# Coverting 2-year Transition Probabilities to 1-year Probabilities

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Notation

$$\begin{aligned} h_t &\in \{bad, good\} \\ gg &\equiv \Pr\left(h_{t+2} = good \middle| h_t = good\right), \\ bb &\equiv \Pr\left(h_{t+2} = bad \middle| h_t = bad\right), \\ g &\equiv \Pr\left(h_{t+1} = good \middle| h_t = good\right), \\ b &\equiv \Pr\left(h_{t+1} = bad \middle| h_t = bad\right). \end{aligned}$$

Derivation:

$$\begin{aligned} 1 - bb &= \Pr\left(h_{t+2} = good \middle| h_t = bad\right) \\ &= (1-b)\, g + b\,(1-b) \\ \Rightarrow g &= \frac{1 - bb - b\,(1-b)}{1-b}. \end{aligned} \tag{1}$$

Plug equation (1) into

$$gg = g^2 + (1-g)(1-b),$$

to get

$$\begin{aligned} gg &= \left[\frac{1 - bb - b\,(1-b)}{1-b}\right]^2 + \left[1 - \frac{1 - bb - b\,(1-b)}{1-b}\right](1-b) \\ &= \frac{[1 - bb - b\,(1-b)]^2}{[1-b]^2} + [1 - b - (1 - bb - b\,(1-b))] \\ &= \frac{(1-bb)^2 - 2\,(1-bb)\,b\,(1-b) + b^2\,(1-b)^2}{(1-b)^2} + \left(bb - b^2\right) \\ &= \frac{1 - 2bb + bb^2 - 2b + 2b^2 + 2bb \cdot b - 2bb \cdot b^2 + b^2 - 2b^3 + b^4}{(1-b)^2} \\ &\quad + \frac{bb - 2bb \cdot b + bb \cdot b^2 - b^2 + 2b^3 - b^4}{(1-b)^2} \\ &= \frac{1 - bb + bb^2 - 2b + 2b^2 - bb \cdot b^2}{(1-b)^2}. \end{aligned}$$

Or

$$\begin{aligned} 1-bb+bb^2-2b+2b^2-bb\cdot b^2 &= gg\left(1-b\right)^2 \\ &= gg-2b\cdot gg+gg\cdot b^2, \end{aligned}$$

which is a quadratic in $b$:

$$\left[1-bb-gg+bb^2\right]+\left[2gg-2\right]b+\left[2-gg-bb\right]b^2=0,$$

or

$$\begin{aligned} Ab^2+Bb+C &= 0, \\ A &\equiv 2-gg-bb>0, \\ B &\equiv 2\left(gg-1\right)<0 \\ C &\equiv 1-bb-gg+bb^2. \end{aligned} \tag{2}$$

It follows that

$$b=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

The bigger (and more sensible) root is

$$b=\frac{-B+\sqrt{B^2-4AC}}{2A}. \tag{3}$$

Plugging this number into equation (1) yields $g$.

A second way to derive equation (2) to plug equation (1) into

$$\begin{aligned} 1-gg &= \Pr\left(h_{t+2}=bad|\, h_t=good\right) \\ &= g\left(1-g\right)+\left(1-g\right)b \\ &= \left(g+b\right)\left(1-g\right), \end{aligned}$$

yielding

$$\begin{aligned} 1-gg &= \left[\frac{1-bb-b\left(1-b\right)}{1-b}+b\right]\left[1-\frac{1-bb-b\left(1-b\right)}{1-b}\right] \\ &= \left[\frac{1-bb-b\left(1-b\right)+b\left(1-b\right)}{1-b}\right]\left[\frac{1-b-1+bb+b-b^2}{1-b}\right] \\ &= \frac{\left(1-bb\right)\left(bb-b^2\right)}{\left(1-b\right)^2}, \end{aligned}$$

or

$$\left(1-gg\right)\left(1-2b+b^2\right) = bb\left(1-bb\right) - b^2\left(1-bb\right),$$

or

$$1-gg-bb+bb^2-2\left(1-gg\right)b+\left(1-gg+1-bb\right)b^2 = 0,$$

and we can proceed as before.

By way of example, consider

$$\begin{aligned} gg &= 0.7, \\ bb &= 0.86. \end{aligned}$$

Plugging this into equations (2) and (3), we get:

$$\begin{aligned} A &\equiv 2-0.7-0.86 = 0.44 \\ B &\equiv 2\left(0.7-1\right) = -0.6 \\ C &\equiv 1-0.86-0.7+0.86^2 = 0.1796, \\ b &= \frac{0.6+\sqrt{0.36-4\cdot 0.44\cdot 0.1796}}{0.88} \\ &= \frac{0.6+\sqrt{0.043904}}{0.88} = 0.91992, \\ g &= \frac{1-0.86-0.91992\cdot 0.08008}{0.08008} = 0.82833. \end{aligned}$$

We can check the solution:

$$\begin{aligned} gg &= g^2+\left(1-g\right)\left(1-b\right) \\ &= 0.82833^2+0.17167\cdot 0.08008 \\ &= 0.69988 \approx 0.7, \\ bb &= b^2+\left(1-b\right)\left(1-g\right) \\ &= 0.91992^2+0.17167\cdot 0.08008 \\ &= 0.86000. \end{aligned}$$