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Debt Dynamics with Fixed Issuance Costs

Abstract

We investigate equilibrium debt dynamics for a firm that cannot commit to a future debt policy and is subject to a fixed restructuring cost. We formally characterize equilibria when the firm is not required to repurchase outstanding debt prior to issuing additional debt. For realistic values of issuance costs and debt maturity, the no-commitment policy generates tax benefits that are similar to those obtained by a benchmark policy with commitment. For positive but arbitrarily small issuance costs, there are maturities for which shareholders extract essentially the entire claim to cash-flows.

JEL Classification Codes: G12; G32; G33

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1 Introduction

Enticed by low interest rates and incentivized by government intervention in the corporate debt market, firms have been issuing debt at unprecedented levels over the past decade.[1] Such growth underscores the role of debt as a key and recurrent source of corporate financing. Because firms make repeated financing decisions, issuance costs are a crucial determinant of their leverage dynamics, together with taxes and the cost of financial distress (see, e.g., Graham and Harvey (2001) and Graham (2022)).

In this paper, we investigate a firm's dynamic debt policy when shareholders are subject to debt restructuring costs and cannot commit to future restructuring policies. In a framework without commitment, shareholders have an incentive to issue additional debt over time both to increase tax benefits and to extract wealth from existing debtholders. Anticipating this incentive, creditors account for both future dilutions and a higher default probability when pricing debt. This anticipation reduces the proceeds from debt issuances. As noted by, e.g., DeMarzo (2019), issuance costs make it more expensive for the firm to tap the debt market, and thus act as a commitment device that mitigates the manager's desire to issue debt too aggressively. As such, issuance costs may help shareholders extract tax benefits from debt.

We study a firm that, at any time, can issue additional debt or repurchase outstanding debt at market prices by paying a fixed cost. Consistent with the "leverage ratchet" effect of Admati, DeMarzo, Hellwig, and Pfleiderer (2018), we first show that in any Markov Perfect Equilibrium (MPE), firms never find it optimal to repurchase outstanding debt. We can therefore restrict the search for equilibria to strategies that involve only debt issuances. Within this class, we focus on "barrier" strategies that are characterized by a single state variable—the firm's income-to-debt ratio—which can fall into one of three regions. In the lowest region, shareholders choose to default rather than service outstanding debt. In the highest region, firms find it optimal to issue additional debt so that the new income-to-debt ratio immediately jumps into the middle region. In the middle region,

[1] For instance, in the U.S., the stock of non-financial corporate bonds more than doubled from \$3.2 trillions at the end of 2009 to \$6.7 trillions at the end of 2021. Source: Board of Governors of the Federal Reserve System, Nonfinancial Corporate Business, Table B.103.

firms choose to service existing debt, but not to issue additional debt.

We then provide a formal characterization of the equilibrium. We show that the existence of an MPE depends on the interaction between issuance costs and debt maturity. For a given maturity, we analytically identify an issuance cost threshold above which shareholders find it optimal to never issue debt. Intuitively, this is because the present value of issuance costs exceed the present value of additional net tax benefits. For issuance costs below this threshold, we find MPEs in which the firm issues debt and shareholders extract positive tax benefits with magnitude that varies with maturity. These results highlight the role that issuance costs play as a commitment device in mitigating debt issuance, in turn allowing the firm to extract positive tax benefits.

However, as we consider even lower issuance costs, we find a threshold below which an MPE in barrier strategies no longer exists. That is, there we numerically identify a region of parameters for which issuance costs are no longer an effective commitment device. This result may at first be surprising because when debt-restructuring costs are low, shareholders would be able to extract positive tax benefits if they could commit to a future debt issuance policy. However, without commitment, we find that these policies are not *incentive-compatible*. Intuitively, this occurs because lower rollover costs induce the manager to issue debt more aggressively. This action reduces the price creditors are willing to pay for new debt, which in turn reduces issuance proceeds, and thus the option value to keep the firm operating. As such, the manager chooses to default at higher values of the income-to-debt ratio. This creates a "vicious cycle" of creditors offering lower debt prices and the manager liquidating the firm at a higher default boundary, eventually leading to a situation with no equilibrium in barrier strategies.

To quantify the loss in shareholder value due to the firm's inability to commit to a particular strategy, we also explore an otherwise identical model in which shareholders can commit to a future debt adjustment policy. When we calibrate the model to debt maturities and issuance costs consistent with empirical observation, we find that the tax benefits to debt for the policy without commitment are only slightly lower than those for the policy with commitment. Moreover, we show that when the issuance cost parameter approaches zero, there are maturities for which shareholders can extract close to 100% of the claim to the firm's cash flow for both cases with and without commitment. Therefore, even in the presence of arbitrarily small issuance costs, our model implies that shareholders' inability to commit to a dynamic capital structure policy may have only a small impact on the tax benefits that the firm can extract.

To our knowledge, this is the first paper to provide a formal characterization of MPEs when debt issuance is subject to a fixed cost, and firms are not forced to repurchase all outstanding debt prior to issuing new debt. This is in contrast to most of the existing literature, which assumes that the firm must call (i.e., repurchase) all of its outstanding debt prior to issuing new debt, an assumption that rarely holds in practice.[2] The main challenge of this problem comes from the analysis of off-equilibrium deviations. Specifically, when the income-to-debt ratio falls in the restructuring region, shareholders rebalance towards a target that depends on the value of the income-to-debt ratio. That is, the target is characterized by a function whose domain is the entire restructuring region. We provide a verification argument for the existence of an MPE and derive necessary and sufficient conditions for debt issuance to indeed be optimal in the restructuring region. In contrast, when a firm is forced to call all outstanding debt prior to issuing new debt, the issuance decision is always made with zero debt outstanding. In this case, the debt restructuring function reduces to a single point, which greatly simplifies the analysis.

Our paper builds upon the quickly evolving literature that examines dynamic capital structure decisions of firms. There are only a few tractable frameworks in this literature, due to the difficulty of valuing assets in an economy in which current prices depend on the firm's future debt issuance policy.[3] Most relevant to our work is DeMarzo and He (2021), who investigate leverage dynamics without commitment in the absence of restructuring costs. They show that the unique MPE is characterized by a locally deterministic process in which new debt is issued in all states of nature,

[2]See, e.g., Fischer, Heinkel, and Zechner (1989), Leland (1998), Goldstein, Ju, and Leland (2001), Strebulaev (2007), Danis, Rettl, and Whited (2014), Hugonnier, Malamud, and Morellec (2015), and Dangl and Zechner (2020).

[3]See, e.g., Leland and Toft (1996), Leland (1998), Brunnermeier and Yogo (2009), Cheng and Milbradt (2012), He and Xiong (2012a,b), Chen, Xu, and Yang (2012), Brunnermeier and Oehmke (2013), Décamps and Villeneuve (2014), Diamond and He (2014), He and Milbradt (2014, 2016), Hugonnier, Malamud, and Morellec (2015), Abel (2016), Huang, Oehmke, and Zhong (2019), Della Seta, Morellec, and Zucchi (2020), and DeMarzo, He, and Tourre (2021).

even when the firm is near default. Due to this aggressive policy, this model generates no tax benefits to debt regardless of the debt's maturity.

In contrast, we investigate a framework in which debt issuance is subject to a fixed adjustment cost. There is strong empirical evidence that it is costly for firms to issue debt (see, e.g., Altınkılıc and Hansen (2000), Yasuda (2005), Kim, Palia, and Saunders (2008), and Ivashina (2009)). In particular, Leary and Roberts (2005), Strebulaev (2007), and Morellec, Nikolov, and Schürhoff (2012) show that fixed issuance costs of debt can explain many patterns in the data, such as the infrequent adjustments observed in firm financing and the discrete size of debt issues. Consistent with this evidence, we consider a model with fixed issuance costs in which, in a no-commitment equilibrium, firms issue discrete amounts of debt at infrequent intervals and extract positive tax benefits, unlike the continuous debt-adjustment policy in DeMarzo and He (2021).

A vast literature has studied dynamic capital structure choice in the presence of issuance costs.[4] Closely related to our work is Goldstein, Ju, and Leland (2001), who show that the firm can extract positive tax benefits from debt in a model with perpetual callable debt and commitment. We extend their analysis to the case of finite-maturity non-callable debt and no-commitment, and explore the interplay between maturity and issuance costs. We also use a variation of their policy to assess how much tax benefits are lost due to the inability to commit to a debt issuance strategy. Several explanations have been proposed to explain commitment in this framework. For example, firms can limit future debt issuance by specifying restrictive covenants in their bond indentures (see, e.g., Roberts and Sufi (2009)). Alternatively, commitment can arise because of the repeated interaction between the issuer and credit markets. In this setting, reputation concerns provide shareholders the incentive to abide by the commitment policy if debtholders were to punish any deviation by pricing all future debt issuances according to the DeMarzo and He (2021) equilibrium. This idea is further developed by Malenko and Tsoy (2020), who investigate a model of debt issuance and repurchases without restructuring costs. They focus on non-Markovian barrier policies that satisfy a credibility

[4]See, e.g., Kane, Marcus, and McDonald (1984, 1985), Fischer, Heinkel, and Zechner (1989), Titman and Tsyplakov (2007), Strebulaev (2007), Morellec, Nikolov, and Schürhoff (2012), Hennessy and Whited (2007), Gomes and Schmid (2012), Bolton, Chen, and Wang (2011), Hugonnier, Malamud, and Morellec (2015), Bolton, Wang, and Yang (2020), Benzoni, Garlappi, and Goldstein (2020), and many others.

constraint when EBIT follows a jump-diffusion process and they identify a time-consistent debt policy outside the Markov class. Aside from the inclusion of jumps and the absence of issuance costs, their proposed time-consistent policy coincides with our benchmark policy with commitment.

Finally, Dangl and Zechner (2020) also study the interplay between issuance costs and maturity when the firm chooses how to refinance expiring debt. Similar to our findings, Dangl and Zechner (2020) highlight a tradeoff between the costs of higher rollover frequencies and the benefits of increased flexibility associated with shorter maturity debt. Unlike Dangl and Zechner (2020), we focus on non-callable debt that does not have to be repurchased prior to additional debt being issued. Moreover, we formally derive necessary and sufficient conditions for the existence of an MPE in barrier strategies. Finally, we quantify the difference in available tax benefits between models with and without commitment.

The predictions of our model are in line with the empirical literature that investigates the capital structure and maturity decisions of firms. Because we specify fixed restructuring costs, our model predicts that firms issue debt in discrete (rather than continuous) amounts, consistent with observation. Moreover, our model generates both persistence in leverage and a negative correlation between profitability and leverage, consistent with, e.g., Titman and Wessels (1988) and Frank and Goyal (2014). Our model captures these features because, when firms are in the inaction region, higher profitability increases equity values while debt outstanding remains constant, leading to lower leverage, and vice-versa. Also consistent with our model's predictions are van Binsbergen, Graham, and Yang (2010), and Korteweg (2010), who document that firms are able to extract tax benefits to debt. Moreover, our findings are consistent with Barclay and Smith (1995) and Stohs and Mauer (1996), who report that firms are not indifferent toward debt maturity choice. Fama and French (2002), Baker and Wurgler (2002), and Welch (2004) provide evidence that shocks to capital structures are persistent, and Leary and Roberts (2005) attribute this persistence to the presence of adjustment costs. Finally, Graham and Harvey (2001) report survey evidence that 45% of CFOs are concerned with the tax advantage of interest deductibility, suggesting that firms are not indifferent to capital structure choices.

The rest of the paper proceeds as follows. Section 2 presents the model and provides necessary and sufficient conditions for the existence of barrier-strategy MPEs. In Section 3, we illustrate the properties of the MPEs and compare them to those of a policy with commitment. Section 4 discusses the lack of existence of single-barrier MPEs and Section 5 concludes. Appendix A and the Online Appendix contain proofs and additional results.

2 The Model

The firm. We investigate an economy in which all agents are risk neutral and the discount rate ($r > 0$) is exogenous. The representative firm in the economy generates cash flows characterized by an exogenously specified EBIT process Y_t with dynamics:

$$\frac{dY_t}{Y_t} = \mu\, dt + \sigma\, dW_t, \tag{1}$$

for some constants $\mu < r$ and $\sigma > 0$, where dW_t denotes increments of a standard Brownian motion. The value V_t of the claim to EBIT is:

$$V_t = \mathbb{E}_t \left[\int_t^{\infty} e^{-r(s-t)}\, Y_s\, ds \right] = \frac{Y_t}{r - \mu}. \tag{2}$$

Due to the linear relation between Y_t and V_t in equation (2), we can choose either as the exogenous state variable. We choose Y_t for consistency with the existing dynamic capital structure literature.

The firm dynamically chooses its leverage to maximize shareholders' value. Specifically, at any time, the firm can either adjust its capital structure by retiring or issuing bonds at market value, or do nothing. All outstanding bonds pay coupons at a continuous rate $c > 0$, and amortize at a rate ξ. We denote by F_t the date-t outstanding face value of debt. During the interval $(t, t + dt)$, debtholders receive cash flows $(c+\xi)F_t\, dt$ as long as the firm operates. As in the benchmark case of DeMarzo and He (2021), we assume that c and ξ are exogenous parameters and that debtholders' recovery in default is zero. The firm's problem is fully characterized by the state variables Y_t and F_t.

We assume that the firm faces a fixed cost to adjust its debt. In general, these costs can have both a fixed and a proportional component. However, we know from DeMarzo and He (2021) that the MPE with proportional costs would still be characterized by a continuous debt issuance policy in which the net tax benefit to shareholders is zero. In contrast, Leary and Roberts (2005), Strebulaev (2007), and Morellec, Nikolov, and Schürhoff (2012) show that fixed issuance costs can capture many patterns in the data, such as the infrequent adjustments observed in firm financing and the discrete size of debt issues. Therefore, we restrict our attention to fixed adjustment costs.

The dynamics for the face value of debt are endogenously determined in that, by paying a fixed adjustment cost βY_t, the firm can adjust its capital structure by repurchasing or issuing bonds at market prices. The adjustment cost is fixed in that it is independent of the size of debt adjustment. As a result of debt adjustments, the face value of the firm's debt evolves according to the process

$$dF_t = -\xi F_{t\text{-}}\, dt + A_t\, F_{t\text{-}}\, dN_t,$$

where $A_t = \left(\frac{F_t - F_{t\text{-}}}{F_{t\text{-}}}\right)$ is the fractional change in outstanding debt at a restructuring time t, and dN_t is a counting process which increases by unity each time debt is restructured. Both the size (A_t) and timing (dN_t) of debt issuance, are chosen by the firm.

The firm's EBIT Y_t is subject to a corporate tax rate $\tau \in [0, 1)$ and coupon payments are tax deductible. Hence, the instantaneous cash flow to shareholders, i.e., the dividend $\delta(F_s, Y_s)$, is given by

$$\delta(F_s, Y_s) \equiv (1-\tau)Y_s - (c(1-\tau)+\xi)F_s. \tag{3}$$

Markov strategies. Because the firm faces a positive debt adjustment cost, we consider Markov strategies in which management makes one of the following three choices, based only on the knowledge of the current value of the state variables (F_t, Y_t): (i) default; (ii) restructure the amount of debt outstanding; or (iii) do neither, i.e., inaction, and simply service outstanding debt. More formally, denote by $\mathcal{D}$ and $\mathcal{R}$ the sets of state variables (F_t, Y_t) at which default and debt restructuring occur, respectively. Then a Markov strategy (**a**) consists of a default time

$\tau_b(\mathbf{a}) = \inf\{t \geq 0 : (F_t, Y_t) \in \mathcal{D}\}$ and a debt adjustment function $A_t(\mathbf{a}) \equiv A(F_t, Y_t|\mathbf{a}) \geq -1$, for $(F_t, Y_t) \in \mathcal{R}$. The adjustment function $A_t(\mathbf{a})$ determines the face value of debt upon restructuring, that is, $F_t = F_{t-}(1 + A_t(\mathbf{a}))$. A special case consists of Markov strategies that can be fully characterized by a single state variable, the income-to-debt-ratio $y_t = Y_t/F_t$. We refer to such strategies as "reduced" Markov strategies. Note that any strategy that involves continuous debt adjustments would lead to an infinite accumulation of restructuring costs, and can therefore be ruled out.

Debt valuation. We denote by $P_t(\mathbf{a}) \equiv P(F_t, Y_t|\mathbf{a}) \geq 0$ the date-t present value of the outstanding debt claim per unit of face value F_t when creditors anticipate that management will use the Markov strategy $(\mathbf{a})$. Since each bond pays a coupon c per unit of face value and amortizes at the rate ξ, absence of arbitrage opportunities implies[5]

$$P_t(\mathbf{a}) \equiv P(F_t, Y_t|\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{a})} e^{-(r+\xi)(s-t)}(c+\xi)\,ds\right], \tag{4}$$

where $\tau_b(\mathbf{a}) = \inf\{s \geq t : (F_s, Y_s) \in \mathcal{D}\}$, with $\mathcal{D}$ denoting the default region. The exponential amortization rate ξ implies that, conditional upon no default, bonds have an expected maturity $1/\xi$. If a strategy $(\mathbf{a})$ is reduced Markov, then $P(F_t, Y_t|\mathbf{a}) = P(y_t|\mathbf{a})$, where $y_t = Y_t/F_t$ is the income-to-debt ratio.[6]

Equity valuation without commitment. Absent commitment, creditors will lend money to the firm only if the debt contract is *incentive compatible* in that shareholders would never want to deviate from the default and issuance strategy that creditors use to price the bonds at issuance. If creditors conjecture that the firm will use a Markov strategy $(\mathbf{a})$ but management instead uses

[5] The absence of arbitrage opportunities requires that $P_t(\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_m \wedge \tau_b(\mathbf{a})} e^{-r(s-t)} c\,ds + \mathbf{1}_{\{\tau_m < \tau_b(\mathbf{a})\}} e^{-r(\tau_m - t)}\right]$ on the set $\{\tau_m \wedge \tau_b(\mathbf{a}) > t\}$ where τ_m is an exponential random variable with mean $m = 1/\xi$ that denotes the bond maturity. Integrating inside the expectation against the conditional distribution $\mathbb{P}_t[\tau_m \in ds\,|\tau_m > t] = \mathbf{1}_{\{s>t\}} e^{-\xi(s-t)}\xi ds$ shows that on the set $\{\tau_b(\mathbf{a}) > t\}$ the market price of an individual bond issued by the firm is given by equation (4).

[6] More formally, if a strategy $(\mathbf{a})$ is reduced Markov, the debt price function $P(F, Y)$ is homogeneous of degree 0 in F and Y, i.e., $P(1, y = Y/F) \equiv P(y) = P(F, Y)$.

another strategy (**s**), then the value of equity is

$$E_t(\mathbf{s},\mathbf{a}) \equiv E(F_t, Y_t|\mathbf{s},\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{s})} e^{-r(u-t)}\left(\underbrace{\delta(F_u, Y_u)}_{\text{Dividend}}\, du + \underbrace{\left(P_u(\mathbf{a})\, A_u(\mathbf{s})F_{u^-} - \beta Y_u\right)}_{\substack{\text{Cash inflow/outflow from}\\ \text{debt restructuring net of cost}}} dN_u(\mathbf{s})\right)\right],$$

where $(\tau_b(\mathbf{s}), A_u(\mathbf{s}))$ are the default time and the restructuring process associated with strategy (**s**), $\delta(F_u, Y_u)$ is the after-tax cash flow defined in equation (3), $P_u(\mathbf{a})$ is the price of debt per unit of face value defined in equation (4), and $dN_u(\mathbf{s})$ is a counting process that increases by one at the time of debt restructuring. Finally, note that $A_u(\mathbf{s})F_{u^-} = \left(F_u - F_{u^-}\right)$, and hence the term $\left(P_u(\mathbf{a})\, A_u(\mathbf{s})F_{u^-} - \beta Y_u\right)$ represents the amount distributed to shareholders after debt restructuring, net of the fixed restructuring cost βY_u. If, in addition, both strategies (**s**, **a**) are reduced Markov strategies, that is, they can be characterized by the single state variable y_t, then equity value depends on the state variables (F_t, Y_t) only through the ratio $y_t \equiv Y_t/F_t$ and we define the scaled equity function[7]

$$e_t(\mathbf{s},\mathbf{a}) \equiv e(y_t|\mathbf{s},\mathbf{a}) = \frac{E(F_t, Y_t|\mathbf{s},\mathbf{a})}{F_t}.$$

2.1 Characterization of Markov perfect equilibria

Here we define the concept of Markov perfect equilibrium and characterize its properties.

Definition 1 (Markov perfect equilibrium) *A Markov Perfect Equilibrium (MPE) is a strategy* ($\mathbf{a} \in \mathcal{M}$) *such that*

$$E_t(\mathbf{a},\mathbf{a}) = \sup_{\mathbf{s}\in\mathcal{M}} E_t(\mathbf{s},\mathbf{a}), \quad t \geq 0,$$

where $\mathcal{M}$ denotes the space of feasible Markov strategies.[8] *A reduced MPE is an MPE in the class of reduced Markov strategies.*

The definition formalizes the notion that, in an MPE, shareholders' best response to the creditors'

[7]More formally, if a strategy (**a**) is reduced Markov, the equity price function $E(F, Y)$ is homogeneous of degree 1 in F and Y, i.e., $E(1, y = Y/F) \equiv e(y) = \frac{E(F,Y)}{F}$.

[8]Feasible Markov strategies are Markov strategies that satisfy the integrability condition (I.7) in the Online Appendix.

conjectured strategy $(\mathbf{a})$ is the strategy $(\mathbf{a})$ itself. Therefore, if $(\mathbf{a})$ is an MPE, shareholders have no incentive to deviate to a different strategy $(\mathbf{s})$. We next provide two properties of MPE that are useful to construct a candidate equilibrium.

Property 1 (Leverage ratchet effect) *In any feasible strategy that is an MPE, shareholders never repurchase debt.*

Property 1 generalizes the "leverage ratchet" effect (see, e.g., Admati, DeMarzo, Hellwig, and Pfleiderer (2018) and DeMarzo and He (2021)) to the case of fixed restructuring costs and allows us to restrict the search of equilibria within strategies that involve only debt *issuance*. As in Admati, DeMarzo, Hellwig, and Pfleiderer (2018), debt repurchases reduce shareholder wealth because (i) they reduce the option value to default in states of the world where the firm would have defaulted absent such a repurchase; (ii) they transfer more resources to creditors via reverse dilution; and (iii) they reduce tax shields. In our setting, restructuring costs make debt repurchases more costly and therefore even less attractive. Proposition II.1 in the Online Appendix formalizes this argument and provides a rigorous proof.

Property 2 (Constant default boundary) *If a strategy* $(\mathbf{a})$ *is a reduced MPE then there is a constant value of the income-to-debt ratio* $y_b(\mathbf{a}) > 0$ *such that the equity value* $e(y|\mathbf{a}) \equiv e_t(\mathbf{a}, \mathbf{a}) = 0$ *in the default region* $y \leq y_b(\mathbf{a})$.

Property 2, which follows from Corollary II.2 in the Online Appendix, implies that in any reduced MPE the default region is characterized by a constant, rather than a time-varying default boundary. Taken together, these two properties allow us to restrict the search of MPEs, but not the set of possible deviations, to the subset of debt issuance and default strategies characterized by a constant default threshold.

2.2 Barrier strategies

A class of strategies of particular interest within the set identified in Section 2.1 is the class of *barrier* strategies that can be characterized by two parameters and one function. The two parameters are the default boundary (y_b) and the debt-issuance boundary (y_u), with the restriction ($y_u > y_b$). These two parameters demarcate three regions. The region $y_t \leq y_b$ is the *default region*, where it is optimal for management to immediately default, rather than continue to service outstanding debt. The region $y_t \in (y_b, y_u)$ is the *inaction region*, where it is optimal for management to neither default nor change the level of outstanding debt. Finally, the region $y_t \geq y_u$ is the *restructuring region*, where it is optimal for shareholders to issue debt in sufficient amounts so that the post-issuance income-to-debt ratio immediately jumps to the inaction region. The function $\mathcal{Y}(y_t) : [y_u, \infty) \to (y_b, y_u)$ is the *restructuring function* associated with the parameters y_b and y_u. Immediately after a debt issuance, the income-to-debt ratio jumps from $y_{t-} \in [y_u, \infty)$ to $y_{t+} \equiv \mathcal{Y}(y_{t-}) \in (y_b, y_u)$.[9] An important special case is $y^* \equiv \mathcal{Y}(y_u)$, which is the target income-to-debt ratio chosen at the upper boundary of the inaction region. In this section, we focus on such strategies to construct a *candidate* MPE.

Note that, under a barrier strategy, if the state variable "begins" in the inaction region, then the entire *restructuring region* is inaccessible, except for y_u. However, as we discuss in Section 2.3, the characterization of a barrier equilibrium relies on a verification argument for all off-equilibrium values $y_t \in (y_u, \infty)$. Applying Itô's lemma to the income-to-debt ratio $y_t = Y_t / F_t$, we find:

$$\begin{aligned} dy_t &= \frac{dY_t}{F_{t^-}} - \frac{Y_t}{F_{t^-}^2} (-\xi F_{t^-}\, dt) + \left(\frac{Y_t}{F_{t^-}(1 + A_t)} - \frac{Y_t}{F_{t^-}} \right) \mathbf{1}_{\{y_{t^-} \geq y_u\}} \\ &= y_{t^-} \left[(\mu + \xi)\, dt + \sigma\, dW_t \right] + \underbrace{(\mathcal{Y}(y_{t^-}) - y_{t^-})}_{\leq 0} \mathbf{1}_{\{y_{t^-} \geq y_u\}}, \end{aligned} \tag{5}$$

where, consistent with the focus on barrier strategies, we have replaced the counting process (dN_t) with an indicator function that equals one if and only if the current state vector is in the restructuring region, and where we have used the relation $\mathcal{Y}(y_{t^-}) = \frac{y_{t^-}}{1+A_t}$.

Debt valuation in barrier strategies. In a barrier strategy ($\mathbf{a}$), the default stopping time $\tau_b(\mathbf{a})$ in the definition of the bond price in equation (4) denotes the first time that the income-to-debt ratio y_t hits the default boundary $y_b(\mathbf{a})$ from above, that is, $\tau_b(\mathbf{a}) := \inf\{s > t : y_s \leq y_b(\mathbf{a})\}$. Because

[9] Corollary II.3 in the Online Appendix shows that in any MPE in barrier strategies shareholders never find it optimal to jump either to the default region or to another location in the restructuring region.

barrier strategies are reduced Markov strategies, we can write the bond price in equation (4) as a function of the income-to-debt ratio (y_t) only:

$$P_t(\mathbf{a}) \equiv P(y_t|\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{a})} e^{-(r+\xi)(s-t)}(c+\xi)\,ds\right]. \tag{6}$$

The firm may issue additional debt after date-t, which impacts the default time $\tau_b(\mathbf{a})$, but not the validity of the pricing equation (6). For ease of notation, in what follows, we will suppress the dependence of the debt value on the strategy ($\mathbf{a}$). Given the zero-recovery assumption, the bond is worthless in the default region, i.e., $P(y_t) = 0$, for all $y_t \in (0, y_b]$. Similarly, debt issuance occurs when the current income-to-debt ratio is in the restructuring region $[y_u, \infty)$. To preclude arbitrage opportunities, since debt-issuance times are predictable, we require that the bond price does not jump at the time of issuance, that is,

$$P(y_t) = P(\mathcal{Y}(y_t)) \qquad \forall y_t \in [y_u, \infty).$$

Standard arguments relying on Itô's lemma and the continuity of the bond price at the issuance boundary imply that, for values of $y_t \in (y_b, y_u)$, the debt price in equation (6) is the unique solution $P(y)$ to the following ordinary differential equation:

$$(c+\xi) - (r+\xi)\,P(y) + (\mu+\xi)y\,P'(y) + \frac{\sigma^2}{2}y^2\,P''(y) = 0, \quad y \in (y_b, y_u), \tag{7}$$

subject to the boundary conditions

$$\begin{aligned} P(y) &= 0 && 0 \le y \le y_b \\ P(y) &= P(\mathcal{Y}(y)) && y \ge y_u. \end{aligned}$$

The solution is

$$P(y) = \begin{cases} 0 & \text{if} \quad 0 \le y \le y_b \\ \left(\frac{c+\xi}{r+\xi}\right)\left(1 + A_\pi\, y^{\Theta} + B_\pi\, y^{\Pi}\right) & \text{if} \quad y_b < y < y_u \\ \left(\frac{c+\xi}{r+\xi}\right)\left(1 + A_\pi\, \mathcal{Y}(y)^{\Theta} + B_\pi\, \mathcal{Y}(y)^{\Pi}\right) & \text{if} \quad y \ge y_u, \end{cases}$$

with $A_\pi \equiv \frac{y_u^{\Pi} - y^{*\Pi}}{y_b^{\Pi}\left(y_u^{\Theta} - y^{*\Theta}\right) + y_b^{\Theta}\left(y^{*\Pi} - y_u^{\Pi}\right)} \le 0$ and $B_\pi \equiv \frac{y^{*\Theta} - y_u^{\Theta}}{y_b^{\Pi}\left(y_u^{\Theta} - y^{*\Theta}\right) + y_b^{\Theta}\left(y^{*\Pi} - y_u^{\Pi}\right)} \le 0$, and where the exponents (Θ, Π) are the two solutions of the quadratic equation

$$\frac{\sigma^2}{2}\lambda(\lambda - 1) + (\mu + \xi)\lambda - (r + \xi) = 0. \tag{8}$$

Because both A_π and B_π are non-positive and ($y_b < y^* < y_u$), the bond price function is strictly concave in the inaction region (y_b, y_u), with $P'(y_b) > 0$, $P'(y^*) \ge 0$, and $P'(y_u) \le 0$. Intuitively, an increase in the income to debt ratio y raises the value of debt claim. However, as y approaches the restructuring boundary (y_u), bondholders anticipate that their claim will be diluted by debt issuance. This explains the negative slope of the debt price function at y_u.

Equity valuation in barrier strategies. Consider the equity value that prevails when creditors correctly anticipate that management will use a barrier strategy $\mathbf{a} = (y_b, y_u, \mathcal{Y}(y))$. In the restructuring region $[y_u, \infty)$, let $E(F_t, Y_t)$ be the equity value just prior to debt restructuring and $E(F_t^*, Y_t)$ the value just after debt restructuring to the new face value F_t^*. To preclude arbitrage, the value of the equity claim just prior to debt restructuring must equal the equity claim just after restructuring plus any cash flows received from debt issuance:

$$E(F_t, Y_t) = \underbrace{E(F_t^*, Y_t)}_{\text{Post-issuance equity value}} + \underbrace{(F_t^* - F_t)\; P\big(F_t^*, Y_t\big)}_{\text{Proceeds from debt issuance}} - \underbrace{\beta Y_t}_{\text{Issuance costs}}, \tag{9}$$

where, for ease of notation, we suppress the dependence of claim values on the barrier strategy (**a**). The second term on the right-hand side of equation (9) is the cash flow to equity from the debt issuance of size $(F_t^* - F_t)$, which is priced by creditors at the *post-adjustment* price $P\left(F_t^*, Y_t\right)$. Defining $y_t \equiv (Y_t/F_t)$ and $z_t \equiv \left(Y_t/F_t^*\right)$, we use the homogeneity property of the equity and debt claims to identify the expressions $E(F_t^*, Y_t) = F_t^*\, e\,(z_t)$, $P\left(F_t^*, Y_t\right) = P\left(1, \frac{Y_t}{F_t^*}\right) \equiv P(z_t)$. As such, we can rewrite equation (9) as:

$$e(y_t) \;\equiv\; \frac{E(F_t, Y_t)}{F_t} \;=\; \left(\frac{y_t}{z_t}\right) e(z_t) + \left(\frac{y_t}{z_t} - 1\right) P(z_t) - \beta y_t \;\equiv\; \Phi(y_t, z_t), \qquad y_t \in [y_u, \infty). \quad (10)$$

Inside the inaction region, i.e., for $y_t \in (y_b, y_u)$, standard results relying on Itô's lemma show that the scaled equity value $e(y)$ is the solution to the following ordinary differential equation

$$\delta(y) - (r+\xi)\, e(y) + (\mu+\xi) y\, e'(y) + \frac{\sigma^2}{2} y_t^2\, e''(y) = 0, \qquad \text{for all } \; y \in (y_b, y_u), \quad (11)$$

with $\delta(y) \equiv \delta\left(1, \frac{Y}{F}\right) = \frac{\delta(F,Y)}{F}$, subject to the boundary conditions

$$e(y) = 0 \qquad\qquad 0 \le y \le y_b$$

$$e(y) = \frac{y}{\mathcal{Y}(y)} e(\mathcal{Y}(y)) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)) - \beta y \qquad y \ge y_u.$$

The solution to this problem is

$$e(y) = \begin{cases} 0 & \text{if } \; 0 \le y \le y_b \\ \hat{e}(y) + A_\varepsilon\, y_t^\Theta + B_\varepsilon\, y^\Pi & \text{if } \; y_b < y < y_u \\ \frac{y}{\mathcal{Y}(y)} \left[\hat{e}(\mathcal{Y}(y)) + A_\varepsilon\, \mathcal{Y}(y)^\Theta + B_\varepsilon\, \mathcal{Y}(y)^\Pi\right] + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)) - \beta y & \text{if } \; y \ge y_u, \end{cases} \quad (12)$$

with $\hat{e}(y_t)$ denoting the levered claim to EBIT,

$$\hat{e}(y) = -\frac{c(1-\tau)+\xi}{r+\xi} + \left(\frac{1-\tau}{r-\mu}\right) y, \quad (13)$$

and where the constants $(A_\varepsilon,\ B_\varepsilon)$ are the unique solutions to the value-matching conditions

$$e(y_b) = 0 \tag{14}$$

$$e(y_u) = \frac{y_u}{y^*} e(y^*) + \left(\frac{y_u}{y^*} - 1\right) P(y^*) - \beta y_u, \quad \text{with } y^* \equiv \mathcal{Y}(y_u). \tag{15}$$

Equations (12)–(15) determine the equity value under any barrier strategy $\mathbf{a} = (y_b, y_u, \mathcal{Y}(y))$. The next section provides necessary and sufficient conditions for a barrier strategy ($\mathbf{a}$) to be an MPE.

2.3 Characterization of MPEs in barrier strategies

In a barrier-strategy MPE, in addition to the value-matching conditions (14)–(15), the equity value function $e(y_t)$ must satisfy smooth-pasting conditions at the default and restructuring boundaries $(y_b,\ y_u)$, i.e.,

$$e'(y_b) = 0 \tag{16}$$

$$e'(y_u) = \frac{e(y^*) + P(y^*)}{y^*} - \beta. \tag{17}$$

The smooth pasting condition (16), which is derived by differentiating (14) with respect to y_b, guarantees that y_b is the default threshold that maximizes equity value under the limited liability constraint. Similarly, the smooth pasting condition (17), which is derived by differentiating (15) with respect to y_u, guarantees that y_u is chosen optimally.

Moreover, in any MPE the restructuring function $\mathcal{Y}(y)$ maximizes equity value at the time of restructuring, that is,

$$\mathcal{Y}(y) = \operatorname*{argmax}_{z \in (y_b, y_u)} \left\{ \frac{y}{z} e(z) + \left(\frac{y}{z} - 1\right) P(z) - \beta y \right\}, \quad y \geq y_u. \tag{18}$$

For the special case $(y = y_u)$, we denote $y^* \equiv \mathcal{Y}(y_u)$.

Finally, the default threshold y_b of an MPE with debt issuance cannot exceed the threshold $y_{b,\text{NI}}$ of a "no-issuance" MPE in which shareholders find it optimal to never issue debt, derived below in equation (26). The intuition for this result is as follows. If creditors price debt according to a policy (**a**) with default threshold such that $y_b > y_{b,\mathrm{NI}}$, then it is in shareholders' best interest not to choose (**a**), because there are policies under which equity value would be higher, implying that (**a**) cannot be an MPE. For example, since under policy (**a**) the equity claim $e(y|\mathbf{a}) = 0$ for all $y \in [0, y_b]$, the value of equity under this strategy would be lower than that in a no-issuance MPE for all $y \in [y_{b,\mathrm{NI}}, y_b]$. Moreover, Lemma II.10 in the Online Appendix shows that the equity function is increasing and convex in $y \in (y_b, y_u)$ and decreasing in y_b. Thus, the equity claim in a no-issuance MPE would be higher than that under policy (**a**) for all $y \in [y_{b,\mathrm{NI}}, y_u]$. As equation (26) shows, the default threshold $y_{b,\mathrm{NI}}$ in a no-issuance MPE coincides with the default barrier of an MPE without issuance costs (see Proposition 4, equation (26) in DeMarzo and He (2021)) and with the default barrier in the Leland (1994) model, extended for finite maturity.

In summary, necessary (but not sufficient) conditions for a strategy (**a**) to be an MPE in barrier strategies are: (i) the value-matching conditions (14)–(15), (ii) the smooth pasting conditions (16), (iii) the optimality of the restructuring function (18) and (iv) the requirement that $y_b < y_{b,\mathrm{NI}}$. The next proposition formalizes these necessary conditions, which provide a blueprint for the construction of a *candidate* MPE in barrier strategies. Lemma II.9 in the Online Appendix proves these conditions.

Proposition 1 (Necessary conditions for barrier-strategy MPE) *Assume that the barrier strategy* $\mathbf{a} = (y_b, y_u, \mathcal{Y}(y))$ *is an MPE. Then the following conditions are satisfied:*

1. *Default boundary:* $y_b < y_{b,NI}$, *with* $y_{b,NI} = \frac{\Pi}{\Pi-1}\frac{r-\mu}{r+\xi}\left(c + \frac{\xi}{1-\tau}\right)$ *and* Π *given by the negative root of the quadratic equation (8).*

2. *Limited liability:* $e(y|\mathbf{a}) = \max\{\phi(y|\mathbf{a}), 0\} = 0$ *for* $y \in (0, y_b]$, *where* $\phi(y|\mathbf{a})$ *is the equity continuation value*

$$\phi(y|\mathbf{a}) \equiv \sup_{z\geq 0} \Phi(y, z|\mathbf{a}) = \sup_{z\in\mathbb{R}_+} \left\{ \frac{y}{z} e(z|\mathbf{a}) + \left(\frac{y}{z} - 1\right) P(z|\mathbf{a}) - \beta y \right\}.$$

3. *Equity valuation in the restructuring region:* $e(y|\mathbf{a}) = \max\{\phi(y|\mathbf{a}), 0\} > 0$ *for* $y \in [y_u, \infty)$.

4. Value-matching and smooth-pasting at the default boundary y_b:

$$e(y_b|\mathbf{a}) = e'(y_b|\mathbf{a}) = 0.$$

5. Value-matching and smooth-pasting at the restructuring boundary y_u:

$$e(y|\mathbf{a}) = \frac{y}{\mathcal{Y}(y)} e(\mathcal{Y}(y)|\mathbf{a}) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y, \quad y \geq y_u$$

$$e'(y|\mathbf{a}) = \frac{e(\mathcal{Y}(y)|\mathbf{a}) + P(\mathcal{Y}(y)|\mathbf{a})}{\mathcal{Y}(y)} - \beta, \quad y \geq y_u.$$

6. Optimality of restructuring:

$$\{\mathcal{Y}(y)\} = \underset{z \in (y_b, y_u)}{\operatorname{argmax}} \left\{ \frac{y}{z} e(z|\mathbf{a}) + \left(\frac{y}{z} - 1\right) P(z|\mathbf{a}) - \beta y \right\}, \quad y \geq y_u. \tag{19}$$

To insure that such a candidate strategy is indeed an MPE, we also need to verify that shareholders have no incentive to deviate from the strategy conjectured by creditors. Intuitively, in a barrier-strategy MPE it must be that: (i) in the *default region* ($y \leq y_b$), default dominates restructuring or inaction; (ii) in the *inaction region* ($y_b < y < y_u$), inaction dominates defaulting or restructuring; and (iii) in the *restructuring region* ($y \geq y_u$), restructuring into the inaction region dominates inaction, default, and restructuring to any point in the default or restructuring regions.

In an MPE, the shareholders' strategy ($\mathbf{s}$) coincides with the conjectured strategy ($\mathbf{a}$) of bondholders, i.e., ($\mathbf{s} = \mathbf{a}$) in Definition 1. Therefore, in what follows we condition the claim values to the equilibrium strategy ($\mathbf{a}$). As a first step, we investigate the value $\Phi(y_t, z|\mathbf{a})$, defined in equation (10), of a barrier strategy ($\mathbf{a}$) in which management decides to change the level of outstanding debt so that the firm's income-to-debt ratio immediately jumps from y_t to z. If ($\mathbf{a}$) is an MPE then $e(y|\mathbf{a}) \geq \Phi(y_t, z|\mathbf{a})$ for all $y, z \in [0, y_u]$ and $\Phi(y_t, z|\mathbf{a}) \leq \Phi(y_t, \mathcal{Y}(y_t)|\mathbf{a})$ for all $z \in [y_b, y_u]$ and $y_t \geq y_u$.

Second, we also need to show that inaction is dominated by immediate debt issuance in the restructuring region. To provide intuition, we compare two strategies. The first strategy is to follow the proposed restructuring policy by immediately issuing an amount of debt as described in equation (10) above. The second strategy is to wait a period dt (in turn, receiving the cash flows owed to shareholders), and then issue an amount of debt according to equation (10). Hence, for a proposed strategy to be an equilibrium, for all values of (Y_t, F_t) such that $y_t \equiv \frac{Y_t}{F_t} \in [y_u, \infty)$, it must be that:

$$E(F_t, Y_t|\mathbf{a}) \geq \delta(F_t, Y_t)\, dt + e^{-r\,dt}\, \mathbb{E}_t\left[E(F_t + dF_t, Y_t + dY_t|\mathbf{a})\right]. \tag{20}$$

In the special case of $F_t = 0$, equation (20) implies that the equity value of an unlevered firm that issues debt in the future must be higher than the value of an equivalent firm that never issues debt:

$$E(0, Y_t|\mathbf{a}) \geq \left(\frac{1-\tau}{r-\mu}\right) Y_t. \tag{21}$$

This condition will prove important in Section 3 when we numerically investigate the existence of MPEs and compute the associated net tax benefit of debt.

Using Itô's lemma and recalling that $E(F_t, Y_t|\mathbf{a}) = F_t e(y_t|\mathbf{a})$ and $\delta(F_t, Y_t) = F_t \delta(y_t)$, equation (20) simplifies to:

$$\delta(y_t) - (r+\xi)\, e(y_t|\mathbf{a}) + (\mu+\xi) y\, e'(y_t|\mathbf{a}) + \frac{\sigma^2}{2} y_t^2\, e''(y_t|\mathbf{a}) \leq 0. \tag{22}$$

Note that this equation differs from equation (11) only due to the inequality sign. Equation (20) is reminiscent of the optimal strategy of an agent whose only available gamble is associated with an expected loss, and whose only decision is to choose when to stop playing. As such, we refer to equation (22) as the "supermartingale condition." While the above intuition contemplates only a particular deviation from the candidate equilibrium policy, the following proposition provides a formal characterization of the necessary and sufficient conditions for a barrier strategy to be an MPE.

Proposition 2 (Verification argument) *Consider a barrier strategy* $\mathbf{a} = (y_b, y_u, \mathcal{Y}(y))$ *that satisfies the conditions of Proposition 1 and let* $\Phi(y, z|\mathbf{a})$ *be the equity value when the income-to-debt ratio is* $y \geq 0$ *and the firm restructures to a target* $z \geq 0$*, defined in equation* (10). *Then such a* *strategy* $(\mathbf{a})$ *satisfies*

$$\sup_{z\in[y_b,y_u]} \Phi(y,z|\mathbf{a}) = \sup_{z\geq 0} \Phi(y,z|\mathbf{a}), \quad y \geq 0, \tag{23}$$

and constitutes an MPE if and only if

$$e(y|\mathbf{a}) \geq \Phi(y,z|\mathbf{a}), \quad (y,z) \in [0,y_u]^2, \tag{24}$$

and the following condition holds for all $y \geq y_u$*:*

$$\delta(y) - (r+\xi)\, e(y|\mathbf{a}) + (\mu+\xi) y\, e'(y|\mathbf{a}) + \frac{\sigma^2}{2} y_t^2\, e''(y|\mathbf{a}) \leq 0, \tag{25}$$

or, equivalently,

$$\delta(y) - (r-\mu)\, e(y|\mathbf{a}) + (\mu+\xi)\, P(y|\mathbf{a}) + \frac{1}{2}\sigma^2 y\, P'(y|\mathbf{a}) \leq 0.$$

Equation (23) states that it is never optimal for shareholders to issue debt so that the post-issuance income-to-debt ratio falls either into the default region, $y < y_b$, or into the restructuring region $y > y_u$.[10] The condition in equation (24) guarantees that it is not optimal to issue debt when the income to debt ratio y is inside the inaction region (y_b, y_u). Equation (25), which is equivalent to the supermartingale condition that we derived heuristically in equation (22), provides an off-equilibrium condition guaranteeing that it is optimal for shareholders to issue debt immediately when the income-to-debt ratio falls in the restructuring region, $y \geq y_u$.

A key requirement for the verification argument in Proposition 2 is to allow for *any* possible off-equilibrium Markov strategies, including strategies outside of the barrier class. To achieve this generality, the Online Appendix establishes that a strategy $(\mathbf{a})$ is an MPE if and only if the induced equity value can be written as an optimal stopping time problem over *all* stopping times associated with Markov strategies, not just those in the barrier class (Lemmas II.4 and II.5). The proof of Proposition 2 exploits this stopping time representation and applies results from the stochastic control literature (e.g., Lamberton and Zervos (2013)) to obtain a formal characterization of the

[10] Corollary II.3 in the Online Appendix provides a formal proof of this claim.

MPE.

2.4 No-issuance equilibrium

In this section we identify necessary and sufficient conditions for an MPE in which shareholders default strategically and find it optimal to never issue additional debt in the future. This is a special case of the barrier strategy MPE derived in the previous section in which the restructuring boundary is infinite: $y_u \to \infty$.

We first investigate the case in which shareholders are not allowed to issue debt in the future, but can default strategically. In this case, the equity value is given by:

$$e_{\text{NI}}(y_t) = \sup_{\tau} \mathbb{E}_t \left[\int_t^{\tau} e^{-(r+\xi)(s-t)} \, \delta(\overline{y}_s) \, ds \right], \quad \text{with } \overline{y}_t \equiv Y_t/\overline{F}_t.$$

Because shareholders cannot issue any debt in the future, the face value of debt evolves via

$$d\overline{F}_t = -\xi \overline{F}_t dt.$$

Itô's lemma implies that the income-to-debt ratio $\overline{y}_t$ dynamics follow:

$$\frac{d\overline{y}_t}{\overline{y}_t} = (\mu + \xi) \, dt + \sigma \, dW_t.$$

It is well known (see, e.g., DeMarzo and He (2021, Propositions 4 and 6)) that in this case the default boundary, which satisfies the smooth pasting condition, is

$$y_{b,\text{NI}} \equiv \left(\frac{\Pi}{\Pi - 1} \right) \left(\frac{r - \mu}{r + \xi} \right) \left(\frac{c(1-\tau) + \xi}{1 - \tau} \right), \tag{26}$$

and that the values of debt and equity per unit face value of debt F_t are:

$$P_{\text{NI}}(y) = \begin{cases} 0 & \text{if} \quad 0 \le y \le y_{b,\text{NI}} \\ \left(\frac{c+\xi}{r+\xi}\right)\left(1 - \left(\frac{y}{y_{b,\text{NI}}}\right)^{\Pi}\right) & \text{if} \quad y_t > y_{b,\text{NI}}, \end{cases} \tag{27}$$

$$e_{\text{NI}}(y) = \begin{cases} 0 & \text{if} \quad 0 \le y \le y_{b,\text{NI}} \\ \hat{e}(y) - \frac{1-\tau}{(r-\mu)\Pi}\left(\frac{y}{y_{b,\text{NI}}}\right)^{\Pi} y_{b,\text{NI}} & \text{if} \quad y > y_{b,\text{NI}}, \end{cases} \tag{28}$$

where $\hat{e}(y)$ is the value of the levered claim to EBIT defined in equation (13), and Π is the negative root of the quadratic equation (8).

The debt and equity values in equations (27)–(28) are derived under the assumption that shareholders do not issue any debt in the future. In order for this no-issuance strategy to be an MPE, it follows from Proposition 2 that, for any value of the income-to-debt ratio y, shareholders find it optimal not to issue debt, that is:

$$\underbrace{e_{\text{NI}}(y)}_{\text{No issue}} \ge \underbrace{\sup_{z \ge 0}\left\{\frac{y}{z} e_{\text{NI}}(z) + \left(\frac{y}{z} - 1\right) P_{\text{NI}}(z) - \beta y\right\}}_{\text{Issue}} \equiv \phi_{\text{NI}}(y), \tag{29}$$

Since the debt and equity prices in equations (27)–(28) do not depend on β, condition (29) is equivalent to

$$\beta \ge \beta^*(\xi) \equiv \sup_{(y,z)\in[y_{b,\text{NI}},\infty]^2}\left\{\frac{e_{\text{NI}}(z)}{z} - \frac{e_{\text{NI}}(y)}{y} + \left(\frac{1}{z} - \frac{1}{y}\right) P_{\text{NI}}(z)\right\}. \tag{30}$$

As we show in Appendix A.2, the incentive-compatibility condition (30) holds for any $y \ge y_{b,\text{NI}}$ if and only if it holds for $y \to \infty$. That is, $\phi_{\text{NI}}(\infty) = \sup_{y\in[y_{b,\text{NI}},\infty]} \phi_{\text{NI}}(y)$. This is because the benefit of issuing debt is highest when there is no debt in place. Hence, it is necessary to check only that shareholders will not want to issue debt for $y \to \infty$. Checking that issuance is not optimal at $y \to \infty$ is also sufficient since, for finite values of y, the benefit to issue debt is smaller and shareholder will still not want to issue debt.

The case $y \to \infty$ corresponds to a generalization of the Leland (1994) model in which the issuance of finite maturity debt can occur only at time zero when the firm is unlevered, that is, $F_0 = 0$. Using this property, we can find the threshold $\beta^*(\xi)$ of debt issuance costs above which the MPE involves no debt issuance. Specifically, letting $y \to \infty$ in equation (30), we find that

$$\beta^*(\xi) = \sup_{z \in [y_{b,\text{NI}}, \infty]} \left\{ \frac{e_{\text{NI}}(z)}{z} - \frac{1-\tau}{r-\mu} + \frac{P_{\text{NI}}(z)}{z} \right\}. \tag{31}$$

The solution y^*_{NI} to the maximization problem in equation (31) identifies the optimal income-to-debt ratio for a firm that, as in Leland (1994), starts unlevered and issues debt only once at time 0. It is given by:

$$y^*_{\text{NI}} = \left(\frac{\tau c}{\tau c + (-\Pi)(c+\xi)} \right)^{\frac{1}{\Pi}} y_{b,\text{NI}}. \tag{32}$$

Using equations (27), (28) and (32), we can rewrite equation (31) as:

$$\beta^*(\xi) = \left(\frac{1-\tau}{r-\mu} \right) \left(\frac{\tau c}{\tau c - \Pi(c+\xi)} \right)^{\frac{\Pi-1}{\Pi}} \left[\frac{(c+\xi)(1-\Pi)}{c(1-\tau)+\xi} - 1 \right]. \tag{33}$$

Therefore, the no-issuance strategy is an MPE if and only if debt issuance costs are higher than the threshold $\beta^*(\xi)$. For $\beta > \beta^*(\xi)$, the restructuring barrier y_u in the barrier strategy MPE converges to infinity, $y_u \to \infty$, and the restructuring target $y^* \to y^*_{\text{NI}}$, where y^*_{NI} is defined in equation (32). An important special case is $\xi = 0$, for which we obtain

$$\overline{\beta} = \left(\frac{\tau}{r-\mu} \right) \left(\frac{\tau - \Pi^*}{\tau} \right)^{\frac{1}{\Pi^*}}, \quad \text{where } \Pi^* \equiv \Pi_{(\xi=0)} = \frac{\frac{\sigma^2}{2} - \mu - \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2r\sigma^2}}{\sigma^2} < 0. \tag{34}$$

If $\beta > \overline{\beta}$, then regardless of debt maturity $1/\xi$, the MPE corresponds to a policy in which there is no future debt issuance (i.e., $y_u = \infty$).

2.5 Dynamic debt issuance with commitment

So far we have focused on Markov-perfect equilibria in which shareholders do not have the ability to commit to a future debt issuance policy. To assess the value of commitment to shareholders, here we investigate a benchmark model in which shareholders can commit to a future restructuring policy. Specifically, we consider a variation of the Goldstein, Ju, and Leland (2001) framework in which debt maturity ($1/\xi$) is finite and shareholders commit to an optimal restructuring threshold y_u and target y^*, but not to a default boundary y_b, which is chosen to be consistent with limited liability. Commitment to the restructuring policies (y_u, y^*) could arise, for instance, by the firm specifying restrictive covenants in its bond indenture that limit future debt issuances (see, e.g., Roberts and Sufi (2009)). Alternatively, our benchmark policy can be motivated based on the repeated interactions between the issuer and the credit market. In this context, reputation concerns provide shareholders the incentive to abide by the commitment policy if the punishment is that, following a deviation, debt would always be priced according to the no-commitment equilibrium of DeMarzo and He (2021). Such a "grim-trigger" punishment is a credible threat because debtholders observe the size of the debt issuance, pay fair value for their claim at the restructuring date, and are therefore indifferent to the firm's debt issuance policy.

For consistency with our no-commitment framework, we modify the Goldstein, Ju, and Leland (2001) setting to the case in which the firm faces fixed restructuring costs and does not need to repurchase all of its outstanding debt prior to issuing new debt.[11] Furthermore, we extend their model to allow for finite maturity. As in Goldstein, Ju, and Leland (2001), we define the commitment policy $(y_b, y^*, y_u)^{\text{GJL}}$ that maximizes the enterprise value at the beginning of the firm's life, subject to the constraint that the default decision is incentive compatible, i.e., y_b satisfies a smooth-pasting condition $e'(y_b) = 0$. More formally, the commitment policy $(y_b, y^*, y_u)^{\text{GJL}}$ solves the following problem:

$$(y_b, y^*, y_u)^{\text{GJL}} = \underset{0 \le y_b \le y^* \le y_u}{\operatorname{argmax}} \frac{e(y^*) + P(y^*)}{y^*} - \beta, \quad \text{such that} \quad e'(y_b) = 0,$$

where the debt and equity prices, $P(y)$ and $e(y)$, satisfy the ODEs (7) and (11), for $y \in (y_b, y_u)$,

[11] In the Goldstein, Ju, and Leland (2001) model restructuring costs are proportional. However, because the firm is constrained to repurchase all debt before issuing new debt, in their model, there is de facto a fixed issuance cost.

subject to the boundary conditions

$$P(y_b) = 0 \quad \text{(zero debt recovery at default)}$$

$$P(y_u) = P(y^*) \quad \text{(no jump at issuance)}$$

$$e(y_b) = 0 \quad \text{(zero equity recovery at default)}$$

$$e(y_u) = \frac{y_u}{y^*} e(y^*) + \left(\frac{y_u}{y^*} - 1 \right) P(y^*) - \beta y_u \quad \text{(equity value at issuance).}$$

3 Results

In this section, we investigate the properties of the MPE in barrier strategies derived from our model. Table 1 reports the coefficients for the baseline calibration of the model. We set the annual risk-free rate to $r = 4\%$, consistent with the average three-month constant maturity U.S. Treasury yield over the 1990–2020 period. We fix the drift of the EBIT dynamics in equation (1) to $\mu = 0$, which yields a price-dividend ratio $1/(r - \mu) = 25$. Finally we set the EBIT volatility coefficient to $\sigma = 22\%$, consistent with values used in the dynamic capital structure literature (e.g., Leland (1994)). We assume that corporate profits are taxed at a rate $\tau = 20\%$ and we fix the coupon rate $c = r$.[12]

The key parameters of our model are the debt issuance cost parameter β and the inverse maturity parameter ξ. In our benchmark case, we calibrate them to match two empirical facts: (i) an average debt maturity of three to seven years (e.g., Choi, Hackbarth, and Zechner (2018)) and (ii) debt issuance costs in the range of one to two percent of the amount issued (e.g., Altınkılıc and Hansen (2000)). Within these ranges, we focus on a maturity of five years (i.e., $\xi = 0.2$), and a debt issuance cost parameter β that generates a 1% fee on the amount raised. From equation (9), issuance costs at a restructuring time t equal $\beta Y_t = \beta y_u F_t$. Moreover, the amount of debt raised is $(F_t^* - F_t)\, P\big(Y_t/F_t^*\big)$. Noting that $y^* F_t^* = y_u F_t = Y_t$ and simplifying, we choose β so that:

$$\beta = 0.01 \times \left(\frac{1}{y^*} - \frac{1}{y_u} \right) P(y^*),$$

[12]Choosing c so that the bond is priced at par at issuance generates nearly identical results.

where $P(y^*)$, which equals $P(y_u)$, is the MPE debt issuance price when debt maturity is five years. We obtain a value $\beta = 0.065$, which corresponds to 0.26% of asset value.

3.1 MPEs in barrier strategies

Figure 1 partitions the parameter space (β, ξ) into two regions separated by the blue line labeled "existence threshold." The points to the right of this threshold represent MPEs in that they satisfy the necessary and sufficient conditions of Proposition 2. Within this region, the red line labeled "no-issuance threshold" further partitions barrier-strategy MPEs into two sub-regions: one region with a *finite* restructuring boundary y_u, and one region with $y_u = \infty$, implying that firms in this region choose not to issue debt in the future. The no-issuance threshold $\beta^*(\xi)$ is determined analytically by equation (33). Figure 2 illustrates policy parameters for the baseline case of a five-year debt maturity, $\xi = 0.2$, as a function of β. As the issuance cost parameter β approaches $\beta^*(\xi = 0.2)$ from below, the values of the default threshold and restructuring target (y_b, y^*) converge to those of the generalized Leland (1994) model given in equations (26) and (32), and the restructuring threshold y_u approaches infinity. That is, when β is sufficiently high so that the threshold y_u goes to infinity, both the no-commitment model and the model with commitment converge to the Leland model.

The intuition for the three regions in Figure 1 is the following: For any given maturity $(1/\xi)$, and for sufficiently large values of β (i.e., the region to the right of the no-issuance threshold in Figure 1), issuance costs exceed the tax-benefits from debt and, in equilibrium, the firm finds it optimal not to issue debt in the future. A special case in this region is $\overline{\beta} = \lim_{\xi \to 0} \beta^*(\xi)$, defined in equation (34) and denoted by the red dotted line. This parameter value refers to the lowest issuance cost at which the upper restructuring boundary y_u is infinite regardless of maturity.

For intermediate levels of β (the region between the existence and no-issuance thresholds), the tax benefit from debt exceeds issuance costs and the MPE is characterized by a barrier strategy with a finite restructuring boundary y_u. This finding highlights the role of issuance costs as a commitment device that mitigates shareholders' incentive to issue debt too aggressively, and allows the firm to extract positive tax benefits.

However, for values of β that fall to the left of the existence threshold in Figure 1, no barrier-strategy MPE exists, because at least one of the conditions of Proposition 2 is violated. Figure 2 provides intuition for this result in the special case $\xi = 0.2$, which corresponds to a five-year maturity; this is the case highlighted by the horizontal black-dotted segment in Figure 1. Specifically, without commitment, Panels B and C show that the manager issues debt more aggressively as the issuance cost parameter β decreases. Indeed, both the restructuring boundary y_u and the target y^* drop with issuance costs. As shown in Panel D, this more aggressive policy leads to a reduction in bond price. However, the impact of a change in β on the default boundary y_b in Panel A is U-shaped. On the one hand, a lower β reduces issuance costs. On the other hand, it also induces more debt issuance, which lowers the debt price and therefore the proceeds associated with debt issuance. When β is sufficiently close to $\beta^*(\xi = 0.2)$, the first effect dominates, which increases the option value of ownership, thus resulting in a lower default boundary y_b. As β is decreased even further, however, the price of new debt falls to a point where the second effect dominates, leading to a higher default boundary y_b. Indeed, the value of the default boundary at the left-most point of the red line in Panel A of Figure 2 is only slightly lower than the value of the default boundary at the right-most point, which corresponds to the default boundary of the no-issuance MPE. Reducing β further (to the left of the existence threshold) leads to a failure of the supermartingale condition (25), implying that the candidate policy $(y_b, y_u, \mathcal{Y}(\cdot))$ is not an MPE.[13]

Figure 2 also shows that the no-commitment results are in sharp contrast to those with commitment. First, for all parameter values (β, ξ), a barrier strategy with commitment always exists. Moreover, lower values of β offer larger tax benefits to debt. This can be seen in Panel A of Figure 2 in which the default boundary with commitment monotonically increases with β (dashed-blue line). That is, with commitment, lower restructuring costs are associated with larger option values to maintain ownership, and thus, higher equity values. Panels B and C show that the restructuring boundary y_u and the target y^* with commitment are more conservative in the commitment case than in the MPE. This more conservative strategy implies higher debt issuance prices across all

[13] Hugonnier, Malamud, and Morellec (2015) also find a threshold value of issuance costs below which an MPE no longer exists in a model with proportional issuance costs and callable debt.

values of the issuance cost parameter (Panel D).

3.2 Equilibrium restructuring policy

For the baseline parameters $\beta = 0.065$ and $\xi = 0.2$, Figure 3 shows the restructuring function $\mathcal{Y}(y) : [y_u, \infty) \to (y_b, y_u)$, defined in equation (18). In the MPE, the firm issues debt at $y = y_u$, bringing the income-to-debt ratio to the value $y^* \in (y_b, y_u)$, denoted by the black dot in the figure. Off-equilibrium, i.e., $y > y_u$, the target income-to-debt ratio $\mathcal{Y}(y)$ is an increasing function of y. Note that in the limit $y \to \infty$ the function $\mathcal{Y}(y)$ converges to the income-to-debt ratio that maximizes the total enterprise value firm value $(e(z) + P(z))/z$, i.e., $\mathcal{Y}(\infty)$ represents the optimal initial debt issuance of an unlevered firm. The dependence of the restructuring target $\mathcal{Y}(y) \in (y_b, y_u)$ on y is a key feature of our model, and is due to our assumption that the firm is not required to repurchase (i.e., call) all outstanding debt prior to issuing new debt. This assumption contrasts with much of the existing literature, which assumes all outstanding debt must be called prior to any new debt issuance. This assumption implies that the debt issuance decision is always made with zero debt outstanding ($y \to \infty$) and thus the function $\mathcal{Y}(y)$, for all $y \in (y_u, \infty)$, reduces to a single value, $\mathcal{Y}(\infty)$.

3.3 Tax benefits to debt, issuance costs, and debt maturity

In this section, we quantify the tax benefits to debt as a function of restructuring costs (β) and debt maturity (ξ). To construct this measure, we first consider the total enterprise value (TEV) for an initially unlevered firm that chooses its optimal leverage by issuing an amount of debt F^*:

$$TEV^L \equiv E(0, Y_0) = \sup_{F^* \geq 0} \Big(E(F^*, Y_0) + P(F^*, Y_0) - \beta Y_0 \Big), \tag{35}$$

where F^* is chosen optimally given initial conditions $(F_0 = 0, Y_0)$, i.e., $F^* = Y_0/y^*$, where $y^* = \mathcal{Y}(y \to \infty)$ from equation (19). We then relate the total enterprise value of the optimally levered firm to that of an unlevered firm that never issues any debt, that is,

$$TEV^U = (1-\tau)\left(\frac{Y_0}{r-\mu}\right).$$

The differences between TEV^L and TEV^U represent the net benefits to leverage. Hence, following the literature, e.g., van Binsbergen, Graham, and Yang (2010) and Korteweg (2010), we define the net tax benefit as:

$$NTB = \frac{TEV^L - TEV^U}{TEV^U}. \tag{36}$$

Note that $TEV^L \equiv E(0, Y_0)$, defined in equation (35), includes the proceeds from the initial optimal debt placement, net of issuance costs. Hence, NTB denotes the net tax benefit of debt accrued to shareholders when they choose an optimal initial leverage. Moreover, the unlevered enterprise value TEV^U equals the equity value in the no-issuance MPE described in Section 2.4 for $y \to \infty$, that is, $TEV^U \equiv E_{\text{NI}}(0, Y_0)$, where $E_{\text{NI}}(0, Y_0)/Y_0 = \lim_{y\to\infty} e_{\text{NI}}(y)/y$. Therefore, we can interpret NTB as the net tax benefit accrued to shareholders relative to a no-issuance benchmark.[14]

Figure 4 shows that the tax benefit is zero at the boundary of the MPE existence region. This is because in our numerical analysis we find that, for any given amortization rate ξ, as β approaches the MPE existence threshold in Figure 1 the supermartingale condition (21) at $y \to \infty$ holds as an equality, i.e., $E(0, Y_t) = \left(\frac{1-\tau}{r-\mu}\right) Y_t$. This result implies that, on the MPE existence threshold, the shareholders of an unlevered firm are indifferent between issuing debt or not. Hence, by equation (36), the net tax benefit is zero.

In contrast, for (β, ξ) values that lie inside the MPE existence region, the net tax benefit is strictly positive with magnitude that varies with debt maturity. In particular, the debt maturity that maximizes tax benefits increases with issuance costs, as highlighted by the red markers. This is due to a tradeoff between debt-issuance fees and tax benefits net of bankruptcy costs. A shorter maturity makes debt less risky and thus allows the manager to increase leverage and extract more

[14] Equation (36) also applies to the special case $\beta = 0$, in which the tax benefit is zero, regardless of debt maturity. See equation (31) in DeMarzo and He (2021),evaluated in the limit $y \to \infty$.

tax-benefit. This is evident from equation (5) in which the drift of the EBIT process y increases with ξ. Hence, for $\xi \to \infty$ debt becomes virtually risk-free. However, a shorter maturity also implies more frequent rollovers, leading to higher restructuring costs.

To provide a benchmark for the MPE, we also compute the tax benefit in equation (36) for the case of the commitment policy (blue markers in Figure 4). As the figure shows, the value of tax benefit associated with the MPE (red markers) is very close to that of the commitment policy (blue markers). An alternative way to assess the value of commitment is to compute the equity share of the claim to EBIT, $E(0, Y_0)/V_0$, which, from equations (35) and (36), equals $(1 - \tau) \times (1 + NTB)$. We find that, along the (β, ξ) points that maximize the tax benefit in the MPE, the firm extracts most of the value of the EBIT claim. Indeed, as β is lowered, for both cases with and without commitment, the equity share $E(0, Y_0)/V_0$ approaches 1, which, from equation (36), corresponds to a tax benefit equal to $\tau/(1 - \tau) = 0.25$ in our calibration. These results show that the presence of even arbitrarily small issuance costs can break the irrelevance of capital structure and maturity choices found in DeMarzo and He (2021) that arise in an MPE without issuance costs.

Finally, Figure 5 shows the net tax benefit in the MPE as a function of the issuance cost β when the inverse maturity parameter is fixed at the baseline value $\xi = 0.2$, which corresponds to a five-year debt maturity. The plot shows that the net tax benefit is a hump-shaped function of the issuance cost parameter β. The right-most point in the figure corresponds to the issuance cost threshold $\beta^*(\xi = 0.2)$, derived in equation (33), beyond which debt issuance is too costly and there is no issuance in the MPE. At that point, the net tax benefit is zero. As the issuance cost drops below $\beta^*(\xi = 0.2)$ shareholders issue debt in equilibrium, leading to a positive net tax benefit that first increases as issuance becomes cheaper. However, as β continues to decline, the effectiveness of issuance costs as a commitment device weakens. As a result, the MPE involves progressively more debt issuance compared to the benchmark policy with commitment. Such debt accumulation increases expected default costs and thus erodes the net tax benefit. Finally, the left-most point in the figure corresponds to the issuance cost threshold below which an MPE fails to exists (the blue line in Figure 1) and the net tax benefit is zero.

The red marker in Figure 5 denotes the tax benefit for the baseline parameter $\beta = 0.065$, which reflects a fractional cost of 1% of the debt amount issued. At this point, dynamic debt issuance in the MPE increases enterprise value by 5.2% compared to an unlevered firm. In the commitment case, for the same value of β and ξ, the tax benefit is very close at 5.7% (the blue marker). For comparison, van Binsbergen, Graham, and Yang (2010) and Korteweg (2010) report values of the net tax benefit ranging from 3.5% to 5.5% of asset value. Our estimate of the net tax benefit in the MPE falls within this range.

In sum, when the restructuring cost β and the inverse maturity ξ are calibrated to match empirical observation, we find realistic estimates of the net tax benefit that are only slightly impacted by shareholders' inability to commit to a future issuance policy.

4 On the lack of existence of barrier-strategy MPEs

One of the most puzzling findings of our model is that, as seen in Figure 1, there are combinations of (β, ξ) parameters for which a barrier-strategy MPE does not exist. This raises the question of whether, for those parameters, an MPE would exist if we either consider a broader class of strategies or modify some features of the model. Here we briefly discuss two possible modifications that might change the nature of the equilibrium.

First, for parameter values to the left of the existence threshold in Figure 1, we find that a barrier strategy MPE does not exist because the supermartingale condition in equation (25) fails. One way to modify our model that might circumvent this failure is to consider *multi-barrier* strategies that allow for more than a single inaction region. In this setting, a firm that begins with, say, zero debt could jump into the inaction region associated with the highest income-to-debt ratios y_t (i.e., lowest leverage ratios) before transitioning into inaction regions with lower values of y_t. Hence, multi-barrier policies could reduce the amount of debt issuance expected by creditors, increase the debt price, and in turn help shareholders extract a positive tax shield. A possible direction of future work is to characterize such strategies and verify whether multi-barrier MPEs exist.

Second, the lack of existence of MPEs for small values of the issuance cost parameter β may be due to the continuous-time specification of our model. Note that when β is set to zero at the outset, our framework is identical to that of DeMarzo and He (2021). Yet, for a fixed maturity ξ, we find that an MPE does not exist in the limit $\beta \to 0$ and therefore our model does not converge to theirs. To gain intuition for this discrepancy, consider a discrete-time model with horizon T, time step Δt, and therefore $N = \frac{T}{\Delta t}$ time intervals. For any positive value of β, a policy in which shareholders issue debt in each period, as in DeMarzo and He (2021), would generate issuance costs proportional to βN. In the continuous time limit, $\Delta t \to 0$, N goes to infinity, implying an infinite accumulation of issuance costs for any positive value of β. Therefore, a continuous issuance policy cannot be an equilibrium if $\beta > 0$. In contrast, when Δt is finite, the issuance costs βN goes to zero in the limit $\beta \to 0$. Thus, in a discrete-time version of our model, it is possible that the MPE converges to the smooth issuance policy of DeMarzo and He (2021). In particular, there could be MPEs for values of β below the existence threshold in Figure 1. Interestingly, in such equilibria there could be positive tax benefit even in the case of $\beta = 0$, because the inability to issue continuously effectively provides a commitment mechanism over a finite time interval. DeMarzo, He, and Tourre (2021) explore this mechanism in a model of sovereign debt issuance and conclude that, when time intervals are discrete, there are gains from trade between an impatient sovereign and a patient lender. However, such gains are entirely dissipated when trading occurs continuously. These findings echo the result that a durable goods monopolist can extract positive rents when production is fixed over discrete-time intervals, while in continuous-time the Coase conjecture (Coase (1972)) of a competitive equilibrium holds (Stokey (1981)).

5 Conclusion

In the absence of both debt issuance costs and the ability to commit to a specific financing policy, DeMarzo and He (2021) prove a "Modigliani-Miller" irrelevance result in which dynamic debt issuance adds no value in equilibrium. In practice, however, it is costly for firms to issue debt. Moreover, the literature has shown that by specifying fixed debt issuance costs, models can capture empirical regularities observed in debt dynamics such as the lumpiness of leverage adjustments and the negative correlation between profitability and leverage. Motivated by this evidence, in this paper we investigate equilibrium leverage dynamics when a firm is subject to a fixed restructuring cost and cannot commit to a future debt policy. We provide a formal characterization of Markov Perfect Equilibria in barrier strategies when the firm is not required to repurchase outstanding debt prior to issuing additional debt. We show that the interaction between issuance costs and debt maturity determines a region in which equilibria exist and debt tax benefits are positive with magnitude that depends on maturity. These results highlight the role of issuance costs as a commitment device that mitigates a manager's incentive to issue debt too aggressively, thus allowing the firm to extract positive tax benefits. When issuance costs decline, their effectiveness as a commitment device weakens. Indeed, for any maturity, we numerically identify an issuance cost threshold below which the debt contract is no longer incentive compatible, implying that a barrier-strategy MPE no longer exists.

When we calibrate our model to empirically relevant issuance costs and debt maturities, we find realistic estimates of the net tax benefit in the MPE that are only slightly lower than those associated with a commitment benchmark policy. Furthermore, as the issuance cost parameter approaches zero, there are maturities for which shareholder can extract close to 100% of the firm's EBIT for both cases with and without commitment. Thus, even for vanishing (but strictly positive) issuance costs, not being able to commit to a dynamic capital structure policy may have only a small impact on a firm's ability to extract tax benefits to debt.

Appendix A

A.1 Proof of Proposition 2

By construction we have:

$$P(y|\mathbf{a}) = P(\mathcal{Y}(y)|\mathbf{a}), \qquad y \geq y_u \tag{A.1}$$

$$e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \sup_{z\in[y_b,y_u]} \Phi(y, z|\mathbf{a}) \qquad y \geq y_u, \tag{A.2}$$

and, since the scaled equity value function is differentiable at y_u, we deduce from Lemma II.10.i) in the Online Appendix that the function $e(y|\mathbf{a})$ is globally convex, non-decreasing, and strictly positive on the interval $[y_b, \infty)$.

To prove the claim in equation (23), note that, a direct calculation using the above expressions shows that it is not optimal for shareholders to restructure from y to a point $z \geq y_u$. In fact, for $z \geq y_u$ we have

$$\begin{aligned}
\Phi(y, z|\mathbf{a}) &= \frac{y}{z}e(z) + \left(\frac{y}{z} - 1\right) P(z) - \beta y \\
&= \frac{y}{z}\Phi(z, \mathcal{Y}(z)|\mathbf{a}) + \left(\frac{y}{z} - 1\right) P(z) - \beta y \\
&= \frac{y}{z}\left\{\frac{z}{\mathcal{Y}(z)}e(\mathcal{Y}(z)) + \left(\frac{z}{\mathcal{Y}(z)} - 1\right) P(\mathcal{Y}(z)) - \beta z\right\} + \left(\frac{y}{z} - 1\right) P(z) - \beta y \\
&= \frac{y}{\mathcal{Y}(z)}e(\mathcal{Y}(z)) + \left(\frac{y}{\mathcal{Y}(z)} - \frac{y}{z}\right) P(\mathcal{Y}(z)) - \beta y + \left(\frac{y}{z} - 1\right) P(\mathcal{Y}(z)) - \beta y \\
&= \underbrace{\frac{y}{\mathcal{Y}(z)}e(\mathcal{Y}(z)) + \left(\frac{y}{\mathcal{Y}(z)} - 1\right) P(\mathcal{Y}(z)) - \beta y}_{\equiv\Phi(y,\mathcal{Y}(z)|\mathbf{a})} - \beta y \\
&= \Phi(y, \mathcal{Y}(z)|\mathbf{a}) - \beta y \\
&< \Phi(y, \mathcal{Y}(y)|\mathbf{a}),
\end{aligned} \tag{A.3}$$

where the first equality follows from the definition of $\Phi(y, z|\mathbf{a})$; the second equality uses condition (A.2); the third equality follows from the definition of $\Phi(x, \mathcal{Y}(z))$; and the fourth equality uses the no-jump condition (A.1) at $z \geq y_u$. The inequality at the end of equation (A.3) shows that restructuring to $\mathcal{Y}(y) < y_u$ dominates restructuring to $z \geq y_u$. Furthermore, restructuring from y to a point $z < y_b$ is also suboptimal, since $\Phi(y, z|\mathbf{a}) = -y\beta < 0$ for all $z \leq y_b$. Hence, the claim in equation (23) follows.

Now, since the function $e(y|\mathbf{a})$ is convex it follows from Proposition II.3 in the Online Appendix that $(\mathbf{a})$ is an equilibrium if and only if the convex function $v(y) \equiv e(y|\mathbf{a}) - \hat{e}(y)$, with $\hat{e}(y)$ defined in equation (13), is a weak solution to the Hamilton-Jacobi-Bellman equation (II.14) of the Online Appendix. On the interval $[0, y_b]$ we have $v(y) = -\hat{e}(y)$ and

$$\mathcal{O}v(dy) = \ \delta(y_b)\, dy \leq \delta(y_0)\, dy < 0,$$

where $\mathcal{O}v(dy)$ denotes the measure defined in equation (II.13) of the Online Appendix. Hence $v(y)$ is a weak solution on that interval if and only if equation (24) holds for all $y \leq y_b$. On the interval (y_b, y_u) we have that $\mathcal{O}v(dy) \equiv 0$ and it follows that $v(y)$ is a weak solution on that interval if and only if equation (24) holds for $y \in [y_b, y_u]$. Finally, since $e(y|\mathbf{a}) > 0$ for $y > y_b$ we have that

$$v(y) = e(y|\mathbf{a}) - \hat{e}(y) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) - \hat{e}(y) = \psi(y|\mathbf{a})$$

for all $y \in [y_u, \infty)$ and it follows that $v(y)$ is a weak solution on that interval if and only if the restriction of the measure

$$\mathcal{O}v(dy) = \mathcal{O}e(dy|\mathbf{a}) - \mathcal{O}\hat{e}(dy) = \mathcal{O}e(dy|\mathbf{a}) + \delta(y)dy \tag{A.4}$$

to that interval is non-positive. To complete the proof we now provide an expression for this measure. First observe that as a result of Condition (18) and Milgrom and Segal (2002, Corollary 4) we have that

$$e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \sup_{z \in [y_b, y_u]} \Phi(y, z|\mathbf{a})$$

is continuously differentiable at all points of $[y_u, \infty)$ and satisfies

$$e'(y|\mathbf{a}) = s(\mathcal{Y}(y)|\mathbf{a}) - \beta = s(y|\mathbf{a}) = \frac{1}{y}\left(e(y|\mathbf{a}) + P(\mathcal{Y}(y)|\mathbf{a})\right), \qquad y \geq y_u, \tag{A.5}$$

where the last two equalities equality follow from (A.1) and

$$s(y|\mathbf{a}) \equiv \frac{1}{y}\left(e(y|\mathbf{a}) + P(y|\mathbf{a})\right),$$

denotes the enterprise value of the firm per unit of cash flow. Since $e(y|\mathbf{a})$ is convex, $\mathcal{Y}(y)$ is continuous on $y \geq y_u$, and $P(y|\mathbf{a})$ is continuous on $[0, y_u]$ we have this derivative is continuous as well as non-decreasing and, therefore, absolutely continuous. Combining this property with equation (A.5) we obtain

$$e''(dy|\mathbf{a}) = s'(y|\mathbf{a})dy = \frac{1}{y}\left(e'(y|\mathbf{a}) + P'(y|\mathbf{a}) - s(y|\mathbf{a})\right)dy = \frac{1}{y}P'(y|\mathbf{a})dy. \tag{A.6}$$

Using equations (A.5) and (A.6), it follows that the non-positivity of equation (A.4) is equivalent to the supermartingale condition in equation (22):

$$\begin{aligned} 0 &\geq \delta(y) - (r-\mu)\, e(y|\mathbf{a}) + (\mu+\xi)\, P(y|\mathbf{a}) + \frac{1}{2}\sigma^2 y\, P'(y|\mathbf{a}) \\ &= \delta(y) - (r+\xi)\, e(y|\mathbf{a}) + (\mu+\xi) y\, e'(y|\mathbf{a}) + \frac{\sigma^2}{2} y^2\, e''(y|\mathbf{a}). \end{aligned}$$

□

A.2 No-issuance equilibrium

Here we show that $\sup_{y\in[y_{b,\mathrm{NI}},\infty]} \phi_{\mathrm{NI}}(y) = \phi_{\mathrm{NI}}(\infty)$, where $\phi_{\mathrm{NI}}(y)$ is defined in equation (29). By definition, $\phi_{\mathrm{NI}}(y)$ is the equity value at y upon restructuring to an optimal income-to-debt level $z^*(y)$, that is,

$$\begin{aligned} \phi_{\mathrm{NI}}(y) &\equiv \sup_{z\geq 0}\left\{\frac{y}{z}e_{\mathrm{NI}}(z) + \left(\frac{y}{z} - 1\right) P_{\mathrm{NI}}(z) - \beta y\right\} \\ &= y\left(\frac{e_{\mathrm{NI}}(z^*(y)) + P_{\mathrm{NI}}(z^*(y))}{z^*(y)} - \beta\right) - P_{\mathrm{NI}}(z^*(y)), \end{aligned} \tag{A.7}$$

No-arbitrage implies the value matching conditions:

$$P_{\text{NI}}(y) = P_{\text{NI}}(z^*(y)) \tag{A.8}$$

$$e_{\text{NI}}(y) = \frac{y}{z^*(y)} e_{\text{NI}}(z^*(y)) + \left(\frac{y}{z^*(y)} - 1\right) P_{\text{NI}}(z^*(y)) - \beta y \tag{A.9}$$

Optimality of $z^*(y)$ implies that the smooth pasting condition at y must hold, that is,

$$e'_{\text{NI}}(y) = \frac{e_{\text{NI}}(z^*(y)) + P_{\text{NI}}(z^*(y))}{z^*(y)} - \beta, \tag{A.10}$$

which is equivalent to the optimality condition in equation (17). To prove equation (A.10), suppose that the state vector is (F, Y) and the firm decides to issue debt. The value of equity is then

$$E_{\text{NI}}(F, Y) = \sup_{F^*} \{E_{\text{NI}}(F^*, Y) + (F^* - F) P_{\text{NI}}(F^*, Y) - \beta Y\} \equiv \sup_{F^*} \Phi_{\text{NI}}(F, Y, F^*)$$

Note that $F^* = F^*(F, Y)$, and hence $E_{\text{NI}}(F, Y) = \Phi_{\text{NI}}(F, Y, F^*(F, Y))$. By the envelope theorem we have

$$\frac{\partial E_{\text{NI}}(F, Y)}{\partial F} = \frac{\partial \Phi_{\text{NI}}(F, Y, F^*)}{\partial F} + \underbrace{\frac{\partial \Phi_{\text{NI}}(F, Y, F^*)}{\partial F^*} \frac{\partial F^*(F, Y)}{\partial F}}_{=0} = -P_{\text{NI}}(F^*, Y).$$

In the scaled space, $y = Y/F$, $z^*(y) = Y/F^*$, $P_{\text{NI}}(F, Y) = P_{\text{NI}}(y)$, and $E_{\text{NI}}(F, Y) = F e_{\text{NI}}(y)$, so we have

$$\frac{\partial E_{\text{NI}}(F, Y)}{\partial F} = e_{\text{NI}}(y) - y e'_{\text{NI}}(y) = -P_{\text{NI}}(z^*(y))$$

which can be rewritten as

$$e'_{\text{NI}}(y) = \frac{e_{\text{NI}}(y) + P_{\text{NI}}(z^*(y))}{y}. \tag{A.11}$$

Using the value-matching conditions (A.8)–(A.9) for debt and equity in equation (A.11) we obtain

$$e'_{\text{NI}}(y) = \frac{e_{\text{NI}}(z^*(y)) + P_{\text{NI}}(z^*(y))}{z^*(y)} - \beta,$$

which is equation (A.10). Using condition (A.10), we can write (A.7) as follows:

$$\phi_{\text{NI}}(y) = ye'_{\text{NI}}(y) - P_{\text{NI}}(y).$$

Since $e_{\text{NI}}(y)$ is an increasing and convex function and $P_{\text{NI}}(y) \le \frac{c+\xi}{r+\xi}$ we have that

$$\sup_{y\in[y_{b,\text{NI}},\infty]} \phi_{\text{NI}}(y) = \sup_{y\in[y_{b,\text{NI}},\infty]} \left\{ye'_{\text{NI}}(y) - P_{\text{NI}}(y)\right\} = \phi_{\text{NI}}(\infty).$$

□

Parameter	Symbol	Value
Annual risk-free rate	r	0.04
Annual coupon rate	c	0.04
Annual EBIT drift	μ	0
Annual EBIT volatility	σ	0.22
Corporate tax rate	τ	0.2
Inverse debt maturity	ξ	0.2
Fixed issuance cost	β	0.065

Table 1: Baseline model coefficients. The table shows the values of the model coefficients in the baseline calibration. An inverse debt maturity $\xi = 0.2$ corresponds to a five-year expected maturity. The fixed issuance cost $\beta = 0.065$ corresponds to a 1% of the debt amount issued.

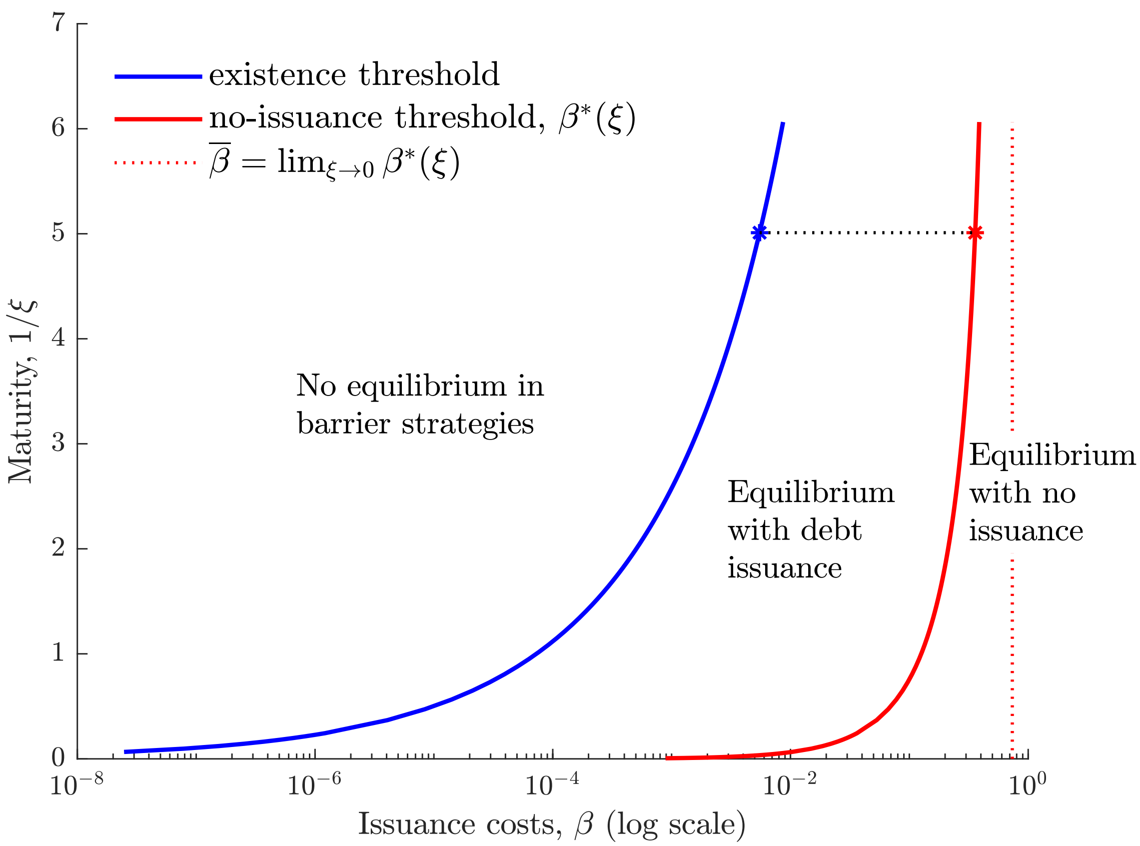

Figure 1: Existence of barrier-strategy MPEs. This figure identifies the regions in the space (β, ξ) of issuance costs and inverse-maturity parameters for which barrier-strategy MPEs exist. In the region to the left of the existence threshold (blue line) there is no MPE in barrier strategies. To the right of the no-issuance threshold (red line) it is optimal for the firm not to issue debt. The black dotted segment highlights the β values for which there is an MPE with debt issuance and debt maturity is five years, $\xi = 0.2$.

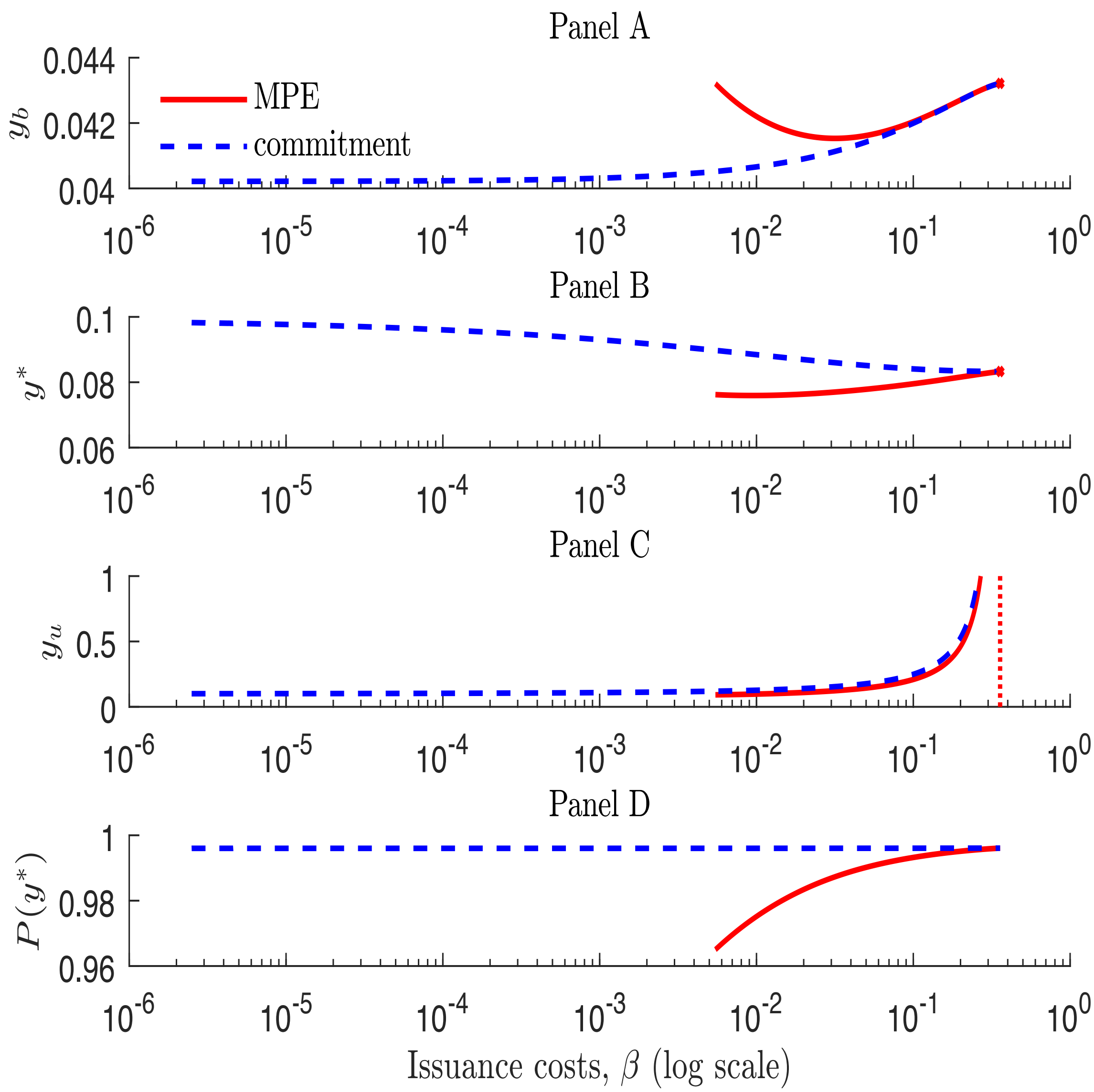

Figure 2: Debt issuance policies and values for a fixed maturity. This figure shows the default boundary y_b (Panel A); the post-issuance target y^* (Panel B); the restructuring boundary y_u (Panel C); and the value of debt $P(y^*)$ (Panel D) as a function of the issuance cost β. The red lines correspond to the no-commitment case (MPE) while the blue lines refer to the case of commitment. The two red markers in the top two panels denote the values of y_b and y^* in the no-issuance MPE. The vertical dotted line in the next panel refers to $\beta^*(\xi = 0.2)$, which is the smallest value of β for which the issuance boundary is $y_u = \infty$ when $\xi = 0.2$. The inverse maturity parameter is fixed at $\xi = 0.2$, corresponding to a debt maturity of 5 years.

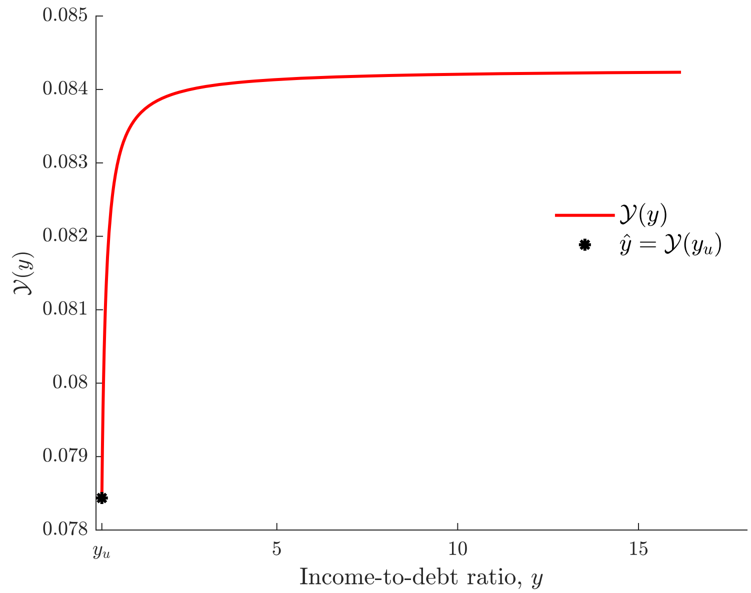

Figure 3: Restructuring function $\mathcal{Y}(y)$. This figure shows the restructuring function $\mathcal{Y}(y)$, $y \geq y_u$, defined in equation (18). As in the baseline calibration, we set $\beta = 0.065$ and $\xi = 0.2$, which reflect a fractional cost of 1% of the debt amount issued and a maturity of five years. The black dot denotes the restructuring point $(y_u, y^* = \mathcal{Y}(y_u))$.

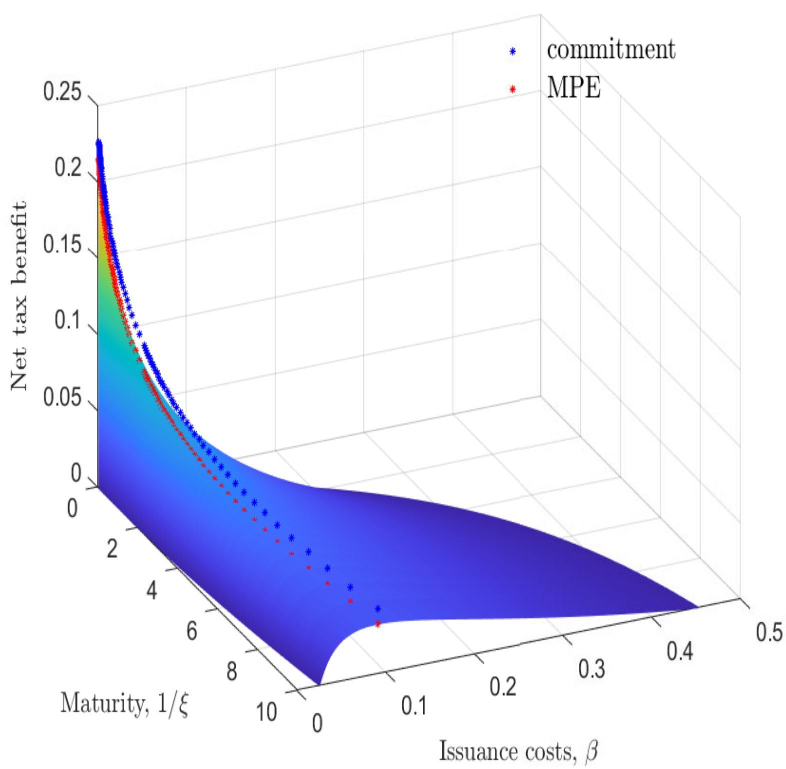

Figure 4: Net tax benefit. This figure shows the net tax benefit of debt for values of (β, ξ) that belong to the MPE existence region of Figure 1. The red markers correspond to the set (β, ξ) at which the tax benefit in the MPE is highest. For the same set of (β, ξ) points, the blue markers show the tax benefit under the benchmark policy with commitment.

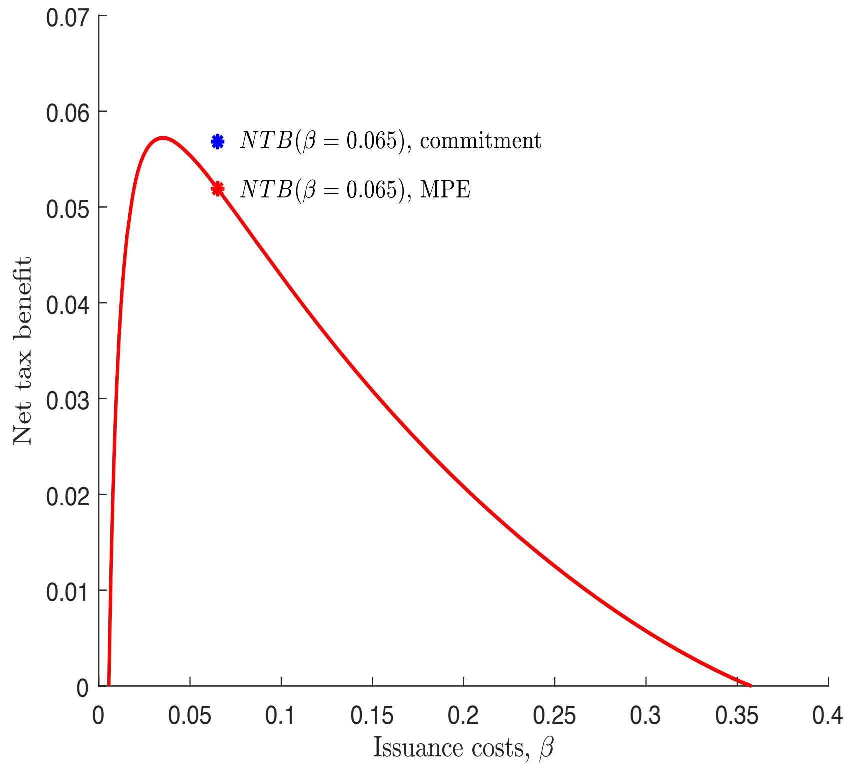

Figure 5: Net tax benefit for a five-year debt maturity. This figure shows the net tax benefit in the MPE as a function of the issuance cost β when debt maturity is fixed at five years ($\xi = 0.2$). The red marker denotes the tax benefit for $\beta = 0.065$, which reflects a fractional cost of 1% of the debt amount issued. For the same values of β and ξ, the blue marker denotes the tax benefit associated with the commitment benchmark policy.

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Debt Dynamics with Fixed Issuance Costs

Online Appendix

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I Formulation

I.1 The firm

The firm operates in continuous time and generates cash flows at rate $Y_t \geq 0$. The value of the unlevered firm is defined as

$$V_t = \mathbb{E}_t \int_t^\infty e^{-r(s-t)}(1-\tau)Y_s ds$$

where $r > 0$ denotes the risk-free rate, $\tau \in [0, 1)$ denotes the corporate tax rate, and the expectation is with respect to the risk-neutral probability measure $\mathbb{P}$. We assume that the cash flow process Y_t evolves according to

$$dY_t = Y_t \left(\sigma dW_t + \mu dt\right) \tag{I.1}$$

for some constants $\sigma > 0$ and $0 \leq \mu < r$ where W_t is a risk-neutral Brownian motion. As a result, the value of the unlevered firm is explicitly given by

$$V_t = \left(\frac{1-\tau}{r-\mu}\right) Y_t$$

and thus also evolves as a geometric Brownian motion. This shows that we may equivalently use V_t or Y_t as a state variable and we choose later so stay in line with the existing dynamic capital structure literature .

I.2 Debt contracts

The firm's debt takes the form of a continuum of ex-ante identical, exponentially maturing bonds that have equal seniority and pay coupons at rate $c > 0$. The mass of existing bonds at date $t \geq 0$ defines the face value F_t of the firm's debt and we assume that each bond matures independently of all others with intensity $\xi > 0$ so that debtholders receive payments at rate $(c + \xi)F_t$ as long as the firm operates. The recovery of bonds in default is assumed to be zero.

The firm can at adjust its capital structure all times by retiring or issuing bonds at market value but is subject to a fixed adjustment cost βY_t with $\beta > 0$. As a result, the face value of the firm's debt evolves according to

$$dF_t = -\xi F_{t-} dt + dI_t$$

where I_t is a process with initial value $I_{0-} = 0$ whose increments capture changes in the capital structure of the firm.

I.3 Strategies

As long as $\beta > 0$ any adjustment process I_t whose paths has intervals of continuity leads to an infinite accumulation of adjustment costs. Building on this observation we define a default and adjustment strategy as a pair

$$\mathbf{s} \equiv \{\tau_b(\mathbf{s}), I(\mathbf{s})\}$$

where $\tau_b(\mathbf{s})$ is a stopping time that represents the time of default and $I_t(\mathbf{s})$ is a discrete process of the form

$$I_t(\mathbf{s}) = \sum_{s \in \mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s \leq t\}} \Delta I_s(\mathbf{s}) = \sum_{s \in \mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s \leq t\}} A_s(\mathbf{s}) F_{s-}$$

where $\mathcal{A}(\mathbf{s})$ is a thin set whose elements represent the moments at which the firm restructures its capital and $A_t(\mathbf{s}) \geq -1$ is a predictable process that represents the relative size of the adjustment conditional on a restructuring at date $t \geq 0$.

We will for the most part focus on Markov equilibria in which the state summarized by the variables F_t and Y_t that determine the cash flows of all stakeholders. Accordingly, a strategy is said to be *Markovian* if

$$\mathcal{A}(\mathbf{s}) = \{t \geq 0 : (F_{t-}, Y_t) \in \mathcal{R}\},$$

$$\tau_b(\mathbf{s}) = \inf\{t \geq 0 : (F_t, Y_t) \in \mathcal{D}\},$$

and

$$A_t(\mathbf{s}) = \mathbf{1}_{\{(F_{t-},Y_t)\in\mathcal{R}\}} A\left(F_{t-}, Y_t\right).$$

for some closed disjoint subsets $\mathcal{D}, \mathcal{R}$ of $\mathbb{R}_+^2$ and some function $A = A(\cdot|\mathbf{s}) : \mathcal{R} \to [-1, \infty)$. If in addition

$$\begin{aligned}
\mathcal{D} &= \left\{(F, Y) \in \mathbb{R}_+^2 : Y/F \in \bar{\mathcal{D}}\right\}, \\
\mathcal{R} &= \left\{(F, Y) \in \mathbb{R}_+^2 : Y/F \in \bar{\mathcal{R}}\right\}, \\
A\left(F, Y\right) &= a\left(Y/F\right)
\end{aligned}$$

for some closed disjoint subsets $\bar{\mathcal{D}}, \bar{\mathcal{R}}$ of $\mathbb{R}_+$ and some function $a = a(\cdot|\mathbf{s}) : \bar{\mathcal{R}} \to [-1, \infty)$ then we say that $\mathbf{s}$ is *reduced* Markovian. Throughout we denote by $\mathcal{S}_0$ the set of all default and adjustment strategies, by $\mathcal{M} \subset \mathcal{S}_0$ the subset of Markovian strategies, and by $\mathcal{M}_r \subset \mathcal{M}$ the subset of reduced Markovian strategies.

I.4 Debt valuation

Denote by $P_t(\mathbf{s}) \geq 0$ the value of an individual bond issued by the firm *when creditors anticipate that management will use the strategy* $\mathbf{s} \in \mathcal{S}_0$. The absence of arbitrage opportunities requires that

$$P_t(\mathbf{s}) = \mathbb{E}_t\left[\int_t^{\tau_m \wedge \tau_b(\mathbf{s})} e^{-r(s-t)} c ds + \mathbf{1}_{\{\tau_m < \tau_b(\mathbf{s})\}} e^{-r(\tau_m - t)}\right]$$

on the set $\{\tau_m \wedge \tau_b(\mathbf{s}) > t\}$ where τ_m is an exponential random variable with mean $m = 1/\xi$ that represents the maturity of the individual bond under consideration. Integrating inside the expectation against the conditional distribution

$$\mathbb{P}_t\left[\tau_m \in ds \,|\tau_m > t\right] = \mathbf{1}_{\{s>t\}} e^{-\xi(s-t)} \xi ds$$

then shows that on the set $\{\tau_b(\mathbf{s}) > t\}$ the market price of an individual bond issued by the firm is given by

$$P_t(\mathbf{s}) = \mathbb{E}_t \left[\int_t^{\tau_b(\mathbf{s})} e^{-\rho(s-t)}(c+\xi)ds \right] \leq \frac{c+\xi}{\rho} \tag{I.2}$$

with the maturity-adjusted discount rate $\rho \equiv r + \xi$.

If $\mathbf{s}$ is Markovian then the right hand side of the above identity only depends on the path of the pair (F_t, Y_t). As a result, the bond price

$$P_t(\mathbf{s}) = P(F_t, Y_t|\mathbf{s})$$

is a *bounded* function of these variables and, since the adjustment times are predictable, we have that this function satisfies the no-jump condition

$$P(F, Y|\mathbf{s}) = P\left(F\left(1 + A(F, Y)\right), Y|\mathbf{s}\right), \qquad (F, Y) \in \mathcal{R}(\mathbf{s}), \tag{I.3}$$

which guarantees that the market price of the bond does not react to the occurence of an anticipated restructuring of the firm's capital. By the same token, if $\mathbf{s}$ is reduced Markovian then the right hand side of (I.2) only depends on the path of the invers leverage process y_t. In that case, the bond price $P_t(\mathbf{s}) = P(y_t|\mathbf{s})$ is a *bounded* function of y_t and the absence of arbitrage requires that this function satisfies

$$P(y|\mathbf{s}) = P\left(\frac{y}{1+a(y)} \middle| \mathbf{s} \right), \qquad y \in \bar{\mathcal{R}}(\mathbf{s}), \tag{I.4}$$

which again guarantees that the market price of the bond does not react to the occurence of an anticipated restructuring.

I.5 Equity valuation without commitment

Because default and capital adjustments are decided upon after debt has been issued management may have incentives to deviate from the policy conjectured by creditors. Absent commitment this implies that creditors will only accept to lend money to the firm if the debt contract is incentive compatible in the sense that management never wants to deviate from the strategy that creditors use to price the bonds at issuance.

If creditors conjecture that the firm will use $\mathbf{a} \in \mathcal{S}_0$ but management instead uses another strategy $\mathbf{s} \in \mathcal{S}_0$ then the value of equity is

$$E_t(\mathbf{s}, \mathbf{a}) \equiv \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{s})} e^{-r(s-t)} \left(\delta(F_s, Y_s)ds + P_s(\mathbf{a})dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s})\right)\right] \tag{I.5}$$

subject to the cash flow dynamics (I.1) and

$$dF_s = -\xi F_{s-}dt + dI_s(\mathbf{s}) \tag{I.6}$$

where

$$\delta(F_s, Y_s) \equiv (1-\tau)Y_s - (\xi + c(1-\tau))\, F_s$$

is the instantaneous cash flow to equity holders and

$$N_t(\mathbf{s}) \equiv \sum_{s \in \mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s \leq t\}}$$

is the counting process induced by the restructuring times of $\mathbf{s}$. To formalize the notion of an equilibrium let $\mathcal{S}$ denote the set of strategies such that

$$\Lambda(\mathbf{s}) \equiv \mathbb{E}\left[\int_0^{\tau_b(\mathbf{s})} e^{-rs}\left(F_s(\mathbf{s}) + Y_s\right) ds + \left(|\Delta F_s(\mathbf{s})| + Y_s\right) dN_s(\mathbf{s})\right] < \infty \tag{I.7}$$

We then have the following:

Definition I.1 *A subgame perfect equilibrium* (SPE) *is a strategy* $\mathbf{a} \in \mathcal{S}$ *such that*

$$E_t\left(\mathbf{a}, \mathbf{a}\right) = \sup_{\mathbf{s} \in \mathcal{S}} E_t\left(\mathbf{s}, \mathbf{a}\right), \qquad t \geq 0.$$

A Markov perfect equilibrium (MPE) *is a* SPE *in* $\mathcal{M}$ *while a reduced Markov Perfect Equilibrium* (rMPE) *is a* SPE *in* $\mathcal{M}_r$.

II Results

II.1 Characterization of SPEs

As a first step towards the construction of equilibria for our default and restructuring game we derive a version of the one shot deviation principle.

Lemma II.1 *A strategy* $\mathbf{a} \in \mathcal{S}$ *is a* SPE *if and only if*

$$E_t(\mathbf{a},\mathbf{a}) = \sup_{\mathbf{s}\in\mathcal{S}} \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{s})\wedge\tau_b(\mathbf{s})} e^{-r(s-t)}\delta(F_s,Y_s)ds \tag{II.1}$$

$$+\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}e^{-r(\theta_t(\mathbf{s})-t)}\left(E_{\theta_t(\mathbf{s})}(\mathbf{a},\mathbf{a})+P_{\theta_t(\mathbf{s})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s})-\beta Y_{\theta_t(\mathbf{s})}\right)\Bigg]$$

where the stopping time

$$\theta_t(\mathbf{s}) \equiv \inf\{s\geq t : s\in\mathcal{A}(\mathbf{s})\} = \inf\{s\geq t : dI_s(\mathbf{s})\neq 0\}$$

denotes the time of the first restructuring prescribed by the strategy $\mathbf{s}\in\mathcal{S}$ *on or after an arbitrary date* $t\geq 0$.

Proof. Assume that $\mathbf{a}\in\mathcal{S}$ is a SPE, let $\mathbf{s}\in\mathcal{S}$ and consider for each fixed starting point $t\geq 0$ the one-shot deviation $\mathbf{s}_t$ obtained by following $\mathbf{s}$ until $\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})$ and then reverting to $(\mathbf{a})$. Using the equilibrium property of $(\mathbf{a})$ together with the law of iterated expectations we deduce that

$$E_t(\mathbf{a},\mathbf{a}) = \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{a})\wedge\tau_b(\mathbf{a})} e^{-r(s-t)}\delta(F_s,Y_s)ds$$

$$+\mathbf{1}_{\{\theta_t(\mathbf{a})<\tau_b(\mathbf{a})\}}e^{-r(\theta_t(\mathbf{a})-t)}\left(E_{\theta_t(\mathbf{a})}(\mathbf{a},\mathbf{a})+P_{\theta_t(\mathbf{a})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{a})}(\mathbf{a})-\beta Y_{\theta_t(\mathbf{a})}\right)\Bigg]$$

$$\geq E_t(\mathbf{s}_t,\mathbf{a}) = \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{s})\wedge\tau_b(\mathbf{s})} e^{-r(s-t)}\delta(F_s,Y_s)ds$$

$$+\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}e^{-r(\theta_t(\mathbf{s})-t)}\left(E_{\theta_t(\mathbf{s})}(\mathbf{a},\mathbf{a})+P_{\theta_t(\mathbf{s})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s})-\beta Y_{\theta_t(\mathbf{s})}\right)\Bigg]$$

and the necessity of (II.1) follows from the arbitrariness of $\mathbf{s} \in \mathcal{S}$. To establish the converse assume that $\mathbf{a} \in \mathcal{S}$ satisfies (II.1). Since never restructuring and defaulting at the first time that the cash flow becomes negative is feasible we have that $E_t(\mathbf{a}, \mathbf{a}) \geq 0$ at all times. Using this property and iterating (II.1) forward we deduce that

$$E_t(\mathbf{a}, \mathbf{a}) \geq \mathbb{E}_t \left[\int_t^{\tau_b(\mathbf{s}) \wedge \theta_{n,t}(\mathbf{s})} e^{-r(s-t)} \left(\delta(F_s, Y_s) ds + P_s(\mathbf{a}) dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s}) \right) \right]$$

where $\theta_{n,t}(\mathbf{s})$ is the time of the nth restructuring after $t \geq 0$. let $Z_{n,t}$ denote the random variable inside the conditional expectation. Since the bond price is bounded and $\delta(F, Y)$ is a linear function we have that

$$\sup_{n \geq 1} |Z_{n,t}| \leq C_0(t) \int_0^{\tau_b(\mathbf{s})} e^{-rs} \left((F_s + Y_s)\, ds + (|\Delta I_s(\mathbf{s})| + Y_s) dN_s(\mathbf{s}) \right)$$

for some deterministic function $C_0(t) > 0$ and it follows from (I.7) that the right hand side has finite expectation. Letting the number of restructuring rounds $n \to \infty$ and appealing to the dominated convergence theorem then gives

$$E_t(\mathbf{a}, \mathbf{a}) \geq \mathbb{E}_t \left[\int_t^{\tau_b(\mathbf{s})} e^{-r(s-t)} \left(\delta(F_s, Y_s) ds + P_s(\mathbf{a}) dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s}) \right) \right]$$

and the desired result now follows from (I.5). □

Lemma II.1 provides a characterization of SPEs in terms of a stochastic control problem in which the controlled process is two dimensional. To further simplify the construction of equilibria we now provide an alternative characterization that only involves a one dimensional controlled process.

Corollary II.1 *A default and adjustment strategy* $\mathbf{a} \in \mathcal{S}$ *is a* SPE *if and only if the scaled equity* *value process* $e_t(\mathbf{a}) = E_t(\mathbf{a},\mathbf{a})/F_t$ *satisfies*

$$e_t(\mathbf{a}) = \sup_{\mathbf{s}\in\mathcal{S}} \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{s})\wedge\tau_b(\mathbf{s})} e^{-\rho(s-t)}\delta(y_s)ds \tag{II.2}$$

$$+ \mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}} e^{-\rho(\theta_t(\mathbf{s})-t)}\Big((1+A_{\theta_t(\mathbf{s})}(\mathbf{s}))e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-}\Big)\Bigg]$$

with the discount rate $\rho = r + \xi$ *and the cash flow function* $\delta(y) \equiv \delta(1, y)$.

Proof. The result follows from Lemma II.1 by noting that we have

$$F_s = e^{-\xi(s-t)}F_t, \qquad \text{for all } s \in [t, \theta_t(\mathbf{s})),$$

and therefore

$$\begin{aligned}
&E_{\theta_t(\mathbf{s})}(\mathbf{a},\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta Y_{\theta_t(\mathbf{s})} \\
&\quad = F_{\theta_t(\mathbf{s})-}\left(\frac{F_{\theta_t(\mathbf{s})}}{F_{\theta_t(\mathbf{s})-}}e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})\frac{\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s})}{F_{\theta_t(\mathbf{s})-}} - \beta y_{\theta_t(\mathbf{s})-}\right) \\
&\quad = F_{\theta_t(\mathbf{s})-}\Big(\big(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\big)\, e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-}\Big) \\
&\quad = e^{-\xi(\theta_t(\mathbf{s})-t)}F_t\Big(\big(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\big)\, e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-}\Big)
\end{aligned}$$

where the second equality follows from the definition of $A_t(\mathbf{s}) \geq -1$ as the relative size of the debt adjustment. □

Proposition II.1 (Leverage ratchet effect) *If* $\mathbf{a} \in \mathcal{S}$ *is a* MPE *then* $I_t(\mathbf{a})$ *is a non decreasing process.*

Proof. Assume that $\mathbf{a} \in \mathcal{M}$ is a MPE in which

$$\mathbb{P}\left[\{s \in \mathcal{A}(\mathbf{a}) : dI_s(\mathbf{a}) < 0\}\right] \neq 0.$$

To show that this leads to a contradiction consider the deviation $\hat{\mathbf{a}} \in \mathcal{S}_0$ defined by the default time

$\tau_b(\hat{\mathbf{a}}) \equiv \tau_b(\mathbf{a})$ and the face value process

$$F_t(\hat{\mathbf{a}}) \equiv \sup_{0\le u\le t} \left\{ e^{\xi(u-t)} F_u(\mathbf{a}) \right\}.$$

Standard results in the theory of Skorokhod reflection problems (see for example Kruk, Lehoczky, Ramanan, and Shreve (2007) and references therein) show that we have

$$0 \le \Delta I_t(\hat{\mathbf{a}}) = (F_t(\mathbf{a}) - F_{t-}(\hat{\mathbf{a}}))^+ \le \Delta I_t(\mathbf{a})^+ \tag{II.3a}$$

$$0 = (F_t(\mathbf{a}) - F_t(\hat{\mathbf{a}}))\, \Delta I_t(\hat{\mathbf{a}}). \tag{II.3b}$$

Using these properties we show in Lemma II.2 below that $\hat{\mathbf{a}} \in \mathcal{S}$ defines a feasible deviation and it remains to show that this deviation is profitable.

Denote by $P(F, Y) = P(F, Y|\mathbf{a})$ the bond price function induced by the assumed Markov perfect equilibrium and by $\bar{c} \equiv c(1-\tau) + \xi$ the after tax cost of debt per unit of face value. A direct calculation using (I.5) then shows

$$\begin{aligned}
E_0(\hat{\mathbf{a}}, \mathbf{a}) - E_0(\mathbf{a}, \mathbf{a}) = \mathbb{E}\Bigg[& \int_0^{\tau_b(\mathbf{a})} e^{-rs} \bar{c} \left(F_s(\mathbf{a}) - F_s(\hat{\mathbf{a}}) \right) ds \\
& + \int_0^{\tau_b(\mathbf{a})} e^{-rs} \left(P(F_s(\hat{\mathbf{a}}), Y_s) dI_s(\hat{\mathbf{a}}) - \beta Y_s dN_s(\hat{\mathbf{a}}) \right) \\
& - \int_0^{\tau_b(\mathbf{a})} e^{-rs} \left(P(F_s(\mathbf{a}), Y_s) dI_s(\mathbf{a}) - \beta Y_s dN_s(\mathbf{a}) \right) \Bigg] \\
\ge \mathbb{E}\Bigg[& \int_0^{\tau_b(\mathbf{a})} e^{-rs} \left(\bar{c} G_s ds - P\left(F_{s-}(\mathbf{a}), Y_s\right) \left(dG_s + \xi G_{s-} ds\right) \right) \Bigg] \\
= \mathbb{E}\Bigg[& \int_0^{\tau_b(\mathbf{a})} e^{-rs} (\bar{c} - \xi P\left(F_s(\mathbf{a}), Y_s\right)) G_s ds + \int_0^{\tau_b(\mathbf{a})} G_{s-} d\left(e^{-rs} P\left(F_s(\mathbf{a}), Y_s\right) \right) \Bigg]
\end{aligned} \tag{II.4}$$

where $G_t \equiv F_t(\mathbf{a}) - F_t(\hat{\mathbf{a}})$ is the difference between the face value processes associated with the two strategies, the inequality follows from (II.3b) and (I.3), and the last equality follows from the fact that

$$P(F_0, Y_0|\mathbf{a})G_0 = G_{\tau_b(\mathbf{a})}P\left(F_{\tau_b(\mathbf{a})}(\mathbf{a}), Y_{\tau_b(\mathbf{a})}\right) = 0.$$

Now, since the process

$$e^{-\rho t}P(F_t(\mathbf{a}), Y_t) + \int_0^t e^{-\rho s}(c+\xi)ds$$

is by construction a martingale on the stochastic interval $[\![0, \tau_b(\mathbf{a})]\!]$ we have that there exists a local martingale M_t such that

$$d(e^{-rt}P\left(F_t(\mathbf{a}), Y_t\right)) = e^{-rt}\left(\xi P\left(F_{t-}(\mathbf{a}), Y_t\right) - (c+\xi)\right)dt + e^{-rt}dM_t.$$

Substituting this evolution into (II.4) then gives

$$E_0\left(\hat{\mathbf{a}}, \mathbf{a}\right) - E_0\left(\mathbf{a}, \mathbf{a}\right) \geq \mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs}G_{s-}dM_s - \int_0^{\tau_b(\mathbf{a})} e^{-rs}c\tau G_s ds\right] \tag{II.5}$$

and the desired result now follows from Lemma II.3 below and the fact that G_t is non positive by construction. □

Lemma II.2 *The strategy* $\hat{\mathbf{a}} \in \mathcal{S}$.

Proof. Using (II.3) we deduce that the deviation $(\hat{\mathbf{a}})$ satisfies $dN_t(\hat{\mathbf{a}}) \leq dN_t(\mathbf{a})$ as well as $|\Delta F_t(\hat{\mathbf{a}})| \leq |\Delta F_t(\mathbf{a})|$ and it follows that

$$\mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(\left(|\Delta F_s(\hat{\mathbf{a}})| + Y_s\right)dN_s(\hat{\mathbf{a}}) - \left(|\Delta F_s(\mathbf{a})| + Y_s\right)dN_s(\mathbf{a})\right)\right] \leq 0. \tag{II.6}$$

On the other hand, Itô's formula implies that we have

$$F_s(\hat{\mathbf{a}}) - F_s(\mathbf{a}) = \int_0^s e^{\xi(u-s)}\left(dI_u(\hat{\mathbf{a}}) - dI_u(\mathbf{a})\right)$$

and therefore

$$\begin{aligned}\int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(F_s(\hat{\mathbf{a}})-F_s(\mathbf{a})\right)ds &= \int_0^{\tau_b(\mathbf{a})} ds\, e^{-\rho s}\left\{\int_0^s e^{\xi u}\left(dI_u(\hat{\mathbf{a}})-dI_u(\mathbf{a})\right)\right\}\\ &\leq \int_0^{\tau_b(\mathbf{a})} ds\, e^{-\rho s}\left\{\int_0^s e^{\xi u}|\Delta F_u(\mathbf{a})|dN_u(\mathbf{a})\right\}\\ &= \int_0^{\tau_b(\mathbf{a})} e^{-ru}dN_u(\mathbf{a})|\Delta F_u(\mathbf{a})|\left\{\int_u^{\tau_b(\mathbf{a})} e^{-\rho(s-u)}ds\right\}\\ &\leq \frac{1}{\rho}\int_0^{\tau_b(\mathbf{a})} e^{-ru}|\Delta F_u(\mathbf{a})|dN_u(\mathbf{a}). \end{aligned} \tag{II.7}$$

where the first inequality follows from (II.3b). Combining (II.6) and (II.7) then shows that $\Lambda(\hat{\mathbf{a}}) \leq C_0\Lambda(\mathbf{a})$ for some $C_0 > 0$ and the desired result follows. □

Lemma II.3 *The process*

$$U_t \equiv \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs}G_{s-}dM_s$$

that appears in (II.5) has expected value zero.

Proof. Denote by $P_t(\mathbf{a}) = P(F_t(\mathbf{a}), Y_t)$ the bond price along the path of the assumed equilibrium. Itô's formula implies that

$$U_t = e^{-r\theta}G_\theta P_\theta(\mathbf{a}) + \int_0^\theta e^{-rs}\left((c+\xi)G_s ds - P_s(\mathbf{a})\left(dI_s(\mathbf{a})-dI_s(\hat{\mathbf{a}})\right)\right)\Big|_{\theta\equiv\tau_b(\mathbf{a})\wedge t}$$

and it thus follows from the uniform boundedness of the bond price process, the non positivity of G_t and (II.3a) that we have

$$\begin{aligned}\frac{|U_t|}{C_1} &\leq e^{-rt\wedge\tau_b(\mathbf{a})}|G_{t\wedge\tau_b(\mathbf{a})}| + \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs}\left((c+\xi)|G_s|ds + |dI_s(\mathbf{a})-dI_s(\hat{\mathbf{a}})|\right)\\ &= \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs}\left((r-c)G_s ds + dI_s(\hat{\mathbf{a}}) - dI_s(\mathbf{a}) + |dI_s(\mathbf{a})-dI_s(\hat{\mathbf{a}})|\right)\\ &\leq \int_0^{\tau_b(\mathbf{a})} e^{-rs}\left((F_s(\hat{\mathbf{a}})+F_s(\mathbf{a}))\,|r-c|ds + |\Delta F_s(\mathbf{a})|dN_s(\mathbf{a})\right)\end{aligned}$$

for some constant $C_1 > 0$. This in turn implies that

$$\mathbb{E}\left\{\sup_{t\geq 0}|U_t|\right\} \leq C_2\left(\Lambda(\mathbf{a}) + \Lambda(\hat{\mathbf{a}})\right) < \infty$$

for some constant $C_2 > 0$ where the second inequality follows from the fact that ($\mathbf{a}$) and ($\hat{\mathbf{a}}$) are both feasible by Lemma II.2. This shows that the local martingale U_t is a uniformly integrable martingale and the desired result follows. □

II.2 Recursive optimal stopping representation

The following lemma shows that the search for Markov equilibria is equivalent to solving a recursive optimal stopping problem.

Lemma II.4 *A Markovian strategy* $\mathbf{a} \in \mathcal{M} \cap \mathcal{S}$ *is a* MPE *if and only if the induced equity value function satisfies*

$$E(F, Y|\mathbf{a}) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_{F,Y}\left[\int_0^\theta e^{-rt}\delta(\bar{F}_t, Y_t)dt + e^{-r\theta}R\left(\bar{F}_\theta, Y_\theta|\mathbf{a}\right)^+\right] \tag{II.8}$$

subject to (I.1) *and the uncontrolled dynamics*

$$d\bar{F}_t = -\xi\bar{F}_t dt$$

where the reward function is defined by

$$R(F, Y|\mathbf{a}) \equiv \sup_{G\in\mathbb{R}_+}\left\{E(G, Y|\mathbf{a}) + (G - F)\,P(G, Y|\mathbf{a}) - \beta Y\right\} \tag{II.9}$$

and $\mathcal{T}$ *denotes the set of stopping times.*

Proof of necessity. Assume that $\mathbf{a} \in \mathcal{M} \cap \mathcal{S}$ is a MPE and denote by

$$R(F, Y, G|\mathbf{a}) \equiv E(G, Y|\mathbf{a}) + (G - F)\,P(G, Y|\mathbf{a}) - \beta Y$$

the objective function on the right hand side of (II.9). Since $(\tau_b(\mathbf{a}), \theta_0(\mathbf{a}))$ are stopping times it follows from (II.9) and Lemma II.1 that

$$E(F,Y|\mathbf{a}) = \mathbb{E}_{F,Y}\Bigg[\int_0^{\tau_b(\mathbf{a})\wedge\theta_0(\mathbf{a})} e^{-rt}\delta(\bar{F}_t,Y_t)dt$$

$$+ e^{-r\theta_0(\mathbf{a})}\mathbf{1}_{\{\theta_0(\mathbf{a})<\tau_b(\mathbf{a})\}}R\left(\bar{F}_{\theta_0(\mathbf{a})},Y_{\theta_0(\mathbf{a})},\bar{F}_{\theta_0(\mathbf{a})}\left(1+A(\bar{F}_{\theta_0(\mathbf{a})},Y_{\theta_0(\mathbf{a})})\right)\big|\,\mathbf{a}\right)\Bigg]$$

$$\le \sup_{(\tau,\theta)\in\mathcal{T}^2}\mathbb{E}_{F,Y}\left[\int_0^{\tau\wedge\theta} e^{-rt}\delta(\bar{F}_t,Y_t)dt + \mathbf{1}_{\{\theta<\tau\}}e^{-r\theta}R\left(\bar{F}_\theta,Y_\theta|\mathbf{a}\right)\right]$$

$$\le \sup_{(\tau,\theta)\in\mathcal{T}^2}\mathbb{E}_{F,Y}\left[\int_0^{\tau\wedge\theta} e^{-rt}\delta(\bar{F}_t,Y_t)dt + \mathbf{1}_{\{\theta<\tau\}}e^{-r\theta}R\left(\bar{F}_\theta,Y_\theta|\mathbf{a}\right)^+\right]$$

$$\le \sup_{\zeta\in\mathcal{T}}\mathbb{E}_{F,Y}\left[\int_0^{\zeta} e^{-rt}\delta(\bar{F}_t,Y_t)dt + e^{-r\zeta}R\left(\bar{F}_\zeta,Y_\zeta|\mathbf{a}\right)^+\right]$$

To establish the reverse inequality let

$$R_n(F,Y|\mathbf{a}) \equiv \sup_{0\le G\le n} R(F,Y,G|\mathbf{a})$$

and consider the sequence $(\mathbf{s}_n)_{n=1}^\infty$ of one shot deviations defined by

$$\theta_0(\mathbf{s}_n) \equiv \sigma + \mathbf{1}_{\left\{R_n(\bar{F}_\sigma,Y_\sigma|\mathbf{a})\le 0\right\}}\infty,$$

$$\tau_b(\mathbf{s}_n) \equiv \mathbf{1}_{\left\{R_n(\bar{F}_\sigma,Y_\sigma|\mathbf{a})\le 0\right\}}\sigma + \mathbf{1}_{\left\{R_n(\bar{F}_\sigma,Y_\sigma|\mathbf{a})> 0\right\}}(\sigma + q_\sigma\circ\tau_b(\mathbf{a})),$$

and

$$\bar{F}_{\theta_0(\mathbf{s}_n)}\left(1+A_{\theta_0(\mathbf{s}_n)}(\mathbf{s}_n)\right) = \text{argmax}_{0\le G\le n}R\left(\bar{F}_{\theta_0(\mathbf{s}_n)},Y_{\theta_0(\mathbf{s}_n)},G|\mathbf{a}\right)$$

where σ is an arbitrary but fixed stopping time, and q_σ denotes the Markov shift operator. It is easily seen that $\mathbf{s}_n\in\mathcal{S}$ is a feasible deviation for each $n\ge 1$. Therefore, it follows from Lemma II.1

and the specification of $\mathbf{s}_n$ that we have

$$E(F, Y|\mathbf{a}) \geq \mathbb{E}_{F,Y}\Bigg[\int_0^{\tau_b(\mathbf{s}_n)\wedge\theta_0(\mathbf{s}_n)} e^{-rt}\delta(\bar{F}_t, Y_t)dt$$

$$+ e^{-r\theta_0(\mathbf{s}_n)}\mathbf{1}_{\{\theta_0(\mathbf{s}_n)<\tau_b(\mathbf{s}_n)\}}R_n(\bar{F}_{\theta_0(\mathbf{s}_n)}, Y_{\theta_0(\mathbf{s}_n)}|\mathbf{a})\Bigg]$$

$$= \mathbb{E}_{F,Y}\left[\int_0^{\sigma} e^{-rt}\delta(\bar{F}_t, Y_t)dt + e^{-r\sigma}R_n(\bar{F}_\sigma, Y_\sigma|\mathbf{a})^+\right].$$

Letting $n \to \infty$ on both sides and invoking the monotone convergence theorem to justify the interchange of limit and expectation then gives

$$E(F, Y|\mathbf{a}) \geq \mathbb{E}_{F,Y}\left[\int_0^{\sigma} e^{-rt}\delta(\bar{F}_t, Y_t)dt + e^{-r\sigma}R(\bar{F}_\sigma, Y_\sigma|\mathbf{a})^+\right]$$

and the result follows by taking the supremum over $\sigma \in \mathcal{T}$. □

Proof of sufficiency. Assume that $\mathbf{a} \in \mathcal{M} \cap \mathcal{S}$ satisfies (II.8) and let $\mathbf{s} \in \mathcal{S}$ be fixed. Because $\tau_b(\mathbf{s}) \wedge \theta_t(\mathbf{s})$ is a stopping time this implies that we have

$$E(F_t, Y_t|\mathbf{a}) \geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} e^{-r(s-t)}\delta(\bar{F}_s, Y_s)ds$$

$$+ e^{-r(\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})-t)}R\left(\bar{F}_{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})}, Y_{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})}\middle|\,\mathbf{a}\right)^+\Bigg]$$

$$\geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} e^{-r(s-t)}\delta(\bar{F}_s, Y_s)ds$$

$$+ e^{-r(\theta_t(\mathbf{s})-t)}\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}R\left(\bar{F}_{\theta_t(\mathbf{s})}, Y_{\theta_t(\mathbf{s})}\middle|\,\mathbf{a}\right)\Bigg]$$

$$\geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} e^{-r(s-t)}\delta(\bar{F}_s, Y_s)ds$$

$$+ e^{-r(\theta_t(\mathbf{s})-t)}\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}R\left(\bar{F}_{\theta_t(\mathbf{s})}, Y_{\theta_t(\mathbf{s})}, \bar{F}_{\theta_t(\mathbf{s})}\left(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\right)\middle|\,\mathbf{a}\right)\Bigg]$$

and the required result now follows from Lemma II.1, the arbitrariness of $\mathbf{s} \in \mathcal{S}$ and the definition of the function $R(F, Y, G|\mathbf{a})$. □

The next result specializes Lemma II.4 to the case of rMPEs and will serve as a basis for most of our results on barrier strategies.

Lemma II.5 *A strategy* $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ *is a* rMPE *if and only if the induced equity value function satisfies*

$$e(y|\mathbf{a}) = \sup_{\theta \in \mathcal{T}} \mathbb{E}_y \left[\int_0^\theta e^{-\rho t} \delta(\bar{y}_t) dt + e^{-\rho\theta} \phi\left(\bar{y}_\theta|\mathbf{a}\right)^+ \right] \tag{II.10}$$

$$= \hat{e}(y) + \sup_{\theta \in \mathcal{T}} \mathbb{E}_y \left[e^{-\rho\theta} \psi\left(\bar{y}_\theta|\mathbf{a}\right) \right] \tag{II.11}$$

subject to

$$d\bar{y}_t = \bar{y}_t \left(\sigma dW_t + (\xi + \mu)\, dt\right)$$

where

$$\hat{e}(y) \equiv \mathbb{E}_y \left[\int_0^\infty e^{-\rho t} \delta(\bar{y}_t) dt \right] = \frac{\delta(0)}{\rho} + \frac{\delta(y) - \delta(0)}{r - \mu}$$

denotes the equity value associated with never defaulting or restructuring, and the reward functions are defined by

$$\phi(y|\mathbf{a}) \equiv \sup_{z \geq 0} \Phi(y, z|\mathbf{a}) = \sup_{z \in \mathbb{R}_+} \left\{ \frac{y}{z} e(z|\mathbf{a}) + \left(\frac{y}{z} - 1 \right) P(z|\mathbf{a}) - \beta y \right\}$$

$$\psi(y|\mathbf{a}) \equiv \phi(y|\mathbf{a})^+ - \hat{e}(y).$$

In particular, if $(\mathbf{a})$ *is a* rMPE *then the induced scaled equity value is nonnegative, convex, and differentiable at all points where* $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+$.

Proof. Equation (II.10) follows from Lemma II.4 by noting that

$$R(\bar{F}_t, Y_t|\mathbf{a})/\bar{F}_t = \sup_{z \geq 0} \left\{ \frac{Y_t}{z\bar{F}_t} e\left(z|\mathbf{a}\right) + \left(\frac{Y_t}{z\bar{F}_t} - 1 \right) P\left(z|\mathbf{a}\right) \right\} - \frac{\beta Y_t}{\bar{F}_t}$$

$$= \sup_{z \geq 0} \left\{ \frac{\bar{y}_t}{z} e\left(z|\mathbf{a}\right) + \left(\frac{\bar{y}_t}{z} - 1 \right) P\left(z|\mathbf{a}\right) \right\} - \beta \bar{y}_t = \phi(\bar{y}_t|\mathbf{a})$$

and $\bar{F}_t = e^{-\xi t}F_0$. To see that (II.10) is equivalent to (II.11) it suffices to observe that the no-action equity value function satisfies the Dynkin identity

$$\hat{e}(y) - \mathbb{E}_y\left[e^{-\rho\zeta}\hat{e}(\bar{y}_\zeta)\right] = \mathbb{E}_y\left[\int_0^\zeta e^{-\rho t}\delta(\bar{y}_t)dt\right]$$

for all stopping times $\zeta \in \mathcal{T}$. Setting $\theta \equiv 0$ in (II.10) shows that the equity value function is nonnegative. On the other hand, we have that $\psi(y|\mathbf{a})$ is convex as the supremum of a family of affine functions and it thus follows from Alvarez (2003, Theorem 5) and Lamberton and Zervos (2013, Corollary 7.5) that $v(y) \equiv e(y|\mathbf{a}) - \hat{e}(y)$ is differentiable at all points of the set

$$\{y \geq 0 : v(y|\mathbf{a}) = \psi(y|\mathbf{a})\} = \{y \geq 0 : e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+\}.$$

Since the function $\hat{e}(y)$ is linear this in turn implies that $e(y|\mathbf{a})$ is also convex and differentiable at all points of this set and the proof is complete. □

Corollary II.2 *If $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ is a* rMPE *then the induced scaled equity value function is nondecreasing and there exists and constant $0 \leq y_b(\mathbf{a}) < \infty$ such that*

$$e(y|\mathbf{a}) = 0 = \phi(y|\mathbf{a})^+$$

at all points $y \leq y_b(\mathbf{a})$.

Proof. Assume that $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ is a rMPE and observe that since $e(y|\mathbf{a}) \geq \hat{e}(y)$ we have $e(y|\mathbf{a}) > 0$ for all sufficiently large y and thus $\bar{\mathcal{D}}(\mathbf{a}) \neq \mathbb{R}_+$. Let

$$y_b(\mathbf{a}) \equiv \sup\{y \geq 0 : y \in \bar{\mathcal{D}}(\mathbf{a})\}.$$

Since the scaled equity value function is nonnegative and not identically zero we have that $y_b(\mathbf{a}) < \infty$ and that $e'_+(z|\mathbf{a}) > 0$ at some point $z > y_b(\mathbf{a})$. Together with the convexity afforded by Lemma II.5 this implies that the scaled equity value is nondecreasing and it follows by continuity that $e(y|\mathbf{a}) = 0 \geq \phi(y|\mathbf{a})^+$ for all $y \leq y_b(\mathbf{a})$. □

Corollary II.3 *If* $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ *is a* rMPE *then*

$$e(y|\mathbf{a}) = \sup_{z \in \mathcal{C}(\mathbf{a})} \Phi\left(y, z|\mathbf{a}\right),$$

$$\left\{\mathcal{Y}(y) \equiv y/\left(1 + a(y|\mathbf{a})\right)\right\} = \text{argmax}_{z \in \mathcal{C}(\mathbf{a})} \Phi\left(y, z|\mathbf{a}\right)$$

for all $y \in \bar{\mathcal{R}}(\mathbf{a})$ *where* $\mathcal{C}(\mathbf{a}) \equiv \mathbb{R}_+ \backslash (\bar{\mathcal{D}} \cup \bar{\mathcal{R}})(\mathbf{a})$. *Furthermore, the scaled equity value is differentiable and satisfies*

$$e'(y|\mathbf{a}) = \frac{\partial \Phi}{\partial y}(y, \mathcal{Y}(y)|\mathbf{a})$$

at all points of the restructuring region $\bar{\mathcal{R}}(\mathbf{a})$.

Proof. If $y \in \bar{\mathcal{R}}(\mathbf{a})$ lies then it follows from (II.10) that

$$e(y|\mathbf{a}) \geq \phi(y|\mathbf{a})^+ = \sup_{z \geq 0} \Phi(y, z|\mathbf{a})^+$$

and from (II.2) that

$$e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \frac{y}{\mathcal{Y}(y)} e(\mathcal{Y}(y)|\mathbf{a}) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y$$

Combining the two shows that we have

$$\bar{\mathcal{R}}(\mathbf{a}) \subseteq \{y \geq 0 : e(y|\mathbf{a}) = \phi(y|\mathbf{a}) \geq 0\} \tag{II.12}$$

$$\mathcal{Y}(y) \in \mathcal{Z} = \text{argmax}_{z \geq 0} \Phi\left(y, z|\mathbf{a}\right)$$

and the first part will follow if we can show that the maximizer is unique and lies in $\mathcal{C}(\mathbf{a})$. Suppose to the contrary that $y \in \bar{\mathcal{R}}(\mathbf{a})$ is such that

$$\sup_{z \geq 0} \Phi(y, z|\mathbf{a}) = \Phi(y, z^*|\mathbf{a}).$$

for some $z^* \notin \mathcal{C}(\mathbf{a})$. If $z^* \in \bar{\mathcal{D}}(\mathbf{a})$ then it follows from (II.12) that we have

$$e(y|\mathbf{a}) = \Phi(y, z^*|\mathbf{a}) = -\beta y < 0$$

which contradicts the nonnegativity of the scaled equity value function. On the other hand, if $z^* \in \bar{\mathcal{R}}(\mathbf{a})$ then

$$\begin{aligned} e(y|\mathbf{a}) &= \Phi(y, z^*|\mathbf{a}) \\ &= \frac{y}{z^*} e(z^*|\mathbf{a}) + \left(\frac{y}{z^*} - 1\right) P(z^*|\mathbf{a}) - \beta y \\ &= \frac{y}{z^*} \Phi(z^*, \mathcal{Y}(z^*)|\mathbf{a}) + \left(\frac{y}{z^*} - 1\right) P(z^*|\mathbf{a}) - \beta y \\ &= \Phi(y, \mathcal{Y}(z^*)) + \left(\frac{y}{z^*} - 1\right) (P(z^*|\mathbf{a}) - P(\mathcal{Y}(z^*)|\mathbf{a})) - \beta y \\ &= \Phi(y, \mathcal{Y}(z^*)) - \beta y < \Phi(y, \mathcal{Y}(z^*)) \end{aligned}$$

where the third equality follows from (II.2), the fifth equality follows from the no jump condition (I.4) and the inequality follows from the strict positivity of the fixed cost. This contradicts the fact that $e(y|\mathbf{a}) = \phi(y|\mathbf{a})$ over $\bar{\mathcal{R}}(\mathbf{a})$ and thus establishes that $\mathcal{Z} \subseteq \mathcal{C}(\mathbf{a})$. To complete the proof observe that

$$e(y|\mathbf{a}) = \phi(y|\mathbf{a}) = \sup_{z \in \mathcal{C}(\mathbf{a})} \Phi(y, z|\mathbf{a})$$

is differentiable at all points of $\bar{\mathcal{R}}(\mathbf{a})$ as a result of (II.12) and Lemma II.5, and apply Milgrom and Segal (2002, Corollary 4.iii)). □

II.3 The HJB equation

If $v : \mathbb{R}_+ \to \mathbb{R}$ is a convex function then its one sided derivatives $v'_\pm(y)$ are nondecreasing functions of finite variation, and its second distributional derivative is a positive measure that we denote by $v''(dy)$. Consider now the measure

$$\mathcal{O}v(dy) = [(\xi + \mu) y v'_-(y) - \rho v(y)] dy + \frac{1}{2} \sigma^2 y^2 v''(dy). \tag{II.13}$$

Lamberton and Zervos (2013) show that the solution to (II.11) is intimately related to the set of functions that solve the HJB equation

$$\max\{\mathcal{O}v(y), \psi(y|\mathbf{a}) - v(y)\} = 0 \tag{II.14}$$

in the distributional sense. To make this result precise we start by formally defining the type of weak solutions we are interested in.

Definition II.1 *A function $v : (0,\infty) \to \mathbb{R}$ is a solution to (II.14) in the sense of distributions if it is convex and such that*

i) $v(y) \geq \psi(y|\mathbf{a})$ *for all* $y \geq 0$

ii) $\mathcal{O}v(dy)$ *is a non positive measure on* $\mathbb{R}_+$

iii) $\mathcal{O}v(dy)$ *does not charge the set* $\{y \geq 0 : v(y) > \psi(y|\mathbf{a})\}$

Proposition II.2 $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ *is a* rMPE *if and only if*

$$v(y) \equiv e(y|\mathbf{a}) - \hat{e}(y)$$

solves (II.14) in the sense of distributions subject to the boundary conditions

$$\limsup_{y\downarrow 0} y^{-\Pi} v(y) = \limsup_{y\downarrow 0} y^{-\Pi} \psi(y|\mathbf{a}) < \infty, \tag{II.15}$$

$$\limsup_{y\uparrow\infty} y^{-\Theta} v(y) = \limsup_{y\uparrow\infty} y^{-\Theta} \psi(y|\mathbf{a}) < \infty, \tag{II.16}$$

where $\Pi < 0$ and $\Theta > 1$ denote the two solutions to (II.22).

Proof. This follows from Lemma II.5 and Lamberton and Zervos (2013, Theorems 6.3|4) using the fact that in our case the state space is the positive real line with inaccessible boundaries and the reward function is convex and thus continuous. □

II.4 Barrier strategies

In view of Proposition II.1 and Corollary II.2 we can essentially restrict the search for rMPEs (but not the set of possible deviations) to the subset of default and adjustment strategies such that $dI_t(\mathbf{a}) \geq 0$ and

$$\tau_b(\mathbf{a}) = \inf\{t \geq 0 : y_t \leq y_b(\mathbf{a})\}$$

for some constant default threshold $y_b(\mathbf{a}) > 0$. A class of strategies of particular interest within that subset is the class of *barrier* strategies which is illustrated in Figure II.1 and formally defined as follows.

Definition II.2 *A strategy* $\mathbf{a} \in \mathcal{M}_r$ *is a barrier strategy if*

$$\mathcal{D}(\mathbf{a}) = \left\{(F, Y) \in \mathbb{R}_+^2 : Y \leq y_b F\right\}$$

$$\mathcal{R}(\mathbf{a}) = \left\{(F, Y) \in \mathbb{R}_+^2 : Y \geq y_u F\right\}$$

and

$$A(F, Y|\mathbf{a}) = \frac{y}{\mathcal{Y}(y)} - 1 \,\Bigg|_{y=\frac{Y}{F}}$$

for some $0 < y_b \leq y_u$ *and some function* $\mathcal{Y} : [y_u, \infty) \to (y_b, y_u)$ *that determines the target level of inverse leverage after the adjustment. In what follows we denote the set of barrier strategies by* $\mathcal{B}$.

Remark 1 The requirement that $\mathcal{Y}(y)$ takes values in $(y_b(\mathbf{a}), y_u(\mathbf{a}))$ rather than in $[0, y]$ is without loss of generality for equilibrium purposes since adjustments that move the state to a point inside $(\mathcal{D} \cup \mathcal{R})(\mathbf{a})$ are strictly suboptimal as long as the fixed cost of adjustment is strictly positive.

Assume that the firm follows a barrier strategy $\mathbf{a} \in \mathcal{B}$. Then (I.6) implies that the face value of debt satisfies

$$dF_t = -\xi F_t dt + \mathbf{1}_{\{y_{t-} \geq y_u(\mathbf{a})\}} \left(\frac{Y_t}{\mathcal{Y}(y_{t-})} - F_{t-}\right), \qquad \text{on } \{t < \tau_b(\mathbf{a})\}$$

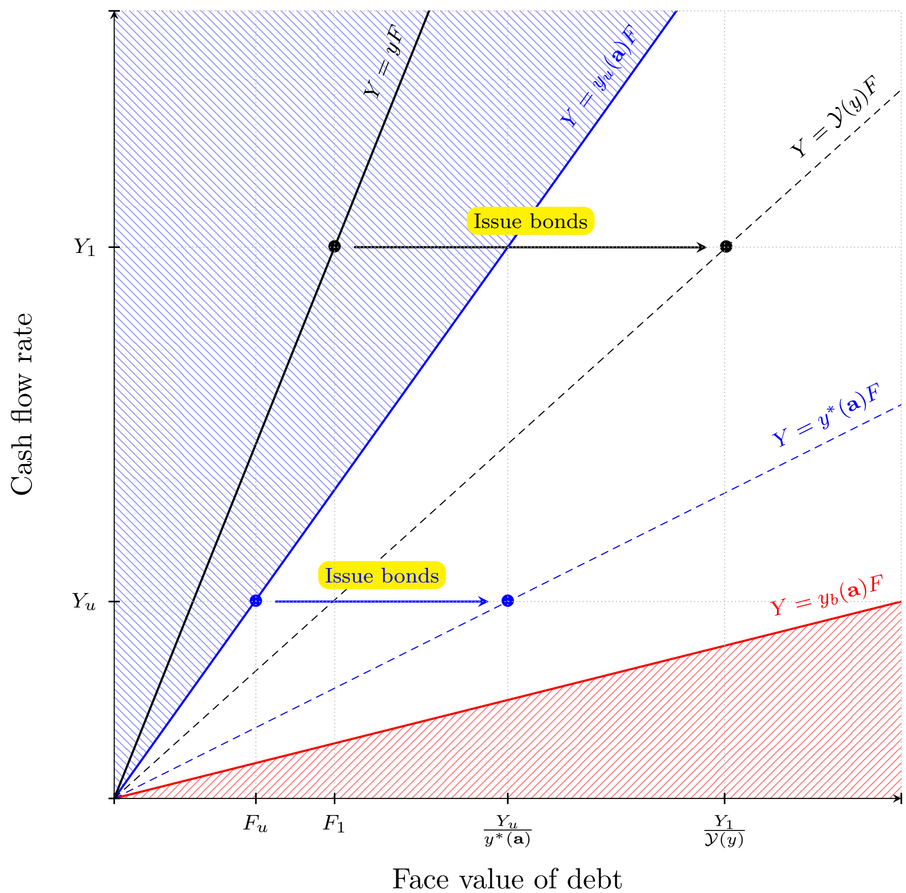

Figure II.1: Illustration of a barrier strategy. In the figure ▨ represents the default region $\mathcal{D}(\mathbf{a})$, ▧ represents the restructuring region $\mathcal{R}(\mathbf{a})$, the complement ▭ represents the continuation region, and the arrows indicate increases in the face value of debt that move the state from the restructuring region to the continuation region.

and it follows that the induced inverse leverage process is an autonomous Markov process that evolves according to

$$dy_t = y_{t-}\sigma dW_t + y_{t-}(\xi + \mu)\, dt + \mathbf{1}_{\{y_{t-} \geq y_u(\mathbf{a})\}} \left(\mathcal{Y}(y_{t-}) - y_{t-}\right) \tag{II.17}$$

until the first time that it reaches the barrier level $y_b(\mathbf{a})$ where the strategy requires shareholders to file for bankruptcy.

Before proceeding with the computation of the security values induced by a barrier strategy we first prove that all barrier strategies are feasible.

Lemma II.6 $\mathcal{B} \subseteq \mathcal{S}$.

Proof. Fix a barrier strategy $\mathbf{a} \in \mathcal{B}$. Since the pair (F_t, Y_t) forms a Markov process we have that $\Lambda(\mathbf{a}) = \Lambda(F_0, Y_0)$ for some (possibly infinite) function $\Lambda : \mathbb{R}_+^2 \to \mathbb{R} \cup \{\infty\}$ that satisfies the boundary conditions

$$\Lambda(F, Y) = 0, \qquad (F, Y) \in \mathcal{D}(\mathbf{a}), \tag{II.18}$$

$$\Lambda(F, Y) = Y\left(1 + \frac{1}{\mathcal{Y}(y)}\right) + \Lambda\left(\frac{Y}{\mathcal{Y}(y)}, Y\right), \qquad (F, Y) \in \mathcal{R}(\mathbf{a}). \tag{II.19}$$

On the other hand, a standard calculation using Girsanov's theorem and the law of iterated expectations shows that

$$\Lambda(F, Y) = \lambda(y) Y, \qquad (F, Y) \in \mathbb{R}_+ \backslash (\mathcal{D} \cup \mathcal{R})(\mathbf{a})$$

for some function $\lambda : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ that satisfies

$$\lambda(y) = G(y) + H(y)\left(1 + \frac{1}{\mathcal{Y}(y_u(\mathbf{a}))} - \frac{1}{y} + \lambda(\mathcal{Y}(y_u(\mathbf{a})))\right) \tag{II.20}$$

with $H, G : \mathbb{R}_+ \to \mathbb{R}_+$ uniformly bounded and such that

$$\min\{G(y), 1 - H(y)\} > 0, \quad y \in \mathcal{C}(\mathbf{a}) \equiv (y_b(\mathbf{a}), y_u(\mathbf{a})). \tag{II.21}$$

Combining (II.18), (II.19), and (II.20) we deduce that the strategy is feasible if and only if the constant

$$\lambda(\mathcal{Y}(y_u(\mathbf{a}))) = \frac{G(\mathcal{Y}(y_u(\mathbf{a}))) + H(\mathcal{Y}(y_u(\mathbf{a})))}{1 - H(\mathcal{Y}(y_u(\mathbf{a})))}$$

is finite and the desired result now follows from (II.21) since the point $\mathcal{Y}(y_u(\mathbf{a}))$ lies by assumption in the set $\mathcal{C}(\mathbf{a})$. □

Let now $\mathbf{a} \in \mathcal{B}$ be a barrier strategy and denote by

$$\mathcal{L}f(y) \equiv y\,(\xi + \mu)\, f'(y) + \frac{1}{2}\sigma^2 y^2 f''(y)$$

the differential operator associated with the continuous part of this stochastic differential equation.

Standard arguments relying on Itô's lemma and the continuity of the bond price at the issuance point show that the function

$$P(y_t|\mathbf{a}) = P_t(\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{a})} e^{-\rho(s-t)}\,(c+\xi)\,ds\right]$$

is the unique solution to

$$\begin{aligned} \rho P(y) &= \mathcal{L}P(y) + c + \xi, && y \in (y_b(\mathbf{a}), y_u(\mathbf{a})), \\ P(y) &= P(\mathcal{Y}(y)), && y \geq y_u(\mathbf{a}), \\ P(y) &= 0, && y \leq y_b(\mathbf{a}) \end{aligned}$$

in the space of functions that are bounded on $\mathbb{R}_+$ and twice continuously differentiable on the continuation region $\mathcal{C}(\mathbf{a}) \equiv (y_b(\mathbf{a}), y_u(\mathbf{a}))$.

To describe the solution to this differential problem denote by $\Theta > 1$ and $\Pi < 0$ the solutions to the quadratic equation

$$-\rho + (\xi + \mu)\,y + \frac{1}{2}\sigma^2 y(y-1) = 0 \tag{II.22}$$

induced by the continuous part of (II.17); and by $y^*(\mathbf{a}) \equiv \mathcal{Y}(y_u(\mathbf{a}))$ the level of inverse leverage to which the firm moves upon reaching from the inside the right boundary of the continuation region.

Lemma II.7 *Assume that* $\mathbf{a} \in \mathcal{B}$ *is a barrier strategy. Then*

$$P_t(\mathbf{a}) = P(y_t|\mathbf{a}) \equiv \mathbf{1}_{\{y_t \in \mathcal{C}(\mathbf{a})\}}\pi(y|\mathbf{a}) + \mathbf{1}_{\{y_t \geq y_u(\mathbf{a})\}}\pi\left(\mathcal{Y}(y_t)|\mathbf{a}\right)$$

with the function

$$\pi(y|\mathbf{a}) \equiv \frac{1}{\rho}\,(c+\xi)\left(1 + A_\pi(\mathbf{a})y^\Theta + B_\pi(\mathbf{a})y^\Pi\right),$$

and the constants

$$A_\pi(\mathbf{a}) \equiv \frac{y_u(\mathbf{a})^\Pi - y^*(\mathbf{a})^\Pi}{y_b(\mathbf{a})^\Pi\left(y_u(\mathbf{a})^\Theta - y^*(\mathbf{a})^\Theta\right) + y_b(\mathbf{a})^\Theta\left(y^*(\mathbf{a})^\Pi - y_u(\mathbf{a})^\Pi\right)},$$

$$B_\pi(\mathbf{a}) \equiv \frac{y^*(\mathbf{a})^\Theta - y_u(\mathbf{a})^\Theta}{y_b(\mathbf{a})^\Pi\left(y_u(\mathbf{a})^\Theta - y^*(\mathbf{a})^\Theta\right) + y_b(\mathbf{a})^\Theta\left(y^*(\mathbf{a})^\Pi - y_u(\mathbf{a})^\Pi\right)}.$$

In particular, the bond price function is strictly concave on $\mathcal{C}(\mathbf{a})$ *with* $P'(y_b(\mathbf{a})|\mathbf{a}) > 0$, $P'(y^*(\mathbf{a})|\mathbf{a}) \geq 0$, *and* $P'(y_u(\mathbf{a})|\mathbf{a}) \leq 0$.

Proof. The first part follows by direct calculation and the second by noting that since $\Theta > 1$, $\Pi < 0$, and $y_b(\mathbf{a}) \leq y^*(\mathbf{a}) \leq y_u(\mathbf{a})$ we have $A_\pi(\mathbf{a}), B_\pi(\mathbf{a}) \leq 0$. □

Remark 2 The derivatives of the bond price at the points $y^*(\mathbf{a})$ and $y_u(\mathbf{a})$ are either both non zero or both equal to zero depending on whether the length of the continuation region $|\mathcal{C}(\mathbf{a})|$ is strictly positive or zero. In the latter case, the bond price function coincides with the solution that obtains when imposing a reflecting boundary condition at the upper threshold.

Consider now the scaled equity value that prevails when creditors correctly anticipate that management will use the barrier strategy $\mathbf{a}$:

$$e_t(\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{a})} e^{-\rho(s-t)}\left(\delta(y_s)ds + \left(A_s(\mathbf{a})\left(e_s(\mathbf{a}) + P_s(\mathbf{a})\right) - \beta y_{s-}\right)dN_s(\mathbf{a})\right)\right].$$

When $\mathbf{a}$ is a barrier strategy (or more generally a reduced Markov strategy) all the terms in the conditional expectation only depend on the path of the Markov process described by (II.17). As a result, $e_t(\mathbf{a}) = e(y_t|\mathbf{a})$ for some deterministic function and standard results show that this function is the unique solution to

$$\rho e(y) = \mathcal{L}e(y) + \delta(y), \qquad y \in \mathcal{C}(\mathbf{a}), \tag{II.23}$$

$$e(y) = 0, \qquad y \leq y_b(\mathbf{a}),$$

$$e(y) = \frac{y}{\mathcal{Y}(y)}e(\mathcal{Y}(y)) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right)P(\mathcal{Y}(y)|\mathbf{a}) - \beta y, \qquad y \geq y_u(\mathbf{a}), \tag{II.24}$$

in the space of functions that are finite on $\mathbb{R}_+$ and twice continuously differentiable on the continuation region $\mathcal{C}(\mathbf{a})$.

Lemma II.8 *Assume that* $\mathbf{a} \in \mathcal{B}$ *is a barrier strategy. Then*

$$e_t(\mathbf{a}) = e(y_t|\mathbf{a}) \equiv \mathbf{1}_{\{y_t \in \mathcal{C}(\mathbf{a})\}} \varepsilon(y_t|\mathbf{a}) + \mathbf{1}_{\{y_t \geq y_u(\mathbf{a})\}} \overline{\varepsilon}(y_t|\mathbf{a})$$

where

$$\varepsilon(y|\mathbf{a}) \equiv \hat{e}(y) + A_\varepsilon(\mathbf{a}) y^\Theta + B_\varepsilon(\mathbf{a}) y^\Pi$$

$$\overline{\varepsilon}(y|\mathbf{a}) \equiv \frac{y}{\mathcal{Y}(y)} \varepsilon(\mathcal{Y}(y)|\mathbf{a}) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y$$

and $(A_\varepsilon(\mathbf{a}), B_\varepsilon(\mathbf{a}))$ *are the unique solutions to the value matching conditions*

$$\varepsilon(y_b(\mathbf{a})|\mathbf{a}) = 0,$$

$$\varepsilon(y_u(\mathbf{a})|\mathbf{a}) = \overline{\varepsilon}(y_u(\mathbf{a})|\mathbf{a}),$$

at the endpoints of the continuation region.

Proof. Follows by direct calculation. □

II.5 rMPEs in barrier strategies

We start with a result that specializes the differential characterization of Proposition II.2 to the case of barrier strategies.

Proposition II.3 *A barrier strategy is a* rMPE *if and only if the induced equity value function* $e(y|\mathbf{a})$ *is a solution to* (II.14) *in the sense of distributions.*

Proof. Combining Corollary II.3, Lemma II.7, and Lemma II.8 we deduce that there exists a constant $k > 0$ such that

$$|e(y|\mathbf{a})| \vee |\phi(y|\mathbf{a})| \leq k\left(1 + |y|\right), \qquad y \geq 0.$$

Since $\Pi < 0$ and $\Theta > 1$ this implies that we have

$$\lim_{y \downarrow 0} y^{-\Pi} f(y) = \lim_{y \uparrow \infty} y^{-\Theta} f(y) = 0, \qquad \text{for } f \in \{e(\cdot|\mathbf{a}), \phi(\cdot|\mathbf{a})^+\}.$$

This shows that the boundary conditions (II.15) and (II.16) hold for any barrier strategy and the desired result now follows from Proposition II.2. □

The next result provides a set of *necessary* conditions for a barrier strategy to form an equilibrium. To state the result let

$$s(y|\mathbf{a}) \equiv \frac{1}{y}\left(e(y|\mathbf{a}) + P(y|\mathbf{a})\right)$$

denote the value of the firm per unit of cash flow.

Lemma II.9 *Assume that the barrier strategy* $\mathbf{a} \in \mathcal{B}$ *is a* rMPE. *Then the following conditions are satisfied:*

i) $y_b(\mathbf{a}) \leq y_{b,NI}$, *where* $y_{b,NI} = \frac{\Pi}{\Pi - 1}\frac{r-\mu}{\rho}\left(c + \frac{\xi}{1-\tau}\right)$,

ii) $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ = 0$ *on* $(0, y_b(\mathbf{a})]$,

iii) $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ > 0$ *on* $[y_u(\mathbf{a}), \infty)$,

iv) Smooth pasting and value matching at the default boundary:

$$e'(y_b(\mathbf{a})|\mathbf{a}) = e(y_b(\mathbf{a})|\mathbf{a}) = 0. \tag{II.25}$$

v) Smooth pasting and value matching at restructuring points:

$$e'(y|\mathbf{a}) = s(\mathcal{Y}(y)|\mathbf{a}) - \beta = s(y|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}). \tag{II.26}$$

vi) Optimality of restructuring:

$$\{\mathcal{Y}(y)\} = \operatorname{argmax}_{z \geq 0} \Phi(y, z|\mathbf{a}) = \operatorname{argmax}_{z \in \mathcal{C}(\mathbf{a})} \Phi(y, z|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}).$$

Proof of i). This follows by observing that if $\mathbf{a}$ is an rMPE with $y_b(\mathbf{a}) > y_{b,\text{NI}}$ then $e(y|\mathbf{a}) = 0 < e_{\text{NI}}(y)$ with $e_{\text{NI}}(y)$ the equity-value in a no-issuance rMPE, defined in equation (28) of the main text, for all $y \in (y_{b,\text{NI}}, y_b(\mathbf{a}))$. This contradicts (II.2). □

Proof of ii). If $\mathbf{a} \in \mathcal{B}$ is a rMPE then it follows from Lemma II.5 and the definition of the strategy that we have $e(y|\mathbf{a}) = 0 \geq \phi(y|\mathbf{a})^+$ for all $y \leq y_b(\mathbf{a})$. □

Proof of iv). Since by definition $e(y|\mathbf{a}) = 0$ for all $y \leq y_b(\mathbf{a})$ it follows from Lemma II.5 that $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ = 0$ over that region. This in turn implies that the scaled equity value function is differentiable at all points $y \leq y_b(\mathbf{a})$ and the desired result follows by noting that $e'_-(y|\mathbf{a}) = 0$ at any such point. □

Proof of iii). Since by definition $e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) \leq \phi(y|\mathbf{a})$ for $y \geq y_u(\mathbf{a})$ it follows from Lemma II.5 that we have

$$0 \leq e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \phi(y|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}).$$

To see that the inequality is strict note that due to Item iv) the scaled equity value function solves (II.23) subject to (II.25). In particular,

$$\lim_{y \downarrow y_b(\mathbf{a})} \frac{1}{2}\sigma^2 y^2 e''(y|\mathbf{a}) = -\delta(y_b(\mathbf{a})) > 0$$

where the strict inequality follows from Item i) and the definition of the no-issuance default threshold. The above inequality implies that we have $e(y|\mathbf{a}) > 0$ in a right neighborhood of $y_b(\mathbf{a})$ and thus for all $y > y_b(\mathbf{a})$ by convexity. □

Proof of vi). This follows directly from Corollary II.3. □

Proof of v). Since $e(y|\mathbf{a}) = \phi(y|\mathbf{a}) > 0$ for all $y \geq y_u(\mathbf{a})$ by Item iii) it follows from Lemma II.5 that the scaled equity value function, and thus also $\phi(y|\mathbf{a})$, is differentiable at all points $y \geq y_u(\mathbf{a})$. On the other hand, by Item vi) we have that $\mathcal{Y}(y)$ is the unique maximizer of the function $z \mapsto \Phi(y, z|\mathbf{a})$ over the compact set $\mathcal{C}(a)$ and the validity of (II.26) now follows from Milgrom and Segal (2002, Corollary 4) and (II.24). □

Lemma II.10 *Assume that the barrier strategy* $\mathbf{a} \in \mathcal{B}$ *satisfies Conditions i) and iv) of Lemma II.9. Then*

i) $e(y|\mathbf{a})$ *is nonnegative, nondecreasing, convex on the interval* $[0, y_u(\mathbf{a}))$ *and strictly positive on the interval* $(y_b(\mathbf{a}), y_u(\mathbf{a}))$

ii) $e(y|\mathbf{a}) \geq e_{NI}(y)$ *for all* $y \leq y_u(\mathbf{a})$ *if and only if* $y_b(\mathbf{a}) \leq y_{b,NI}$.

Proof of i). Since $e(y|\mathbf{a})$ solves (II.23) subject to value matching and smooth pasting at the default boundary the uniqueness of the solution to second order differential equations implies that the constants in Lemma II.8 can be expressed as

$$A_\varepsilon(\mathbf{a}) = \frac{y_b(\mathbf{a})^{-\Theta}(1-\tau)\left(y_b(\mathbf{a}) - y_{b,\mathrm{NI}}\right)(\Pi - 1)}{(r-\mu)(\Theta - \Pi)} \geq 0,$$

$$B_\varepsilon(\mathbf{a}) = \frac{y_b(\mathbf{a})^{-\Pi}(1-\tau)(y_b(\mathbf{a})(\Theta-1)\Pi + y_{b,\mathrm{NI}}\Theta(1-\Pi))}{(r-\mu)\Pi(\Pi - \Theta)} \geq 0.$$

Therefore, $e(y|\mathbf{a})$ is convex on the interval $[0, y_u(\mathbf{a}))$ and remaining claims in the statement follow by observing that because

$$\lim_{y \downarrow y_b(\mathbf{a})} \frac{1}{2}\sigma^2 y^2 e''(y) = -\delta(y_b(\mathbf{a})) > 0$$

we must have $\min\{e, e'\}(y|\mathbf{a}) > 0$ in a right neighbourhood of $y_b(\mathbf{a})$ and thus over the whole interval since the scaled equity value is convex. □

Proof of ii). The necessity of the condition is clear since in its absence $e(y|\mathbf{a}) = 0 < e_{\mathrm{NI}}(y)$ for all $y \leq (y_{b,\mathrm{NI}}, y_b(\mathbf{a}))$. Now assume that $y_b(\mathbf{a}) \leq y_{b,\mathrm{NI}}$. If $y_u(\mathbf{a}) \leq y_{b,\mathrm{NI}}$ then the result follows from Item i)) since $e_{\mathrm{NI}}(y) = 0$ on $[0, y_{b,\mathrm{NI}}]$. Assume from now on that $y_u(\mathbf{a}) > y_{b,\mathrm{NI}}$. Proceeding as in the first part of the proof shows that

$$w(y) = e(y|\mathbf{a}) - e_{\mathrm{NI}}(y) = A_\varepsilon(\mathbf{a})y^\Theta + \mathbf{1}_{\{y > y_{b,\mathrm{NI}}\}}\bar{B}(\mathbf{a})y^\Pi, \qquad y \in [y_{b,\mathrm{NI}}, y_u(\mathbf{a}))$$

where $A_\varepsilon(\mathbf{a}) \geq 0$ and

$$\bar{B}(\mathbf{a}) \equiv \frac{y_{b,\mathrm{NI}}^{1-\Pi}(1-\tau)}{(r-\mu)\Pi} + \frac{y_b(\mathbf{a})^{-\Pi}(1-\tau)(y_b(\mathbf{a})(\Theta-1)\Pi + y_{b,\mathrm{NI}}\Theta(1-\Pi))}{(r-\mu)(\Pi-\Theta)\Pi}$$

Noting that $0 = \bar{B}(\mathbf{a})\big|_{y_{b(\mathbf{a})=y_{b,\mathrm{NI}}}}$ and

$$\frac{d\bar{B}(\mathbf{a})}{dy_b(\mathbf{a})} = \frac{(1-\tau)(\Pi-1)(y_b(\mathbf{a})(\Theta-1) - y_{b,\mathrm{NI}}\Theta)}{y_b(\mathbf{a})^{1+\Pi}(r-\mu)(\Theta-\Pi)} \geq 0, \qquad y_b(\mathbf{a}) \leq y_{b,\mathrm{NI}}$$

we deduce that $\bar{B}(\mathbf{a}) \leq 0$. This implies that $w(y)$ is non decreasing on $[y_{b,\mathrm{NI}}, y_u(\mathbf{a})]$ and the thesis follows by observing that $w(y_{b,\mathrm{NI}}) = e(y_{b,\mathrm{NI}}|\mathbf{a}) \geq 0$. □

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