\right)\right]\left[\left(1-\pi_u\right)E - \sum_s \int h(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right)\right] \qquad (68)
$$

$$
q(z,K,E,X,\mu,z') = \beta \frac{c\left(K,z,K,E,X,\mu\right)^{\sigma}}{c\left(\left(1-\delta\right)K + i\left(K,z,K,E,X,\mu\right), z', L\left(z,K,E,X,\mu\right)\right)^{\sigma}} \qquad (69)
$$

$$
\begin{aligned}
& c\left(K,z,K,E,X,\mu\right) + i\left(K,z,K,E,X,\mu\right) + A\left(z,K,E,X,\mu\right) \qquad (70) \\
= \quad & \sum_s \int e^z sF\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right]\mu\left(s, dl \times dj\right) \qquad (71)
\end{aligned}
$$

$$
\sum_s \int k(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right) = K \qquad (72)
$$

$$
\sum_s \int v(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right) = H\left[A\left(z,K,E,X,\mu\right), U\left(z,K,E,X,\mu\right), A\left(z,K,E,X,\mu\right), U\left(z,K,E,X,\mu\right)\right] \qquad (73)
$$

$$
U\left(z,K,E,X,\mu\right) = X + E - \sum_s \int h(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right) + \sum_s \int f(s,l,j,z,K,E,X,\mu)\mu\left(s, dl \times dj\right) \qquad (74)
$$

$$
\left(K', E', X', \mu'\right) = L\left(z,K,E,X,\mu\right) \qquad (75)
$$

is given by:

$$K' = (1-\delta) K + i (K, z, K, E, X, \mu) \tag{76}$$

$$E' = G [A (z, K, E, X, \mu), U (z, K, E, X, \mu), A (z, K, E, X, \mu), U (z, K, E, X, \mu)] \tag{77}$$

$$X' = U (z, K, E, X, \mu) - G [A (z, K, E, X, \mu), U (z, K, E, X, \mu), A (z, K, E, X, \mu), U (z, K, E, X, \mu)] \tag{78}$$

$$\mu' (s', \mathcal{L} \times \mathcal{J}) = \sum_s \int_{\{(l,j): n(s,l,j,z,K,E,X,\mu) \in \mathcal{L} \text{ and } v(s,l,j,z,K,E,X,\mu) \in \mathcal{J}\}} Q (s, s') \mu (s, dl \times dj) + \varrho \psi (s') \mathcal{I} (\mathcal{L} \times \mathcal{J}) \tag{79}$$

2 Characterization of establishments' decision rules

It will be convenient to truncate the establishments' problem to a finite horizon T. The truncated problem is given by

$$\begin{aligned} W^{t,T}(s, l, j, z, K, E, X, \mu) \quad = \quad & \max_{\{f,h,k,n,v\}} \{ e^z s F (n, k) + p^u (z, K, E, X, \mu) f - p^e (z, K, E, X, \mu) h \\ & - r^k (z, K, E, X, \mu) k - p^v (z, K, E, X, \mu) v \\ & + \mathcal{E} \left[\sum_{s'} q (z, K, E, X, \mu, z') W^{t+1,T} (s', l', j', z', K', E', X', \mu') Q (s, s') \mid z \right] \Bigg\} \end{aligned}$$

subject to

$$n \leq l + h - f \tag{80}$$

$$\pi_n l \leq f \tag{81}$$

$$f \leq l \tag{82}$$

$$h \leq j \tag{83}$$

$$l' = n \tag{84}$$

$$j' = v \tag{85}$$

$$(K', E', X', \mu') = L (z, K, E, X, \mu) \tag{86}$$

for $t = 0, 1, ..., T$, where

$$W^{T+1,T}(s, l, j, z, K, E, X, \mu) = 0.$$

In what follows, it will be assumed that

$$\mathcal{E} [q (z, K, E, X, \mu, z') p^u (z', L (z, K, E, X, \mu)) \mid z] \leq p^u (z, K, E, X, \mu),$$

and that

$$p^u (z, K, E, X, \mu) \leq p^e (z, K, E, X, \mu),$$

since these are properties that will be satisfied in equilibrium.

Let $\xi_{t,T}(s,l,j,z,K,E,X,\mu)$, $\alpha_{t,T}(s,l,j,z,K,E,X,\mu)$, $\chi_{t,T}(s,l,j,z,K,E,X,\mu)$, and $\eta_{t,T}(s,l,j,z,K,E,X,\mu)$ be the (non-negative) Lagrange multipliers for constraints (80)-(83), respectively. From the first-order and envelope conditions we get for $t = 0, 1, ..., T$, the following.

$$e^z sF_k \left[n_{t,T}(s,l,j,z,K,E,X,\mu), k_{t,T}(s,l,j,z,K,E,X,\mu)\right] \leq r^k \left(z,K,E,X,\mu\right), \; (= \text{ if } k_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \tag{87}$$

$$n_{t,T}(s,l,j,z,K,E,X,\mu) = l + h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu) \tag{88}$$

$$\begin{aligned} & p^u \left(z,K,E,X,\mu\right) - \xi_{t,T}(s,l,j,z,K,E,X,\mu) + \alpha_{t,T}(s,l,j,z,K,E,X,\mu) - \chi_{t,T}(s,l,j,z,K,E,X,\mu) \\ \leq \; & 0, \; (= \text{ if } f_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \end{aligned} \tag{89}$$

$$-p^e \left(z,K,E,X,\mu\right) + \xi_{t,T}(s,l,j,z,K,E,X,\mu) - \eta_{t,T}(s,l,j,z,K,E,X,\mu) \leq 0, \; (= \text{ if } h_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \tag{90}$$

$$\begin{aligned} & e^z sF_n \left[n_{t,T}(s,l,j,z,K,E,X,\mu), k_{t,T}(s,l,j,z,K,E,X,\mu)\right] - \xi_{t,T}(s,l,j,z,K,E,X,\mu) \\ & + \mathcal{E}\left\{\sum_{s'} q\left(z,K,E,X,\mu,z'\right) \times \right. \\ & \left. W_l^{t+1,T}\left(s', n_{t,T}(s,l,j,z,K,E,X,\mu), v_{t,T}(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right\} \\ \leq \; & 0, \; (= \text{ if } n_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \end{aligned} \tag{91}$$

$$\begin{aligned} & \mathcal{E}\left\{\sum_{s'} q\left(z,K,E,X,\mu,z'\right) \times \right. \\ & \left. W_j^{t+1,T}\left(s', n_{t,T}(s,l,j,z,K,E,X,\mu), v_{t,T}(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right\} \\ \leq \; & p^v \left(z,K,E,X,\mu\right), \; (= \text{ if } v_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \end{aligned} \tag{92}$$

$$\alpha_{t,T}(s,l,j,z,K,E,X,\mu)\left[f_{t,T}(s,l,j,z,K,E,X,\mu) - \pi_n l\right] = 0 \tag{93}$$

$$\chi_{t,T}(s,l,j,z,K,E,X,\mu)\left[l - f_{t,T}(s,l,j,z,K,E,X,\mu)\right] = 0 \tag{94}$$

$$\eta_{t,T}(s,l,j,z,K,E,X,\mu)\left[j - h_{t,T}(s,l,j,z,K,E,X,\mu)\right] = 0 \tag{95}$$

$$W_l^{t,T}(s,l,j,z,K,E,X,\mu) = \xi_{t,T}(s,l,j,z,K,E,X,\mu) - \pi_n \alpha_{t,T}(s,l,j,z,K,E,X,\mu) + \chi_{t,T}(s,l,j,z,K,E,X,\mu) \tag{96}$$

$$W_j^{t,T}(s,l,j,z,K,E,X,\mu) = \eta_{t,T}(s,l,j,z,K,E,X,\mu) \tag{97}$$

$$W_l^{T+1,T}(s,l,j,z,K,E,X,\mu) = 0 \tag{98}$$

$$W_j^{T+1,T}(s,l,j,z,K,E,X,\mu) = 0 \tag{99}$$

Lemma 1 *Let* $\left\{\xi_{t,T}(0,l,j,z,K,E,X,\mu)\right\}_{t=0}^{T}$ *be any sequence of functions satisfying that*

$$\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right) p^u\left(z', L\left(z,K,E,X,\mu\right)\right) \mid z\right] \leq \xi_{t,T}(0,l,j,z,K,E,X,\mu) \leq p^u\left(z,K,E,X,\mu\right), \; \textit{for } t < T$$

and that

$$\xi_{T,T}(0,l,j,z,K,E,X,\mu) \leq p^u\left(z,K,E,X,\mu\right)$$

Then, $\left\{\xi_{t,T}(0,l,j,z,K,E,X,\mu)\right\}_{t=0}^{T}$ *together with*

$$n_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$k_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$f_{t,T}(0,l,j,z,K,E,X,\mu) = l$$

$$h_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$v_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$\alpha_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$\eta_{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

$$\chi_{t,T}(0,l,j,z,K,E,X,\mu) = p^u\left(z,K,E,X,\mu\right) - \xi_{t,T}(0,l,j,z,K,E,X,\mu)$$

$$W_l^{t,T}(0,l,j,z,K,E,X,\mu) = p^u\left(z,K,E,X,\mu\right)$$

$$W_j^{t,T}(0,l,j,z,K,E,X,\mu) = 0$$

for $t = 0,...,T$*, satisfy equations (87)-(99).*

Proof. It is a straightforward verification. ■

To simplify the subsequent analysis it will be convenient to define two functions $\hat{k}\left(n,z,s,r\right)$ and $\hat{F}_n\left(n,z,s,r\right)$ as follows. For $n > 0$ and $s > 0$, they are given by:

$$e^z s F_k\left[n,\hat{k}\left(n,z,s,r\right)\right] = r,$$

$$\hat{F}_n\left(n,z,s,r\right) = e^z s F_n\left[n,\hat{k}\left(n,z,s,r\right)\right].$$

Observe that $\hat{F}_n\left(n,z,s,r\right)$ is strictly decreasing, that $\lim_{n\to\infty}\hat{F}_n\left(n,z,s,r\right) = 0$, and that $\lim_{n\to 0}\hat{F}_n\left(n,z,s,r\right) = +\infty$.

Lemma 2 *Suppose that* $s > 0$ *and that* $l + j > 0$*. Then,*

$$k_{t,T}(s,l,j,z,K,E,X,\mu) = \hat{k}\left[n_{t,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right] \quad (100)$$

$$n_{t,T}(s,l,j,z,K,E,X,\mu) = l + h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu) \quad (101)$$

$$\begin{aligned} & p^u\left(z,K,E,X,\mu\right) - \xi_{t,T}(s,l,j,z,K,E,X,\mu) + \alpha_{t,T}(s,l,j,z,K,E,X,\mu) - \chi_{t,T}(s,l,j,z,K,E,X,\mu) \\ \leq \; & 0, \; (= \text{ if } f_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \end{aligned} \quad (102)$$

$$-p^e\left(z,K,E,X,\mu\right) + \xi_{t,T}(s,l,j,z,K,E,X,\mu) - \eta_{t,T}(s,l,j,z,K,E,X,\mu) \leq 0, \; (= \text{ if } h_{t,T}(s,l,j,z,K,E,X,\mu) > 0) \quad (103)$$

$$
\begin{aligned}
& \hat{F}_n\left(n_{t,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right) \\
& +\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \left.\xi_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right\} \\
& -\pi_n\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \left.\alpha_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right\} \\
& +\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \left.\chi_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right\} \\
& +Q\left(s,0\right)\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)p^u\left(z',L\left(z,K,E,X,\mu\right)\right)\mid z\right] \\
= \;& \xi_{t,T}(s,l,j,z,K,E,X,\mu),\ \textit{for } t<T
\end{aligned}
\tag{104}
$$

$$
\xi_{T,T}(s,l,j,z,K,E,X,\mu)=\hat{F}_n\left(n_{T,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right)
\tag{105}
$$

$$
\begin{aligned}
& \mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \left.\eta_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right\} \\
\leq \;& p^v\left(z,K,E,X,\mu\right),\ \left(=\ \textit{if } v_{t,T}(s,l,j,z,K,E,X,\mu)>0\right),\ \textit{for } t<T
\end{aligned}
\tag{106}
$$

$$
v_{T,T}(s,l,j,z,K,E,X,\mu)=0
\tag{107}
$$

$$
\alpha_{t,T}(s,l,j,z,K,E,X,\mu)\left[f_{t,T}(s,l,j,z,K,E,X,\mu)-\pi_n l\right]=0
\tag{108}
$$

$$
\chi_{t,T}(s,l,j,z,K,E,X,\mu)\left[l-f_{t,T}(s,l,j,z,K,E,X,\mu)\right]=0
\tag{109}
$$

$$
\eta_{t,T}(s,l,j,z,K,E,X,\mu)\left[j-h_{t,T}(s,l,j,z,K,E,X,\mu)\right]=0.
\tag{110}
$$

Moreover, there is no loss of generality in assuming that

$$
\eta_{t,T}(s,l,j,z,K,E,X,\mu)=\max\left\{\xi_{t,T}(s,l,j,z,K,E,X,\mu)-p^e\left(z,K,E,X,\mu\right),0\right\}
\tag{111}
$$

$$
\chi_{t,T}(s,l,j,z,K,E,X,\mu)=0
\tag{112}
$$

$$
\alpha_{t,T}(s,0,j,z,K,E,X,\mu)=\xi_{t,T}(s,0,j,z,K,E,X,\mu)-p^e\left(z,K,E,X,\mu\right)
\tag{113}
$$

Proof. *Equations (99)-(110) follow from equations (87)-(99), Lemma 1 and the fact that F satisfies the Inada conditions (and therefore that $n_{t,T}(s,l,j,z,K,E,X,\mu)>0$ and $k_{t,T}(s,l,j,z,K,E,X,\mu)>0$).*

Need to show that $\eta_{t,T}(s,l,j,z,K,E,X,\mu)$ can be restricted as in equation (111).

First, consider the case in which $j>0$.

If $h_{t,T}(s,l,j,z,K,E,X,\mu)>0$, from equation (103) we have that

$$
\xi_{t,T}(s,l,j,z,K,E,X,\mu)-p^e\left(z,K,E,X,\mu\right)=\eta_{t,T}(s,l,j,z,K,E,X,\mu)\geq 0,
$$

where the inequality follows from the fact that $\eta_{t,T}(s,l,j,z,K,E,X,\mu)$ is a Lagrange multiplier.

If $h_{t,T}(s,l,j,z,K,E,X,\mu)=0$, from equations (103) and (110) we have that:

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)-p^e\left(z,K,E,X,\mu\right)\leq\eta_{t,T}(s,l,j,z,K,E,X,\mu)=0.$$

Hence, when $j>0$, $\eta_{t,T}(s,l,j,z,K,E,X,\mu)$ must satisfy equation (111).

When $j=0$, equation (110) imposes no restriction on $\eta_{t,T}(s,l,0,z,K,E,X,\mu)$. The only restrictions (from equations (103) and (106)) are that $\left\{\eta_{t,T}(s,l,0,z,K,E,X,\mu)\right\}_{t=0}^{T}$ must satisfy for $t\leq T$ that

$$\xi_{t,T}(s,l,0,z,K,E,X,\mu)-p^e\left(z,K,E,X,\mu\right)\leq\eta_{t,T}(s,l,0,z,K,E,X,\mu),$$

and for $t<T$ that

$$\begin{aligned}&\mathcal{E}\Bigg\{\sum_{s'>0}q\left(z,K,E,X,\mu,z'\right)\times\\&\eta_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),0,z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\Big\}\\\leq\;&p^v\left(z,K,E,X,\mu\right),\ \textit{for}\ (s,l,j,z,K,E,X,\mu)\ \textit{such that}\ v_{t,T}(s,l,j,z,K,E,X,\mu)=0.\end{aligned}$$

Thus there is no loss of generality in restricting $\eta_{t,T}(s,l,0,z,K,E,X,\mu)$ as in equation (111).

Finally, we need to show that $\chi_{t,T}(s,l,j,z,K,E,X,\mu)$ can be restricted as in equation (112).

If $l>0$, equation (112) must hold because of equation (109) and because F satisfies the Inada conditions (and therefore that $n_{t,T}(s,l,j,z,K,E,X,\mu)>0$ and $k_{t,T}(s,l,j,z,K,E,X,\mu)>0$).

If $l=0$, equation (109) imposes no restriction on $\chi_{t,T}(s,l,j,z,K,E,X,\mu)$ and equation (108) imposes no restriction on $\alpha_{t,T}(s,l,j,z,K,E,X,\mu)$. Observe that equation (102) is satisfied under equations (112) and (113). Also, observe that the variables

$$\alpha_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)$$

and

$$\chi_{t+1,T}\left(s',n_{t,T}(s,l,j,z,K,E,X,\mu),v_{t,T}(s,l,j,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)$$

that enter equation (104) have $n_{t,T}(s,l,j,z,K,E,X,\mu)>0$ (because F satisfies the Inada conditions) and therefore equations (112) and (113) do not apply to them.

Thus there is no loss of generality in assuming that equations (112) and (113) hold. ■

Lemma 3 *Suppose that $s>0$ and that $l+j>0$. Then,*

$$k_{t,T}(s,l,j,z,K,E,X,\mu)=\hat{k}\left[n_{t,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right]\tag{114}$$

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=l+h_{t,T}(s,l,j,z,K,E,X,\mu)-f_{t,T}(s,l,j,z,K,E,X,\mu)\tag{115}$$

$$p^e\left(z,K,E,X,\mu\right)\leq\xi_{t,T}(s,l,j,z,K,E,X,\mu),\ \textit{if}\ h_{t,T}(s,l,j,z,K,E,X,\mu)>0\tag{116}$$

$$
\begin{aligned}
& \hat{F}_n\left(n_{t,T}(s,l,j,z,K,E,X,\mu), z, s, r^k\left(z,K,E,X,\mu\right)\right) \\
& +\left(1-\pi_n\right)\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \left.\xi_{t+1,T}\left(s', n_{t,T}(s,l,j,z,K,E,X,\mu), v_{t,T}(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right\} \\
& +\pi_n\mathcal{E}\left[\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right) p^u\left(z', L\left(z,K,E,X,\mu\right)\right) Q\left(s,s'\right) \mid z\right] \\
& +Q\left(s,0\right)\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right) p^u\left(z', L\left(z,K,E,X,\mu\right)\right) \mid z\right] \\
= \; & \xi_{t,T}(s,l,j,z,K,E,X,\mu), \;\; \textit{for } t<T \qquad (117)
\end{aligned}
$$

$$
\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \hat{F}_n\left(n_{T,T}(s,l,j,z,K,E,X,\mu), z, s, r^k\left(z,K,E,X,\mu\right)\right) \qquad (118)
$$

$$
\begin{aligned}
& \mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
& \max[\xi_{t+1,T}\left(s', n_{t,T}(s,l,j,z,K,E,X,\mu), v_{t,T}(s,l,j,z,K,E,X,\mu), z', L\left(z,K,E,X,\mu\right)\right) \\
& \qquad \left.-p^e\left(z', L\left(z,K,E,X,\mu\right)\right), 0] Q\left(s,s'\right) \mid z\right\} \\
\leq \; & p^v\left(z,K,E,X,\mu\right), \; \left(= \;\; \textit{if } v_{t,T}(s,l,j,z,K,E,X,\mu)>0\right), \; \textit{for } t<T \qquad (119)
\end{aligned}
$$

$$
v_{T,T}(s,l,j,z,K,E,X,\mu) = 0 \qquad (120)
$$

$$
\xi_{t,T}(s,l,j,z,K,E,X,\mu) \geq p^u\left(z,K,E,X,\mu\right) \qquad (121)
$$

$$
\left[\xi_{t,T}(s,l,j,z,K,E,X,\mu) - p^u\left(z,K,E,X,\mu\right)\right]\left[f_{t,T}(s,l,j,z,K,E,X,\mu) - \pi_n l\right] = 0 \qquad (122)
$$

$$
\max\left[\xi_{t,T}(s,l,j,z,K,E,X,\mu) - p^e\left(z,K,E,X,\mu\right), 0\right]\left[j - h_{t,T}(s,l,j,z,K,E,X,\mu)\right] = 0 \qquad (123)
$$

Moreover, there is no loss of generality in assuming that

$$
h_{t,T}(s,l,j,z,K,E,X,\mu)\left[f_{t,T}(s,l,j,z,K,E,X,\mu) - \pi_n l\right] = 0. \qquad (124)
$$

Proof. Equations (114)-(123) are a straigthward consequence of Lemma 2.

Need to show that $h_{t,T}(s,l,j,z,K,E,X,\mu)$ and $f_{t,T}(s,l,j,z,K,E,X,\mu)$ can be restricted as in equation (124).

First consider the case in which $p^e\left(z,K,E,X,\mu\right) > p^u\left(z,K,E,X,\mu\right)$.

Suppose that $h_{t,T}(s,l,j,z,K,E,X,\mu) > 0$ and that $f_{t,T}(s,l,j,z,K,E,X,\mu) > \pi_n l$.

From equations (116) and (122)

$$
p^e\left(z,K,E,X,\mu\right) \leq \xi_{t,T}(s,l,j,z,K,E,X,\mu) = p^u\left(z,K,E,X,\mu\right),
$$

which is a contradiction. Hence, when $p^e\left(z,K,E,X,\mu\right) > p^u\left(z,K,E,X,\mu\right)$, equation (124) must hold.

Now consider the case in which $p^e\left(z,K,E,X,\mu\right) = p^u\left(z,K,E,X,\mu\right)$.

Suppose that $h_{t,T}(s,l,j,z,K,E,X,\mu) > 0$ and that $f_{t,T}(s,l,j,z,K,E,X,\mu) > \pi_n l$.

Observe, from equation (115), that

$$
n_{t,T}(s,l,j,z,K,E,X,\mu) = l + h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu)
$$

and, from equation (122), that:

$$p^u(z,K,E,X,\mu) = p^e(z,K,E,X,\mu) = \xi_{t,T}(s,l,j,z,K,E,X,\mu) \tag{125}$$

1) Suppose that $h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu) \geq -\pi_n l$.

Let

$$\begin{aligned}
\hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) &= n_{t,T}(s,l,j,z,K,E,X,\mu) - l + \pi_n l \\
\hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) &= \pi_n l
\end{aligned}$$

Observe that

$$l + \hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) - \hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) = n_{t,T}(s,l,j,z,K,E,X,\mu),$$

that

$$\begin{aligned}
\hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) &= n_{t,T}(s,l,j,z,K,E,X,\mu) - l + \pi_n l \\
&< n_{t,T}(s,l,j,z,K,E,X,\mu) - l + f_{t,T}(s,l,j,z,K,E,X,\mu) \\
&= h_{t,T}(s,l,j,z,K,E,X,\mu) \leq j
\end{aligned}$$

and that

$$\begin{aligned}
\hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) &= n_{t,T}(s,l,j,z,K,E,X,\mu) - l + \pi_n l \\
&= l + h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu) - l + \pi_n l \\
&\geq 0.
\end{aligned}$$

From equation (125) we then know that equations (114)-(124) hold.

2) Suppose that $h_{t,T}(s,l,j,z,K,E,X,\mu) - f_{t,T}(s,l,j,z,K,E,X,\mu) \leq -\pi_n l$.

Let

$$\begin{aligned}
\hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) &= 0 \\
\hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) &= l - n_{t,T}(s,l,j,z,K,E,X,\mu).
\end{aligned}$$

Observe that

$$l + \hat{h}_{t,T}(s,l,j,z,K,E,X,\mu) - \hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) = n_{t,T}(s,l,j,z,K,E,X,\mu),$$

that

$$\begin{aligned}
\hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) &= l - n_{t,T}(s,l,j,z,K,E,X,\mu) \\
&= f_{t,T}(s,l,j,z,K,E,X,\mu) - h_{t,T}(s,l,j,z,K,E,X,\mu) \geq \pi_n l
\end{aligned}$$

and that

$$\hat{f}_{t,T}(s,l,j,z,K,E,X,\mu) = l - n_{t,T}(s,l,j,z,K,E,X,\mu) < l.$$

From equation (125) we then know that equations (114)-(124) hold. ■

Lemma 4 considers the case in which $t = T$.

Lemma 4 *Suppose that $s > 0$, $l > 0$ and $j = 0$. Then,*

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \max\left\{\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right],p^u\,(z,K,E,X,\mu)\right\}$$

Proof. Since $j = 0$, $h_{T,T}(s,l,j,z,K,E,X,\mu) = 0$ and equations (116) and (123) impose no restrictions on $\xi_{T,T}(s,l,j,z,K,E,X,\mu)$. Moreover,

$$n_{T,T}(s,l,j,z,K,E,X,\mu) = l - f_{T,T}(s,l,j,z,K,E,X,\mu) \tag{126}$$

1) Consider the case that $\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right] \geq p^u\,(z,K,E,X,\mu)$.

Suppose that $f_{T,T}(s,l,j,z,K,E,X,\mu) > \pi_n l$. Then, from equations (122), (118), (126), and the fact that $\hat{F}_n$ is strictly decreasing in its first argument, we have that:

$$\begin{aligned}
p^u\,(z,K,E,X,\mu) &= \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\
&= \hat{F}_n\left[l - f_{T,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\,(z,K,E,X,\mu)\right] \\
&> \hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right],
\end{aligned}$$

which is a contradiction. Then, $f_{T,T}(s,l,j,z,K,E,X,\mu) = \pi_n l$ and

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right] \geq p^u\,(z,K,E,X,\mu)\,.$$

2) Consider the case that $\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right] < p^u\,(z,K,E,X,\mu)$.

Suppose that $f_{T,T}(s,l,j,z,K,E,X,\mu) = \pi_n l$. Then, from equations (118) and (126),

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right] < p^u\,(z,K,E,X,\mu)\,.$$

But this contradicts equation (121).

Hence, $f_{T,T}(s,l,j,z,K,E,X,\mu) > \pi_n l$ and, from equation (122),

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = p^u\,(z,K,E,X,\mu) > \hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\,(z,K,E,X,\mu)\right].$$

■

Lemma 5 *Suppose that $s > 0$, $l = 0$ and $j > 0$. Then,*

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \max\left\{\hat{F}_n\left[j,z,s,r^k\,(z,K,E,X,\mu)\right],p^e\,(z,K,E,X,\mu)\right\}$$

Proof. Since $l = 0$, $f_{T,T}(s,l,j,z,K,E,X,\mu) = 0$ and equation (122) imposes no restriction on $\xi_{T,T}(s,l,j,z,K,E,X,\mu)$. Moreover,

$$n_{T,T}(s,l,j,z,K,E,X,\mu) = h_{T,T}(s,l,j,z,K,E,X,\mu) > 0 \tag{127}$$

because F satisfies the Inada conditions.

1) Consider the case that $\hat{F}_n\left[j,z,s,r^k\,(z,K,E,X,\mu)\right] \geq p^e\,(z,K,E,X,\mu)\,.$

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu) < j$. Then, from equations (123), (118), (127), (116) and the fact that $\hat{F}_n$ is strictly decreasing in its first argument, we have that

$$\begin{aligned} p^e(z,K,E,X,\mu) &= \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\ &= \hat{F}_n\left[h_{T,T}(s,l,j,z,K,E,X,\mu), z, s, r^k(z,K,E,X,\mu)\right] \\ &> \hat{F}_n\left[j, z, s, r^k(z,K,E,X,\mu)\right], \end{aligned}$$

which is a contradiction. Then, $h_{T,T}(s,l,j,z,K,E,X,\mu) = j$ and

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \hat{F}_n\left[j, z, s, r^k(z,K,E,X,\mu)\right] \geq p^e(z,K,E,X,\mu).$$

2) Consider the case that $\hat{F}_n\left[j, z, s, r^k(z,K,E,X,\mu)\right] < p^e(z,K,E,X,\mu)$.

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu) = j$. Then, from equations (127) and (118), we have that

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \hat{F}_n\left[j, z, s, r^k(z,K,E,X,\mu)\right] < p^e(z,K,E,X,\mu).$$

But this contradicts equation (116).

Hence, $h_{T,T}(s,l,j,z,K,E,X,\mu) < j$ and from equations (116) and (123):

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = p^e(z,K,E,X,\mu) > \hat{F}_n\left[j, z, s, r^k(z,K,E,X,\mu)\right].$$

■

Lemma 6 *Suppose that $s > 0$, $l > 0$ and $j > 0$. Then,*

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu) = \min\left\{\begin{array}{c} \max\left\{\hat{F}_n\left[(1-\pi_n)l + j, z, s, r^k(z,K,E,X,\mu)\right], p^e(z,K,E,X,\mu)\right\}, \\ \max\left\{\hat{F}_n\left[(1-\pi_n)l, z, s, r^k(z,K,E,X,\mu)\right], p^u(z,K,E,X,\mu)\right\} \end{array}\right\}$$

Proof. a) First consider the case in which

$$\begin{aligned} &\max\left\{\hat{F}_n\left[(1-\pi_n)l + j, z, s, r^k(z,K,E,X,\mu)\right], p^e(z,K,E,X,\mu)\right\} \\ \geq\ &\max\left\{\hat{F}_n\left[(1-\pi_n)l, z, s, r^k(z,K,E,X,\mu)\right], p^u(z,K,E,X,\mu)\right\} \end{aligned} \tag{128}$$

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu) > 0$. Then, from Lemma 3, $f_{T,T}(s,l,j,z,K,E,X,\mu) = \pi_n l$.

From equations (116), (118) and the fact that $\hat{F}_n$ is strictly decreasing in its first argument, we have that

$$\begin{aligned} p^e(z,K,E,X,\mu) &\leq \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\ &= \hat{F}_n\left((1-\pi_n)l + h_{T,T}(s,l,j,z,K,E,X,\mu), z, s, r^k(z,K,E,X,\mu)\right) \\ &< \hat{F}_n\left((1-\pi_n)l, z, s, r^k(z,K,E,X,\mu)\right) \end{aligned}$$

and that

$$\hat{F}_n\left[(1-\pi_n)l + j, z, s, r^k(z,K,E,X,\mu)\right] < \hat{F}_n\left((1-\pi_n)l, z, s, r^k(z,K,E,X,\mu)\right),$$

which contradict equation (128).

Thus, $h_{T,T}(s,l,j,z,K,E,X,\mu)=0$ and equation (126) holds.

Considering the two cases in the proof of Lemma 4 we conclude that.

$$\begin{aligned}\xi_{T,T}(s,l,j,z,K,E,X,\mu) &= \max\left\{\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^u\left(z,K,E,X,\mu\right)\right\} \\ &\leq \max\left\{\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^e\left(z,K,E,X,\mu\right)\right\}.\end{aligned}$$

b) Consider the case in which

$$\begin{aligned}&\max\left\{\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^e\left(z,K,E,X,\mu\right)\right\} \\ < \;&\max\left\{\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^u\left(z,K,E,X,\mu\right)\right\} \qquad (129)\end{aligned}$$

Suppose that $f_{T,T}(s,l,j,z,K,E,X,\mu)>\pi_n l$. Then, from Lemma 3, $h_{T,T}(s,l,j,z,K,E,X,\mu)=0$.

From equations (122) (118), the fact that $p^e\left(z,K,E,X,\mu\right)\geq p^u\left(z,K,E,X,\mu\right)$ and the fact that $\hat{F}_n$ is strictly decreasing in its first argument, we have that

$$\begin{aligned}p^e\left(z,K,E,X,\mu\right) &\geq p^u\left(z,K,E,X,\mu\right) \\ &= \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\ &= \hat{F}_n\left(l-f_{T,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right) \\ &> \hat{F}_n\left((1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right) \\ &> \hat{F}_n\left((1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right),\end{aligned}$$

which contradict equation (129).

Thus, $f_{T,T}(s,l,j,z,K,E,X,\mu)=\pi_n l$ and

$$n_{T,T}(s,l,j,z,K,E,X,\mu)=(1-\pi_n)\,l+h_{T,T}(s,l,j,z,K,E,X,\mu)$$

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu)=0$.

Then, from equations (123) and (118),

$$\begin{aligned}p^e\left(z,K,E,X,\mu\right) &\geq \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\ &= \hat{F}_n\left((1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right) \\ &> \hat{F}_n\left((1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right)\end{aligned}$$

which contradicts equation (129). Hence, $h_{T,T}(s,l,j,z,K,E,X,\mu)>0$

b.1) Consider the case that $\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]\geq p^e\left(z,K,E,X,\mu\right)$.

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu)<j$. Then, from equations (116), (123), (118), and the fact that $\hat{F}_n$ is strictly decreasing in its first argument, we have that

$$\begin{aligned}p^e\left(z,K,E,X,\mu\right) &= \xi_{T,T}(s,l,j,z,K,E,X,\mu) \\ &= \hat{F}_n\left[(1-\pi_n)\,l+h_{T,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right] \\ &> \hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],\end{aligned}$$

which contradicts equation (129). Then, $h_{T,T}(s,l,j,z,K,E,X,\mu)=j$ and

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu)=\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]\geq p^e\left(z,K,E,X,\mu\right).$$

b.2) Consider the case that $\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]<p^e\left(z,K,E,X,\mu\right).$

Suppose that $h_{T,T}(s,l,j,z,K,E,X,\mu)=j$. Then, from equation (118) we have that

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu)=\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]<p^e\left(z,K,E,X,\mu\right).$$

But this contradicts equation (116).

Hence, $h_{T,T}(s,l,j,z,K,E,X,\mu)<j$ and from equations (116) and (123):

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu)=p^e\left(z,K,E,X,\mu\right)>\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right].$$

In case b) we thus conclude that

$$\begin{aligned}\xi_{T,T}(s,l,j,z,K,E,X,\mu) &= \max\left\{\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^e\left(z,K,E,X,\mu\right)\right\}\\ &< \max\left\{\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^u\left(z,K,E,X,\mu\right)\right\}\end{aligned}$$

■

Lemma 7 *Suppose that $s>0$, $l+j>0$. Then,*

$$\xi_{T,T}(s,l,j,z,K,E,X,\mu)=\min\left\{\begin{array}{c}\max\left\{\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^e\left(z,K,E,X,\mu\right)\right\},\\ \max\left\{\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^u\left(z,K,E,X,\mu\right)\right\}\end{array}\right\}$$

Proof. *If follows from Lemmas 4, 5, and 6.* ■

The following assumption will be helpful in stating subsequent Lemmas.

Assumption 3: For every $s>0$, $l+j>0$, $\xi_{t,T}(s,l,j,z,K,E,X,\mu)$ is given by

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)=\min\left\{\begin{array}{c}\max\left\{\Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^e\left(z,K,E,X,\mu\right)\right\},\\ \max\left\{\Lambda_{t,T}\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right],p^u\left(z,K,E,X,\mu\right)\right\}\end{array}\right\}\qquad(130)$$

for some continuous function $\Lambda_{t,T}$ that is strictly decreasing in its first argument and strictly increasing in s.

Lemma 8 *Let $s>0$, $l+j>0$. Suppose that $\xi_{t,T}(s,l,j,z,K,E,X,\mu)$ satisfies Assumption 2 and that*

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)>p^e\left(z,K,E,X,\mu\right).$$

Then,

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)=\Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right].$$

Proof. a) Consider the case that $j>0$.

Suppose that $\xi_{t,T}(s,l,j,z,K,E,X,\mu)\neq\Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]$. Since $\xi_{t,T}(s,l,j,z,K,E,X,\mu)>p^e\left(z,K,E,X,\mu\right)\geq p^u\left(z,K,E,X,\mu\right)$, from Assumption 2 we have that

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)=\Lambda_{t,T}\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right]>p^e\left(z,K,E,X,\mu\right).$$

But since $\Lambda_{t,T}$ is strictly decreasing in its first argument,

$$\begin{aligned}\xi_{t,T}(s,l,j,z,K,E,X,\mu) &= \Lambda_{t,T}\left[(1-\pi_n)\,l,z,s,r^k(z,K,E,X,\mu)\right] \\ &> \max\left\{\Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k(z,K,E,X,\mu)\right],p^e(z,K,E,X,\mu)\right\},\end{aligned}$$

which contradicts Assumption 2.

b) Consider the case that $j=0$.

Suppose that $\xi_{t,T}(s,l,j,z,K,E,X,\mu) \neq \Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k(z,K,E,X,\mu)\right]$. Since $\xi_{t,T}(s,l,j,z,K,E,X,\mu) > p^e(z,K,E,X,\mu)$, from Assumption 2 the only possible value left is

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu) = p^u(z,K,E,X,\mu).$$

But $\xi_{t,T}(s,l,j,z,K,E,X,\mu) > p^e(z,K,E,X,\mu) \geq p^u(z,K,E,X,\mu)$. A contradiction. ■

Lemma 9 *Suppose that $\xi_{t,T}$ satisfies Assumption 2. Let $s>0$ and $l+j>0$. Then*

$$\min\left\{\xi_{t,T}(s,l,j,z,K,E,X,\mu),p^e(z,K,E,X,\mu)\right\} = \min\left\{\xi_{t,T}(s,l,0,z,K,E,X,\mu),p^e(z,K,E,X,\mu)\right\}$$

Proof. Since $p^e(z,K,E,X,\mu) \geq p^u(z,K,E,X,\mu)$, from equation (130) we have that

$$\xi_{t,T}(s,l,0,z,K,E,X,\mu) = \max\left\{\Lambda_{t,T}\left[(1-\pi_n)\,l,z,s,r^k(z,K,E,X,\mu)\right],p^u(z,K,E,X,\mu)\right\}.$$

Hence equation (130) can be written as follows:

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu) = \min\left\{\begin{array}{c}\max\left\{\Lambda_{t,T}\left[(1-\pi_n)\,l+j,z,s,r^k(z,K,E,X,\mu)\right],p^e(z,K,E,X,\mu)\right\}, \\ \xi_{t,T}(s,l,0,z,K,E,X,\mu)\end{array}\right\}. \tag{131}$$

Observe that the first term of the min operator in equation (131) is greater than or equal to $p^e(z,K,E,X,\mu)$. Therefore,

$$\begin{aligned}\xi_{t,T}(s,l,j,z,K,E,X,\mu) &< p^e(z,K,E,X,\mu) \Rightarrow \\ \xi_{t,T}(s,l,j,z,K,E,X,\mu) &= \xi_{t,T}(s,l,0,z,K,E,X,\mu) < p^e(z,K,E,X,\mu)\end{aligned}$$

Suppose now that $\xi_{t,T}(s,l,j,z,K,E,X,\mu) \geq p^e(z,K,E,X,\mu)$. From equation (131) we know that

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu) \leq \xi_{t,T}(s,l,0,z,K,E,X,\mu).$$

Hence,

$$p^e(z,K,E,X,\mu) \leq \xi_{t,T}(s,l,j,z,K,E,X,\mu) \Rightarrow p^e(z,K,E,X,\mu) \leq \xi_{t,T}(s,l,0,z,K,E,X,\mu).$$

■

Observe from equation (119) that for $t<T$, $v_{t,T}(s,l,j,z,K,E,X,\mu)$ depends on (s,l,j) only through s and $n_{t,T}(s,l,j,z,K,E,X,\mu)$. This motivates the following definition:

Definition 10 *Let $t<T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. For every $n \geq 0$ and $s>0$ (implicitely) define $\hat{v}_{t,T}(n,s,z,K,E,X,\mu) \geq 0$ as follows:*

$$\begin{aligned}&\mathcal{E}\left\{\sum_{s'>0} q(z,K,E,X,\mu,z') \times\right. \\ &\left.\max[\xi_{t+1,T}(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L(z,K,E,X,\mu)) - p^e(z',L(z,K,E,X,\mu)),0]Q(s,s') \mid z\right\} \\ \leq\ & p^v(z,K,E,X,\mu)\,,\ (=\ \textit{if}\ \hat{v}_{t,T}(n,s,z,K,E,X,\mu)>0)\end{aligned} \tag{132}$$

The following definition will help characterize $\hat{v}_{t,T}$.

Definition 11 *Let $t < T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. For every $s > 0$, define $\bar{v}_{t,T}(s, z, K, E, X, \mu)$ as follows:*

$$\begin{aligned} & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', 0, \bar{v}_{t,T}(s, z, K, E, X, \mu), z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\} \\ = \; & p^{v}\left(z, K, E, X, \mu\right) \end{aligned} \tag{133}$$

From Assumption 2 and Lemma 8, observe that $\bar{v}_{t,T}(s, z, K, E, X, \mu) > 0$ is uniquely determined.

Lemma 12 *Let $t < T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. Then, for every $n \geq 0$ and $s > 0$:*

$$\hat{v}_{t,T}(n, s, z, K, E, X, \mu) = \left\{ \begin{array}{c} \bar{v}_{t,T}(s, z, K, E, X, \mu) - (1 - \pi_n)n, \text{ if } (1 - \pi_n)n \leq \bar{v}_{t,T}(s, z, K, E, X, \mu) \\ 0, \text{ otherwise} \end{array} \right\},$$

Proof. It is a direct consequence of Assumption 2, Lemma 8, Definition 10 and Definition 11. ■

Lemma 13 *Let $t < T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. Then, for every $n \geq 0$ and $s > 0$:*

$$\begin{aligned} & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', n, \hat{v}_{t,T}(n, s, z, K, E, X, \mu), z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\} \\ = \; & \min\left\{ \begin{array}{c} \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ \left. \max[\xi_{t+1,T}\left(s', n, 0, z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\}, \\ p^{v}\left(z, K, E, X, \mu\right) \end{array} \right\} \end{aligned} \tag{134}$$

Proof. a) If $(1 - \pi_n)n \leq \bar{v}_{t,T}(s, z, K, E, X, \mu)$:

$$\begin{aligned} & p^{v}\left(z, K, E, X, \mu\right) \\ = \; & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', 0, \bar{v}_{t,T}(s, z, K, E, X, \mu), z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\} \\ = \; & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', n, \hat{v}_{t,T}(n, s, z, K, E, X, \mu), z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\} \\ \geq \; & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', 0, \left(1 - \pi_n\right)n, z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\} \\ = \; & \mathcal{E}\left\{\sum_{s'>0} q\left(z, K, E, X, \mu, z'\right) \times \right. \\ & \left. \max[\xi_{t+1,T}\left(s', n, 0, z', L\left(z, K, E, X, \mu\right)\right) - p^{e}\left(z', L\left(z, K, E, X, \mu\right)\right), 0]Q\left(s, s'\right) \mid z\right\}. \end{aligned} \tag{135}$$

where the first equality follows from Definition 11, the second equality follows from Lemmas 8 and 12, the inequality follows from Lemma 8 and Assumption 2 (in particular, the fact that $\Lambda_{t+1,T}$ is strictly decreasing in its first argument), and the last equality follows from Lemma 8.

b) If $(1-\pi_n)n > \bar{v}_{t,T}(s,z,K,E,X,\mu)$, from Lemma 12 we have that $\hat{v}_{t,T}(n,s,z,K,E,X,\mu) = 0$. From Definition 10 we then have that:

$$\begin{aligned} & p^v(z,K,E,X,\mu) \\ \geq\; & \mathcal{E}\left\{\sum_{s'>0} q(z,K,E,X,\mu,z') \times \right. \\ & \left. \max[\xi_{t+1,T}(s',n,0,z',L(z,K,E,X,\mu)) - p^e(z',L(z,K,E,X,\mu)),0]Q(s,s') \mid z\right\}. \end{aligned} \tag{136}$$

∎

Lemma 14 *Let $t<T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. Then, for every $n\geq 0$ and $s>0$:*

$$\begin{aligned} & \mathcal{E}\left\{\sum_{s'>0} q(z,K,E,X,\mu,z')\,\xi_{t+1,T}(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L(z,K,E,X,\mu))\,Q(s,s') \mid z\right\} \\ =\; & \min\left\{\begin{array}{c} \mathcal{E}\left\{\sum_{s'>0} q(z,K,E,X,\mu,z') \times \right. \\ \left. \max[\xi_{t+1,T}(s',n,0,z',L(z,K,E,X,\mu)) - p^e(z',L(z,K,E,X,\mu)),0]Q(s,s') \mid z\right\}, \\ p^v(z,K,E,X,\mu) \end{array}\right\} \\ & + \mathcal{E}\left\{\sum_{s'>0} q(z,K,E,X,\mu,z') \min[\xi_{t+1,T}(s',n,0,z',L(z,K,E,X,\mu)), p^e(z',L(z,K,E,X,\mu))]Q(s,s') \mid z\right\}. \end{aligned} \tag{137}$$

Proof. Observe that for any positive numbers a and b :

$$a = \max\{a-b,0\} + \min\{a,b\}. \tag{138}$$

Hence,

$$\begin{aligned} & \xi_{t+1,T}(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L(z,K,E,X,\mu)) \\ =\; & \max[\xi_{t+1,T}(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L(z,K,E,X,\mu)) - p^e(z',L(z,K,E,X,\mu)),0] \\ & + \min[\xi_{t+1,T}(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L(z,K,E,X,\mu)), p^e(z',L(z,K,E,X,\mu))]. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\xi_{t+1,T}\left(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right\} \\
=\ &\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
&\left.\max[\xi_{t+1,T}\left(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\} \\
&+\left\{\mathcal{E}\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
&\left.\min[\xi_{t+1,T}\left(s',n,\hat{v}_{t,T}(n,s,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right),p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)]Q\left(s,s'\right)\mid z\right\} \\
=\ &\min\left\{\begin{array}{c}
\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
\left.\max[\xi_{t+1,T}\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\}, \\
p^{v}\left(z,K,E,X,\mu\right)
\end{array}\right\} \\
&+\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\min[\xi_{t+1,T}\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right),p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)]Q\left(s,s'\right)\mid z\right\}.
\end{aligned}$$

where the first equality follows from equation (138), and the second equality follows from Lemmas 9 and 13. ■

Lemma 15 *Let $t<T$ and suppose that $\xi_{t+1,T}$ satisfies Assumption 2. Then, for every $s>0$ and $l+j>0$:*

$$\begin{aligned}
&\xi_{t,T}(s,l,j,z,K,E,X,\mu) \qquad (139)\\
=\ &\min\left\{\begin{array}{c}
\max\left\{\begin{array}{c}
\hat{F}_{n}\left[\left(1-\pi_{n}\right)l+j,z,s,r^{k}\left(z,K,E,X,\mu\right)\right]+\left(1-\pi_{n}\right)\Omega_{t,T}\left(\left(1-\pi_{n}\right)l+j,s,z,K,E,X,\mu\right) \\
+\Psi\left(z,K,E,X,\mu\right),p^{e}\left(z,K,E,X,\mu\right)
\end{array}\right\}, \\
\max\left\{\begin{array}{c}
\hat{F}_{n}\left[\left(1-\pi_{n}\right)l,z,s,r^{k}\left(z,K,E,X,\mu\right)\right]+\left(1-\pi_{n}\right)\Omega_{t,T}\left(\left(1-\pi_{n}\right)l,s,z,K,E,X,\mu\right) \\
+\Psi\left(z,K,E,X,\mu\right),p^{u}\left(z,K,E,X,\mu\right)
\end{array}\right\}
\end{array}\right\}.
\end{aligned}$$

where

$$\begin{aligned}
&\Omega_{t,T}\left(n,s,z,K,E,X,\mu\right) \qquad (140)\\
=\ &\min\left\{\begin{array}{c}
\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right. \\
\left.\max[\xi_{t+1,T}\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\}, \\
p^{v}\left(z,K,E,X,\mu\right)
\end{array}\right\} \\
&+\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\min[\xi_{t+1,T}\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right),p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)]Q\left(s,s'\right)\mid z\right\}.
\end{aligned}$$

and where

$$\begin{aligned}
\Psi\left(z,K,E,X,\mu\right) &= \pi_{n}\mathcal{E}\left[\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right] \\
&\quad+Q\left(s,0\right)\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\mid z\right] \qquad (141)
\end{aligned}$$

Proof. Since $\xi_{t+1,T}$ satisfies Assumption 2, it follows that $\Omega_{t,T}\left(n,s,z,K,E,X,\mu\right)$ is weakly decreasing in n and, therefore, that

$$\hat{F}_{n}\left[n,z,s,r^{k}\left(z,K,E,X,\mu\right)\right]+\left(1-\pi_{n}\right)\Omega_{t,T}\left(n,s,z,K,E,X,\mu\right)+\Psi\left(z,K,E,X,\mu\right) \qquad (142)$$

is strictly decreasing in n.

Also, from Definition 10, observe that $n_{t,T}(s,l,j,z,K,E,X,\mu)$ and $v_{t,T}(s,l,j,z,K,E,X,\mu)$ in equation (119) satisfy that

$$v_{t,T}(s,l,j,z,K,E,X,\mu) = \hat{v}_{t,T}\left[n_{t,T}(s,l,j,z,K,E,X,\mu),s,z,K,E,X,\mu\right].$$

Using Lemma 14 and following exactly the same arguments as in the proofs of Lemmas 4, 5 and 6 (with equation (142) taking the place of $\hat{F}_n\left[n,z,s,r^k\left(z,K,E,X,\mu\right)\right]$ and equation (117) taking the place of equation (118)) then leads to equation (139). ■

Lemma 16 *For $t = 0,...,T-1$, $\xi_{t,T}(s,l,j,z,K,E,X,\mu)$ satisfies equation (139) for every $s > 0$ and $l+j > 0$. Moreover, $\xi_{t,T}(s,l,j,z,K,E,X,\mu)$ is decreasing in l and j.*

Proof. From Lemma 7, $\xi_{T,T}(s,l,j,z,K,E,X,\mu)$ satisfies Assumption 2 (with $\hat{F}_n$ playing the role of $\Lambda_{T,T}$). The claim then follows by induction. ■

Lemma 17 *Let $t \leq T-1$ and $s > 0$. Define $\underline{n}_{t,T}(s,z,K,E,X,\mu) \leq \bar{n}_{t,T}(s,z,K,E,X,\mu)$ as follows:*

$$\begin{aligned} p^e\left(z,K,E,X,\mu\right) &= \hat{F}_n\left[\underline{n}_{t,T}(s,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right] \\ &+ \left(1-\pi_n\right)\Omega_{t,T}\left(\underline{n}_{t,T}(s,z,K,E,X,\mu),s,z,K,E,X,\mu\right) + \Psi\left(z,K,E,X,\mu\right), \end{aligned} \tag{143}$$

$$\begin{aligned} p^u\left(z,K,E,X,\mu\right) &= \hat{F}_n\left[\bar{n}_{t,T}(s,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right] \\ &+ \left(1-\pi_n\right)\Omega_{t,T}\left(\bar{n}_{t,T}(s,z,K,E,X,\mu),s,z,K,E,X,\mu\right) + \Psi\left(z,K,E,X,\mu\right), \end{aligned} \tag{144}$$

where $\Omega_{t,T}\left(n,s,z,K,E,X,\mu\right)$ is given by equation (140) and $\Psi\left(z,K,E,X,\mu\right)$ is given by equation (141).

Then,

$$n_{t,T}(s,l,j,z,K,E,X,\mu) = \max\left\{\begin{array}{c} \min\left\{\left(1-\pi_n\right)l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\} \\ \min\left\{\left(1-\pi_n\right)l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\} \end{array}\right\}. \tag{145}$$

Proof. From Lemma (16), $\xi_{t,T}$ satisfies equation (139)

From equation (117), Lemma (14) and the fact that $n_{t,T}(s,l,j,z,K,E,X,\mu)$ and $v_{t,T}(s,l,j,z,K,E,X,\mu)$ satisfy that

$$v_{t,T}(s,l,j,z,K,E,X,\mu) = \hat{v}_{t,T}\left[n_{t,T}(s,l,j,z,K,E,X,\mu),s,z,K,E,X,\mu\right],$$

we have that

$$\begin{aligned} \xi_{t,T}(s,l,j,z,K,E,X,\mu) &= \hat{F}_n\left[n_{t,T}(s,l,j,z,K,E,X,\mu),z,s,r^k\left(z,K,E,X,\mu\right)\right] \\ &+.\left(1-\pi_n\right)\Omega_{t,T}\left(n_{t,T}(s,l,j,z,K,E,X,\mu),s,z,K,E,X,\mu\right) \\ &+\Psi\left(z,K,E,X,\mu\right). \end{aligned} \tag{146}$$

a) Suppose that $\left(1-\pi_n\right)l > \underline{n}_{t,T}(s,z,K,E,X,\mu)$. Then, since $\hat{F}_n$ is strictly decreasing in n and $\Omega_{t,T}$ is decreasing in n, we have that

$$\begin{aligned} & \hat{F}_n\left[\left(1-\pi_n\right)l,z,s,r^k\left(z,K,E,X,\mu\right)\right] + \left(1-\pi_n\right)\Omega_{t,T}\left(\left(1-\pi_n\right)l,s,z,K,E,X,\mu\right) + \Psi\left(z,K,E,X,\mu\right) \\ < \; & p^e\left(z,K,E,X,\mu\right) \\ \leq \; & \max\left\{\begin{array}{c} \hat{F}_n\left[\left(1-\pi_n\right)l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right] + \left(1-\pi_n\right)\Omega_{t,T}\left(\left(1-\pi_n\right)l+j,s,z,K,E,X,\mu\right) \\ +\Psi\left(z,K,E,X,\mu\right),p^e\left(z,K,E,X,\mu\right) \end{array}\right\}. \end{aligned}$$

Since $p^u(z,K,E,X,\mu) \leq p^e(z,K,E,X,\mu)$, frome equation (139) we have that

$$\begin{aligned} & \xi_{t,T}(s,l,j,z,K,E,X,\mu) \\ = \quad & \max\left\{ \begin{array}{c} \hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) \\ +\Psi(z,K,E,X,\mu), p^u(z,K,E,X,\mu) \end{array} \right\}. \end{aligned} \tag{147}$$

a.1) Suppose that $\hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu) > p^u(z,K,E,X,\mu)$.

Then, from equation (147),

$$\begin{aligned} & \xi_{t,T}(s,l,j,z,K,E,X,\mu) \\ = \quad & \hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu). \end{aligned}$$

From equations (146) and (144), and using that $\hat{F}_n$ is strictly decreasing in n and that $\Omega_{t,T}$ is decreasing in n, we have that

$$n_{t,T}(s,l,j,z,K,E,X,\mu) = (1-\pi_n)l < \bar{n}_{t,T}(s,z,K,E,X,\mu). \tag{148}$$

a.2) Suppose that $\hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu) \leq p^u(z,K,E,X,\mu)$.

Then, from equation (147),

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu) = p^u(z,K,E,X,\mu).$$

From equations (146) and (143), and using that $\hat{F}_n$ is strictly decreasing in n and that $\Omega_{t,T}$ is decreasing in n, we have that

$$n_{t,T}(s,l,j,z,K,E,X,\mu) = \bar{n}_{t,T}(s,z,K,E,X,\mu) \leq (1-\pi_n)l. \tag{149}$$

From equations (148) and (149) we then have that

$$(1-\pi_n)l > \underline{n}_{t,T}(s,z,K,E,X,\mu) \Rightarrow n_{t,T}(s,l,j,z,K,E,X,\mu) = \min\left\{(1-\pi_n)l, \bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}. \tag{150}$$

b) Suppose that $(1-\pi_n)l \leq \underline{n}_{t,T}(s,z,K,E,X,\mu)$. Then, since $\hat{F}_n$ is strictly decreasing in n and $\Omega_{t,T}$ is decreasing in n, we have that

$$\begin{aligned} & \hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu) \\ \geq \quad & p^e(z,K,E,X,\mu), \end{aligned}$$

and that

$$\begin{aligned} & \hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu) \\ \geq \quad & \hat{F}_n\left[(1-\pi_n)l+j,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l+j,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu). \end{aligned}$$

Hence,

$$\begin{aligned} & \hat{F}_n\left[(1-\pi_n)l,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l,s,z,K,E,X,\mu) + \Psi(z,K,E,X,\mu) \\ \geq \quad & \max\left\{ \begin{array}{c} \hat{F}_n\left[(1-\pi_n)l+j,z,s,r^k(z,K,E,X,\mu)\right] + (1-\pi_n)\Omega_{t,T}((1-\pi_n)l+j,s,z,K,E,X,\mu) \\ +\Psi(z,K,E,X,\mu), p^e(z,K,E,X,\mu) \end{array} \right\}. \end{aligned}$$

From equation (139) we then have that

$$\begin{aligned} & \xi_{t,T}(s,l,j,z,K,E,X,\mu) \\ = \ & \max\left\{ \begin{array}{c} \hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\,(z,K,E,X,\mu)\right]+(1-\pi_n)\,\Omega_{t,T}\,((1-\pi_n)\,l+j,s,z,K,E,X,\mu) \\ +\Psi\,(z,K,E,X,\mu)\,,p^e\,(z,K,E,X,\mu) \end{array} \right\} \end{aligned} \tag{151}$$

b.1) Suppose that $\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\,(z,K,E,X,\mu)\right]+(1-\pi_n)\,\Omega_{t,T}\,((1-\pi_n)\,l+j,s,z,K,E,X,\mu)+\Psi(z,K,E,X,\mu) > p^e\,(z,K,E,X,\mu)$.

Then, from equation (151),

$$\begin{aligned} & \xi_{t,T}(s,l,j,z,K,E,X,\mu) \\ = \ & \hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\,(z,K,E,X,\mu)\right]+(1-\pi_n)\,\Omega_{t,T}\,((1-\pi_n)\,l+j,s,z,K,E,X,\mu)+\Psi\,(z,K,E,X,\mu)\,. \end{aligned}$$

From equations (146) and (143), and using that $\hat{F}_n$ is strictly decreasing in n and that $\Omega_{t,T}$ is decreasing in n, we have that

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=(1-\pi_n)\,l+j<\underline{n}_{t,T}(s,z,K,E,X,\mu). \tag{152}$$

b.2) Suppose that $\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\,(z,K,E,X,\mu)\right]+(1-\pi_n)\,\Omega_{t,T}\,((1-\pi_n)\,l+j,s,z,K,E,X,\mu)+\Psi(z,K,E,X,\mu) \leq p^e\,(z,K,E,X,\mu)$.

Then, from equation (151),

$$\xi_{t,T}(s,l,j,z,K,E,X,\mu)=p^e\,(z,K,E,X,\mu)\,.$$

From equations (146) and (143), and using that $\hat{F}_n$ is strictly decreasing in n and that $\Omega_{t,T}$ is decreasing in n, we have that

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=\underline{n}_{t,T}(s,z,K,E,X,\mu)\leq(1-\pi_n)\,l+j. \tag{153}$$

From equations (152) and (153) we then have that

$$(1-\pi_n)\,l\leq\underline{n}_{t,T}(s,z,K,E,X,\mu)\Rightarrow n_{t,T}(s,l,j,z,K,E,X,\mu)=\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}. \tag{154}$$

c) Need to show that equation (145) holds.

c.1) Consider the case that

$$\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}<\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}. \tag{155}$$

Suppose that $(1-\pi_n)\,l\leq\underline{n}_{t,T}(s,z,K,E,X,\mu)$.

Since $\underline{n}_{t,T}(s,z,K,E,X,\mu)\leq\bar{n}_{t,T}(s,z,K,E,X,\mu)$, it follows that $(1-\pi_n)\,l\leq\bar{n}_{t,T}(s,z,K,E,X,\mu)$. Hence, equation (155) becomes

$$\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}<(1-\pi_n)\,l.$$

But $(1-\pi_n)\,l\leq\underline{n}_{t,T}(s,z,K,E,X,\mu)$ and $j\geq0$. A contradiction.

Thus, $(1-\pi_n)\,l>\underline{n}_{t,T}(s,z,K,E,X,\mu)$ and from equation (150) we conclude that

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}>\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}.$$

c.2) Consider the case that

$$\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}>\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}. \tag{156}$$

Suppose that $(1-\pi_n)\,l>\underline{n}_{t,T}(s,z,K,E,X,\mu)$.

Since $j\geq 0$, it follows that $(1-\pi_n)\,l+j>\underline{n}_{t,T}(s,z,K,E,X,\mu)$. Hence, equation (156) becomes

$$\underline{n}_{t,T}(s,z,K,E,X,\mu)>\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}.$$

But $\underline{n}_{t,T}(s,z,K,E,X,\mu)\leq\bar{n}_{t,T}(s,z,K,E,X,\mu)$ and $(1-\pi_n)\,l>\underline{n}_{t,T}(s,z,K,E,X,\mu)$. A contradiction.

Thus, $(1-\pi_n)\,l\leq\underline{n}_{t,T}(s,z,K,E,X,\mu)$ and from equation (154) we conclude that

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}>\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}.$$

c.3) Consider the case that

$$\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}=\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}.$$

Equations (150) and (154) then imply that

$$n_{t,T}(s,l,j,z,K,E,X,\mu)=\min\left\{(1-\pi_n)\,l+j,\underline{n}_{t,T}(s,z,K,E,X,\mu)\right\}=\min\left\{(1-\pi_n)\,l,\bar{n}_{t,T}(s,z,K,E,X,\mu)\right\}.$$

From cases c1), c2) and c3), we conclude that equation (145) holds. ■

Lemma 18 *Let $t\leq T-1$ and $s>0$. Define $\bar{v}_{t,T}(s,z,K,E,X,\mu)$ as follows:*

$$\begin{aligned}
&\mathcal{E}\left\{\sum_{s'>0}q\left(z,K,E,X,\mu,z'\right)\times\right.\\
&\left.\max[\xi_{t+1,T}\left(s',0,\bar{v}_{t,T}(s,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\}\\
=\;&p^{v}\left(z,K,E,X,\mu\right)
\end{aligned} \tag{157}$$

Then,

$$v_{t,T}(s,l,j,z,K,E,X,\mu)=\max\left\{\bar{v}_{t,T}(s,z,K,E,X,\mu)-(1-\pi_n)n_{t,T}(s,l,j,z,K,E,X,\mu),0\right\}$$

where $n_{t,T}(s,l,j,z,K,E,X,\mu)$ is given by equation (145).

Proof. *By Lemma (16), $\xi_{t+1,T}$ satisfies Assumption 2. Therefore, Lemma (12) applies. The claim then follows from Lemma (17) and the fact that*

$$v_{t,T}(s,l,j,z,K,E,X,\mu)=\hat{v}_{t,T}\left[n_{t,T}(s,l,j,z,K,E,X,\mu),s,z,K,E,X,\mu\right].$$

■

Lemma 19 *Let $\xi(s,l,j,z,K,E,X,\mu)$ be the Lagrange mulplier function for the establishments' problem with infinite planning horizon (i.e. for $T=\infty$). Then, for every $s>0$ and $l+j>0$:*

$$\begin{aligned}
&\xi(s,l,j,z,K,E,X,\mu)\\
=\;&\min\left\{\begin{array}{c}
\max\left\{\begin{array}{c}\hat{F}_n\left[(1-\pi_n)\,l+j,z,s,r^k\left(z,K,E,X,\mu\right)\right]+(1-\pi_n)\,\Omega\left((1-\pi_n)\,l+j,s,z,K,E,X,\mu\right)\\+\Psi\left(z,K,E,X,\mu\right),p^{e}\left(z,K,E,X,\mu\right)\end{array}\right\},\\
\max\left\{\begin{array}{c}\hat{F}_n\left[(1-\pi_n)\,l,z,s,r^k\left(z,K,E,X,\mu\right)\right]+(1-\pi_n)\,\Omega\left((1-\pi_n)\,l,s,z,K,E,X,\mu\right)\\+\Psi\left(z,K,E,X,\mu\right),p^{u}\left(z,K,E,X,\mu\right)\end{array}\right\}
\end{array}\right\}.
\end{aligned} \tag{158}$$

where

$$\Omega\left(n,s,z,K,E,X,\mu\right) \tag{159}$$

$$= \min\left\{\begin{array}{c}\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right.\\ \left.\max[\xi\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\},\\ p^{v}\left(z,K,E,X,\mu\right)\end{array}\right\}$$
$$+\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\min[\xi_{t+1,T}\left(s',n,0,z',L\left(z,K,E,X,\mu\right)\right),p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right)]Q\left(s,s'\right)\mid z\right\}.$$

and where

$$\begin{aligned}\Psi\left(z,K,E,X,\mu\right) &= \pi_n\mathcal{E}\left[\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)Q\left(s,s'\right)\mid z\right]\\ &\quad +Q\left(s,0\right)\mathcal{E}\left[q\left(z,K,E,X,\mu,z'\right)p^{u}\left(z',L\left(z,K,E,X,\mu\right)\right)\mid z\right]\end{aligned} \tag{160}$$

Moreover, $\xi(s,l,j,z,K,E,X,\mu)$ is decreasing in l and j.

Proof. The claim follows from Lemma 16 and the fact that $\lim_{T\to\infty}\xi_{0,T}=\xi$ and that $\lim_{T\to\infty}\xi_{1,T}=\xi$ (see Easley and Spulber (1)). ■

Lemma 20 *Let $n(s,l,j,z,K,E,X,\mu)$, $h(s,l,j,z,K,E,X,\mu)$, $f(s,l,j,z,K,E,X,\mu)$, $k(s,l,j,z,K,E,X,\mu)$ and $v(s,l,j,z,K,E,X,\mu)$ be optimal decision rules for the establishments' problem with infinite planning horizon (i.e. for $T=\infty$). Let $\xi(s,l,j,z,K,E,X,\mu)$ be given by equation (158), $\Omega\left(n,s,z,K,E,X,\mu\right)$ be given by equation (159) and $\Psi\left(z,K,E,X,\mu\right)$ be given by equation (160).*

Define $\underline{n}(s,z,K,E,X,\mu)$, $\bar{n}(s,z,K,E,X,\mu)$ and $\bar{v}(s,z,K,E,X,\mu)$ as follows:

$$\begin{aligned}p^{e}\left(z,K,E,X,\mu\right) &= \hat{F}_n\left[\underline{n}(s,z,K,E,X,\mu),z,s,r^{k}\left(z,K,E,X,\mu\right)\right]\\ &\quad +\left(1-\pi_n\right)\Omega\left(\underline{n}(s,z,K,E,X,\mu),s,z,K,E,X,\mu\right)+\Psi\left(z,K,E,X,\mu\right),\end{aligned}$$

$$\begin{aligned}p^{u}\left(z,K,E,X,\mu\right) &= \hat{F}_n\left[\bar{n}(s,z,K,E,X,\mu),z,s,r^{k}\left(z,K,E,X,\mu\right)\right]\\ &\quad +\left(1-\pi_n\right)\Omega\left(\bar{n}(s,z,K,E,X,\mu),s,z,K,E,X,\mu\right)+\Psi\left(z,K,E,X,\mu\right),\end{aligned}$$

$$\begin{aligned}&\mathcal{E}\left\{\sum_{s'>0} q\left(z,K,E,X,\mu,z'\right)\times\right.\\ &\left.\max[\xi\left(s',0,\bar{v}(s,z,K,E,X,\mu),z',L\left(z,K,E,X,\mu\right)\right)-p^{e}\left(z',L\left(z,K,E,X,\mu\right)\right),0]Q\left(s,s'\right)\mid z\right\}\\ =\;& p^{v}\left(z,K,E,X,\mu\right)\end{aligned}$$

Then, for every $s>0$ and $l+j>0$:

$$n(s,l,j,z,K,E,X,\mu)=\max\left\{\begin{array}{c}\min\left\{\left(1-\pi_n\right)l+j,\underline{n}(s,z,K,E,X,\mu)\right\}\\ \min\left\{\left(1-\pi_n\right)l,\bar{n}(s,z,K,E,X,\mu)\right\}\end{array}\right\},$$

$$h(s,l,j,z,K,E,X,\mu)=\max\left\{n(s,l,j,z,K,E,X,\mu)-l,0\right\}$$

$$f(s,l,j,z,K,E,X,\mu) = \max\{l - n(s,l,j,z,K,E,X,\mu), 0\}$$

$$e^z s F_k \left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] = r^k(z,K,E,X,\mu),$$

$$v(s,l,j,z,K,E,X,\mu) = \max\{\bar{v}(s,z,K,E,X,\mu) - (1-\pi_n) n(s,l,j,z,K,E,X,\mu), 0\}.$$

Proof. It is a direct consequence of Lemma 3, Lemma 16, Lemma 17, Lemma 18, Lemma 19 and the fact that $\lim_{T\to\infty}\xi_{0,T} = \xi$, $\lim_{T\to\infty}\xi_{1,T} = \xi$, $\lim_{T\to\infty} n_{0,T} = n$, $\lim_{T\to\infty} v_{0,T} = v$ (see Easley and Spulber (1)). ■

3 Finding a RCE

3.1 The myopic social planner's problem

The problem of the myopic social planner facing a stochastic process $\left(\hat{A},\hat{U},\hat{L}\right)$ is given by the following Bellman equation:

$$V(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}) = \max\left\{\frac{C^{1-\sigma}-1}{1-\sigma} + \varphi U + \beta\mathcal{E}\left[V\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right) \mid z\right]\right\}$$

subject to

$$C + I + A \leq \sum_s \int e^z s F\left[n(s,l,j), k(s,l,j)\right]\mu(s, dl\times dj) \tag{161}$$

$$\sum_s \int k(s,l,j)\mu(s,dl\times dj) \leq K \tag{162}$$

$$\sum_s \int v(s,l,j)\mu(s,dl\times dj) \leq H\left(A,U,\hat{A},\hat{U}\right) \tag{163}$$

$$U \leq X + E - \sum_s\int h(s,l,j)\mu(s,dl\times dj) + \sum_s\int f(s,l,j)\mu(s,dl\times dj) \tag{164}$$

$$\sum_s\int h(s,l,j)\mu(s,dl\times dj) \leq (1-\pi_u)E \tag{165}$$

$$n(s,l,j) = l + h(s,l,j) - f(s,l,j) \tag{166}$$

$$h(s,l,j) \leq j \tag{167}$$

$$\pi_n l \leq f(s,l,j) \tag{168}$$

$$f(s,l,j) \leq l \tag{169}$$

$$K' = (1-\delta)K + I$$

$$E' = G\left(A,U,\hat{A},\hat{U}\right)$$

$$X' = U - G\left(A,U,\hat{A},\hat{U}\right)$$

$$\mu'(s',\mathcal{L}\times\mathcal{J}) = \sum_s \int_{\{(l,j):\ n(s,l,j)\in\mathcal{L}\text{ and }v(s,l,j)\in\mathcal{J}\}} Q(s,s')\mu(s,dl\times dj) + \varrho\psi(s')\mathcal{I}(\mathcal{L}\times\mathcal{J})$$

$$\hat{A} = \hat{A}\left(z,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)$$

$$\hat{U} = \hat{U}\left(z,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)$$

$$\left(\hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) = \hat{L}\left(z, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right).$$

The MSP's decision rules are $C = C^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $I = I^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $n = n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $k = k^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $f = f^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $h = h^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $v = v^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $U = U^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $A = A^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$.

3.2 Solution to MSP's problem

The necessary and sufficient conditions for a solution $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ to the MSP's problem with exogenous stochastic process $\left(\hat{A}, \hat{U}, \hat{L}\right)$ is that there exist Lagrange multipliers $\lambda^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $r^{k,m}(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $p^{v,m}(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $p^{u,m}\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $p^{e,m}(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) - p^{u,m}(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $\xi^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$, $\eta^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $\alpha^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ and $\chi^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$ (for constraints (161)-(169) respectively) such that equations (170)-(199) hold.

$$C^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)^{-\sigma} = \lambda^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) \tag{170}$$

$$1 = \mathcal{E}\left[\beta \frac{\lambda^m\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right)}{\lambda^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)}\left[1 - \delta + r^{k,m}\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right)\right] \mid z\right] \tag{171}$$

$$\begin{aligned} & e^z s F_k\left[n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), k^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)\right] \\ \leq \; & r^{k,m}\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right), \; \left(= \text{ if } k^m\left(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) > 0\right) \end{aligned} \tag{172}$$

$$\begin{aligned} 0 \;=\; & \varphi \lambda^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)^{-1} - p^{u,m}\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) \\ & + p^{v,m}\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right) H_u\left[A^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right), U^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), \hat{A}, \hat{U}\right] \\ & + (1 - \pi_u) G_u\left[A^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right), U^m(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), \hat{A}, \hat{U}\right] \times \\ & \mathcal{E}\left\{\beta \frac{\lambda^m\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right)}{\lambda^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)} \times \right. \\ & \left.\left[p^{e,m}\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) - p^{u,m}\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right)\right] \mid z\right\} \\ & + \mathcal{E}\left[\beta \frac{\lambda^m\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right)}{\lambda^m\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)} p^{u,m}\left(z', K', E', X', \mu', \hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) \mid z\right] \end{aligned} \tag{173}$$

$$
\begin{aligned}
1 \;=\; & p^{v,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)H_a\left[A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),\hat{A},\hat{U}\right] \\
& +(1-\pi_u)\,G_a\left[A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),\hat{A},\hat{U}\right]\times \\
& \mathcal{E}\left\{\beta\frac{\lambda^m\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)}{\lambda^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)}\times\right. \\
& \left.\left[p^{e,m}\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)-p^{u,m}\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)\right]\mid z\right\}
\end{aligned}
\tag{174}
$$

$$
\begin{aligned}
& \mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)}{\lambda^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)}Q\left(s,s'\right)\times\right. \\
& \left.\eta^m(s',n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}')\mid z\right\} \\
\leq\; & p^{v,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\left(=\text{ if } v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)>0\right)
\end{aligned}
\tag{175}
$$

$$
\begin{aligned}
& e^z sF_n\left[n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),k^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\right] \\
& +\mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)}{\lambda^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)}Q\left(s,s'\right)\times\right. \\
& \left.\xi^m\left(s',n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)\mid z\right\} \\
& -\pi_n\mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)}{\lambda^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)}Q\left(s,s'\right)\times\right. \\
& \left.\alpha^m\left(s',n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)\mid z\right\} \\
& \mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)}{\lambda^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)}Q\left(s,s'\right)\times\right. \\
& \left.\chi^m\left(s',n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),z',K',E',X',\mu',\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)\mid z\right\} \\
\leq\; & \xi^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),\ \left(=\text{ if } n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})>0\right)
\end{aligned}
\tag{176}
$$

$$
\begin{aligned}
& -p^{e,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)+\xi^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}) \\
\leq\; & \eta^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})\ \left(=\text{ if } h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)>0\right)
\end{aligned}
\tag{177}
$$

$$
\begin{aligned}
& p^{u,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)-\xi^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})+\alpha^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}) \\
& -\chi^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})\leq 0\ \left(=\text{ if } f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)>0\right)
\end{aligned}
\tag{178}
$$

$$
n^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)=l+h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)-f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \tag{179}
$$

$$
h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\leq j \tag{180}
$$

$$
\pi_n l\leq f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \tag{181}
$$

$$f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\leq l \tag{182}$$

$$\eta^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})\left[j-h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\right]=0 \tag{183}$$

$$\alpha^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})\left[f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)-\pi_n l\right]=0 \tag{184}$$

$$\chi^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})\left[l-f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\right]=0 \tag{185}$$

$$\sum_s\int h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right)\leq\left(1-\pi_u\right)E \tag{186}$$

$$\begin{aligned}0 \quad = \quad & \left[p^{e,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)-p^{u,m}\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\right] \\ & \left[\left(1-\pi_u\right)E-\sum_s\int h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right)\right]\end{aligned} \tag{187}$$

$$\begin{aligned}& C^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)+I^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)+A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \\ = \quad & \sum_s\int e^z sF\left[n^m(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),k^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\right]\mu\left(s,dl\times dj\right)\end{aligned} \tag{188}$$

$$\sum_s\int k^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right)=K \tag{189}$$

$$\begin{aligned}& \sum_s\int v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right) \\ = \quad & H\left[A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),\hat{A},\hat{U}\right]\end{aligned} \tag{190}$$

$$\begin{aligned}U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}) \quad = \quad & X+E-\sum_s\int h^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right) \\ & +\sum_s\int f^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\mu\left(s,dl\times dj\right)\end{aligned} \tag{191}$$

$$K'=\left(1-\delta\right)K+I^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \tag{192}$$

$$E'=G\left[A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),\hat{A},\hat{U}\right] \tag{193}$$

$$X'=U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu})-G\left[A^m\left(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right),U^m(z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}),\hat{A},\hat{U}\right] \tag{194}$$

$$\mu'\left(s',\mathcal{L}\times\mathcal{J}\right)=\sum_s\int_{\mathcal{B}(s,\mathcal{L}\times\mathcal{J})}Q\left(s,s'\right)\mu\left(s,dl\times dj\right)+\varrho\psi\left(s'\right)\mathcal{I}\left(\mathcal{L}\times\mathcal{J}\right), \tag{195}$$

where

$$\mathcal{B}\left(s,\mathcal{L}\times\mathcal{J}\right)=\left\{(l,j):n^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\in\mathcal{L}\text{ and }v^m\left(s,l,j,z,K,E,X,\mu,\hat{K},\hat{E},\hat{X},\hat{\mu}\right)\in\mathcal{J}\right\}. \tag{196}$$

$$\hat{A}=\hat{A}\left(z,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \tag{197}$$

$$\hat{U}=\hat{U}\left(z,\hat{K},\hat{E},\hat{X},\hat{\mu}\right) \tag{198}$$

$$\left(\hat{K}',\hat{E}',\hat{X}',\hat{\mu}'\right)=\hat{L}\left(z,\hat{K},\hat{E},\hat{X},\hat{\mu}\right). \tag{199}$$

3.3 Characterization of myopic planner's decision rules

Proposition 21 *Let* $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ *be the solution to the MSP's with exogenous stochastic process* $\left(\hat{A}, \hat{U}, \hat{L}\right)$. *Then, there exist thresholds* $\underline{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$, $\bar{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ *and* $\overline{v}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$ *and a shadow capital price function* $r^k\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right)$ *such that, for every* $s > 0$ *and* $l + j > 0$*:*

$$n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max \left\{ \begin{array}{c} \min\left\{(1-\pi_n)\, l + j, \underline{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right\} \\ \min\left\{(1-\pi_n)\, l, \bar{n}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right\} \end{array} \right\},$$

$$\begin{aligned} & v^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) \\ = \; & \max\left\{\overline{v}^m(s, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) - (1-\pi_n) n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), 0\right\}. \end{aligned}$$

$$h^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) - l, 0\right\}$$

$$f^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = \max\left\{l - n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), 0\right\}$$

$$\begin{aligned} & e^z s F_k \left[n^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}), k^m(s, l, j, z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})\right] \\ = \; & r^k\left(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu}\right), \end{aligned}$$

Proof. In the economy in which $\left(\hat{A}, \hat{U}, \hat{L}\right)$ trully represent exogenous productivity shocks to the recruitment technology the Welfare Theorems apply. In this case the problem described in Section 3.1 is the social planner's problem and its solution can be descentralized as a recursive competitive equilibrium in which prices are functions of the state $(z, K, E, X, \mu, \hat{K}, \hat{E}, \hat{X}, \hat{\mu})$. The claim then follows from analyzing the associated establishments' problem using identical arguments as those in the proof of Proposition 20. ■

3.4 Construction of a RCE

Proposition 22 *Let* $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ *be the solution to the MSP's with exogenous stochastic process* $\left(\hat{A}, \hat{U}, \hat{L}\right)$.

Suppose that

$$\hat{A}\,(z, K, E, X, \mu) = A^m\,(z, K, E, X, \mu, K, E, X, \mu)\,, \tag{200}$$

$$\hat{U}\,(z, K, E, X, \mu) = U^m\,(z, K, E, X, \mu, K, E, X, \mu)\,, \tag{201}$$

and that

$$\left(\hat{K}', \hat{E}', \hat{X}', \hat{\mu}'\right) = \hat{L}\,(z, K, E, X, \mu) \tag{202}$$

satisfy the following conditions:

$$\hat{K}' = (1-\delta)\, K + I^m\,(z, K, E, X, \mu, K, E, X, \mu)\,, \tag{203}$$

$$\begin{aligned} \hat{E}' \;\; = \;\; & G\left[A^m\,(z, K, E, X, \mu, K, E, X, \mu)\,, U^m\,(z, K, E, X, \mu, K, E, X, \mu)\,,\right. \\ & \left. A^m\,(z, K, E, X, \mu, K, E, X, \mu)\,, U^m\,(z, K, E, X, \mu, K, E, X, \mu)\right], \end{aligned} \tag{204}$$

$$\hat{X}' = U^m(z, K, E, X, \mu, K, E, X, \mu) - \hat{E}', \tag{205}$$

$$\hat{\mu}'(s', \mathcal{L} \times \mathcal{J}) = \sum_s \int_{\mathcal{B}(s,\mathcal{L}\times\mathcal{J})} Q(s, s')\, \mu(s, dl \times dj) + \varrho\psi(s')\, \mathcal{I}(\mathcal{L} \times \mathcal{J}), \tag{206}$$

where

$$\mathcal{B}(s, \mathcal{L} \times \mathcal{J}) = \{(l, j) : n^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \in \mathcal{L} \text{ and } v^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \in \mathcal{J}\}. \tag{207}$$

Then, there exists a RCE $\{B,\ W,\ R,\ c,\ i,\ m,\ n,\ k,\ f,\ h,\ v,\ a,\ b,\ d,\ u,\ A,\ U,\ L,\ \Pi,\ r^k,\ r^u,\ p^u,\ p^e,\ p^v,\ q\}$ *such that*

$$\begin{aligned}
c(K, z, K, E, X, \mu) &= C^m(z, K, E, X, \mu, K, E, X, \mu) \\
i(K, z, K, E, X, \mu) &= I^m(z, K, E, X, \mu, K, E, X, \mu) \\
n(s, l, j, z, K, E, X, \mu) &= n^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\
f(s, l, j, z, K, E, X, \mu) &= f^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\
h(s, l, j, z, K, E, X, \mu) &= h^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\
v(s, l, j, z, K, E, X, \mu) &= v^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) \\
A(z, K, E, X, \mu) &= A^m(z, K, E, X, \mu, K, E, X, \mu) \\
U(z, K, E, X, \mu) &= U^m(z, K, E, X, \mu, K, E, X, \mu).
\end{aligned}$$

Proof. *Since* $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ *is a solution to the MSP's problem with exogenous stochastic process* $\left(\hat{A}, \hat{U}, \hat{L}\right)$ *we know that there exist Lagrange multipliers* $\left\{\lambda^m, r^{k,m}, p^{v,m}, p^{u,m}, p^{e,m} - p^{u,m}, \xi^m, \eta^m, \alpha^m, \chi^m\right\}$ *such that equations (170)-(199) hold. Evaluate these equations at* $(\hat{K}, \hat{E}, \hat{X}, \hat{\mu}) = (K, E, X, \mu)$ *and use equations (200)-(207) to get equations (208)-(230).*

$$C^m(z, K, E, X, \mu, K, E, X, \mu)^{-\sigma} = \lambda^m(z, K, E, X, \mu, K, E, X, \mu) \tag{208}$$

$$\begin{aligned}
1 \quad = \quad & \mathcal{E}\left\{\beta \frac{\lambda^m\left(z', \hat{L}(z, K, E, X, \mu), \hat{L}(z, K, E, X, \mu)\right)}{\lambda^m(z, K, E, X, \mu, K, E, X, \mu)} \times \right. \\
& \left. \left[1 - \delta + r^{k,m}\left(z', \hat{L}(z, K, E, X, \mu), \hat{L}(z, K, E, X, \mu)\right)\right] \mid z \right\}
\end{aligned} \tag{209}$$

$$\begin{aligned}
& e^z s F_k\left[n^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu), k^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu)\right] \\
\leq \quad & r^{k,m}(z, K, E, X, \mu, K, E, X, \mu), \ (= \ \text{if } k^m(s, l, j, z, K, E, X, \mu, K, E, X, \mu) > 0)
\end{aligned} \tag{210}$$

$$
\begin{aligned}
0 &= \varphi\lambda^{m}(z,K,E,X,\mu,K,E,X,\mu)^{-1} - p^{u,m}(z,K,E,X,\mu,K,E,X,\mu) \\
&+p^{v,m}(z,K,E,X,\mu,K,E,X,\mu)\,H_{u}\left[A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu),\right. \\
&\qquad \left. A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu)\right] \\
&+(1-\pi_{u})\,G_{u}\left[A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu),\right. \\
&\qquad \left. A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu)\right]\times \\
&\mathcal{E}\left\{\beta\frac{\lambda^{m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)}{\lambda^{m}(z,K,E,X,\mu,K,E,X,\mu)}\times\right. \\
&\left.\left[p^{e,m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)-p^{u,m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)\right]\mid z\right\} \\
&+\mathcal{E}\left[\beta\frac{\lambda^{m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)}{\lambda^{m}(z,K,E,X,\mu,K,E,X,\mu)}p^{u,m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)\mid z\right] \qquad (211)
\end{aligned}
$$

$$
\begin{aligned}
1 &= p^{v,m}(z,K,E,X,\mu,K,E,X,\mu)\,H_{a}\left[A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu),\right. \\
&\qquad \left. A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu)\right] \\
&+(1-\pi_{u})\,G_{a}\left[A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu),\right. \\
&\qquad \left. A^{m}(z,K,E,X,\mu,K,E,X,\mu),U^{m}(z,K,E,X,\mu,K,E,X,\mu)\right]\times \\
&\mathcal{E}\left\{\beta\frac{\lambda^{m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)}{\lambda^{m}(z,K,E,X,\mu,K,E,X,\mu)}\times\right. \\
&\left.\left[p^{e,m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)-p^{u,m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)\right]\mid z\right\} \qquad (212)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^{m}\left(z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right)}{\lambda^{m}(z,K,E,X,\mu,K,E,X,\mu)}Q(s,s')\times\right. \\
&\eta^{m}(s',n^{m}(s,l,j,z,K,E,X,\mu,K,E,X,\mu),v^{m}(s,l,j,z,K,E,X,\mu,K,E,X,\mu), \\
&\qquad \left. z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu))\mid z\right\} \\
\leq\;& p^{v,m}(z,K,E,X,\mu,K,E,X,\mu),\ \ (=\ \ \mathit{if}\ v^{m}(s,l,j,z,K,E,X,\mu,K,E,X,\mu)>0) \qquad (213)
\end{aligned}
$$

$$
\begin{aligned}
& e^z sF_n \left[n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu), k^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\right] \\
& +\mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)\right)}{\lambda^m\left(z,K,E,X,\mu,K,E,X,\mu\right)}Q\left(s,s'\right)\times\right. \\
& \xi^m\left(s',n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu), v^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right),\right. \\
& \qquad\qquad \left.\left. z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)\right) \mid z\right\} \\
& -\pi_n\mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)\right)}{\lambda^m\left(z,K,E,X,\mu,K,E,X,\mu\right)}Q\left(s,s'\right)\times\right. \\
& \alpha^m(s',n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu), v^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right), \\
& \qquad\qquad \left. z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)) \mid z\right\} \\
& \mathcal{E}\left\{\sum_{s'}\beta\frac{\lambda^m\left(z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)\right)}{\lambda^m\left(z,K,E,X,\mu,K,E,X,\mu\right)}Q\left(s,s'\right)\times\right. \\
& \chi^m(s',n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu), v^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right), \\
& \qquad\qquad \left. z',\hat{L}\left(z,K,E,X,\mu\right),\hat{L}\left(z,K,E,X,\mu\right)) \mid z\right\} \\
\leq\ & \xi^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right),\ (=\ \mathit{if}\ n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)>0) \qquad (214)
\end{aligned}
$$

$$
\begin{aligned}
& -p^{e,m}\left(z,K,E,X,\mu,K,E,X,\mu\right)+\xi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu) \\
\leq\ & \eta^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\ \ (=\ \mathit{if}\ h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)>0) \qquad (215)
\end{aligned}
$$

$$
\begin{aligned}
& p^{u,m}\left(z,K,E,X,\mu,K,E,X,\mu\right)-\xi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)+\alpha^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu) \\
& -\chi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\leq 0\ \ (=\ \mathit{if}\ f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)>0) \qquad (216)
\end{aligned}
$$

$$
\begin{aligned}
n^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right) &= l+h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right) \\
&\quad -f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right) \qquad (217)
\end{aligned}
$$

$$
h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\leq j \qquad (218)
$$

$$
\pi_n l\leq f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right) \qquad (219)
$$

$$
f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\leq l \qquad (220)
$$

$$
\eta^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\left[j-h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\right]=0 \qquad (221)
$$

$$
\alpha^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\left[f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)-\pi_n l\right]=0 \qquad (222)
$$

$$
\chi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\left[l-f^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\right]=0 \qquad (223)
$$

$$
\sum_s\int h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\mu\left(s,dl\times dj\right)\leq\left(1-\pi_u\right)E \qquad (224)
$$

$$
\begin{aligned}
0 &= \left[p^{e,m}\left(z,K,E,X,\mu,K,E,X,\mu\right)-p^{u,m}\left(z,K,E,X,\mu,K,E,X,\mu\right)\right] \\
&\quad \left[\left(1-\pi_u\right)E-\sum_s\int h^m\left(s,l,j,z,K,E,X,\mu,K,E,X,\mu\right)\mu\left(s,dl\times dj\right)\right] \qquad (225)
\end{aligned}
$$

$$
\begin{aligned}
& C^m(z,K,E,X,\mu,K,E,X,\mu)+I^m(z,K,E,X,\mu,K,E,X,\mu)+A^m(z,K,E,X,\mu,K,E,X,\mu) \\
= & \sum_s \int e^z sF\left[n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu),k^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\right]\mu(s,dl\times dj) \qquad (226)
\end{aligned}
$$

$$
\sum_s \int k^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\,\mu(s,dl\times dj)=K \qquad (227)
$$

$$
\begin{aligned}
& \sum_s \int v^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\,\mu(s,dl\times dj) \qquad (228) \\
= & H\left[A^m(z,K,E,X,\mu,K,E,X,\mu),U^m(z,K,E,X,\mu,K,E,X,\mu),\right. \\
& \left. A^m(z,K,E,X,\mu,K,E,X,\mu),U^m(z,K,E,X,\mu,K,E,X,\mu)\right] \qquad (229)
\end{aligned}
$$

$$
\begin{aligned}
U^m(z,K,E,X,\mu,K,E,X,\mu) = & X+E-\sum_s \int h^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\,\mu(s,dl\times dj) \\
& +\sum_s \int f^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\,\mu(s,dl\times dj) \qquad (230)
\end{aligned}
$$

Define

$$A(z,K,E,X,\mu)=A^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$U(z,K,E,X,\mu)=U^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$L(z,K,E,X,\mu)=\hat{L}(z,K,E,X,\mu)$$

$$r^k(z,K,E,X,\mu)=r^{k,m}(z,K,E,X,\mu,K,E,X,\mu)$$

$$r^u(z,K,E,X,\mu)=\varphi\lambda^m(z,K,E,X,\mu,K,E,X,\mu)^{-1}$$

$$p^v(z,K,E,X,\mu)=p^{v,m}(z,K,E,X,\mu,K,E,X,\mu)$$

$$p^u(z,K,E,X,\mu)=p^{u,m}(z,K,E,X,\mu,K,E,X,\mu)$$

$$p^e(z,K,E,X,\mu)=p^{e,m}(z,K,E,X,\mu,K,E,X,\mu)$$

$$q(z,K,E,X,\mu,z')=\beta\frac{\lambda^m\left[z',\hat{L}(z,K,E,X,\mu),\hat{L}(z,K,E,X,\mu)\right]}{\lambda^m(z,K,E,X,\mu,K,E,X,\mu)}$$

$$
\begin{aligned}
\Pi(z,K,E,X,\mu) = & C^m(z,K,E,X,\mu,K,E,X,\mu)+I^m(z,K,E,X,\mu,K,E,X,\mu) \\
& +r^u(z,K,E,X,\mu)U^m(z,K,E,X,\mu,K,E,X,\mu)-r^k(z,K,E,X,\mu)K
\end{aligned}
$$

$$\lambda(\kappa,z,K,E,X,\mu)=\lambda^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$c(\kappa,z,K,E,X,\mu)=C^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$m(\kappa,z,K,E,X,\mu)=U^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$i(\kappa,z,K,E,X,\mu)=I^m(z,K,E,X,\mu,K,E,X,\mu)+r^k(z,K,E,X,\mu)\kappa-r^k(z,K,E,X,\mu)K$$

$$\xi(s,l,j,z,K,E,X,\mu)=\xi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$\alpha(s,l,j,z,K,E,X,\mu)=\alpha^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$\chi(s,l,j,z,K,E,X,\mu) = \chi^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$\eta(s,l,j,z,K,E,X,\mu) = \eta^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$f(s,l,j,z,K,E,X,\mu) = f^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$h(s,l,j,z,K,E,X,\mu) = h^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$k(s,l,j,z,K,E,X,\mu) = k^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$n(s,l,j,z,K,E,X,\mu) = n^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$v(s,l,j,z,K,E,X,\mu) = v^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)$$

$$a(e,x,z,K,E,X,\mu) = \frac{e}{E}A^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$u(e,x,z,K,E,X,\mu) = \frac{e}{E}U^m(z,K,E,X,\mu,K,E,X,\mu)$$

$$\begin{aligned} b(e,x,z,K,E,X,\mu) &= \frac{e}{E}H\left[A^m(z,K,E,X,\mu,K,E,X,\mu), U^m(z,K,E,X,\mu,K,E,X,\mu),\right. \\ &\quad \left. A^m(z,K,E,X,\mu,K,E,X,\mu), U^m(z,K,E,X,\mu,K,E,X,\mu)\right] \end{aligned}$$

$$d(e,x,z,K,E,X,\mu) = \frac{e}{E}\sum_s \int h^m(s,l,j,z,K,E,X,\mu,K,E,X,\mu)\,\mu(s, dl \times dj)$$

In addition, define recursively the following value functions:

$$\begin{aligned} B(\kappa,z,K,E,X,\mu) &= \frac{c(\kappa,z,K,E,X,\mu)^{1-\sigma}-1}{1-\sigma} + \varphi m(\kappa,z,K,E,X,\mu) \\ &\quad + \beta\mathcal{E}\left\{B\left[(1-\delta)\kappa + i(\kappa,z,K,E,X,\mu), z', L(z,K,E,X,\mu)\right] \mid z\right\} \end{aligned}$$

$$\begin{aligned} W(s,l,j,z,K,E,X,\mu) &= e^z sF\left[n(s,l,j,z,K,E,X,\mu), k(s,l,j,z,K,E,X,\mu)\right] \\ &\quad + p^u(z,K,E,X,\mu)\, f(s,l,j,z,K,E,X,\mu) - p^e(z,K,E,X,\mu)\, h(s,l,j,z,K,E,X,\mu) \\ &\quad - r^k(z,K,E,X,\mu)\, k(s,l,j,z,K,E,X,\mu) - p^v(z,K,E,X,\mu)\, v(s,l,j,z,K,E,X,\mu) \\ &\quad + \mathcal{E}\left[\sum_{s'} q(z,K,E,X,\mu,z')\, W\left(s', n(s,l,j,z,K,E,X,\mu), v(s,l,j,z,K,E,X,\mu), z', L(z,K,E,X,\mu)\right) Q(s,s') \mid z\right] \end{aligned}$$

$$\begin{aligned} R(e,x,z,K,E,X,\mu) &= p^e(z,K,E,X,\mu)\, d(e,x,z,K,E,X,\mu) + p^v(z,K,E,X,\mu)\, b(e,x,z,K,E,X,\mu) \\ &\quad + p^u(z,K,E,X,\mu)\left[x + e - d(e,x,z,K,E,X,\mu) - u(e,x,z,K,E,X,\mu)\right] \\ &\quad + r^u(z,K,E,X,\mu)\, u(e,x,z,K,E,X,\mu) - a(e,x,z,K,E,X,\mu) \\ &\quad + \mathcal{E}\left[q(z,K,E,X,\mu,z')\, R\left(\begin{array}{c} G\left[\begin{array}{c} a(e,x,z,K,E,X,\mu), u(e,x,z,K,E,X,\mu), \\ A(z,K,E,X,\mu), U(z,K,E,X,\mu) \end{array}\right], \\ u(e,x,z,K,E,X,\mu) - G\left[\begin{array}{c} a(e,x,z,K,E,X,\mu), u(e,x,z,K,E,X,\mu), \\ A(z,K,E,X,\mu), U(z,K,E,X,\mu) \end{array}\right], \\ z', L(z,K,E,X,\mu) \end{array}\right) \mid z\right] \end{aligned}$$

Using these definitions, the homogeneity of degree one of H and G with respect to (a,u), and the homogeneity of degreee zero of H and G with respect to (A,U), equations (200)-(207) together with equations (208)-(230) imply equations (12)-(50).

■

4 Steady state equilibrium

4.1 Steady state conditions

In order to compute a recursive competitive equilibrium it will be first necessary to compute a steady state equilibrium for the deterministic version of the economy, i.e. one in which the aggregate productivity level z is equal to zero. This section describes the conditions that such steady state equilibrium must satisfy. Using equations (51)-(79), Lemma (19), Lemma (20), the conditon that $z = 0$ and the condition that the vector (K, E, X, μ) is constant over time, we get the following steady state conditions.

$$r = \frac{1}{\beta} - 1 + \delta \tag{231}$$

$$\xi(s,l,j) = \min \left\{ \begin{array}{c} \max \left\{ \hat{F}_n \left[(1-\pi_n) l + j, s, r \right] + (1-\pi_n) \Omega \left((1-\pi_n) l + j, s \right) + \Psi, p^e \right\}, \\ \max \left\{ \hat{F}_n \left[(1-\pi_n) l, s, r \right] + (1-\pi_n) \Omega \left((1-\pi_n) l, s \right) + \Psi, p^u \right\} \end{array} \right\}. \tag{232}$$

$$\Omega(n,s) = \min \left\{ \sum_{s'>0} \beta \max[\xi(s', n, 0) - p^e, 0] Q(s, s'), p^v \right\} + \sum_{s'>0} \beta \min[\xi_{t+1,T}(s', n, 0), p^e] Q(s, s') \tag{233}$$

$$\Psi = \pi_n \sum_{s'>0} \beta p^u Q(s, s') + Q(s, 0) \beta p^u \tag{234}$$

$$p^e = \hat{F}_n \left[\underline{n}(s), s, r \right] + (1-\pi_n) \Omega(\underline{n}(s), s) + \Psi, \tag{235}$$

$$p^u = \hat{F}_n \left[\bar{n}(s), s, r \right] + (1-\pi_n) \Omega(\bar{n}(s), s) + \Psi, \tag{236}$$

$$p^v = \sum_{s'>0} \beta \max[\xi(s', 0, \bar{v}(s)) - p^e, 0] Q(s, s') \tag{237}$$

$$n(s,l,j) = \max \left\{ \min \left\{ (1-\pi_n) l + j, \underline{n}(s) \right\}, \min \left\{ (1-\pi_n) l, \bar{n}(s) \right\} \right\} \tag{238}$$

$$h(s,l,j) = \max \left\{ n(s,l,j) - l, 0 \right\} \tag{239}$$

$$f(s,l,j) = \max \left\{ l - n(s,l,j), 0 \right\} \tag{240}$$

$$sF_k \left[n(s,l,j), k(s,l,j) \right] = r, \tag{241}$$

$$v(s,l,j) = \max \left\{ \bar{v}(s) - (1-\pi_n) n(s,l,j), 0 \right\}. \tag{242}$$

$$p^u = \varphi c^\sigma + p^v H_u(A,U,A,U) + \beta p^u + (1-\pi_u) G_u(A,U,A,U) \beta \left[p^e - p^u \right] \tag{243}$$

$$1 = p^v H_a(A,U,A,U) + (1-\pi_u) G_a(A,U,A,U) \beta \left[p^e - p^u \right] \tag{244}$$

$$\sum_s \int h(s,l,j) \mu(s, dl \times dj) \leq (1-\pi_u) G[A,U,A,U] \tag{245}$$

$$0 = \left[p^e - p^u \right] \left[(1-\pi_u) G[A,U,A,U] - \sum_s \int h(s,l,j) \mu(s, dl \times dj) \right] \tag{246}$$

$$\sum_s \int v(s,l,j) \mu(s, dl \times dj) = H[A,U,A,U] \tag{247}$$

$$\sum_s \int n(s,l,j)\mu(s,dl\times dj)+U=1 \tag{248}$$

$$c+\delta K+A=\sum_s \int e^z sF\left[n(s,l,j),k(s,l,j)\right]\mu(s,dl\times dj) \tag{249}$$

$$\sum_s \int k(s,l,j)\mu(s,dl\times dj)=K \tag{250}$$

$$E=G\left[A,U,A,U\right] \tag{251}$$

$$X=U-G\left[A,U,A,U\right] \tag{252}$$

$$\mu(s',\mathcal{L}\times\mathcal{J})=\sum_s \int_{\{(l,j):n(s,l,j)\in\mathcal{L}\text{ and }v(s,l,j)\in\mathcal{J}\}} Q(s,s')\mu(s,dl\times dj)+\varrho\psi(s')\mathcal{I}(\mathcal{L}\times\mathcal{J}) \tag{253}$$

Observe from Lemma (22) that if $\{C^m, I^m, n^m, k^m, f^m, h^m, v^m, U^m, A^m\}$ is a solution to the MSP's with exogenous stochastic process $\left(\hat{A},\hat{U},\hat{L}\right)$, conditions (200)-(207) are satisfied, z is identical to zero, and the aggregate state (K,E,X,μ) is constant over time, then equations (231)-(253) must hold.

4.2 Invariant distribution

The following Lemma characterizes a support to the invariant distribution μ that satisfies equation (253).[1]

Lemma 23 *Let M be a natural number satisfying that*

$$(1-\pi_n)^M \max\left\{\bar{n}(s_{\max}),\bar{v}(s_{\max})\right\}<\min\left\{\underline{n}(s_{\min}),\bar{v}(s_{\min})\right\}. \tag{254}$$

Define the set $\mathcal{N}$ as follows:

$$\mathcal{N}=\left\{\underset{s\in S}{\cup}\ \underset{k=0}{\overset{M-1}{\cup}}\left\{(1-\pi_n)^k\underline{n}(s),(1-\pi_n)^k\bar{n}(s),(1-\pi_n)^k\bar{v}(s)\right\}\right\}\cup\{0\}.$$

Then, the set

$$\mathcal{P}=\left\{(s,l,j):\ s\in S,\ l\in\mathcal{N},\ \text{and } j\in\left\{\underset{s'\in S}{\cup}\left\{\max\left[\bar{v}(s')-(1-\pi_n)l,0\right]\right\}\right\}\cup\{0\}\right\}$$

is a support of the invariant distribution μ.

Proof. From equations (238) and (242) we know that an establishment of type (s,l,j) transits to a next-period type (s',l',j'), with s' randomly determined,

$$l'=n(s,l,j), \tag{255}$$

and

$$j'=\max\left\{\bar{v}(s)-(1-\pi_n)l',0\right\}. \tag{256}$$

Define

$$\mathcal{P}^{(0)}=\underset{s\in S}{\cup}\left\{(s,0,0)\right\}.$$

[1] In the statement of the lemma $s_{\max}$ and $s_{\min}$ denote the largest and smallest positive values for s, respectively.

Since establishments are created with $(l, j) = (0, 0)$, $\mathcal{P}^{(0)}$ describes the set of all possible types (s, l, j) of establisments of zero age.

Define

$$\mathcal{N}^{(0)} = \{0\}.$$

Since $n(s, l, j) = 0$ whenever $(l, j) = (0, 0)$, $N^{(0)}$ describes the set of all possible employment levels of establishments of zero age.

Starting from $\mathcal{N}^{(0)}$, define recursively a sequence of sets $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$ as follows:

$$\mathcal{P}^{(m)} = \left\{(s, l, j) \colon \; s \in S, \; l \in \mathcal{N}^{(m-1)}, \text{ and } j \in \left\{ \underset{s_{-1} \in S}{\cup} \{\max\left[\bar{v}\left(s_{-1}\right) - \left(1 - \pi_n\right) l, 0\right]\} \right\} \cup \{0\} \right\}$$

$$\mathcal{N}^{(m)} = \left\{ \underset{s \in S}{\cup} \{\underline{n}\left(s\right), \bar{n}\left(s\right), \bar{v}\left(s\right)\} \right\} \cup \left\{ n \colon \; n = \left(1 - \pi_n\right) n_{m-1} \text{ for some } n_{m-1} \in N^{(m-1)} \right\},$$

for $m = 1, 2, ..., \infty$.

From equations (238), (242), (255) and (256) we know that $\mathcal{P}^{(m)}$ contains the set of all possible types (s, l, j) of establishments of age m, and that $\mathcal{N}^{(m)}$ contains the set of all possible employment levels of establishments of age m.[2]

By induction, it can be shown that:

$$\mathcal{N}^{(m)} = \left\{ \underset{s \in S}{\cup} \; \overset{m-1}{\underset{k=0}{\cup}} \left\{ \left(1 - \pi_n\right)^k \underline{n}\left(s\right), \left(1 - \pi_n\right)^k \bar{n}\left(s\right), \left(1 - \pi_n\right)^k \bar{v}\left(s\right) \right\} \right\} \cup \{0\}, \tag{257}$$

for $m = 1, 2, ..., \infty$.

A direct consequence of equation (257) is that $\mathcal{N}^{(m-1)} \subset \mathcal{N}^{(m)}$, for every $m \geq 1$. Thus, the set $\mathcal{N}^{(m)}$ in fact contains all the possible employment levels of establishments of age m or younger. Moreover,

$$\mathcal{N}^{(m)} / \mathcal{N}^{(m-1)} = \underset{s \in S}{\cup} \left\{ \left(1 - \pi_n\right)^{m-1} \underline{n}\left(s\right), \left(1 - \pi_n\right)^{m-1} \bar{n}\left(s\right), \left(1 - \pi_n\right)^{m-1} \bar{v}\left(s\right) \right\}, \tag{258}$$

for $m = 1, 2, ..., \infty$, where "/" denotes set difference.

In what follows it will be shown that there exists a $M < \infty$ such that $\mathcal{N}^{(M)}$ contains the set of all possible employment levels of establishments of all ages $m = 0, 1, ..., \infty$. To prove this it suffices to show that there exists a $M < \infty$ such that no establishment of age $M + 1$ will choose an employment level in the set $\mathcal{N}^{(M+1)} / \mathcal{N}^{(M)}$, i.e. all establishments of age $M + 1$ will choose an employment level in the set $\mathcal{N}^{(M)}$.[3]

Let M satisfy equation (254). Since $0 < \pi_n < 1$, such a M exists.

Let $(s, l, j) \in \mathcal{P}^{(M+1)}$.

Suppose that $n(s, l, j) \in \mathcal{N}^{(M+1)} / \mathcal{N}^{(M)}$. Since $\mathcal{N}^{(M+1)} / \mathcal{N}^{(M)}$ satisfies equation (258), and M satisfies equation (254), it follows that

$$n(s, l, j) \leq \left(1 - \pi_n\right)^M \max\left\{\bar{n}\left(s_{\max}\right), \bar{v}\left(s_{\max}\right)\right\} < \min\left\{\underline{n}\left(s_{\min}\right), \bar{v}\left(s_{\min}\right)\right\}. \tag{259}$$

[2] Observe that the "max" and "min" operators in equation (238) have been disregarded in the construction of the sets $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$. Thus, the set of actual types of establishments of age m and the set of actual employment levels of establishments of age m are smaller than $\mathcal{P}^{(m)}$ and $\mathcal{N}^{(m)}$, respectively.

[3] This condition is sufficient because whenever an establishment reaches age $M + 1$, its age can be reset to M without consequence. This procedure can be repeated an infinite number of times.

Also, since $n(s,l,j)$ satisfies equation (238) and $(s,l,j) \in \mathcal{P}^{(M+1)}$, we have that

$$n(s,l,j) \in \{\underline{n}(s), \bar{n}(s), (1-\pi_n) l\} \cup \left\{ \underset{s_{-1} \in S}{\cup} \bar{v}(s_{-1}) \right\}. \tag{260}$$

From equation (260) and the last inequality in equation (259), we then have that

$$n(s,l,j) = (1-\pi_n) l.$$

Suppose, first, that $j = 0$.

Suppose that some establishment of age M transits to (s,l,j). From equation (256), this implies that

$$0 = \max\{\bar{v}(s_{-1}) - (1-\pi_n)l, 0\},$$

for some $s_{-1} \in S$.

But, from equation (259)

$$n(s,l,j) = (1-\pi_n) l < \bar{v}(s_{-1}),$$

for all $s_{-1} \in S$. A contradiction.

Hence, $(s,l,j) \in \mathcal{P}^{(M+1)}$ does not correspond to an establishment of age $M+1$.

Suppose now that $j > 0$.

Let s_{-1} be such that $(1-\pi_n) l + j = \bar{v}(s_{-1})$ (since $(s,l,j) \in \mathcal{P}^{(M+1)}$, such an s_{-1} exists).

Then, from equation (238) we have that

$$n(s,l,j) = \max\{\min\{\bar{v}(s_{-1}), \underline{n}(s)\}, \min\{(1-\pi_n) l, \bar{n}(s)\}\},$$

and, therefore, that

$$n(s,l,j) = (1-\pi_n) l \le \bar{n}(s) \text{ and } n(s,l,j) = (1-\pi_n) l \ge \min\{\bar{v}(s_{-1}), \underline{n}(s)\}. \tag{261}$$

The second inequality in equation (261) contradicts equation (259).

We conclude that no establishment of age $M+1$ chooses an employment level in the set $\mathcal{N}^{(M+1)}/\mathcal{N}^{(M)}$. It follows that the set $\mathcal{P}^{(M+1)}$ is a support of the invariant distribution μ. ∎

4.3 Steady state computational algorithm

This section describes the algorithm used to compute a steady state equilibrium. It will be convenient to do so under the functional forms that will be used later on. In particular, the production function F is here assumed to have the following form:

$$F(n,k) = n^{\gamma} k^{\theta}, \tag{262}$$

where $\gamma > 0$, $\theta > 0$, and $\gamma + \theta < 1$. Observe that under this functional form F_k becomes

$$F_k(n,k) = \theta n^{\gamma} k^{\theta - 1} \tag{263}$$

and $\hat{F}_n$ becomes:

$$\hat{F}_n(n,s,r) = s^{\frac{1}{1-\theta}} n^{\frac{\gamma+\theta-1}{1-\theta}} \gamma \left[\frac{\theta}{r}\right]^{\frac{\theta}{1-\theta}}. \tag{264}$$

Lemmas 24-26 provide certain homogeneity results that will be used in the computational algorithm.

Lemma 24 *Suppose that F is given by equation (262). Let $\xi(s,l,j;p^u,p^e,p^v)$, $\Omega(n,s;p^u,p^e,p^v)$, $\Psi(p^u)$, $\underline{n}(s;p^u,p^e,p^v)$, $\bar{n}(s;p^u,p^e,p^v)$, $\bar{v}(s;p^u,p^e,p^v)$, $n(s,l,j;p^u,p^e,p^v)$, $h(s,l,j;p^u,p^e,p^v)$, $f(s,l,j;p^u,p^e,p^v)$, $k(s,l,j;p^u,p^e,p^v)$, $v(s,l,j;p^u,p^e,p^v)$ be the solutions to equations (232)-(242), given prices (p^u,p^e,p^v). Then,*

$$\begin{aligned}
\xi(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda\xi(s,l,j;p^u,p^e,p^v), && (265)\\
\Omega\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}n,s;\lambda p^u,\lambda p^e,\lambda p^v\right) &= \lambda\Omega(n,s;p^u,p^e,p^v) && (266)\\
\Psi(\lambda p^u) &= \lambda\Psi(p^u) && (267)\\
\underline{n}(s;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}\underline{n}(s;p^u,p^e,p^v) && (268)\\
\bar{n}(s;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}\bar{n}(s;p^u,p^e,p^v) && (269)\\
\bar{v}(s;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}\bar{v}(s;p^u,p^e,p^v) && (270)\\
n(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}n(s,l,j;p^u,p^e,p^v) && (271)\\
h(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}h(s,l,j;p^u,p^e,p^v) && (272)\\
f(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}f(s,l,j;p^u,p^e,p^v) && (273)\\
k(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{\gamma}{\gamma+\theta-1}}k(s,l,j;p^u,p^e,p^v) && (274)\\
v(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) &= \lambda^{\frac{1-\theta}{\gamma+\theta-1}}v(s,l,j;p^u,p^e,p^v) && (275)
\end{aligned}$$

for every $\lambda > 0$.

Proof. The claim follows from guessing and verifying equations (265)-(275) in equations (232)-(242). ■

Lemma 25 *Suppose that F is given by equation (262). Let $\mu(p^u,p^e,p^v)$ be the invariant distribution that satisfies equation (253) and $\mathcal{P}(p^u,p^e,p^v)$ be the finite support in Lemma 23, when prices are given by (p^u,p^e,p^v). Then, for every $\lambda > 0$,*

$$(s,l,j) \in \mathcal{P}(p^u,p^e,p^v) \Leftrightarrow \left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right) \in \mathcal{P}(\lambda p^u,\lambda p^e,\lambda p^v) \tag{276}$$

and

$$(s,l,j) \in \mathcal{P}(p^u,p^e,p^v) \Rightarrow \mu(s,l,j;p^u,p^e,p^v) = \mu\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v\right).$$

Proof. Equation (276) is a direct consequence of Lemmas 23 and 24.

From Lemma 23, observe that equation (253) can be written as follows. For every $(s',l',j') \in \mathcal{P}(p^u,p^e,p^v)$,

$$\mu(s',l',j';p^u,p^e,p^v) = \sum_s \sum_{(l,j)\in\mathcal{B}(s,l',j';p^u,p^e,p^v):} Q(s,s')\,\mu(s,l,j;p^u,p^e,p^v) + \varrho\psi(s')\,\mathcal{I}(l',j'), \tag{277}$$

where

$$\mathcal{B}(s,l',j';p^u,p^e,p^v) = \{(l,j) : (s,l,j) \in \mathcal{P}(p^u,p^e,p^v),\ n(s,l,j;p^u,p^e,p^v) = l' \text{ and } v(s,l,j;p^u,p^e,p^v) = j'\}, \tag{278}$$

and where $\mathcal{I}(l',j')=1$ if $(l',j')=(0,0)$, and $\mathcal{I}(l',j')=0$, otherwise.

For the same reason, we have that for every $\left(s',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j'\right)\in\mathcal{P}(\lambda p^u,\lambda p^e,\lambda p^v)$,

$$\begin{aligned}&\mu\left(s',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)\\ =&\sum_s\sum_{\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right)\in\mathcal{B}\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right):}Q(s,s')\,\mu\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v\right)\\ &+\varrho\psi(s')\,\mathcal{I}\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j'\right),\end{aligned}\tag{279}$$

where

$$\begin{aligned}\mathcal{B}\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)=&\left\{\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right):\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right)\in\mathcal{P}(\lambda p^u,\lambda p^e,\lambda p^v),\right.\\ &n(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l'\\ &\left.\text{and } v(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j'\right\}.\end{aligned}\tag{280}$$

Observe, from equation (280) and Lemma 24, that

$$\begin{aligned}\mathcal{B}\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)=&\left\{\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right):(s,l,j)\in\mathcal{P}(p^u,p^e,p^v),\right.\\ &\left.n(s,l,j;p^u,p^e,p^v)=l'\text{ and } v(s,l,j;p^u,p^e,p^v)=j'\right\}.\end{aligned}\tag{281}$$

From equations (278) and (281) we then have that

$$\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right)\in\mathcal{B}\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)\Leftrightarrow(l,j)\in\mathcal{B}(s,l',j';p^u,p^e,p^v)\tag{282}$$

Using equation (282) and the fact that $\mathcal{I}\left(\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j'\right)=\mathcal{I}(l',j')$, equation (279) can be written as follows:

$$\begin{aligned}&\mu\left(s',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)=\\ &\sum_s\sum_{(l,j)\in\mathcal{B}(s,l',j';p^u,p^e,p^v):}Q(s,s')\,\mu\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v\right)+\varrho\psi(s')\,\mathcal{I}(l',j').\end{aligned}$$

But this is the same as equation (277). It follows that

$$\mu\left(s',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l',\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j';\lambda p^u,\lambda p^e,\lambda p^v\right)=\mu(s',l',j';p^u,p^e,p^v).$$

■

Lemma 26 *Suppose that F is given by equation (262). Let $\xi(s,l,j;p^u,p^e,p^v)$, $\Omega(n,s;p^u,p^e,p^v)$, $\Psi(p^u)$, $\underline{n}(s;p^u,p^e,p^v)$, $\bar{n}(s;p^u,p^e,p^v)$, $\bar{v}(s;p^u,p^e,p^v)$, $n(s,l,j;p^u,p^e,p^v)$, $h(s,l,j;p^u,p^e,p^v)$, $f(s,l,j;p^u,p^e,p^v)$, $k(s,l,j;p^u,p^e,p^v)$, $v(s,l,j;p^u,p^e,p^v)$ be the solutions to equations (232)-(242), let $\mu(p^u,p^e,p^v)$ be the invariant distribution that satisfies equation (253) and let $\mathcal{P}(p^u,p^e,p^v)$ be the finite support in Lemma 23, when prices are given by (p^u,p^e,p^v). Then, for every $\lambda>0$,*

$$\sum_s\int n(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,dl\times dj;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{1-\theta}{\gamma+\theta-1}}\sum_s\int n(s,l,j;p^u,p^e,p^v)\mu(s,dl\times dj;p^u,p^e,p^v)\tag{283}$$

$$\sum_s\int h(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,dl\times dj;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{1-\theta}{\gamma+\theta-1}}\sum_s\int h(s,l,j;p^u,p^e,p^v)\mu(s,dl\times dj;p^u,p^e,p^v)\tag{284}$$

$$\sum_s\int v(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,dl\times dj;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{1-\theta}{\gamma+\theta-1}}\sum_s\int v(s,l,j;p^u,p^e,p^v)\mu(s,dl\times dj;p^u,p^e,p^v)\tag{285}$$

$$\sum_s\int k(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,dl\times dj;\lambda p^u,\lambda p^e,\lambda p^v)=\lambda^{\frac{\gamma}{\gamma+\theta-1}}\sum_s\int k(s,l,j;p^u,p^e,p^v)\mu(s,dl\times dj;p^u,p^e,p^v)\tag{286}$$

$$\sum_s \int sn(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)^\gamma k(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)^\theta \mu\left(s,dl\times dj;\lambda p^u,\lambda p^e,\lambda p^v\right) \tag{287}$$
$$= \lambda^{\frac{\gamma}{\gamma+\theta-1}} \sum_s \int sn(s,l,j;p^u,p^e,p^v)^\gamma k(s,l,j;p^u,p^e,p^v)^\theta \mu\left(s,dl\times dj;p^u,p^e,p^v\right)$$

Proof. *We shall prove equation (283). The proofs for equations (284) and (287) are analogous.*

From Lemmas 24 and 25 we have that for every $(s,l,j)\in\mathcal{P}\left(p^u,p^e,p^v\right)$ *:*

$$n(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) = \lambda^{\frac{1-\theta}{\gamma+\theta-1}}n(s,l,j;p^u,p^e,p^v)\mu(s,l,j;p^u,p^e,p^v).$$

Therefore,

$$\sum_{(s,l,j)\in\mathcal{P}(p^u,p^e,p^v)} n(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)$$
$$= \lambda^{\frac{1-\theta}{\gamma+\theta-1}} \sum_{(s,l,j)\in\mathcal{P}(p^u,p^e,p^v)} n(s,l,j;p^u,p^e,p^v)\mu(s,l,j;p^u,p^e,p^v),$$

and from equation (276),

$$\sum_{\left(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j\right)\in\mathcal{P}(\lambda p^u,\lambda p^e,\lambda p^v)} n(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}l,\lambda^{\frac{1-\theta}{\gamma+\theta-1}}j;\lambda p^u,\lambda p^e,\lambda p^v) =$$
$$\lambda^{\frac{1-\theta}{\gamma+\theta-1}} \sum_{(s,l,j)\in\mathcal{P}(p^u,p^e,p^v)} n(s,l,j;p^u,p^e,p^v)\mu(s,l,j;p^u,p^e,p^v).$$

Thus,

$$\sum_{(s,l,j)\in\mathcal{P}(\lambda p^u,\lambda p^e,\lambda p^v)} n(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v)\mu(s,l,j;\lambda p^u,\lambda p^e,\lambda p^v) =$$
$$\lambda^{\frac{1-\theta}{\gamma+\theta-1}} \sum_{(s,l,j)\in\mathcal{P}(p^u,p^e,p^v)} n(s,l,j;p^u,p^e,p^v)\mu(s,l,j;p^u,p^e,p^v),$$

which is equation (283). ■

Two computational algorithms will be described: One for the economy with no externalities and another for the economy with externalities.

4.3.1 Algorithm for economy with no externalities

In the economy with no externalities the recruitment technology is assumed to be given by

$$G\left(a,u,A,U\right) = \frac{u.a}{\left[u^\phi + a^\phi\right]^{\frac{1}{\phi}}},$$
$$H\left(a,u,A,U\right) = \frac{u.a}{\left[u^\phi + a^\phi\right]^{\frac{1}{\phi}}},$$

In this case, we have that

$$G\left(A,U,A,U\right) = H\left(A,U,A,U\right) = \frac{U.A}{\left[U^\phi + A^\phi\right]^{\frac{1}{\phi}}} = U\left\{\frac{1}{\left(\frac{U}{A}\right)^\phi + 1}\right\}^{\frac{1}{\phi}} = U\left\{\left(\frac{U}{A}\right)^\phi + 1\right\}^{-\frac{1}{\phi}},$$

$$G_a(A,U,A,U)=\left\{\frac{1}{1+\left(\frac{A}{U}\right)^{\phi}}\right\}^{\frac{1}{\phi}+1},$$

$$G_u(A,U,A,U)=\left\{\frac{1}{\left(\frac{U}{A}\right)^{\phi}+1}\right\}^{\frac{1}{\phi}+1}.$$

Therefore, equations (243), (244) and (247) become:

$$(1-\beta)\,p^u=\varphi c^{\sigma}+\{p^v+(1-\pi_u)\,\beta\,[p^e-p^u]\}\left\{\frac{1}{\left(\frac{U}{A}\right)^{\phi}+1}\right\}^{\frac{1}{\phi}+1}$$

$$1=\{p^v+(1-\pi_u)\,\beta\,[p^e-p^u]\}\left\{\frac{1}{1+\left(\frac{A}{U}\right)^{\phi}}\right\}^{\frac{1}{\phi}+1}$$

$$\sum_s\int v(s,l,j)\mu\,(s,dl\times dj)=U\left\{\frac{\{p^v+(1-\pi_u)\,\beta\,[p^e-p^u]\}^{\frac{\phi}{1+\phi}}-1}{\{p^v+(1-\pi_u)\,\beta\,[p^e-p^u]\}^{\frac{\phi}{1+\phi}}}\right\}^{\frac{1}{\phi}}$$

The computational algorithm is given by the following steps.

Step 1: Fix $p_0^u=1$.

Step 2: Choose some $p_0^e\geq p_0^u$ (it is convenient for the first choice to be $p_0^e=p_0^u$).

Step 3: Choose some p_0^v

Step 4: Set $j=0$ and solve for the functions $\xi(s,l,0;p_0^u,p_0^e,p_0^v)$, $\Omega(n,s;p_0^u,p_0^e,p_0^v)$ and $\Psi\,(p_0^u)$ that satisfy equations (232)-(234) when $j=0$ and prices are given by (p_0^u,p_0^e,p_0^v). (This can be done through value function iterations).

Step 5: Solve for the function $\xi(s,l,j;p_0^u,p_0^e,p_0^v)$ that satisfy equation (232) when $j>0$ and prices are given by (p_0^u,p_0^e,p_0^v). (Observe that given the output of Step 4, this takes only one value function iteration).

Step 6: For each s, solve for the thresholds $\underline{n}(s;p_0^u,p_0^e,p_0^v)$, $\bar{n}(s;p_0^u,p_0^e,p_0^v)$, $\bar{v}(s;p_0^u,p_0^e,p_0^v)$ that satisfy equations (235)-(237) (This can be done through standard root finding methods),

Step 7: Construct the functions $n(s,l,j;p_0^u,p_0^e,p_0^v)$, $h(s,l,j;p_0^u,p_0^e,p_0^v)$, $f(s,l,j;p_0^u,p_0^e,p_0^v)$, $k(s,l,j;p_0^u,p_0^e,p_0^v)$ and $v(s,l,j;$ $p_0^u,p_0^e,p_0^v)$ as in equations (238)-(242).

Step 8: Construct the finite support $\mathcal{P}\,(p_0^u,p_0^e,p_0^v)$ as in Lemma 23.

Step 9: For every (s,l',j'), construct the set $\mathcal{B}\,(s,l',j';p_0^u,p_0^e,p_0^v)$ as in equation (278) and solve for the invariant distribution $\mu\,(s,l,j;p_0^u,p_0^e,p_0^v)$ that satisfies equation (277) (This can be done recursively).

Step 10: Evaluate $\int v\,(p_0^u,p_0^e,p_0^v)\,d\mu\,(p_0^u,p_0^e,p_0^v)$, $\int n\,(p_0^u,p_0^e,p_0^v)\,d\mu\,(p_0^u,p_0^e,p_0^v)$, $\int h\,(p_0^u,p_0^e,p_0^v)\,d\mu\,(p_0^u,p_0^e,p_0^v)$, $\int k\,(p_0^u,p_0^e,p_0^v)$ $d\mu\,(p_0^u,p_0^e,p_0^v)$ and $\int n(p_0^u,p_0^e,p_0^v)^{\gamma}k(p_0^u,p_0^e,p_0^v)^{\theta}d\mu\,(p_0^u,p_0^e,p_0^v)$.

Step 11: Define

$$\underline{\lambda}\,(p_0^u,p_0^e,p_0^v)=\frac{1}{p_0^v+(1-\pi_u)\,\beta\,[p_0^e-p_0^u]}$$

Find the factor $\lambda>\underline{\lambda}\,(p_0^u,p_0^e,p_0^v)$ that satisfies that

$$\lambda^{\frac{1-\theta}{\gamma+\theta-1}}\sum_s\int v(s,l,j;p_0^u,p_0^e,p_0^v)\mu\,(s,dl\times dj;p_0^u,p_0^e,p_0^v) \tag{288}$$

$$=\left[1-\lambda^{\frac{1-\theta}{\gamma+\theta-1}}\int n\,(p_0^u,p_0^e,p_0^v)\,d\mu\,(p_0^u,p_0^e,p_0^v)\right]\left\{\frac{\lambda^{\frac{\phi}{1+\phi}}\{p_0^v+(1-\pi_u)\,\beta\,[p_0^e-p_0^u]\}^{\frac{\phi}{1+\phi}}-1}{\lambda^{\frac{\phi}{1+\phi}}\{p_0^v+(1-\pi_u)\,\beta\,[p_0^e-p_0^u]\}^{\frac{\phi}{1+\phi}}}\right\}^{\frac{1}{\phi}}$$

(This can be done using standard root finding methods)

Observe that the left hand side of equation (288) is stricly decreasing in λ, while the right hand side is stricly increasing in λ. Morever,

$$\begin{aligned} \lim_{\lambda \to \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)} LHS\left(\lambda\right) &> 0, \lim_{\lambda \to \infty} LHS\left(\lambda\right) = 0 \\ \lim_{\lambda \to \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)} RHS\left(\lambda\right) &= 0, \lim_{\lambda \to \infty} RHS\left(\lambda\right) = 1 \end{aligned}$$

Hence, there exists a unique $\lambda\left(p_0^u, p_0^e, p_0^v\right) > \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)$ that satisfies equation (288).

Step 12: Define $U\left(p_0^u, p_0^e, p_0^v\right)$, $A\left(p_0^u, p_0^e, p_0^v\right)$, $K\left(p_0^u, p_0^e, p_0^v\right)$, $Y\left(p_0^u, p_0^e, p_0^v\right)$, and $c\left(p_0^u, p_0^e, p_0^v\right)$ as follows:

$$U\left(p_0^u, p_0^e, p_0^v\right) = 1 - \lambda\left(p_0^u, p_0^e, p_0^v\right)^{\frac{1-\theta}{\gamma+\theta-1}} \int n\left(p_0^u, p_0^e, p_0^v\right) d\mu\left(p_0^u, p_0^e, p_0^v\right)$$

$$\frac{1}{\lambda\left(p_0^u, p_0^e, p_0^v\right)\left[p_0^v + \left(1-\pi_u\right)\beta\left(p_0^e - p_0^u\right)\right]} = \left\{\frac{1}{1 + \left(\frac{A\left(p_0^u, p_0^e, p_0^v\right)}{U\left(p_0^u, p_0^e, p_0^v\right)}\right)^{\phi}}\right\}^{\frac{1}{\phi}+1}$$

$$K\left(p_0^u, p_0^e, p_0^v\right) = \lambda\left(p_0^u, p_0^e, p_0^v\right)^{\frac{\gamma}{\gamma+\theta-1}} \int k\left(p_0^u, p_0^e, p_0^v\right) d\mu\left(p_0^u, p_0^e, p_0^v\right)$$

$$Y\left(p_0^u, p_0^e, p_0^v\right) = \lambda\left(p_0^u, p_0^e, p_0^v\right)^{\frac{\gamma}{\gamma+\theta-1}} \int sn(p_0^u, p_0^e, p_0^v)^{\gamma} k(p_0^u, p_0^e, p_0^v)^{\theta} d\mu\left(p_0^u, p_0^e, p_0^v\right)$$

$$c\left(p_0^u, p_0^e, p_0^v\right) = Y\left(p_0^u, p_0^e, p_0^v\right) - \delta K\left(p_0^u, p_0^e, p_0^v\right) - A\left(p_0^u, p_0^e, p_0^v\right)$$

Since $\lambda\left(p_0^u, p_0^e, p_0^v\right) > \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)$, observe that $U\left(p_0^u, p_0^e, p_0^v\right) > 0$ and that $A\left(p_0^u, p_0^e, p_0^v\right) > 0$.

Step 13: Evaluate the function

$$\begin{aligned} f\left(p_0^u, p_0^e, p_0^v\right) &= \varphi c\left(p_0^u, p_0^e, p_0^v\right)^{\sigma} \\ &\quad + \lambda\left(p_0^u, p_0^e, p_0^v\right)\left\{p_0^v\left(1-\pi_u\right)\beta\left[p_0^e - p_0^u\right]\right\}\left\{\left(\frac{U\left(p_0^u, p_0^e, p_0^v\right)}{A\left(p_0^u, p_0^e, p_0^v\right)}\right)^{\phi} + 1\right\}^{-\frac{1}{\phi}-1} \\ &\quad - \left(1-\beta\right)\lambda\left(p_0^u, p_0^e, p_0^v\right) p^u \end{aligned}$$

Step 14: In order to satisfy equation (243), go back to Step 3 with a new value for p_0^v until

$$f\left(p_0^u, p_0^e, p_0^v\right) = 0$$

(This can be done using standard root-finding methods)

Step 15: If

$$\lambda\left(p_0^u, p_0^e, p_0^v\right)^{\frac{1-\theta}{\gamma+\theta-1}} \int h\left(p_0^u, p_0^e, p_0^v\right) d\mu\left(p_0^u, p_0^e, p_0^v\right) \leq \left(1-\pi_u\right) G\left[A\left(p_0^u, p_0^e, p_0^v\right), U\left(p_0^u, p_0^e, p_0^v\right), A\left(p_0^u, p_0^e, p_0^v\right), U\left(p_0^u, p_0^e, p_0^v\right)\right], \tag{289}$$

and

$$\begin{aligned} 0 = \left[p_0^e - p_0^u\right]\Big\{&\left(1-\pi_u\right) G\left[A\left(p_0^u, p_0^e, p_0^v\right), U\left(p_0^u, p_0^e, p_0^v\right), A\left(p_0^u, p_0^e, p_0^v\right), U\left(p_0^u, p_0^e, p_0^v\right)\right] \\ &- \lambda\left(p_0^u, p_0^e, p_0^v\right)^{\frac{1-\theta}{\gamma+\theta-1}} \int h\left(p_0^u, p_0^e, p_0^v\right) d\mu\left(p_0^u, p_0^e, p_0^v\right)\Big\}, \end{aligned} \tag{290}$$

(i.e. if equations (245) and (246) are satisfied), then

$$\begin{aligned} p^u &= \lambda\left(p_0^u, p_0^e, p_0^v\right) p_0^u, \\ p^e &= \lambda\left(p_0^u, p_0^e, p_0^v\right) p_0^e, \\ p^v &= \lambda\left(p_0^u, p_0^e, p_0^v\right) p_0^v \end{aligned}$$

are equilibrium prices. (At this point, exit the algorithm).

Step 16: If conditions (289)-(290) are not satisfied, go back to Step 2 with a new guess for p_0^e (the search for p_0^e can be implemented within a standard root finding method).

4.3.2 Algorithm for economy with externalities

In the economy with externalities the recruitment technology is assumed to be given by

$$\begin{aligned} G\left(a,u,A,U\right) &= u\frac{A}{\left[U^{\phi}+A^{\phi}\right]^{\frac{1}{\phi}}}, \\ H\left(a,u,A,U\right) &= a\frac{U}{\left[U^{\phi}+A^{\phi}\right]^{\frac{1}{\phi}}}, \end{aligned}$$

In this case we have that

$$G\left(A,U,A,U\right)=H\left(A,U,A,U\right)=\frac{U.A}{\left[U^{\phi}+A^{\phi}\right]^{\frac{1}{\phi}}}=U\left\{\frac{1}{\left(\frac{U}{A}\right)^{\phi}+1}\right\}^{\frac{1}{\phi}}=U\left\{\left(\frac{U}{A}\right)^{\phi}+1\right\}^{-\frac{1}{\phi}},$$

$$G_a\left(A,U,A,U\right)=0$$

$$G_u\left(A,U,A,U\right)=\frac{A}{\left[U^{\phi}+A^{\phi}\right]^{\frac{1}{\phi}}}$$

$$H_a\left(a,u,A,U\right)=\frac{U}{\left[U^{\phi}+A^{\phi}\right]^{\frac{1}{\phi}}}$$

$$H_u\left(a,u,A,U\right)=0.$$

Therefore, equations (243), (244) and (247) become:

$$\left(1-\beta\right)p^u=\varphi c^{\sigma}+\left(1-\pi_u\right)\beta\left[p^e-p^u\right]\left\{\frac{1}{\left(\frac{U}{A}\right)^{\phi}+1}\right\}^{\frac{1}{\phi}}$$

$$1=p^v\left\{\frac{1}{1+\left(\frac{A}{U}\right)^{\phi}}\right\}^{\frac{1}{\phi}}$$

$$\sum_s\int v(s,l,j)\mu\left(s,dl\times dj\right)=U\left\{\frac{\left(p^v\right)^{\phi}-1}{\left(p^v\right)^{\phi}}\right\}^{\frac{1}{\phi}}$$

The computational algorithm is given by the following steps.

Steps 1-10 are the same as in Section 4.3.1.

Step 11: Define

$$\underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right) = \frac{1}{p_0^v}$$

Find the factor $\lambda > \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)$ that satisfies that

$$\begin{aligned} & \lambda^{\frac{1-\theta}{\gamma+\theta-1}} \sum_s \int v(s,l,j;p_0^u,p_0^e,p_0^v)\mu\left(s,dl\times dj;p_0^u,p_0^e,p_0^v\right) \\ = & \left[1-\lambda^{\frac{1-\theta}{\gamma+\theta-1}} \int n\left(p_0^u,p_0^e,p_0^v\right)d\mu\left(p_0^u,p_0^e,p_0^v\right)\right]\left\{\frac{\lambda^{\phi}\left(p_0^v\right)^{\phi}-1}{\lambda^{\phi}\left(p_0^v\right)^{\phi}}\right\}^{\frac{1}{\phi}} \end{aligned} \tag{291}$$

(This can be done using standard root finding methods)

Observe that the left hand side of equation (291) is stricly decreasing in λ, while the right hand side is stricly increasing in λ. Morever,

$$\begin{aligned} \lim_{\lambda\to\underline{\lambda}\left(p_0^u,p_0^e,p_0^v\right)} LHS\left(\lambda\right) &> 0,\ \lim_{\lambda\to\infty} LHS\left(\lambda\right)=0 \\ \lim_{\lambda\to\underline{\lambda}\left(p_0^u,p_0^e,p_0^v\right)} RHS\left(\lambda\right) &= 0,\ \lim_{\lambda\to\infty} RHS\left(\lambda\right)=1 \end{aligned}$$

Hence, there exists a unique $\lambda\left(p_0^u, p_0^e, p_0^v\right) > \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)$ that satisfies equation (291).

Step 12: Define $U\left(p_0^u, p_0^e, p_0^v\right)$, $A\left(p_0^u, p_0^e, p_0^v\right)$, $K\left(p_0^u, p_0^e, p_0^v\right)$, $Y\left(p_0^u, p_0^e, p_0^v\right)$, and $c\left(p_0^u, p_0^e, p_0^v\right)$ as follows:

$$U\left(p_0^u,p_0^e,p_0^v\right) = 1-\lambda\left(p_0^u,p_0^e,p_0^v\right)^{\frac{1-\theta}{\gamma+\theta-1}}\int n\left(p_0^u,p_0^e,p_0^v\right)d\mu\left(p_0^u,p_0^e,p_0^v\right)$$

$$\frac{1}{\lambda\left(p_0^u,p_0^e,p_0^v\right)p_0^v} = \left\{\frac{1}{1+\left(\frac{A\left(p_0^u,p_0^e,p_0^v\right)}{U\left(p_0^u,p_0^e,p_0^v\right)}\right)^{\phi}}\right\}^{\frac{1}{\phi}}$$

$$K\left(p_0^u,p_0^e,p_0^v\right) = \lambda\left(p_0^u,p_0^e,p_0^v\right)^{\frac{\gamma}{\gamma+\theta-1}}\int k\left(p_0^u,p_0^e,p_0^v\right)d\mu\left(p_0^u,p_0^e,p_0^v\right)$$

$$Y\left(p_0^u,p_0^e,p_0^v\right) = \lambda\left(p_0^u,p_0^e,p_0^v\right)^{\frac{\gamma}{\gamma+\theta-1}}\int sn(p_0^u,p_0^e,p_0^v)^{\gamma}k(p_0^u,p_0^e,p_0^v)^{\theta}d\mu\left(p_0^u,p_0^e,p_0^v\right)$$

$$c\left(p_0^u,p_0^e,p_0^v\right) = Y\left(p_0^u,p_0^e,p_0^v\right)-\delta K\left(p_0^u,p_0^e,p_0^v\right)-A\left(p_0^u,p_0^e,p_0^v\right)$$

Since $\lambda\left(p_0^u, p_0^e, p_0^v\right) > \underline{\lambda}\left(p_0^u, p_0^e, p_0^v\right)$, observe that $U\left(p_0^u, p_0^e, p_0^v\right) > 0$ and that $A\left(p_0^u, p_0^e, p_0^v\right) > 0$.

Step 13: Evaluate the function

$$\begin{aligned} f\left(p_0^u,p_0^e,p_0^v\right) &= \varphi c\left(p_0^u,p_0^e,p_0^v\right)^{\sigma} \\ & +\left(1-\pi_u\right)\beta\lambda\left(p_0^u,p_0^e,p_0^v\right)\left[p_0^e-p_0^u\right]G_u(A\left(p_0^u,p_0^e,p_0^v\right),U\left(p_0^u,p_0^e,p_0^v\right),A\left(p_0^u,p_0^e,p_0^v\right),U\left(p_0^u,p_0^e,p_0^v\right)) \\ & -\left(1-\beta\right)\lambda\left(p_0^u,p_0^e,p_0^v\right)p^u \end{aligned}$$

Steps 14-16 are the same as in Section 4.3.1.

5 Off-steady state dynamics

In this section it will be important to have separate notation for steady state variables. In particular, $\underline{n}^*$, $\bar{n}^*$, and $\bar{v}^*$ will denote steady state threshold functions and μ^* will denote the invariant distribution. From Lemma 23, we know that μ^* has a finite support given by

$$\mathcal{P}^* = \left\{ (s,l,j) : \; s \in S, \; l \in \mathcal{N}^*, \text{ and } j \in \left\{ \underset{s' \in S}{\cup} \{ \max \left[\bar{v}^* (s') - (1-\pi_n) \, l, 0 \right] \} \right\} \cup \{0\} \right\}$$

where

$$\mathcal{N}^* = \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \underline{n}^* (s) \right\} \right\} \cup \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \bar{n}^* (s) \right\} \right\} \cup \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \bar{v}^* (s) \right\} \right\} \cup \{0\} ,$$

and where M is a natural number satisfying that

$$(1-\pi_n)^M \max \left\{ \bar{n}^* (s_{\max}) , \bar{v}^* (s_{\max}) \right\} < \min \left\{ \underline{n}^* (s_{\min}) , \bar{v}^* (s_{\min}) \right\} . \tag{292}$$

In order to analyze off-steady state dynamics it will be useful to define $\underline{n}_t$, $\bar{n}_t$, and $\bar{v}_t$, as the threshold functions chosen at date t. In addition, it will be useful to define the following minimum distance:

$$\varepsilon = \min \;\; |a-b| , \tag{293}$$

subject to

$$a, b \in \mathcal{D}^* \text{ and } a \neq b,$$

where

$$\mathcal{D}^* = \mathcal{N}^* \cup \left\{ \underset{s \in S}{\cup} \; \left\{ (1-\pi_n)^M \, \underline{n}^* (s) , (1-\pi_n)^M \, \bar{n}^* (s) , (1-\pi_n)^M \, \bar{v}^* (s) \right\} \right\} .$$

The following Lemma characterizes the distribution μ_{t+1} under the assumptions that μ_t and the finite history of thressholds $\left\{ \underline{n}_{t-k}, \bar{n}_{t-k}, \bar{v}_{t-k} \right\}_{k=0,1,\ldots,M}$ are sufficiently close to their steady-state values.

Lemma 27 *Let M be defined by equation (292) and ε by equation (293).*

Suppose that

$$\left| \underline{n}_{t-k} (s) - \underline{n}^* (s) \right| \;\; < \;\; \varepsilon/2, \tag{294}$$

$$\left| \bar{n}_{t-k} (s) - \bar{n}^* (s) \right| \;\; < \;\; \varepsilon/2, \tag{295}$$

$$\left| \bar{v}_{t-k} (s) - \bar{v}^* (s) \right| \;\; < \;\; \varepsilon/2 \tag{296}$$

for every s and every $0 \leq k \leq M+1$.

Suppose that the distribution μ_t has a finite support $\mathcal{P}_t$ given by

$$\mathcal{P}_t = \left\{ (s,l,j) : \; s \in S, \; l \in \mathcal{N}_t, \text{ and } j \in \left\{ \underset{s' \in S}{\cup} \{ \max \left[\bar{v}_{t-1} (s') - (1-\pi_n) \, l, 0 \right] \} \right\} \cup \{0\} \right\}$$

where

$$\begin{aligned} \mathcal{N}_t \;\; = \;\; & \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \underline{n}_{t-k-1} (s) \right\} \right\} \cup \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \bar{n}_{t-k-1} (s) \right\} \right\} \cup \\ & \left\{ \underset{s \in S}{\cup} \; \underset{k=0}{\overset{M-1}{\cup}} \left\{ (1-\pi_n)^k \, \bar{v}_{t-k-2} (s) \right\} \right\} \cup \{0\} , \end{aligned}$$

In addition, suppose that for every $(s,l,j) \in \mathcal{P}_t$*:*

$$\mu_t(s,l,j) = \mu^*(s,l^*,j^*),$$

where (s,l^*,j^*) *is the unique element of* $\mathcal{P}^*$ *satisfying that* $|l-l^*| < \varepsilon/2$ *and* $|j-j^*| < \varepsilon/2 + (1-\pi)\varepsilon/2$.

Then, the distribution μ_{t+1} *has a finite support* $\mathcal{P}_{t+1}$ *given by*

$$\mathcal{P}_{t+1} = \left\{(s,l,j) :\ s \in S,\ l \in \mathcal{N}_{t+1},\ \text{and } j \in \left\{\underset{s' \in S}{\cup} \{\max[\bar{v}_t(s') - (1-\pi_n)l, 0]\}\right\} \cup \{0\}\right\}$$

where

$$\begin{aligned}\mathcal{N}_{t+1} \quad = \quad & \left\{\underset{s\in S}{\cup}\ \overset{M-1}{\underset{k=0}{\cup}} \left\{(1-\pi_n)^k \underline{n}_{t-k}(s)\right\}\right\} \cup \left\{\underset{s\in S}{\cup}\ \overset{M-1}{\underset{k=0}{\cup}} \left\{(1-\pi_n)^k \bar{n}_{t-k}(s)\right\}\right\} \cup \\ & \left\{\underset{s\in S}{\cup}\ \overset{M-1}{\underset{k=0}{\cup}} \left\{(1-\pi_n)^k \bar{v}_{t-k-1}(s)\right\}\right\} \cup \{0\},\end{aligned}$$

Moreover, for every $(s,l,j) \in \mathcal{P}_{t+1}$*:*

$$\mu_{t+1}(s,l,j) = \mu^*(s,l^*,j^*)$$

where (s,l^*,j^*) *is the unique element of* $\mathcal{P}^*$ *satisfying that* $|l-l^*| < \varepsilon/2$ *and* $|j-j^*| < \varepsilon/2 + (1-\pi)\varepsilon/2$.

Proof. Observe that the optimal decision rules at period $t-k$ are given by

$$n_{t-k}(s,l,j) = \max\left\{\begin{array}{c}\min\{(1-\pi_n\}\, l + j, \underline{n}_{t-k}(s) \\ \min\{(1-\pi_n)\, l, \bar{n}_{t-k}(s)\}\end{array}\right\} \tag{297}$$

and

$$v_{t-k}(s,l,j) = \max\{\bar{v}_{t-k}(s) - (1-\pi_n)\, n_{t-k}(s,l,j), 0\}, \tag{298}$$

for $k = 0, 1, ..., M+1$.

a) We will first show that $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1}.

Define the sets $\mathcal{A}_t$ and $\mathcal{B}_{t+1}$ as follows:

$$\mathcal{A}_t = \underset{s\in S}{\cup} \left\{(1-\pi_n)^{M-1} \underline{n}_{t-M}(s), (1-\pi_n)^{M-1} \bar{n}_{t-M}(s), (1-\pi_n)^{M-1} \bar{v}_{t-M-1}(s)\right\},$$

$$\mathcal{B}_{t+1} = \{l' : l' = (1-\pi_n)\, l,\ \text{for some } l \in \mathcal{N}_t/\mathcal{A}_t\}. \tag{299}$$

Observe that

$$\mathcal{N}_{t+1} = \mathcal{B}_{t+1} \cup \left\{\underset{s\in S}{\cup} \{\underline{n}_t(s), \bar{n}_t(s), \bar{v}_{t-1}(s)\}\right\}. \tag{300}$$

To show that $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1} it suffices to show that

$$(s,l,j) \in \mathcal{P}_t \Longrightarrow n_t(s,l,j) \in \mathcal{N}_{t+1} \text{ and } v_t(s,l,j) \in \left\{\underset{s'\in S}{\cup} \{\max[\bar{v}_t(s') - (1-\pi_n)\, n_t(s,l,j), 0]\}\right\} \cup \{0\}. \tag{301}$$

Let $(s,l,j) \in \mathcal{P}_t$. Then,

$$s \in S,\ l \in \mathcal{N}_t \text{ and } j = \max[\bar{v}_{t-1}(s') - (1-\pi_n)\, l, 0], \text{ for some } s' \in S. \tag{302}$$

From equation (297) we have that

$$n_t(s,l,j) = \max \left\{ \begin{array}{c} \min \left\{ (1-\pi_n \right\} l + j, \underline{n}_t(s) \right) \\ \min \left\{ (1-\pi_n) l, \bar{n}_t(s) \right\} \end{array} \right\}.$$

Using equation (302), we then have that

$$n_t(s,l,j) = \max \left\{ \begin{array}{c} \min \left\{ \max \left[\bar{v}_{t-1}(s'), (1-\pi_n) l \right], \underline{n}_t(s) \right\} \\ \min \left\{ (1-\pi_n) l, \bar{n}_t(s) \right\} \end{array} \right\}, \tag{303}$$

for some $s' \in S$.

As a consequence,

$$n_t(s,l,j) \in \left\{ (1-\pi_n) l, \bar{n}_t(s), \underline{n}_t(s), \bar{v}_{t-1}(s') \right\}$$

for some $s' \in S$.

From equations (299) and (300) we have that

$$l \in \mathcal{N}_t / \mathcal{A}_t \Rightarrow n_t(s,l,j) \in \mathcal{N}_{t+1}.$$

Suppose that $l \in A_t$. Without loss of generality assume that

$$l = (1-\pi_n)^{M-1} \underline{n}_{t-M}(s'')$$

for some $s'' \in S$ (the cases $l = (1-\pi_n)^{M-1} \bar{n}_{t-M}(s'')$ and $l = (1-\pi_n)^{M-1} \bar{v}_{t-M-1}(s'')$ can be handled in exactly the same way).

Then, equation (303) becomes

$$n_t(s,l,j) = \max \left\{ \begin{array}{c} \min \left\{ \max \left[\bar{v}_{t-1}(s'), (1-\pi_n)^M \underline{n}_{t-M}(s'') \right], \underline{n}_t(s) \right\} \\ \min \left\{ (1-\pi_n)^M \underline{n}_{t-M}(s''), \bar{n}_t(s) \right\} \end{array} \right\}. \tag{304}$$

But from equation (292) and equations (294)-(296), we have that

$$\begin{aligned} (1-\pi_n)^M \underline{n}_{t-M}(s'') &< (1-\pi_n)^M \bar{n}_{t-M}(s'') \\ &\leq (1-\pi_n)^M \bar{n}_{t-M}(s_{\max}) \\ &\leq \bar{v}_{t-1}(s_{\min}) \\ &\leq \bar{v}_{t-1}(s') \end{aligned}$$

and that

$$\begin{aligned} (1-\pi_n)^M \underline{n}_{t-M}(s'') &< (1-\pi_n)^M \bar{n}_{t-M}(s'') \\ &\leq (1-\pi_n)^M \bar{n}_{t-M}(s_{\max}) \\ &\leq \underline{n}_t(s_{\min}) \\ &\leq \underline{n}_t(s) \\ &< \bar{n}_t(s) \end{aligned}$$

Therefore equation (304) becomes

$$\begin{aligned} n_t(s,l,j) &= \max \left\{ \begin{array}{c} \min \left\{ \bar{v}_{t-1}(s'), \underline{n}_t(s) \right\}, \\ (1-\pi_n)^M \underline{n}_{t-M}(s'') \end{array} \right\} \\ &= \min \left\{ \bar{v}_{t-1}(s'), \underline{n}_t(s) \right\}. \end{aligned}$$

Thus, from equation (300), $n_t(s,l,j) \in \mathcal{N}_{t+1}$.

From equation (298), observe that

$$v_t(s,l,j) = \max \left\{ \bar{v}_t(s) - (1-\pi_n) n_t(s,l,j), 0 \right\}$$

Thus,

$$v_t(s,l,j) \in \left\{ \bigcup_{s' \in S} \left\{ \max \left[\bar{v}_t(s') - (1-\pi_n) n_t(s,l,j), 0 \right] \right\} \right\}.$$

Therefore, $\mathcal{P}_{t+1}$ is a support of the distribution μ_{t+1}.

b) To prove the second part of the Proposition it will be convenient to define the following (one-to-one and onto) functions.

For every $(s,l,j) \in \mathcal{P}_t$:

$$\begin{aligned} l_t^*(l,j) &= l^* \\ j_t^*(l,j) &= j^* \end{aligned}$$

where (s,l^*,j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l-l^*| < \varepsilon/2$ and $|[(1-\pi_n)l+j] - [(1-\pi_n)l^*+j^*]| < \varepsilon/2$.

Similarly, for every $(s',l',j') \in \mathcal{P}_{t+1}$:

$$\begin{aligned} l_{t+1}^*(l',j') &= l^* \\ j_{t+1}^*(l',j') &= j^* \end{aligned}$$

where (s',l^*,j^*) is the unique element of $\mathcal{P}^*$ satisfying that $|l'-l^*| < \varepsilon/2$ and $|[(1-\pi_n)l'+j'] - [(1-\pi_n)l^*+j^*]| < \varepsilon/2$.

Observe that, by assumption, we have that for every $(s,l,j) \in \mathcal{P}_t$:

$$\mu_t(s,l,j) = \mu^*(s, l_t^*(l,j), j_t^*(l,j)). \tag{305}$$

We need to show that for every $(s',l',j') \in \mathcal{P}_{t+1}$:

$$\mu_{t+1}(s',l',j') = \mu^*\left(s', l_{t+1}^*(l',j'), j_{t+1}^*(l',j')\right). \tag{306}$$

Let $(s',l',j') \in \mathcal{P}_{t+1}$.

Using equation (305), we have that

$$\mu_{t+1}(s',l',j') = \sum_{(s,l,j) \,\in\, \mathcal{G}_t(l',j')} Q(s,s')\,\mu^*(s, l_t^*(l,j), j_t^*(l,j)) + \varrho\psi(s')\,\mathcal{I}(l',j'),$$

where

$$\mathcal{G}_t(l',j') = \{(s,l,j) \in \mathcal{P}_t \colon\ n_t(s,l,j) = l' \text{ and } v_t(s,l,j) = j'\}.$$

Also observe that

$$\mu^*\left(s', l^*_{t+1}(l',j'), j^*_{t+1}(l',j')\right) = \sum_{(s,l^*,j^*)\ \in\ \mathcal{G}^*(l^*_{t+1}(l',j'),j^*_{t+1}(l',j'))} Q\left(s,s'\right)\mu^*\left(s,l^*,j^*\right) + \varrho\psi\left(s'\right)\mathcal{I}\left(l^*_{t+1}(l',j'), j^*_{t+1}(l',j')\right),$$

where

$$\mathcal{G}^*(l^*_{t+1}(l',j'), j^*_{t+1}(l',j')) = \left\{(s,l^*,j^*) \in \mathcal{P}^*\colon\ n^*(s,l^*,j^*) = l^*_{t+1}(l',j') \text{ and } v^*(s,l^*,j^*) = j^*_{t+1}(l',j')\right\}.$$

To show that equation (306) holds, it then suffices to show that

$$(l',j') = (0,0) \Leftrightarrow \left(l^*_{t+1}(l',j'), j^*_{t+1}(l',j')\right) = (0,0)\,, \tag{307}$$

$$(s,l,j) \in \mathcal{G}_t(l',j') \Longrightarrow (s, l^*_t\left(l,j\right), j^*_t\left(l,j\right)) \in \mathcal{G}^*(l^*_{t+1}(l',j'), j^*_{t+1}(l',j')), \tag{308}$$

$$(s,l^*,j^*) \in \mathcal{G}^*(l^*_{t+1}(l',j'), j^*_{t+1}(l',j')) \Longrightarrow \left(s, [l^*_t]^{-1}(l^*,j^*), [j^*_t]^{-1}(l^*,j^*)\right) \in \mathcal{G}_t(l',j'). \tag{309}$$

where $\left([l^*_t]^{-1}, [j^*_t]^{-1}\right)$ is the inverse function of (l^*_t, j^*_t).

b.1) Proof of equation (307).

It is a direct consequence of how l^*_{t+1} and j^*_{t+1} were defined and equations (294)-(296).

b.2) Proof of equation (308).

Let $(s,l,j) \in \mathcal{G}_t(l',j')$. Then, $(s,l,j) \in \mathcal{P}_t$,

$$l' = \max\left\{\begin{array}{c}\min\left\{\left(1-\pi_n\right)l + j, \underline{n}_t\left(s\right)\right\}\\ \min\left\{\left(1-\pi_n\right)l, \bar{n}_t\left(s\right)\right\}\end{array}\right\},$$

and

$$\left(1-\pi_n\right)l' + j' = \max\left\{\bar{v}_t\left(s\right), \left(1-\pi_n\right)l'\right\}.$$

Observe that $(s, l^*_t\left(l,j\right), j^*_t\left(l,j\right)) \in \mathcal{P}^*$,

$$n^*(s, l^*_t(l,j), j^*_t(l,j)) = \max\left\{\begin{array}{c}\min\left\{\left(1-\pi_n\right)l^*_t(l,j) + j^*_t(l,j), \underline{n}^*\left(s\right)\right\}\\ \min\left\{\left(1-\pi_n\right)l^*_t(l,j), \bar{n}^*\left(s\right)\right\}\end{array}\right\},$$

and

$$\begin{aligned} & \left(1-\pi_n\right)n^*(s, l^*_t(l,j), j^*_t(l,j)) + v^*(s, l^*_t(l,j), j^*_t(l,j)) \\ = \quad & \max\left\{\bar{v}^*\left(s\right), \left(1-\pi_n\right)n^*(s, l^*_t(l,j), j^*_t(l,j))\right\}. \end{aligned}$$

Since

$$\left|\left(1-\pi_n\right)l - \left(1-\pi_n\right)l^*_t(l,j)\right| < \varepsilon/2,$$

$$\left|\left[\left(1-\pi_n\right)l + j\right] - \left[\left(1-\pi_n\right)l^*_t(l,j) + j^*_t(l,j)\right]\right| < \varepsilon/2,$$

$$\left|\underline{n}_t\left(s\right) - \underline{n}^*\left(s\right)\right| < \varepsilon/2,$$

and

$$\left|\bar{n}_t\left(s\right) - \bar{n}^*\left(s\right)\right| < \varepsilon/2,$$

it follows that

$$|n^*(s, l_t^*(l,j), j_t^*(l,j)) - l'| < \varepsilon/2, \tag{310}$$

and, therefore, that

$$|[(1-\pi_n)\, n^*(s, l_t^*(l,j), j_t^*(l,j)) + v^*(s, l_t^*(l,j), j_t^*(l,j))] - [(1-\pi_n)\, l' + j']| < \varepsilon/2. \tag{311}$$

Since $(s', l', j') \in \mathcal{P}_{t+1}$ and $[s', n^*(s, l_t^*(l,j), j_t^*(l,j)), v^*(s, l_t^*(l,j), j_t^*(l,j))] \in \mathcal{P}^*$, equations (310) and (311) imply that

$$l_{t+1}^*(l', j') = n^*(s, l_t^*(l,j), j_t^*(l,j)),$$

$$j_{t+1}^*(l', j') = v^*(s, l_t^*(l,j), j_t^*(l,j)).$$

Since $(s, l_t^*(l,j), j_t^*(l,j)) \in \mathcal{P}^*$ it follows that

$$(s, l_t^*(l,j), j_t^*(l,j)) \in \mathcal{G}^*(l_{t+1}^*(l', j'), j_{t+1}^*(l', j')).$$

b.3) Proof of equation (309).

Let $(s, l^*, j^*) \in \mathcal{G}^*(l_{t+1}^*(l', j'), j_{t+1}^*(l', j'))$. Then, $(s, l^*, j^*) \in \mathcal{P}^*$,

$$\begin{aligned} l_{t+1}^*(l', j') &= n^*(s, l^*, j^*) & (312)\\ &= \max \left\{ \begin{array}{c} \min\left\{(1-\pi_n)\, l^* + j^*, \underline{n}^*(s)\right\} \\ \min\left\{(1-\pi_n)\, l^*, \bar{n}^*(s)\right\} \end{array} \right\}, & (313) \end{aligned}$$

and

$$\begin{aligned} (1-\pi_n)\, l_{t+1}^*(l', j') + j_{t+1}^*(l', j') &= (1-\pi_n)\, n^*(s, l^*, j^*) + v^*(s, l^*, j^*) & (314)\\ &= \max\left\{\bar{v}^*(s), (1-\pi_n)\, l_{t+1}^*(l', j')\right\}. & (315) \end{aligned}$$

Observe that $\left(s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)\right) \in \mathcal{P}_t$,

$$n_t(s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)) = \max \left\{ \begin{array}{c} \min\left\{(1-\pi_n)\, [l_t^*]^{-1}(l^*, j^*) + [j_t^*]^{-1}(l^*, j^*), \underline{n}_t(s)\right\} \\ \min\left\{(1-\pi_n)\, [l_t^*]^{-1}(l^*, j^*), \bar{n}_t(s)\right\} \end{array} \right\},$$

and

$$\begin{aligned} & (1-\pi_n)\, n_t(s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)) + v_t(s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)) \\ = \; & \max\left\{\bar{v}_t(s), (1-\pi_n)\, n_t(s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*))\right\}. \end{aligned}$$

Also, from equation (301), we have that

$$\left[s', n_t\left[s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)\right], v_t\left[s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)\right]\right] \in \mathcal{P}_{t+1}$$

for every s'.

Moreover,

$$\begin{aligned} & l_{t+1}^*(n_t\left[s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)\right], v_t\left[s, [l_t^*]^{-1}(l^*, j^*), [j_t^*]^{-1}(l^*, j^*)\right]) \\ = \; & n^*(s, l^*, j^*) \end{aligned}$$

and

$$\begin{aligned} & j^*_{t+1}(n_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right], v_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right]) \\ = \; & v^*(s, l^*, j^*) \end{aligned}$$

Hence, from equations (312) and (314), we have that

$$\begin{aligned} & l^*_{t+1}(n_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right], v_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right]) \\ = \; & l^*_{t+1}(l', j') \end{aligned}$$

and

$$\begin{aligned} & j^*_{t+1}(n_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right], v_t \left[s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right]) \\ = \; & j^*_{t+1}(l', j') \end{aligned}$$

It follows that

$$l' = n_t(s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)),$$

$$j' = v_t(s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)).$$

Since $\left(s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right) \in \mathcal{P}_t$ it follows that

$$\left(s, [l^*_t]^{-1}(l^*, j^*), [j^*_t]^{-1}(l^*, j^*)\right) \in \mathcal{G}_t(l', j').$$

■

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