Correlation and Asset Correlation in the Structural Portfolio Model

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Abstract

To forecast the default distribution of a credit portfolio, a risk manager often relies on a structural model that contains a measure of correlation. Quite frequently, the manager sets correlation in the model equal to asset correlation. This common practice, it is argued here, may be in error.

Correlation in the portfolio model would be identical to asset correlation under a set of assumptions. This study examines each assumption in detail. Relaxing each creates a potential for difference. The potential is realized in studies that estimate correlation based on default data; those estimates tend to be less than asset correlation. The estimation presented here rejects uniform values of correlation greater than 10.8%. It is shown by example that the difference of correlation values appears to be great enough to produce misleading statements of risk.

Keywords: Asset correlation, portfolio credit risk model, structural credit model, default model, Merton model.

JEL classifications: G32, G21, G28

Introduction

This study calls attention to the common practice that correlation in the structural credit portfolio model is treated as identical to asset correlation. Though often treated as the same, the two concepts are different: correlation in the model ties together events of default, while asset correlation is defined without reference to default. This difference in concept makes possible a meaningful difference in values.

The risk manager might have an inkling that something is not quite right. To calibrate probability of default (***PD***), he optimizes using historical default rates. That is, for ***PD*** estimates, default data comes into play. In the portfolio model, ***PD***s and correlations produce forecasts of joint default rates. These forecasts, also, could be compared to historical data, but the naïve risk manager relies on asset correlation without checking the result. Some risk managers are more careful than this, but not all.

The identification of correlation as asset correlation traces back to the assumptions made by the model. These assumptions are examined in the next three sections, which relax each one in turn. If the assumptions of the model do not hold, correlation and asset correlation differ in concept.

Later sections turn from the difference in concept to a difference of value. A review of the literature shows that estimates of correlation based on default data tend to be less than asset correlation. Based on default data, the present study rejects with customary confidence a value of uniform correlation greater than 10.8%. Thus, the statistical estimate of correlation based on default data is significantly less than values of asset correlation that have appeared in credit portfolio models.

If a credit model employs overstated correlation, it overstates risk. This would be the case for any exposure, but the next-to-last section shows that some exposures are more affected than others. Therefore, if a manager uses the wrong value of correlation, he can be led to engage in the wrong transactions.

The basic structural portfolio credit model

The basic model is based on the insights that Robert Merton put forth over thirty years ago. Since then, numerous advances have been proposed and some have been adopted for particular purposes. For risk control of a large portfolio, however, the basic model is still the most common.

The purpose of the model is to forecast the distribution of loss. Since our interest is the distribution of the default rate, we safely ignore exposure amount, loss given default, maturity, and "marked to market" credit losses not stemming from default.

Everything essential for now can be seen in a portfolio having exposures to only two firms. We assume that the two probabilities of default, $\boldsymbol{PD_1}$ and $\boldsymbol{PD_2}$, are known. Then the joint default rate, ***PDJ***, completely determines the distribution of the default rate:

$$
(1)\qquad
\begin{aligned}
&\Pr[\text{ Default rate} = 100\%\ ] &&= \boldsymbol{PDJ}\\
&\Pr[\text{ Default rate} = 50\%\ ] &&= \boldsymbol{PD_1} + \boldsymbol{PD_2} - 2\,\boldsymbol{PDJ}\\
&\Pr[\text{ Default rate} = 0\%\ ] &&= 1 - \boldsymbol{PD_1} - \boldsymbol{PD_2} + \boldsymbol{PDJ}
\end{aligned}
$$

The task of the portfolio model is to determine ***PDJ***. To do so, it makes three assumptions:

- A firm defaults if and only if it is in "asset shortfall"— that is, if the value of its assets is less than the value of its liabilities. There is an "iron-clad link" between the shortfall event and the default event.
- The value of liabilities is known in advance.
- Asset returns obey the bivariate normal probability distribution in which asset correlation is known in advance.

Given these assumptions, standardized asset returns have a bivariate normal distribution:

$$
(2)\qquad \begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{pmatrix} \sim \boldsymbol{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \boldsymbol{r} \\ \boldsymbol{r} & 1 \end{pmatrix}\right].
$$

We use "asset correlation" specifically to refer to $\boldsymbol{r}$ in Expression (2); $\boldsymbol{r}$ is the correlation between asset returns.

In the model, a firm defaults if the random value of its assets is less than the fixed value of its liabilities. Asset value is a monotonic function of the standardized asset return. Therefore, a firm defaults if its standardized asset return is less than a threshold that is equal to $\Phi^{-1}(\boldsymbol{PD})$, where $\Phi^{-1}(\cdot)$ denotes the inverse standard normal cumulative distribution function.

Both firms default if both standardized asset returns are less than their respective thresholds. Therefore, the joint default rate equals the joint density of standardized returns integrated up to the appropriate limits:

$$
(3)\qquad \boldsymbol{PDJ}(\boldsymbol{PD}_1, \boldsymbol{PD}_2, \boldsymbol{r}) = \int_{-\infty}^{\Phi^{-1}(\boldsymbol{PD}_1)} \int_{-\infty}^{\Phi^{-1}(\boldsymbol{PD}_2)} \boldsymbol{\phi}(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{r})\, d\boldsymbol{A}_2\, d\boldsymbol{A}_1\,,
$$

where $\boldsymbol{\phi}(\cdot,\cdot,\boldsymbol{r})$ denotes the bivariate standard normal probability density function with correlation equal to asset correlation. This calculation is also called the Gauss copula evaluated at ($\boldsymbol{PD_1}$, $\boldsymbol{PD_2}$, $\boldsymbol{r}$), where

$$
(4)\qquad \boldsymbol{G}(\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{\rho}) = \int_{-\infty}^{\Phi^{-1}(\boldsymbol{U}_1)} \int_{-\infty}^{\Phi^{-1}(\boldsymbol{U}_2)} \boldsymbol{\phi}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\rho})\, d\boldsymbol{x}_2\, d\boldsymbol{x}_1\,.
$$

We refer to these expressions and their generalizations to larger portfolios as the "basic structural portfolio credit model" or simply as the "basic model," to distinguish it from more elaborate structural models that have been put forth. The basic model parameters are ***PD***'s and asset correlations. There is a strictly monotonic increasing relation between asset correlation and the joint default rate; greater asset correlation implies greater ***PDJ*** and vice versa.

A related concept is "default correlation", which is the correlation between two events of default. We take note that it is a strictly increasing function of ***PDJ***:

$$(5) \qquad \textbf{\textit{Default correlation}} = \frac{PDJ - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)PD_2(1-PD_2)}}.$$

Though the concept of default correlation may be more familiar than ***PDJ***, we put ***PDJ*** into the spotlight because it is a more direct consequence of the model.

To return, the model makes three assumptions. Taken together, the assumptions depict an uninterrupted flow from asset correlation to asset values to default events and ultimately to joint default events. It might seem intuitive, apparent even, that interruptions to the flow would produce a different, likely lesser, joint default rate. Still, the use of asset correlation in credit models is so common that the matter will be aired out in detail. The next three sections consider the effect on joint default of each of the model assumptions in turn.

Shortfall and default are different

The model assumes the iron-clad link, but the real world is more complicated. Borrowers have many interdependent options. The exercise of one of them, the option to default on a particular obligation, might not accord with a simple theory. The same can be said for the options of actual and potential suppliers of credit, equity, skills, and other productive resources, each of whom can influence an occurrence of default. Some default and non-default events may depend on factors besides the comparison of assets and liabilities.

There are two events of interest: "default without shortfall" and "shortfall without default". To focus on these events, this section does not challenge the other assumptions made by the model, namely, that both the value of liabilities and the value of asset correlation can be known when the model is employed.

"Default without shortfall" resembles a generalization of technical default. It probably includes most defaults reckoned as technical, such as when a covenant violation or an operational error triggers a declaration of default, and also includes other cases. A firm that is not in asset shortfall might default on an obligation because it does not have cash, because it cannot raise cash (as in a liquidity crunch), or because it chooses to retain its cash for strategic reasons. When a firm defaults on a financial obligation, this does not necessarily imply that its asset value is less than its liability value, despite the iron-clad link that is present in the theory.

"Shortfall without default" resembles a generalization of lender forbearance. In lender forbearance, a lender (probably a bank) provides the firm with enough additional cash or credit to enable it to elude default. Other scenarios might also lead to shortfall without default. A firm in asset shortfall might have on hand the cash required to pay small obligations. It might obtain the cash from its operations, from an asset sale, from its closely-associated ("sugar daddy") enterprise, or from equity investors who hope that the firm will soon prosper. Therefore, the fact that a firm fulfills an obligation does not guarantee that its asset value is greater than its liability value.

The question at hand is whether these events, which constitute breaks in the theoretical iron-clad link, increase or decrease the joint default rate (holding fixed $\boldsymbol{PD_1}$ and $\boldsymbol{PD_2}$). The straightforward approach would be to assemble data on the asset shortfall of firms and to measure its dependence on the asset returns or defaults of other firms. But the event of asset shortfall—the cornerstone of the basic model—is rarely observed. In part, this is because the asset value of a firm is rarely observed. Instead, the asset values most commonly used in credit research are inferred from equity market variables and from option pricing theory. The theory accepts that equity owners exercise the option to default if and only if a firm is in asset shortfall. Thus, available data on asset values assume that the iron-clad link is valid. To discover the effect of breaks in the iron-clad link would require a different approach.

A theoretical approach might attempt to infer qualitative effects from the behaviors of participants in the default event. To take an example, consider the event of lender forbearance. Presumably, only the behavior of the bank lender matters, since the troubled firm does not wish to default. The issue becomes the relationship of the bank's behavior to the asset values of firms besides the troubled one. If there is such a relationship, one would expect the bank to behave differently in economic booms than in downturns. But the difference of behavior is not clear, and one can argue that the difference might work either way.

Suppose that there is a period of economic downturn in which many firms are in shortfall and a bank faces the prospect of elevated credit loss. To limit the loss, a bank might withhold additional credit from a troubled firm, that is, during downturns the bank might deny forbearance more frequently than it does in boom periods. On the other hand, the bank might amplify its loyalty to a customer during a stressful period, hoping to be rewarded later by that customer and by new customers who observe this loyal behavior. Arguments for either view might be put forward in a specific situation. The balance of arguments, and the bank's action, would depend on the bank, the firm, the downturn, and the spirit of the times. All one can say is that forbearance might rise or fall in an economic downturn. If the probability of forbearance is related to the asset returns of other firms, it isn't clear whether the relationship is increasing or decreasing.

To summarize the preceding paragraphs, one would like to know the effect of breaks in the iron-clad link on the joint default rate. One would like to use available asset data, but the current data presume that the iron clad link is valid. One would like to reason about the behaviors of people, but the behaviors depend on conflicting motives. It seems that no clear conclusion can be reached.

Given this situation, it seems unlikely that available evidence could reject a statistical null hypothesis of random causation. The null hypothesis is not neutral when with respect to correlation. We show that if asset correlation is positive and if the iron-clad link is broken purely at random, then there is a decline in the joint default rate.

A pedagogic model shows this effect. Of firms in asset shortfall some fraction randomly eludes default. Of firms not in asset shortfall, some fraction randomly undergoes default. We begin with the definitions:

$$\begin{aligned} &\boldsymbol{p^*} = \Pr [\text{ Asset shortfall }] \\ (6)\qquad &\boldsymbol{p^T} = \Pr [\text{ Default} \mid \text{No asset shortfall }] \\ &\boldsymbol{p^F} = \Pr [\text{ No default} \mid \text{Asset shortfall }] \end{aligned}$$

<<Exhibit 1 goes about here>>

Exhibit 1 shows a first step that determines the value of assets and a second that determines default. If the first step produces no shortfall, default occurs at rate $\boldsymbol{p^T}$ (suggesting "technical" default). If the first step produces asset shortfall, there may still be no default. The relevant rate is $\boldsymbol{p^F}$ (suggesting "forbearance"). Combining the two routes, the probability of default equals

$$\begin{aligned} (7)\qquad \boldsymbol{PD} &= \boldsymbol{p^* - p^*\, p^F + (1 - p^*)\, p^T} \\ &= \boldsymbol{p^*} - \textbf{Pr[}\text{Shortfall without default}\textbf{]} + \textbf{Pr[}\text{Default without shortfall}\textbf{]} \end{aligned}$$

The object of interest is the effect on the joint default rate. If only one of the variables on the right hand side of Expression (7) were to change, the effect would be obvious. More default without shortfall would imply more default in general and more joint default in particular. But, as before, the value of $\boldsymbol{PD}$ is assumed to be known; what is at stake is not the rate of default but its cause. In the comparison to be performed, an increase in default without shortfall is accompanied by some combination of more shortfall without default and less shortfall overall. The value of $\boldsymbol{PD}$ is unaffected.

Consider the effect on the joint default rate in the special case that Firm 2 defaults entirely at random. Then, $\boldsymbol{p_2^*} = 0$ and $\boldsymbol{p_2^T} = \boldsymbol{PD_2}$; the joint default rate equals $\boldsymbol{PD_1\, PD_2}$. If asset correlation is positive, this is less than $\boldsymbol{PDJ}$ shown in Expression (3). In this special case, default without shortfall causes a reduction in the joint default rate.

There is a rather strong intuition that this special case would generalize for non-trivial values of $\boldsymbol{p^T}$ and $\boldsymbol{p^F}$. Suppose first that $\boldsymbol{p^T_1} = \boldsymbol{p^F_1} = \boldsymbol{p^F_2} = 0$, and compare a world in which $\boldsymbol{p^T_2} = 0$ to one in which $0 < \boldsymbol{p^T_2} < \boldsymbol{PD_2}$. In the second world, the default of Firm 2 might stem from asset shortfall or it might be completely random. Both types of default would contribute to the frequency of joint defaults. The intuition is that the second type of default produces fewer joint defaults than the first.

Sidebar 1 gives rigor to this intuitive argument. It shows that holding PD_1 and PD_2 fixed, ***any*** increase in ***any*** the four rates $\{p^T_1, p^F_1, p^T_2, p^F_2\}$ causes a reduction in the joint default rate. This includes the increase from zero to positive. Therefore, if the iron-clad link is broken randomly, there is a reduction in joint default. Sidebar 1 also shows that for a large uniform portfolio sensitive to a single risk factor, an increase in either p^T or p^F causes a reduction in value-at-risk.

Future research might reject the null hypothesis that is the basis of the pedagogic model. Suppose it is shown that p_2^F is negatively correlated with the asset return of Firm 1. Then, Firm 2 would be more likely to obtain forbearance when Firm 1 is in shortfall than otherwise. This would depress the joint default rate even more than when p_2^F is specified independently. On the other hand, it might be discovered that p_2^F is positively correlated with the asset return of Firm 1. If this dependence were strong enough, its effect on the joint default rate could overcome the first-order reduction that was demonstrated. It may seem a remote possibility, but there may be some connection between the defaults of firms that is stronger than connection provided by asset correlation, though it is as yet undiscovered.

Frequently, the correlation parameter in a basic credit portfolio model is referred to as "asset correlation." Frequently, specialists have used asset correlation in the structural model to predict the distribution of the default rate, as if this had been shown to give good results. But the events described here drive a wedge between assets and default, and therefore between the joint default rate and the forecast made by the basic model. If the events are significant in real-world default, a model that uses asset correlation may misstate risk. Unless there is an undiscovered connection between defaults, the misstatement would be an overstatement.

Liabilities vary randomly

The model assumes that the value of liabilities is known in advance. To focus on this assumption, this section temporarily accepts the assumption that asset correlation is known in advance, and it temporarily reinstates the iron-clad link. That is, default is to be determined solely by the condition of asset shortfall.

The shortfall condition is more complicated when liabilities vary. First, the appropriate comparison is between the level of assets and the level of liabilities, not between the return of assets and a fixed threshold. For example, if assets equal 100 and liabilities equal 50, default would occur with an asset return of -50% and a liability return of 0%, and it would also occur with an asset return of -75% and a liability return of -50%. Second, the appropriate comparison is between the total economic value of all a firm's assets and the total economic value of all its liabilities, even though the failure to pay would probably involve a financial liability.

Among all a firm's liabilities, many do not have value that is known when the model is employed. The value of a long-dated financial liability would depend on the interest rates that were in force at the model horizon or the time of default. The face amounts of some liabilities change, for example, pension liabilities depend on random factors such as the age at death. Still other liabilities arise at random: tax laws change, lawsuits are filed, and so forth. At the time the portfolio model is employed to assess risk, a substantial fraction of the total economic value of liabilities might be considered random to some degree.

The attempt to analyze the effect of the randomness of liabilities on the joint default rate strikes an immediate difficulty. Generally speaking, data on the economic value of a firm's liabilities do not exist at present. The most accessible data are the face amounts of financial liabilities that appear on the balance sheet, which can contribute little to the topic at hand. It remains to be seen whether fair value accounting per FAS 155 and FAS 156 will improve the situation.

Theorizing in the absence of good data, we first introduce variance into liability values, and then introduce covariance with other random variables. Variance by itself—that is, liabilities that vary independently—acts as a fresh source of idiosyncratic risk. A firm that would otherwise experience asset shortfall might be spared from default by a downward fluctuation of liabilities, and an upward fluctuation might cause the default of an otherwise healthy firm. Sidebar 2 shows this rigorously. If each firm's liability value is statistically independent, the joint default rate is less than the forecast made by the basic structural model. This conclusion holds irrespective of the marginal distribution of liability values.

The reduction of the joint default rate might be overcome by dependencies between the values of assets and liabilities. The following few paragraphs make the case that, though this might happen in practice, it cannot be predicted in theory.

When one introduces dependences, one imagines that correlations are positive in each case. One imagines positive correlation between a firm's assets and its own liabilities for mechanical and strategic reasons. Mechanically, when a firm issues or retires debt, both assets and its liabilities have substantial changes in the same direction. Strategically, if a firm's assets grow enough it is apt to issue more debt to finance still more growth. One imagines positive correlation between the liabilities of different firms, since to some degree they are shaped by the same forces, for example, the introduction of new medicines that increase the medical insurance liabilities at many firms. Finally, one imagines positive correlation between the liabilities of one firm and the assets of another, because a firm would find it easier to expand its liabilities when other firms prosper.

In the simple case where there are only two firms, the four variables of interest have six positive correlations: asset correlation plus the five additional correlations involving a liability value. With respect to the joint default rate, the positive correlations of liabilities have contrary effects. The contrary effect of two of them is shown next.

Into the basic model introduce the correlation between a firm's liabilities and its own assets. In the basic model, Firm 2 defaults if its assets fall below a threshold. If correlation is present, the fall must be greater, on average, to achieve the same $\boldsymbol{PD_2}$. If the correlation with the assets of Firm 1 is to be unaffected, the assets of Firm 1 would fall further, on average. Therefore Firm 1 is more likely to default at the same time that Firm 2 defaults. The joint default rate is greater than in the basic model.

Instead, into the basic model introduce correlation between a firm's liabilities and the assets of another firm. Suppose the liabilities of Firm 2 are correlated with the assets of Firm 1. This correlation has no effect on default of Firm 1. If the assets of Firm 1 decline, the liabilities of Firm 2 tend to decline. This gives the assets of Firm 2 more room to fall, on average, before triggering default. The joint default rate is less than in the basic model.

Since the correlations pull different ways, the net effect is ambiguous until more is known. Adding to the ambiguity, correlation by itself is very unlikely to provide a complete description of the complex dependencies among the four variables of interest. Correlations would be sufficient to describe jointly normal variables, but neither the value of assets nor the value of liabilities is plausibly normal. For example, the values of both assets and liabilities jump when debt is issued or retired. Parameters in place of or in addition to correlation may be required to describe the richness of the dependencies.

To summarize, when liabilities are allowed to vary independently, there is a reduction in the joint default rate. If dependencies are allowed, nothing really can be said; the net effect might be to further reduce the joint default rate, to partly offset the initial reduction, or to overmatch the initial reduction and produce a joint default rate that is greater than forecast by the basic model. The matter could be resolved by the right kind of data.

Asset correlation is a conditional variable

The model assumes that asset correlation is known when the model is run. To focus on this assumption, this section reverts to the idea that default occurs if and only if assets fall below a fixed value of liabilities. This section argues that asset correlation cannot always be known in advance, because it depends on whether firms default. If this argument is accepted, the basic model with asset correlation becomes extremely problematic.

Suppose that for a pair of firms, asset correlation equals 25% if neither defaults, 5% if exactly one of them defaults, and 1% if they both default. A practitioner using the basic model would not know in advance which correlation to use. A naïve practitioner might use historically observed asset correlation, which would most probably approximate the no-defaults level of 25%.

This stylized example may resemble the actual situation. The correlation that would be appropriate for a firm might depend on whether or not the firm becomes distressed during the model horizon. Compared to a firm that is not in distress, the correlation of a firm in financial distress is different ***because*** the firm is in distress. Like any firm, a distressed firm may or may not obtain additional funding, it may or may not agree with unions on cost-saving measures, its customers may or may not remain loyal, and so forth. But for the distressed firm, these idiosyncratic events have relatively greater effect on the firm's value, and may in fact trigger the default event. Because idiosyncratic influences are relatively more important to the distressed firm, its assets exhibit less correlation with other firms.

Separately, as a firm deteriorates it experiences increasing costs of distress. It may begin to pay greater spreads, to give better terms, to lose customers, and so forth. These new costs reduce the return on the firm's assets. This reduction is unrelated to the returns on assets of other firms. As a mechanical matter, there is an increase in the fraction of asset variance stemming from idiosyncratic causes, and this produces a decline in correlation.

This intuition is investigated empirically using Moody's KMV CreditMonitor data on asset values. For each month from August 1993 to September 2006, we calculate the monthly asset return of each firm as follows:

$$\text{(8)} \qquad \boldsymbol{ARET}_{i,t} = \frac{\boldsymbol{ASG}_{i,t} - \boldsymbol{ASG}_{i,t-1} - (\boldsymbol{LIAB}_{i,t} - \boldsymbol{LIAB}_{i,t-1})}{\boldsymbol{ASG}_{i,t-1}},$$

where $\boldsymbol{ASG}_{i,t}$ and $\boldsymbol{LIAB}_{i,t}$ are the values of the assets and liabilities of Firm $\boldsymbol{i}$ at time $\boldsymbol{t}$. The difference of liabilities in the numerator isolates Expression (8) from purely financial transactions such as the issuance or retirement of debt. The numerator expresses the change in the equity belonging to the owners of the firm.

Prior to each known default, we take the correlation between the asset return of the defaulting firm and the asset return of every other firm, using a three-month horizon to promptly detect changes over time. Centered eighteen months prior to default, we calculate 750,713 asset correlations averaging 8.03%. The calculated estimator is biased for small samples; its mean equals $\rho - \rho(1-\rho^2)/2(\mathbf{N}-1)$. With $\mathbf{N} = 3$ and $\rho < 15\%$, this equals approximately $0.75\,\rho$. To correct, we divide each correlation by 0.75 and produce the values shown in Exhibit 2.

<<Exhibit 2 goes about here>>

In Exhibit 2, asset correlation remains fairly constant until about five months before default, when it begins to fall. A basic model using a pre-distress value of asset correlation would overstate the strength of the connections to firms that actually default. In turn, the model would seem to overstate the joint default rate.

In passing, we note that the marginal distribution of asset returns is irrelevant to the joint default rate. Though the marginal distribution of assets affects ***PD***, we have assumed that

PD is known. Converting ***PD***'s into a joint default rate is exclusively the task of the copula that links the marginal distributions. The fixed asset correlation of the Gauss copula cannot handle the systematic variation of correlation that is highlighted in this section.

The basic model requires that asset correlation remain fixed at the historically observed level, but intuition and evidence suggest that if a firm becomes distressed, its asset correlation tends to decline. If this occurs, the joint default rate would be less than forecasted by the basic model using historically observed asset correlation.

Summarizing, we have listed the assumptions made by the basic structural portfolio model, and we have relaxed each in turn. If the iron-clad link is broken at random, or if liabilities vary independently, there is a reduction in the joint default rate, though dependencies could modify this provisional conclusion. Intuition and data suggest that asset correlation tends to decline if a firm deteriorates toward default. All this means that there can be a difference between the joint default rate and the rate calculated by the basic model, and it suggests that the former might be less than the latter.

Estimates of correlation

Empirical evidence shows a difference in correlation values. A review of this evidence is provided by Chernih, Vanduffel, and Henrard. Exhibit 3 reproduces the centerpiece of their comparison of the two approaches.

Exhibit 3.

Estimates of correlation and asset correlation
Source: Chernih, Vanduffel, Henrad, 2006

Source Study	Data Source	Results
Gordy (2002)	S&P	1.5% - 12.5%
Cespedes (2000)	Moody's	10%
Hamerle *et al.* (2003a)		max of 2.3%
Hamerle *et al.* (2003b)	S&P 1982 - 1999	0.4% - 6.04%
Frey *et al.* (2001)	UBS	2.6%, 3.8%, 9.21%
Frey & McNeil (2003)	S&P 1981 - 2000	3.4% - 6.4%
Dietsch & Petey (2004)	Coface 1994 - 2001	0.12% - 10.72%
	AK 1997 - 2001	
Jobst & de Servigny (2004)	S&P 1981 - 2003	intra 14.6%, inter 4.7%
Duellmann & Scheule (2003)	DB 1987 - 2000	0.5% - 6.4%
Jakubik (2006)	BF 1988 - 2003	5.7%

Table 1: *Asset correlations from default data*

Source Study	Data Source	Results
Duellmann *et al.* (2006)	KMV	10.1%
KMV (2001)	Undisclosed	9.46% - 19.98%
Fitch (2005)	Equity	intra 24.09%, inter 20.92%
Lopez (2002)	KMV Software	11.25%

Table 2: *Asset correlations from asset value data*

The authors' Table 1 shows estimates of "asset correlations from default data".[1] Among the studies there is a diversity of estimates. In part, this is because different studies focus on different countries, use different data, employ different models, and have different criteria for determining the value of correlation that provides the best match between the data and the model. The author's Table 2 shows "asset correlations from asset value data". Again there is a diversity of estimates, due at least in part to differences in approach and data source.

[1] Since these estimates do not depend on firms' asset values or returns, the nomenclature "asset correlation", though consistent with common usage, is distracting.

Despite the diversity within each table, there is a rather clear distinction between them. Estimates derived from asset data tend to be 10% or greater, and estimates derived from default data tend to be 10% or less.

To check for statistical significance in a particular set of data, we use a sample drawn from Moody's data to estimate correlation and confidence intervals. We begin by assuming that the default of Firm $\boldsymbol{i}$ is determined by a standard normal latent variable $\boldsymbol{Z_i}$. If all the assumptions of the basic model were true, then $\boldsymbol{Z_i}$ would be none other than Firm $\boldsymbol{i}$'s standardized asset return, but the latent variable formulation does not require this. Firm $\boldsymbol{i}$ defaults if and only if $\boldsymbol{Z_i}$ is less than the default threshold for Firm $\boldsymbol{i}$:

$$(9) \qquad \boldsymbol{Z_i} < \Phi^{-1}(\boldsymbol{PD_i}) .$$

We assume that a single systematic factor $\boldsymbol{Y}$ imparts correlation to the latent $\boldsymbol{Z_i}$ and correlation is constant in time:

$$(10) \qquad \boldsymbol{Z_{i,t}} = \sqrt{\boldsymbol{\rho_i}}\ \boldsymbol{Y_t} + \sqrt{1-\boldsymbol{\rho_i}}\ \boldsymbol{V_{i,t}}; \qquad \boldsymbol{Y_t}, \boldsymbol{V_{i,t}} \sim \ \boldsymbol{i.i.d.}\ \mathrm{N}[0,1] .$$

The conditional probability of default in year $\boldsymbol{t}$ depends on the realization of the systematic risk factor:

$$(11) \qquad \boldsymbol{r_i}(\boldsymbol{y_t}) = \mathbf{Pr}\ [\ \boldsymbol{Z_{i,t}} < \Phi^{-1}(\boldsymbol{PD_i}) \,|\, \boldsymbol{Y_t} = \boldsymbol{y_t}\,] = \Phi\left(\frac{\Phi^{-1}(\boldsymbol{PD_i}) - \sqrt{\boldsymbol{\rho_i}}\ \boldsymbol{y_t}}{\sqrt{1-\boldsymbol{\rho_i}}}\right).$$

Since all firms in a rating grade are treated as statistically identical, Expression (11) can be viewed as the conditionally expected default rate within rating grade $\boldsymbol{i}$ in year $\boldsymbol{t}$. Any firm in grade $\boldsymbol{i}$ has probability of default equal to $\boldsymbol{PD_i}$ and any pair of firms in grade $\boldsymbol{i}$ has correlation equal to $\boldsymbol{\rho_i}$.

Turning now to the data, suppose that in a given year and grade we observe $\boldsymbol{d}$ defaults among $\boldsymbol{n}$ rated firms. This event would be assigned a probability by the binomial model with parameters $\boldsymbol{n}$ and $\boldsymbol{p}$:

$$(12) \qquad \boldsymbol{binprob}(\boldsymbol{d_{i,t}} \,|\, \boldsymbol{n_{i,t}}, \boldsymbol{p_{i,t}}) = \boldsymbol{p_{i,t}}^{\boldsymbol{d_{i,t}}} (1-\boldsymbol{p_{i,t}})^{\boldsymbol{n_{i,t}}-\boldsymbol{d_{i,t}}} \begin{pmatrix} \boldsymbol{n_{i,t}} \\ \boldsymbol{d_{i,t}} \end{pmatrix}$$

The value of $\boldsymbol{p_{i,t}}$ would equal the year's conditionally expected rate, which according to Expression (11) can be any number between 0 and 1. Averaging the all values of $\boldsymbol{p_{i,t}}$, the probability ultimately assigned to the event $\boldsymbol{d_{i,t}}$ is

(13)

$$\boldsymbol{binprob}(\boldsymbol{d}_{i,t} \mid \boldsymbol{n}_{i,t}) =$$

$$\int_{-\infty}^{\infty} \phi(y_t) \left(\Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\ y_t}{\sqrt{1-\rho_i}} \right) \right)^{d_{i,t}} \left(1 - \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\ y_t}{\sqrt{1-\rho_i}} \right) \right)^{n_{i,t} - d_{i,t}} \binom{n_{i,t}}{d_{i,t}} dy_t$$

Assuming each year of data represents an independent observation, the product over time of Expression (13) is the likelihood function of all the data in a rating grade:

(14)

$$L(i, PD_i, \rho_i) =$$

$$\prod_t \int_{-\infty}^{\infty} \phi(y_t) \left(\Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\ y_t}{\sqrt{1-\rho_i}} \right) \right)^{d_{i,t}} \left(1 - \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\ y_t}{\sqrt{1-\rho_i}} \right) \right)^{n_{i,t} - d_{i,t}} \binom{n_{i,t}}{d_{i,t}} dy_t$$

We maximize Expression (14) using the default rates observed annually 1983-2003 for U.S. non-financial obligors in each of eight Moody's rating grades: Baa3, Ba1, Ba2, Ba3, B1, B2, B3, and "C", which we use to designate the union of grades C, Ca, Caa and the "notches," Caa1, Caa2, and Caa3. The resulting estimates appear in Exhibit 4.

Exhibit 4.

Maximum likelihood estimates and asymptotic confidence bounds for ρ

Rating Grade	Total		ML estimates		95% Confidence	
	d	n	PD	ρ	LB	UB
Baa3	11	2,000	0.54%	23.9%	2.6%	69.1%
Ba1	13	1,794	0.72%	2.0%	0.0%	24.9%
Ba2	11	1,660	0.67%	5.3%	0.0%	34.0%
Ba3	69	2,908	2.22%	8.3%	2.9%	20.2%
B1	113	3,240	3.55%	5.8%	1.5%	15.3%
B2	155	2,195	7.58%	10.0%	3.8%	22.6%
B3	210	1,839	12.06%	8.9%	4.0%	18.7%
C	279	1,627	13.90%	6.5%	2.3%	16.2%
Ba3-C	826	11,809	---	5.4%	2.9%	10.8%

Exhibit 4 also shows confidence intervals based on the Chi-square distribution of the asymptotic likelihood ratio. The confidence intervals are wide, and this is especially the case for the three best grades. They tell very little about the value of correlation, perhaps because the grades encompass few defaults and relatively few ratings.

All the confidence intervals share a range of overlap. This suggests the hypothesis that a uniform value of correlation would describe the distribution of default for any pair of firms. The last row of Exhibit 4 reports an estimate of correlation that is constrained to be uniform for rating grades Ba3 through C, which maximizes the likelihood:

(15)

$$\boldsymbol{L}(\{\boldsymbol{PD}_i,\rho\}_i) =$$

$$\prod_{t=1983}^{2003}\int_{-\infty}^{\infty}\phi(y_t)\prod_{i=C}^{Ba3}\left(\Phi\left(\frac{\Phi^{-1}(\boldsymbol{PD}_i)-\sqrt{\rho}\,y_t}{\sqrt{1-\rho}}\right)\right)^{d_{i,t}}\left(1-\Phi\left(\frac{\Phi^{-1}(\boldsymbol{PD}_i)-\sqrt{\rho}\,y_t}{\sqrt{1-\rho}}\right)\right)^{n_{i,t}-d_{i,t}}\binom{n_{i,t}}{d_{i,t}}dy_t$$

Taking this as the most informative model, the 95% confidence interval is still wide, 2.9% to 10.8%, but it is much narrower than the confidence intervals that treat each rating grade as separate. The upper bound distinguishes this estimate of correlation from some, but not all, estimates of asset correlation. For example, the preliminary version of the U.S. version of Basel II minimum capital requirement for wholesale exposures refers to asset value correlation in the range of 12% to 24%.[2] Though we can distinguish correlation from asset correlation, without more information we do not know the source of the difference. Any of the mechanisms that have been discussed, or others, might be responsible in isolation or in concert.

Among the studies cited by Chernih et. al., the closest comparison to the foregoing would be Gordy and Heitfield, who also fit the binomial model to a sample taken from Moody's data, but at the full-letter grade level. Their unrestricted model produces grade-specific estimates of 8.3% (Baa), 11.1% (Ba), 6.7% (B), and 6.3% (Caa), and a uniform estimate of 7.9%. Their simulations indicate that for the binomial model and a single risk factor, the downward bias of maximum likelihood estimates is apt to be less than 10% of the true value of correlation. Their appendix touches on the convergence problems that plague studies like theirs and this one. Demey, Jouanin, Roget, and Roncalli fit the binomial model to S&P data and obtain an overall correlation of 8.3%, which in a correction note they refine to 6.3%. Servigny and Renault proceed in much the same spirit as the discussion of ***PD***s and ***PDJ***s; they utilize counts of events of default and events of joint default; and they comment on the difference between their correlation estimates and those derived from the equity market. Their estimates of default correlation have no direct comparison to the estimates of correlation and asset correlation discussed here.

If the basic structural model, its application, and the data were perfect, a statistical estimate of correlation based on default would equal asset correlation. It seems, though,

[2] It should be noted that the statistical estimate being discussed reflects the distribution of default rates, while the regulatory standard reflects a contribution to bank capital, which is a different and broader concept. There can be no direct inference from one to the other.

that the two differ. The conclusion is that jointly normal asset returns are not the whole story when it comes to understanding and modeling the distribution of default. As a consequence, a model that relies solely on asset correlation can misstate risk.

Incentives

This section shows that the difference highlighted above can have a meaningful effect on a comparison of the risk of two transactions. An example is the comparison between a $80,000,000 exposure to a firm having ***PD*** equal to 0.1%, and a $2,000,000 exposure to a firm having ***PD*** equal to 10%. To keep the focus on default, each transaction is assumed to have maturity of one period and loss given default equal to 100%.

We compare the one-year 99.9th percentile Value-at-Risk for these two transactions in Exhibit 5. Value-at-Risk equals Expression (11) with $\boldsymbol{y_t}$ set equal to $\Phi^{-1}(0.001) = -3.09$.

<<Exhibit 5 goes about here>>

Greater correlation implies greater Value-at-Risk, but it affects the transactions differently. A risk manager using uniform correlation equal to the maximum likelihood estimate of 5.4% perceives the transactions as having nearly equal risk. A manager using uniform correlation equal to 24% perceives the first transaction as more than twice as risky as the second. A manager using 24% for the first transaction and 12% for the second perceives the first transaction as over three times as risky as the second. The comparison between the two transactions is more sensitive to correlation at the 99.99th percentile and less sensitive at the 99th percentile.

If value-at-risk affects incentives through pricing or compensation, the manager using an overstated value of correlation has the incentive to favor exposure to poorer-quality borrowers.

Conclusion

This study has two goals. The first goal is to explain the widely known but poorly understood fact that researchers who estimate correlation with credit models and default data tend to find lower values than found by researchers who use data on firms' asset values. The second goal is to suggest that for the purpose of predicting the distribution of the default rate, historical default rates may provide a better guide than asset correlation and a chain of assumptions.

Users of the basic structural credit portfolio model have often assumed that correlation equals asset correlation. This Credit Forum contribution critiques this assumption. Firms that "should" default may not default, firms that "should not" default may do so nonetheless, firms may default (or may refrain from default) because random liability values as well as random asset values, and asset correlation may decline for firms that deteriorate toward default. All these features have an effect on the joint default rate. Though there might be undiscovered connections that increase the joint default rate, the *prima facie* evidence suggests a reduction.

When researchers estimate correlation, estimates based on credit models and credit data tend to be, in fact, less than estimates based on asset returns. The statistical estimate presented here has an upper bound of 10.8%, which in turn is less than some estimates of asset correlation used in credit models. When it comes to fitting the historical distribution the default rates analyzed here, those values of asset correlation are rejected.

Logically, the event of default depends on more than accounted for in the basic structural portfolio model; more than asset correlation influences the joint default rate. This can, and apparently does, lead to a difference between the value of asset correlation and the value of correlation that provides the best match between the model and default data. The difference in quantities can be practically important for the management of credit risk.

Sidebar 1. In the pedagogic model, increases in p^T or p^F reduce the joint default rate.

Certain mild assumptions are required. Asset correlation is assumed positive. The asset shortfall rates p^*_1 and p^*_2 are assumed less than 50%. A firm in asset shortfall is assumed to be more likely to default than a firm that is not in asset shortfall:

(A) $1 - p^F > p^T$.

In the pedagogic model, asset correlation and asset shortfall rates determine the joint shortfall rate, much as asset correlation and PDs determine PDJ in the basic structural model. Using the compact notation of the Gauss copula, we have for example

(B) **Prob [Both firms experience asset shortfall] = $G(p^*_1, p^*_2, r)$.**

The probability of joint default depends on events in each of four states of joint shortfall, and in the pedagogic model this can be stated without conditioning:

$$\begin{aligned}(C)\quad PDJ(p_1^*, p_1^T, p_1^F, p_2^*, p_2^T, p_2^F, r) &= G(p_1^*, p_2^*, r)(1-p_1^F)(1-p_2^F) \\ &+ (p_1^* - G(p_1^*, p_2^*, r))(1-p_1^F)\,p_2^T \\ &+ (p_2^* - G(p_1^*, p_2^*, r))\,p_1^T(1-p_2^F) \\ &+ (1-p_1^*-p_2^*+G(p_1^*, p_2^*, r))\,p_1^T p_2^T \end{aligned}\ .$$

The model is symmetric in the two firms. We focus on Firm 1. Hold asset correlation fixed and take the total differential of Expression (C):

$$(D)\quad dPDJ = \frac{\partial PDJ}{\partial p_1^*}dp_1^* + \frac{\partial PDJ}{\partial p_1^T}dp_1^T + \frac{\partial PDJ}{\partial p_1^F}dp_1^F\ .$$

To hold PD fixed, take the total differential of Expression (7) and set it equal to zero. This implies that dp^*_1 must obey the following relation to the other differentials:

$$(E)\quad dp_1^* = \frac{p_1^*}{1-p_1^T-p_1^F}dp_1^F - \frac{(1-p_1^*)}{1-p_1^T-p_1^F}dp_1^T\ .$$

If the differentials obey Expression (E), PD_1 is unchanged. We substitute this expression into Expression (D), write explicit derivatives, symbolize $G = G(p^*_1, p^*_2, r)$, and simplify to obtain the following.

$$\begin{aligned}(F)\quad dPDJ &= (1-p_1^T-p_1^F)(p_2^* - G - (1-p_1^*)\frac{\partial G}{\partial p_1^*})dp_1^T + (1-p_1^T-p_1^F)(\frac{\partial G}{\partial p_1^*}p_1^* - G)dp_1^F \\ &= C_1 dp_1^T + C_2 dp_1^F\end{aligned}$$

Next we demonstrate that the coefficients $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are negative. This demonstration has the immediate consequence that any increase in $\boldsymbol{p_1^T}$ or $\boldsymbol{p_1^F}$ produces a reduction in ***PDJ***. Three intermediate results are useful. The proofs are omitted.

(G) $\quad \boldsymbol{G(p_1^*, p_2^*, r)} > \boldsymbol{p_1^* p_2^*}$

(H) $\quad \dfrac{\partial \boldsymbol{G(p_1^*, p_2^*, r)}}{\partial \boldsymbol{p_1^*}} > \boldsymbol{p_2^*}$

(I) $\quad \dfrac{\partial^2 \boldsymbol{G(p_1^*, p_2^*, r)}}{\partial \boldsymbol{p_1^{*2}}} < 0$

Expression (G) says that the joint shortfall rate with positive correlation exceeds the joint shortfall rate with independence. Expression (H) is a statement about the change in the joint shortfall rate given a change in the shortfall rate of Firm 1. If asset correlation were zero, the value of the partial derivative would equal $\boldsymbol{p_2^*}$. Expression (H) says that when asset correlation is positive, the partial is greater than $\boldsymbol{p_2^*}$. Expression (I) says that the joint shortfall rate is a concave function of an individual shortfall rate. Making these substitutions, coefficient $\boldsymbol{C_1}$ is negative:

(J)
$$\begin{aligned} \boldsymbol{C}_1 &= (1 - \boldsymbol{p_1^T} - \boldsymbol{p_1^F})(\boldsymbol{p_2^*} - \boldsymbol{G} - (1 - \boldsymbol{p_1^*})\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{p_1^*}}) \\ &< (1 - \boldsymbol{p_1^T} - \boldsymbol{p_1^F})(\boldsymbol{p_2^*} - \boldsymbol{p_1^*}\boldsymbol{p_2^*} - (1 - \boldsymbol{p_1^*})\boldsymbol{p_2^*}) = 0 \end{aligned}.$$

Coefficient $\boldsymbol{C_2}$ is handled in two steps. First, it would be equal to zero if $\boldsymbol{p^*_1}$ were equal to zero:

(K) $$\boldsymbol{C}_2 = (1 - \boldsymbol{p_1^T} - \boldsymbol{p_1^F})(\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{p_1^*}}\boldsymbol{p_1^*} - \boldsymbol{G}) = (1 - \boldsymbol{p_1^T} - \boldsymbol{p_1^F})(\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{p_1^*}} 0 - 0) = 0\,.$$

Second, its derivative with respect to $\boldsymbol{p^*_1}$ is less than zero:

(L) $$\frac{\partial \boldsymbol{C}_2}{\partial \boldsymbol{p_1^*}} = (1 - \boldsymbol{p_1^T} - \boldsymbol{p_1^F})\frac{\partial^2 \boldsymbol{G}}{\partial \boldsymbol{p_1^{*2}}}\boldsymbol{p_1^*} < 0\,.$$

Therefore, $\boldsymbol{C_2}$ is negative as well as $\boldsymbol{C_1}$. Referring back to Expression (F), a greater rate of either $\boldsymbol{p^T_1}$ or $\boldsymbol{p^F_1}$ produces a reduction of ***PDJ***. In particular, if either or both $\boldsymbol{p^T}$ or $\boldsymbol{p^F}$ have non-zero values, the joint default rate is less than in the basic structural model.

A similar approach produces a similar result for the asymptotic portfolio sensitive to a single risk factor. We symbolize the systematic risk factor by ***Y***. The conditionally expected shortfall rate default rate equals

(M) $$\boldsymbol{cShort(Y)} = \Phi\left(\frac{\Phi^{-1}(\boldsymbol{p^*}) + \sqrt{\boldsymbol{r}}\,\boldsymbol{Y}}{\sqrt{1-\boldsymbol{r}}}\right).$$

where the sign of the systematic risk factor ***Y*** is such that greater levels of ***Y*** cause greater rates of default. The conditionally expected default rate depends on shortfall rates as in Expression (7) in the text:

(N) $$\boldsymbol{cDefault(Y)} = \boldsymbol{cShort(Y)}\,(1-\boldsymbol{p^F}) + (1-\boldsymbol{cShort(Y)})\,\boldsymbol{p^T}.$$

We take the total differential of Expression (N). Again, we hold ***PD*** fixed by imposing the relation of Expression (E). This produces

(O)

$$\boldsymbol{dcDefault(Y)} = \boldsymbol{C}_1 \boldsymbol{dp^T} + \boldsymbol{C}_2 \boldsymbol{dp^F}$$

$$\boldsymbol{C}_1 = 1 - \frac{1-\boldsymbol{p^*}}{\sqrt{1-\boldsymbol{r}}}\boldsymbol{Exp}\left(-\frac{\boldsymbol{r}\left(\Phi^{-1}(\boldsymbol{p^*})\right)^2 + 2\sqrt{\boldsymbol{r}}\,\Phi^{-1}(\boldsymbol{p^*})\boldsymbol{Y} + \boldsymbol{r}\boldsymbol{Y}^2}{2(1-\boldsymbol{r})}\right) - \Phi\left(\frac{\Phi^{-1}(\boldsymbol{p^*}) + \sqrt{\boldsymbol{r}}\,\boldsymbol{Y}}{\sqrt{1-\boldsymbol{r}}}\right).$$

$$\boldsymbol{C}_2 = \frac{\boldsymbol{p^*}}{\sqrt{1-\boldsymbol{r}}}\boldsymbol{Exp}\left(-\frac{\rho\left(\Phi^{-1}(\boldsymbol{p^*})\right)^2 + 2\sqrt{\boldsymbol{r}}\,\Phi^{-1}(\boldsymbol{p^*})\boldsymbol{Y} + \boldsymbol{r}\boldsymbol{Y}^2}{2(1-\boldsymbol{r})}\right) - \Phi\left(\frac{\Phi^{-1}(\boldsymbol{p^*}) + \sqrt{\boldsymbol{r}}\,\boldsymbol{Y}}{\sqrt{1-\boldsymbol{r}}}\right)$$

It can be shown numerically that both $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are negative for combinations that are encountered in practice. At the 99.9th percentile, ***Y*** takes the value 3.09, and both coefficients are negative for $\boldsymbol{p^*} > 0.01\%$ and $\boldsymbol{r} < 60\%$. Within this region, when ***PD*** and asset correlation are held fixed, any increase in either $\boldsymbol{p^T}$ or $\boldsymbol{p^F}$ causes a decline in the default rate at the 99.9th percentile. In particular, non-zero values of either $\boldsymbol{p^T}$ or $\boldsymbol{p^F}$ produce "stress" default rates that are less than produced by the basic structural model.

Sidebar 2. Independent variation of liabilities reduces the joint default rate.

In this Sidebar, default depends on a comparison between the value of assets and the value of liabilities. Given a firm's asset return, there would no longer be certainty regarding the default event, but only a conditional probability of default. We place no restrictions on the distribution of liabilities, but we assume that the value of a firm's liabilities is independent of other variables. Among other things, this assumption implies that the conditional probability of default is monotonic: a greater value of assets can never imply a greater probability of default.

In the basic model, the conditional probability of default is an indicator function equal to 1.0 when the asset return is below the threshold and equal to zero otherwise. The unconditional probability of default equals the expected value of the indicator function:

(A) $$\boldsymbol{PD} = \int_{-\infty}^{\Phi^{-1}(\boldsymbol{PD})} \boldsymbol{\phi(x)dx} = \int_{-\infty}^{\infty} \mathrm{I}[\boldsymbol{x} < \Phi^{-1}(\boldsymbol{PD})]\, \boldsymbol{\phi(x)dx} .$$

If liabilities vary independently, the conditional probability of default, $\boldsymbol{h}(\cdot)$, it is monotonic. We assume that that $\boldsymbol{h}(\cdot)$ is not constant, so that default is not completely divorced from the firm's asset return. We hold its expected value equal to $\boldsymbol{PD}$:

(B) $$\boldsymbol{PD} = \int_{-\infty}^{\infty} \boldsymbol{h(x)\, \phi(x)dx} .$$

To contrast, Expression (A) says that default is a knife-edge event in which a threshold determines default, while Expression (B) loosens the connection from a single threshold to a zone of increasing danger. If a second firm has conditional default rate function $\boldsymbol{j}(\cdot)$, the joint default rate is equal to

(C) $$\boldsymbol{PDJ} = \int_{-\infty}^{\infty} \int_{-\infty}^{-\infty} \boldsymbol{h(x)\, j(y)\, \phi(x, y, r)dxdy} .$$

We will show that the joint default rate in Expression (C) is no greater than the joint default rate in Expression (3). To do this, we show that Expression (C) increases when the function $\boldsymbol{h}(\cdot)$ is replaced by a function $\boldsymbol{h}^*(\cdot)$ that is "more like" the indicator function. Such replacements are possible unless $\boldsymbol{h}(\cdot)$ is the indicator function itself. A similar argument applies to the $\boldsymbol{j}(\cdot)$ function. The conclusion is that if $\boldsymbol{h}(\cdot)$ and $\boldsymbol{j}(\cdot)$ are anything but indicator functions, the joint default rate in Expression (C) is less than the joint default rate in Expression (3). Therefore, if liabilities vary independently, the joint default rate is less than in the basic structural model.

To construct the function $\boldsymbol{h}^*(\cdot)$, begin with these steps:

- Select x_1 and x_2 such that $x_1 < x_2$, $h(x_1) < 1$, and $h(x_2) > 0$
- Select $H_2 \le \text{Min}\, [h(x_2), (1-h(x_2))\, \phi(x_1) / \phi(x_2)]$
- Set $H_1 = H_2\, \phi(x_2) / \phi(x_1)$
- Select a small increment Δx

Construct $\boldsymbol{h}^*(\boldsymbol{x})$ as follows: In the region $(\boldsymbol{x_1}, \boldsymbol{x_1} + \Delta \boldsymbol{x})$, set $\boldsymbol{h}^*(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{H_1}$. In the region $(\boldsymbol{x_2}, \boldsymbol{x_2} + \Delta \boldsymbol{x})$, set $\boldsymbol{h}^*(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{H_2}$. For all other values of $\boldsymbol{x}$, set $\boldsymbol{h}^*(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})$.

When $\boldsymbol{h}(\cdot)$ is replaced by $\boldsymbol{h}^*(\cdot)$, there is no effect on $\boldsymbol{PD}_1$:

$$
\begin{aligned}
\boldsymbol{PD}_1^* &= \int_{-\infty}^{\infty} \boldsymbol{h}^*(x)\,\phi(x)dx \\
\text{(D)} \qquad &= \int_{-\infty}^{\infty} \boldsymbol{h}(x)\,\phi(x)dx + \int_{x_1}^{x_1+\Delta x} \boldsymbol{H}_1\,\phi(x)dx - \int_{x_2}^{x_2+\Delta x} \boldsymbol{H}_2\,\phi(x)dx\,. \\
&= \boldsymbol{PD}_1 + \Delta x\,\boldsymbol{H}_1\,\phi(x_1) - \Delta x\,\boldsymbol{H}_2\,\phi(x_2) = \boldsymbol{PD}_1
\end{aligned}
$$

When $\boldsymbol{h}(\cdot)$ is replaced by $\boldsymbol{h}^*(\cdot)$, there is an increase in $\boldsymbol{PDJ}$. To see this we first perform simplifications:

(E)

$$
\begin{aligned}
\boldsymbol{PDJ}^* &= \int_{-\infty}^{\infty}\int_{-\infty}^{-\infty} \boldsymbol{h}^*(x)\,j(y)\,\phi(x,y,r)dxdy \\
&= \boldsymbol{PDJ} + \int_{-\infty}^{\infty}\int_{x_1}^{x_1+\Delta x} \boldsymbol{H}_1\,j(y)\,\phi(x,y,r)dxdy - \int_{-\infty}^{\infty}\int_{x_2}^{x_2+\Delta x} \boldsymbol{H}_2\,j(y)\,\phi(x,y,r)dxdy \\
&= \boldsymbol{PDJ} + \int_{-\infty}^{\infty} \frac{\phi(x_2)}{\phi(x_1)} \boldsymbol{H}_2\,j(y)\,\phi\!\left(\frac{y - r\,x_1}{\sqrt{1-r^2}}\right)\frac{\phi(x_1)}{\sqrt{1-r^2}}dy - \int_{-\infty}^{\infty} \boldsymbol{H}_2\,j(y)\,\phi\!\left(\frac{y - r\,x_2}{\sqrt{1-r^2}}\right)\frac{\phi(x_2)}{\sqrt{1-r^2}}dy\,. \\
&= \boldsymbol{PDJ} + \frac{\boldsymbol{H}_2\,\phi(x_2)}{\sqrt{1-r^2}}\int_{-\infty}^{\infty} j(y)\left[\phi\!\left(\frac{y - r\,x_1}{\sqrt{1-r^2}}\right) - \phi\!\left(\frac{y - r\,x_2}{\sqrt{1-r^2}}\right)\right]dy
\end{aligned}
$$

It is sufficient to show that the integral is positive. First, we center on the point where the two "bell" curves cross by making the substitution $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{r}\,(\boldsymbol{x_1} + \boldsymbol{x_2})/2$. Partitioning the limits of integration, the integral can be written

$$
\text{(F)} \qquad
\begin{aligned}
&\int_{-\infty}^{0} j(z + r\frac{x_1 + x_2}{2})\left[\phi\!\left(\frac{z + r\,\frac{x_2 - x_1}{2}}{\sqrt{1-r^2}}\right) - \phi\!\left(\frac{z - r\,\frac{x_2 - x_1}{2}}{\sqrt{1-r^2}}\right)\right]dz \\
&+ \int_{0}^{-\infty} j(z + r\frac{x_1 + x_2}{2})\left[\phi\!\left(\frac{z + r\,\frac{x_2 - x_1}{2}}{\sqrt{1-r^2}}\right) - \phi\!\left(\frac{z - r\,\frac{x_2 - x_1}{2}}{\sqrt{1-r^2}}\right)\right]dz\,.
\end{aligned}
$$

Changing the variable of the first integration from $\boldsymbol{z}$ to $-\boldsymbol{z}$, we have

(G)

$$\int_0^{\infty} \boldsymbol{j}(-\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2})\left[\boldsymbol{\phi}\left(\frac{-\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)-\boldsymbol{\phi}\left(\frac{-\boldsymbol{z}-\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)\right]\boldsymbol{dz}$$
$$+\int_0^{-\infty} \boldsymbol{j}(\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2})\left[\boldsymbol{\phi}\left(\frac{\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)-\boldsymbol{\phi}\left(\frac{\boldsymbol{z}-\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)\right]\boldsymbol{dz} .$$

In the second integral in Expression (G), there are two instances of the $\boldsymbol{\phi}(\cdot)$ function. The second instance has greater value, because $\boldsymbol{z}$, $\boldsymbol{r}$ and $(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ are positive. The difference of $\boldsymbol{\phi}(\cdot)$'s in the second integral is therefore negative. It is also equal to the negative of the difference of $\boldsymbol{\phi}(\cdot)$'s in the first integral, because of the symmetry of the $\boldsymbol{\phi}(\cdot)$ function. Therefore we can recombine as follows:

(H)
$$\int_0^{\infty}\left[\boldsymbol{j}(-\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2})-\boldsymbol{j}(\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2})\right]\left[\boldsymbol{\phi}\left(\frac{\boldsymbol{z}-\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)-\boldsymbol{\phi}\left(\frac{\boldsymbol{z}+\boldsymbol{r}\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{2}}{\sqrt{1-\boldsymbol{r}^2}}\right)\right]\boldsymbol{dz}$$

In Expression (H), the difference of $\boldsymbol{\phi}(\cdot)$'s is positive over the range of integration (except at $\boldsymbol{z} = 0$, where the difference is zero). The difference of $\boldsymbol{j}(\cdot)$'s is non-negative, because $\boldsymbol{j}(\cdot)$ is monotonic decreasing, and the difference of $\boldsymbol{j}(\cdot)$'s is positive for some values of $\boldsymbol{z}$, because $\boldsymbol{j}(\cdot)$ is not constant. Therefore the integral in Expression (H) is positive, the integral in Expression (E) is positive, and ***PDJ**** > ***PDJ***.

We have shown that if the conditionally expected default rate functions $\boldsymbol{h}(\cdot)$ and $\boldsymbol{j}(\cdot)$ become "more like" indicator functions, there is an increase in the rate of joint default. Therefore, any departures from the indicator function produce less joint default. Independent variation of liabilities causes the joint default rate to be less than forecast with the basic structural model using asset correlation.

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Exhibit 1.

Asset shortfall and default are different

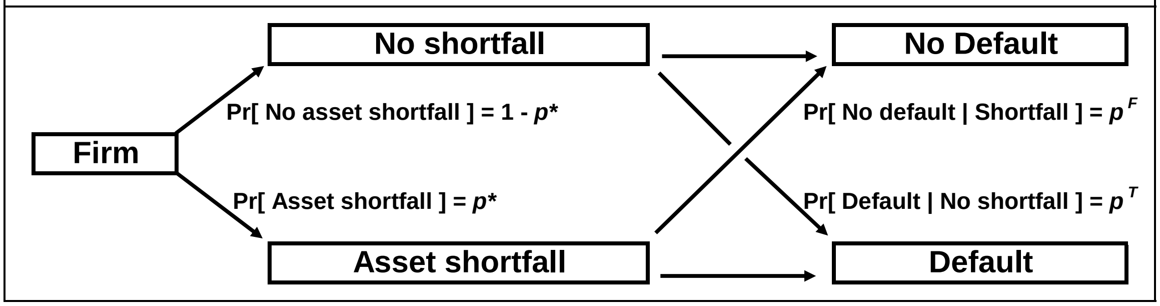
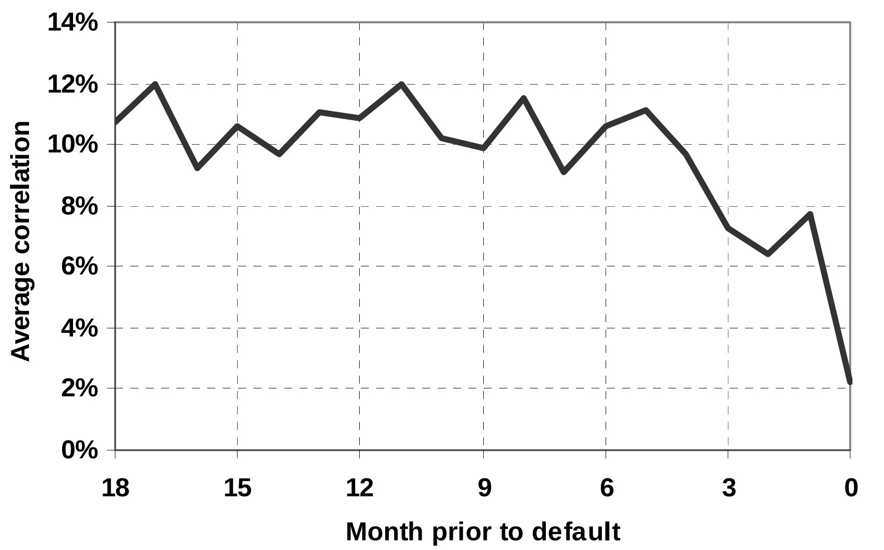


Exhibit 2.

Correlation of asset returns as default approaches



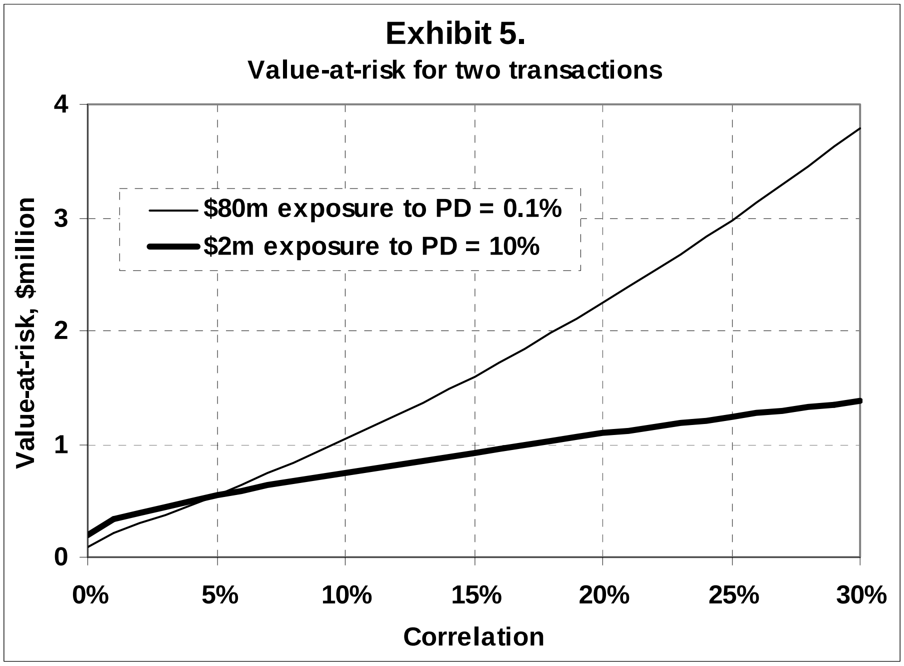


Exhibit 5.
Value-at-risk for two transactions
$80m exposure to PD = 0.1%
$2m exposure to PD = 10%
Value-at-risk, $million
Correlation
0
1
2
3
4
0%
5%
10%
15%
20%
25%
30%