# Incomplete Information and Informative Pricing

Giacomo Rondina*†
Department of Economics
University of California, San Diego

This Version: June 2008.

**Abstract**

This paper studies the information contained in the equilibrium aggregate price level of an economy where firms make output price decisions faced with incomplete information about economy-wide disturbances. It is shown that the ability of the aggregate price to be a sufficient statistics of the underlying aggregate disturbance depends on the degree of strategic complementarity in firms' pricing strategy when firms are allowed to extract information from the aggregate price. As the incentive to price similarily increases, aggregate price goes from a perfect to an imperfect signal of changes in the aggregate state of the economy. This paper contributes to the expanding literature on monetary business cycle and imperfect common knowledge initiated by Woodford (2003a) by applying to a standard macro pricing model a set of techniques in the domain of complex-valued functions (analytic functions) that allow for the derivation of individual prediction formulas in closed form as opposed to the standard recursive form. In addition, it is shown that analytic functions are particularly convenient when solving for the equilibrium fixed point problem in linear environments.

*Contacts: 9500 Gilman Drive, La Jolla, 92093, CA-USA; phone: 001 858 534 1059, E-mail: grondina@ucsd.edu.

†I am especially thankful to Steven Durlauf for his guidance and support. I am also indebted to Buz Brock for his comments and for many discussions on this and related topics. I have benefited from comments by Bruce Hansen, Terry Iverson, Jim Nason, Yongs Shin, Ken West and seminar participants at the ASSA 2008 Winter Meetings, the Atlanta Fed, the Boston Fed, the Fed Board of Governors, Indiana, Penn State, Pompeu Fabra (CREI), St. Louis Fed, Tufts, UC Davis, UC San Diego, UW Madison. Any errors remain my own.

"*We must look at the price system as such a mechanism for communicating information if we want to understand its real function*"

Friedrich A. Hayek, "The use of Knowledge in Society"
*American Economic Review*, 1945, p. 526

# 1 Introduction

One of the major challenges faced by modern macroeconomic research is the formulation of a model for short-run aggregate supply that is both empirically plausible and theoretically sound. In particular, it has proven difficult to develop a micro-founded model of firm pricing strategies whose theoretical predictions are consistent with existing empirical evidence on the behavior of prices both at the individual and at the aggregate level[1]. Empirical studies emphasize the relevance of past realizations in the determination of current aggregate price dynamics. In order for theoretical models to generate a similar prediction, some sort of frictions in price adjustments have to be assumed rather than arise endogenously. Exogenous adjustment frictions imply that expected future realizations become important for current price dynamics. Persistence of prices does arise endogenously in models with exogenous frictions, but these models appear empirically inferior to those in which the relevance of past realizations is directly postulated. This paper develops methods aimed at understanding whether the relevance of past realizations for current aggregate price dynamics may arise endogenously via information asymmetries across agents.

The paper builds on the work of Woodford (2003a) on imperfect common knowledge and price rigidity. Subsequent works by Adams (2006), Hellwig (2005), Kawamoto (2004) and Lorenzoni (2007a, 2007b) have further explored the potential for incomplete information to generate price rigidities. While this new literature has uncovered interesting insights, for example on the role of public and private information in explaining the dynamics of aggregate price, there are still important outstanding theoretical questions to be addressed[2]. The formulation and explicit solution of models of incomplete information remains technically difficult because of the heterogeneity in information and the consequent signal extraction problem involved in the formulation of model-consistent expectations on the part of agents. As a consequence, most recent studies on the subject,

[1] For a discussion of firms pricing models and the dynamics of aggregate price and inflation see for example Fhurer (2005) and Galati and Melick (2006).

[2] For a review of the new generation of monetary business cycle models with imperfect information see Hellwig (2006).

in order to obtain closed-form solutions, are forced to place certain restrictions on the sources of information available to agents. In particular, agents are often not allowed to extract information from variables that emerge in equilibrium[3]. This means, for example, that the aggregate price in these models does not convey information, a role that Hayek advocated as key for the price system in general. In addition, this implies that the persistent heterogeneity of agents' information sets is, in essence, built into these models from the outset.

The objective of this paper is to study equilibria where agents are allowed to extract information both from exogenous and endogenous variables. We formalize this type of equilibria by introducing the notion of an Information Equilibrium (IE), which corresponds to a rational expectations equilibrium where expectations are conditioned on the history of endogenous variables, such as prices and forecast errors. The characterization of the space of Information Equilibria is difficult because a candidate equilibrium has to be consistent with agent's strategies and with agents' information sets upon which the strategies are formulated. To make progress we follow the methodology of Futia (1981) which consists of two steps. We first characterize explicitely two broad families of equilibria under exogenous information, we call them Morris-Shin equilibria and Lucas equilibria. In a Morris-Shin equilibrium the aggregate price lies persistently away from the price under full information. In a Lucas equilibrium the aggregate price is different from the full information price at impact but then it mimics the dynamics under full information in subsequent periods. We then study whether an arbitrary element of each one of these two families belongs also to the space of Information Equilibria. We are able to show that, on the one hand, no Morris-Shin equilibrium can be an Information Equilibrium; on the other hand, every Lucas equilibrium is an Information Equilibrium. We are not able to rule out the possibility that and Information Equilibrium is not a Lucas equilibrium. In this sense, our characterization of the space of Information Equilibrium, although very informative, is only partial. In addition, we show that for any equilibrium aggregate price belonging to the class of Information Equilibria that we are able to characterize, the ability to act as a sufficient statistics for unoservable aggregate disturbances is related to the degree of strategic complementarity in the price setting behavior. We show that this has important implications for the identification of the correct impulse response function of the aggregate price to aggregate disturbances.

We develop our results in the context of a simple pricing model based on the widely used framework synthesized in Woodford (2003b). In the model economy, a large number of intermediate product firms compete monopolistically by setting prices to maximize expected profits. Firms are

[3] Lorenzoni (2007a,b) are important exceptions to this.

free to adjust their pricing strategies in every period at no cost. Intermediate product firms hire labor from households and combine it with a technology which is subjected to a productivity shock that is specific to the firm. Firm-specific productivity disturbances are thus the primary source of heterogeneity across firms in the model. As one would expect, firms' pricing strategies take the form of a markup over real marginal costs. In turn, real marginal costs are a function of the level of aggregate activity, the aggregate price and the specific productivity of firms. The source of strategic complementarity in firms pricing strategies comes from the sort of real rigidities emphasized by Ball and Romer (1990). Under complete information concerning aggregate disturbances, firms correctly anticipate the level of aggregate activity and of the aggregate price. As a result, individual prices differ across firms solely because of the difference in productivity across firms. In contrast, under incomplete information, firms form expectations of the aggregate activity and price. In this case, individual prices differ across firms because of the different expectations formulated on the basis of privately and publicly observed information.

Throughout our analysis we work with analytic functions, i.e. complex-valued functions that are square integrable. Analytic functions are extremely powerful in the implementation of our two-step strategy. On the one hand, they allow for the derivation of prediction formulas under heterogeneous information in closed form and facilitate tremendously the solution of a fixed point problem in linear rational expectations environments. In fact, we are able to derive simple formulas for both Morris-Shin and Lucas equilibrium prices even though the state of the economy is randomly changing over time. On the other hand, the consistency of the information set of a Morris-Shin or a Lucas equilibrium with the definition of an Information Equilibrium can be checked by studying the points where the analytic function that summarizes the information available in equilibrium intersects with the complex plane. The study of such "zeros" is at the heart of our key results.

The remaining of the paper is organized as follows. In Section 2, the model economy is presented and the equilibrium aggregate price is derived in linear form. Section 3 presents the tools in the space of analytic functions needed to solve for prediction formulas and the equilibrium fixed point problem. Section 4 introduces formally the notion of an Information Equilibrium and then characterizes the spaces of Morris-Shin and Lucas equilibria. Section 5 reports and proves the key results that characterize the space of Information Equilibria. Section 6 concludes. Detailed proofs of the results of Sections 2 to 4 are reported in the Technical Appendix available separately.

# 2 The Model

## 2.1 Aggregate Demand, Supply and Equilbrium

**Aggregate Demand** The demand side of the economy consists of a large number of infinitely lived households uniformly distributed on the unit interval indexed $i \in [0,1]$. Households rank alternative consumption-labor sequences according to intertemporal preferences representable by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t \left( \log C_t - \frac{N_t(i)^{1+\psi}}{1+\psi} \right), \ \delta \in (0,1), \psi > 0, \tag{1}$$

where $\delta$ is the subjective discount factor, $C_t$ is the amount of final good consumed by household $i$ and $N_t(i)$ is the amount of time worked. It is assumed that labor services are completely specialized so household $i$ can supply its labor exclusively to an intermediate product firm producing an intermediate good of type $i$.

Households have access to complete markets to insure against idiosyncratic labor income risk. As a result, even though workers receive a different wages and supply different amounts of labor, $C_t$ is the same across households . These set of assumptions lead to a supply of worker-specific labor services that depends positively on the real wage paid by firm $i$ and negatively on the level of consumption $C_t$. Namely,

$$N_t(i) = \left( \frac{W_t(i)}{P_t C_t} \right)^{\frac{1}{\psi}},$$

where $\frac{1}{\psi}$ is the Frisch elasticity of labor supply.

The demand for money balances arises because of a cash-in-advance constraint on the purchase of final good. Notice that, given the complete markets assumption, the demand for money is equal across households as they are ex-ante identical in every period. It is assumed that the nominal interest rate is always positive so that the cash-in-advance constraint is always binding. These set of observations and assumption together with the clearing condition in the money market lead to the relationship $M_t = P_t C_t$ which summarizes the aggregate demand side of the model and where $M_t$ is the process for the supply of money.

Assuming that households can perfectly observe the aggregate state of the economy at period $t$ when formulating their optimal strategies allows one to abstract from issues of incomplete information and asset prices. This assumption is made in order to focus solely on the information interactions arising from the supply side of the economy. It is also necessary to assume that firms cannot observe the prices at which assets are traded in financial markets as they would immediately reveal the state of the economy.

**Aggregate Supply** The production side of the economy consists of a monopolistically competitive intermediate goods sector and a perfectly competitive final good sector. The final good sector produces the composite final good $Y_t$ using intermediate goods $Y_t(i)$ according to the Dixit-Stiglitz CES technology with elasticity $\eta > 1$. Final good firms maximize profits which, in equilibrium, are identically zero. The price emerging in the final good sector takes the standard form

$$P_t = \left( \int_0^1 P_t(i)^{1-\eta} di \right)^{\frac{1}{1-\eta}}, \tag{2}$$

where $P_t(i)$ is the price of the intermediate good $i$. The intermediate product sector is assumed to be populated by a large number of firms indexed by $i \in [0,1]$. Each firm produces a specific product variety, also indexed by $i$, according to a specific labor input technology whose total productivity is proportional to a stochastic factor $A_{it} > 0$,

$$Y_t(i) = \mu A_{it} N_t(i)^{\frac{1}{\mu}}, \quad \mu \geq 1. \tag{3}$$

Firm $i$ is matched with consumer $i$ and acts as a monopsonist in the labor market for its specific labor services by optimally positioning itself on the household $i$ labor supply function. Firm $i$ sets its price $P_t(i)$ and commits itself to sell any amount of its product demanded by the final good sector at that price in period $t$. The pricing problem for the intermediate product firms can then be written as

$$\max_{P_t(i)} \mathbb{E}\left[ P_t(i) Y_t(i) - W_t(i) N_t(i) | \Omega_{it} \right], \tag{4}$$

where $\Omega_{it}$ denotes the information set of firm $i$ at time $t$. Ownership of intermediate goods firms is non-tradable and distributed equally across consumers. In addition, markets for risk sharing on idiosyncratic profits uncertainty are assumed to be absent[4]. This demand is assumed to remain bounded. In equilibrium the market for the final good clears so that $C_t = Y_t$. It follows that the optimal pricing strategy of firm $i$ can be written as a function of three arguments: the aggregate price level, $P_t$, the aggregate production, $Y_t$ and firm-specific productivity, $A_{it}$.

**Equilibrium** In our definition of an aggregate equilibrium for this economy, we follow Hellwig (2005) and Lorenzoni (2007a,b). We focus on symmetric stationary rational expectations equilibria (or symmetric Bayesian-Nash equilibria) in which the prices of intermediate goods are functions only of intermediate firms' information sets at time $t$. A symmetric stationary equilibrium is

[4] The assumption of incomplete markets on privately held information, admittedly not without loss of generality, is common in the recent literature on imperfect information in monopolistic competitive markets, see Lorenzoni (2007a,b).

defined by a set of functions for consumption, money demand, assets demand, labor supply, wages, $P(\cdot)$ and $P(i)(\cdot)$ such that: (i) households maximize utility subject to their budget constraint and the cash in advance constraint; (ii) $P(\cdot)$ and $P(i)(\cdot)$ are such that final good producers and intermediate good producers maximize expected profits; (iii) all markets clear; (iv) individual beliefs are consistent with the distributions emerging in equilibrium. The symmetry of the equilibrium implies in particular that the functions $P(i)(\cdot)$ are identical across intermediate goods producers.

## 2.2 Log-Linearized Economy

It is convenient to express the equilibrium conditions in linear form under the assumption of a stationary rational expectations equilibrium. In what follows, lowercase letters stand for the natural log of their uppercase analogue. We assume that firm $i$ specific productivity $a_{it}$ is the sum of a component $a_t$ that is common across agents and of an idiosyncratic component $v_{it}$ so that

$$a_{it} = a_t + v_{it}. \tag{5}$$

The process for $v_{it}$ is i.i.d. with zero mean and variance $\sigma_v^2$ and the cross sectional distribution of $v_{it}$ is assumed to satisfy $\int_0^1 v_{it} di = 0$. The common component is assumed to be a weakly stationary process given by $a_t = a(L) u_t$. At some point we will allow for a unit root to be present in $a(L)$ without affecting our results. The innovation in aggregate productivity $u_t$ is i.i.d. with zero mean and variance $\sigma_u^2$. Aggregate productivity satisfies $\int_0^1 a_{it} di = a_t$.

The equilibrium aggregate price level $p_t$ in the log-linearized economy is [5]

$$p_t = \int_0^1 p_t(i)\, di. \tag{6}$$

Given a process for money supply, $m_t$, money market clearing implies the canonical quantity equation

$$m_t = p_t + y_t. \tag{7}$$

From the linearized first order conditions from (4), the optimal pricing function for a generic firm $i$ can be written as[6]

$$p_t(i) = \gamma \mathbb{E}(m_t|\Omega_{it}) + (1-\gamma)\mathbb{E}(p_t|\Omega_{it}) - \gamma a_{it}, \tag{8}$$

where

$$\gamma \equiv \frac{\mu(1+\psi)}{1+\eta(\mu(1+\psi)-1)}.$$

[5] This approximated form for the relationship between the log of the aggregate price and the log of individual prices is standard in the monopolistic pricing literature. Its derivation is reported in the technical appendix.

[6] A detailed derivation is reported in the Techincal Appendix

The optimal relative price for firm $i$ at time $t$ is a decreasing function of its own realized productivity, $a_{it}$, and an increasing function of aggregate activity, $y_t = m_t - p_t$. The latter captures the effect of real aggregate demand on the price of factor inputs necessary to operate firm $i$'s technology and the effect of real aggregate supply in the demand that firm $i$ faces from the final good sector. The parameter $\gamma$ represents the elasticity of the optimal real price of firm $i$ to its real marginal costs which in part depend on real aggregate activity, this parameter measures the real rigidity in the model economy in the sense of Ball and Romer (1990). For low values of $\gamma$ the firm relative price is insensitive to the conditions on real marginal costs, in other words the supply is very elastic as real rigidity implies that the firm nominal price is always adjusted in order to stay as close as possible to the aggregate nominal price. In this sense a low value for $\gamma$ is associated with a strong degree of strategic complementarity in nominal price setting. The value of the parameter $\gamma$ determines whether the intermediate products are, respectively, strategic substitutes or strategic complements. In the presence of strategic substitutability (complementarity), a generalized increase in the prices charged by other firms triggers a decrease (increase) in a firm individual price. The assumptions made on the structural parameters of the model ensure that firms' pricing strategies are strategic complements, i.e. $0 < \gamma < 1$.

Notice that, writing the optimal price $p_t(i)$ as in (8), implies that (6) conveniently summarizes the market clearing condition of this stylized economy. Substituting (8) into (9) implies

$$p_t = \gamma \bar{\mathbb{E}}_t(m_t) + (1-\gamma)\bar{\mathbb{E}}_t(p_t) - \gamma a_t, \tag{9}$$

The upper-barred expectational notation represents the average of individual expectations and it is formally defined as

$$\bar{\mathbb{E}}_t(x_t) \equiv \int_0^1 \mathbb{E}_t(x_t|\Omega_{it})\, di. \tag{10}$$

Equation (9) illustrates that the equilibrium aggregate price level depends on aggregate productivity, on the average (or "market") expectations of the nominal aggregate demand - the money supply process under our assumptions - and on the average expectations of the price level it self[7].

**Higher Order Expectations Solution** There exist several approaches to the solution of (9). It is instructive to first solve (9) for $p_t$ using a direct recursive technique as used, for example, by

[7] Both (6) and (10) appeal to a law of large numbers for a continuum of random variables. Judd (1985) shows that while, in general, nothing guarantees that when a continuum of agents draw independent draws from the same distribution $F$ , the sample distribution of realized draws coincides with $F$, it is always possible to construct probabiliy measures which are extensions of the Kolmogorov measure and which satisfy the law of large numbers in the continuum. In what follows we assume that we work with such measures so any random variable emerging from aggregation is well defined almost surely.

Morris and Shin (2002) and by Bacchetta and Van Wincoop (2005). Each firm forms its expectations rationally over the aggregate price[8], so

$$\mathbb{E}_t^i(p_t) = \gamma \mathbb{E}_t^i \bar{\mathbb{E}}_t(m_t) + (1-\gamma)\mathbb{E}_t^i \bar{\mathbb{E}}_t(p_t) - \gamma \mathbb{E}_t^i(a_t). \tag{11}$$

Notice that firms are forming expectations over the market expectations of both $m_t$ and $p_t$. Keynes (1936) called this behavior the *beauty contest* effect. As noted by Morris and Shin (2002), heterogeneity of information breaks the law of iterated expectations in that $\mathbb{E}_t^i \bar{\mathbb{E}}_t(m_t) \neq \mathbb{E}_t^i(m_t)$. As firms attempt to forecast the average forecast of the market they engage in a game of guessing the average guess of the average guess and so on. Proceeding in this fashion, the solution for the price level can be written as the sum of the expectations of increasing order about aggregate productivity and nominal aggregate demand,

$$p_t = \gamma \sum_{j=1}^{\infty} (1-\gamma)^{j-1} \bar{\mathbb{E}}_t^{(j)}(m_t) - \gamma \sum_{j=0}^{\infty} (1-\gamma)^{j} \bar{\mathbb{E}}_t^{(j)}(a_t). \tag{12}$$

The notation $\bar{E}_t^{(j)}$ stands for the $j$-th order expectation, where $\bar{\mathbb{E}}_t^{(0)}(a_t) \equiv a_t$.

### 2.3 Full Information Equilibrium

The full information rate of output in this context can be computed by assuming that firms have complete information about aggregate productivity and about the process for nominal aggregate demand, $m_t$. It follows that $\mathbb{E}_t^i(a_t) = a_t$ and $\mathbb{E}_t^i(m_t) = m_t$, for every $i$. Under these informational assumptions, both higher-order expectational structures in (12) collapse into a single term and the full information rate of output $y_t^*$ and the associated price level $p_t^*$ take the form

$$\begin{aligned} y_t^* &= a_t, \\ p_t^* &= m_t - a_t. \end{aligned}$$

Under full information the rate of output coincides with the aggregate productivity $a_t$. The dynamics of money supply is fully anticipated and entirely allocated to the dynamics of the price level, thereby resulting completely irrelevant for the determination of full information output[9]. The price level corresponding to a full information output also varies over time. This occurs, not only because of the one-to-one effect of the monetary base, but also because productivity has an aggregate

[8] $\mathbb{E}_t^i(p_t)$ is short notation for $\mathbb{E}_t(p_t|\Omega_{it})$.

[9] Uniqueness of an equilibrium realizes because the beliefs off-equilibrium path are specified uniquely by the *form* of the aggregate demand, which is assumed to be common knowledge even when $m_t$ is not perfectly observed.

effect on the marginal cost structures of the intermediate goods sector. As the productivity of the intermediate good sector increases, the final good becomes cheaper and its production increases. However, the increase in production is obtained without affecting the level of aggregate employment which is $n_t = \mu (y_t - a_t)$ so that when $y_t = y_t^*$ one has $n_t^* = 0$. While, under full information, the degree of real rigidity $\gamma$ does not affect the aggregate dynamic behavior of the system, it does affect the cross-sectional distribution of prices, production, hours worked and wages according to

$$\begin{aligned} p_t^*(i) &= -a_t - \gamma v_{it}, & y_t^*(i) &= a_t + \eta\gamma v_{it} \\ n_t^*(i) &= \mu(\eta\gamma - 1) v_{it}, & w_t^*(i) - p_t^* &= a_t + \psi\mu(\eta\gamma - 1) v_{it}, \end{aligned}$$

where $\eta\gamma > 1$. It follows that, when the productivity of firm $i$ is higher than aggregate productivity ($v_{it} > 0$), the real wage for labor of type $i$ is higher than the economy-wide average $a_t$. Basically, when firm $i$ is on average more productive, it is optimal to reduce its relative price, this will increase its demand and therefore its production. The increase in production is obtained with an increase in employment as well as the increased productivity. However, an increase in hours worked is possible only by increaesing real wages as markets for factors are separated. This will limit the reduction in firm $i$'s relative price, hence the real rigidity. The key here is that consumers offer labor according to real wages measured in terms of the final product which they consume. It follows that a decrease in firm $i$'s price will not affect real wages through the price of the final product because of the infinitesimal dimension of the firm. On the other hand, if there is an increase only in aggregate productivity it is optimal for firm $i$ not to change its relative price (i.e. the real rigidity is maximal in this case). To see this, suppose that the nominal price of firm $i$ is decreased, then the demand for its product will increase and the increase in production will be satisfied in part by higher employment. In order to increase the hours worked, the firm will need to offer a higher real wage. Differently from before, as every firm will be reducing nominal prices, real wages will be increased due to a cheaper final product. Nevertheless, production in equilibrium, and thus consumption, will be higher thereby decreasing the marginal utility of consumption and making hours worked less appealing. Overall both effects cancel each other, hours worked do not change and the increase in production will result only from the increase in productivity. In addition, as all firms will be changing their nominal price by the same measure, the real prices across firms will remain unchanged. It follows that a perfeclty observed change in aggregate productivity has no effect on real prices, as anticipated.

# 3 Analytic Function Approach

The characterization of a rational expectations equilibrium in presence of incomplete information requires two steps. First, given a generic form for the equilibrium, an expression for agents' expectations of relevant variables has to be formulated. Second, conditional on such an expression, a fixed point condition has to be solved in order to characterize the generic form of the equilibrium initially specified. In this section we show that both steps can be more conveniently solved using (complex valued) functions instead of recursive formulas and sequences of undetermined coefficients. We present the relevant tools in turn.

## 3.1 Optimal Prediction

The information sets of agents are specified as a sequence of random vectors $\{\mathbf{x}_t\}$ and a filtration $\mathcal{F}$ (i.e. a sequence of $\sigma$-algebras $\Phi$ satisfying $\Phi_t \subset \Phi_{t+1}$) such that $\mathbf{x}_t$ is $\Phi_t$-measurable for every $t$. In a linear stationary environment the filtration $\mathcal{F}$ can be equivalently represented as a sequence of Hilbert spaces spanned by the history of realizations of $\mathbf{x}_t$. It follows that for an agent the history of realizations of the $n \times 1$ covariance stationary random vector, $\mathbf{x}_t$, the information set at the beginning of period $t$, denoted $\Omega_t$, is formally defined by the Hilbert space generated by the random vector $\mathbf{x}$ up to time $t$, $\Omega_t = \mathbb{H}_t(\mathbf{x}) = \underset{j=1,n}{\oplus} \mathbb{H}_t(x_j)$. The operator $\oplus$ denotes the operation of forming the smallest closed subspace containing the elements specified in its argument. We represent the dynamic flow of information in our model using Hilbert spaces. The relative amount of information contained in two Hilbert spaces about a random variable of interest can be measured by the properties of the class of bounded mappings (i.e. an analytic complex valued function) from one space to the other. If there exists a mapping that is analytic and invertible for $|z| \leq 1$, then the two information sets are equal.

Agents want to predict a covariance stationary vector $\mathbf{p}_{t+v}$ ($v \in \mathbb{N}$) by minimizing the mean squared error $\mathbb{E}\left(\hat{\mathbf{p}}_{t+v|t} - \mathbf{p}_{t+v}\right)^2$. The solution of this problem is well known to be the orthogonal projection from the space containing the variable that is to be predicted onto the space spanned by the observed variables available for prediction. When the random variables are jointly normally distributed, the linear projection coincide with the conditional expectation for the random variables, formally,

$$\mathbb{E}\left(\mathbf{p}_{t+v} | \mathbb{H}_t(\mathbf{x})\right) = \sum_{j=0}^{\infty} \Pi_j \mathbf{x}_{t-j},$$

where $\Pi_j$ are the coefficients of the orthogonal projection of $\mathbb{H}_{t+v}(\mathbf{p})$ onto $\mathbb{H}_t(\mathbf{x})$. A powerful method for solving linear projection problems is the recursive representation offered by the Kalman

filter. However, in order to implement the Kalman filter one needs to describe a well defined state for the system under study. Unfortunately, in environments of heterogeneous information the definition of a state is usually problematic[10]. To overcome this difficulty, we take a different route and express the prediction formula using the so called Wiener-Hopf technique[11]. The main advantage of this method lies in its ability to express the prediction formula directly in terms of the information set on which the prediction is based, without defining a state variable. The following result is from Whittle (1983).

**Theorem 1 (Wiener-Hopf Prediction Formula)** *Suppose that* $\{\mathbf{x}_t\}$ *and* $\{\mathbf{p}_t\}$ *are covariance stationary random processes with known statistical properties. The linear least squares estimate* $\hat{p}_{t+v|t}$ *is then provided by the linear combination of the history of* $\mathbf{x}_t$ *denoted by* $\Pi(z)$ *and given by:*

$$\Pi(z) = \left[ z^{-v} \mathbf{g}_{\mathbf{px}}(z) \left( B^*\left(z^{-1}\right)' \right)^{-1} \right]_+ \left(B^*(z) V\right)^{-1}, \tag{13}$$

*where* $g_{\mathbf{px}}(z)$ *is the covariance generating matrix function for* $\mathbf{p}$ *and* $\mathbf{x}$*,* $B^*(z) V$ *is the "canonical" factorization of the autocovariance generating function for* $\mathbf{x}$*, denoted by* $g_{\mathbf{xx}}(z)$*, and the notation* $[\ ]_+$ *instructs to ignore terms in negative powers of* $z$*.*

The expression for $\Pi(z)$, although quite convoluted, posses an intuitive structure. Both the variable to be predicted $\mathbf{p}$ and the system of signals $\mathbf{x}$ can be represented in terms of "primitive" or "structural" innovations. Let the structural representation for the random vector $\mathbf{x}_t$ be

$$\mathbf{x}_t = \underset{m\times n}{B(L)} \varepsilon_t. \tag{14}$$

The lag polynomial matrix $B(L)$ is assumed to be analytic, i.e., all its elements can be represented as a one sided sequence in $L$ whose coefficients are square-summable. In addition it is assumed that the matrix $B(L)$ is normalized so that the variance-covariance matrix of the random vector of disturbances, $\varepsilon_t$, is the identity matrix, $\underset{n\times n}{I}$.The number of rows $m$ correspond to the number of

[10] On this point see Sargent (1991) and Pearlman and Sargent (2005). Interestingly, Woodford (2003) is able to employ the Kalman filtering approach as he defines the state as a linear combination of the infinite higher order expectational terms. It is not clear however whether such a representation can be extended to settings where endogenous variables enter agents information set. In fact, Lorenzoni (2006a) considers endogenous information sets but he is forced to truncate the state space in order to compute an equilibrium.

[11] The Wiener-Hopf technique has been applied to economics by Whiteman (1985) and Taub (1989). Taub (1989) shows that the technique can be used to solve simultaneously for an optimization problem and a signal extraction problem when the former is of the linear-quadratic type.

observable variables (the dimension of $\mathbf{x}_t$) and the number of columns $n$ correspond to the number of structural disturbances (the dimension of $\boldsymbol{\varepsilon}_t$). Some of the structural innovations are common across $\mathbf{p}$ and $\mathbf{x}$ and this allows the prediction of $\mathbf{p}$ from $\mathbf{x}$. The extent of the commonality of the innovations across the two variables is measured by $\mathbf{g_{px}}(z)$. How well are the primitive innovations measured by the observation of $\mathbf{x}$? The answer lies in the "canonical" factorization matrix $B^*(z)$ which extracts from $\mathbf{x}_t$ the optimal inference about the primitive innovations thereby delivering the optimal linear predictor of Theorem 1. Notice that when the problem is one of pure prediction, i.e., one needs to predict the elements of $\mathbf{x}_t$ in the future from the past of $\mathbf{x}_t$ itself, Theorem 1 result returns the prediction formula,

$$\hat{\mathbf{p}}_{t+v|t} = \hat{\mathbf{x}}_{t+v|t} = \left[B^*(L)\,L^{-v}\right]_+ \mathbf{w}_t. \tag{15}$$

This expression has the form of the familiar Wiener-Kolmogorov prediction formula. However, (15) is more general than the version commonly used in macroeconomic time-series analysis because of the non-standard information set on which the prediction is based. The optimal signal extraction is embedded in the mapping between the pair $B(L)$ and $\boldsymbol{\varepsilon}_t$ and the pair $B^*(L)$ and $\mathbf{w}_t$. Usually this mapping is assumed to be a constant. In the present context, because of incomplete heterogeneous information, this mapping is a linear combination of all the elements of the first pair $B(L)$ and $\boldsymbol{\varepsilon}_t$.

The most involved computational step to obtain $\Pi(z)$ is the specification of the canonical matrix $B^*(z)$ satisfying

$$\underset{m\times n}{B(z)}\underset{n\times m}{B\left(z^{-1}\right)'} = \underset{m\times m}{B^*(z)}V\underset{m\times m}{B^*\left(z^{-1}\right)'}.$$

such that $B^*(z)$ is analtyic and invertible for $|z| \leq 1$ and $B^*(0) = 1$. Depending on the properties of the matrix function $B(z)$ several tecniques have been proposed to derive the canonical factorization. Broadly speaking, $B^*(z)$ is different from $B(z)$ when the latter one is non-invertible for $|z| \leq 1$. This can happen for two reasons: (i) the matrix $B(z)$ is rectangular, which means that the signals observed are fewer than the innovations that one would like to know; (ii) the matrix $B(z)$ (or any of its principal minors) is singular at some $|z| \leq 1$. In case (ii), the removal of the singularities can be achieved by the use of a Blaschke factor as showed by Rozanov (1967) and later reproposed by Lippi and Reichlin (1994). Case (i) is usually more involved and no simple results can be found in the existing literature. We report here a novel result that will be extremely useful in the subsequent analysis.

Consider the univariate random process

$$a_t = \rho a_{t-1} + u_t,$$

where $u_t$ is an i.i.d. process with variance $\sigma_u^2 < \infty$ and $|\rho| \leq 1$. Suppose that two signals is available for prediction, namely

$$y_t = a_t + v_t \quad \text{and} \quad x_t = a_t + \varepsilon_t,$$

where $v_t$,$\varepsilon_t$ are both i.i.d. processes with variances $\sigma_v^2, \sigma_\varepsilon^2 < \infty$. In signal-extraction problems is usually convenient to work with the notion of the *precision* of a signal. We define the precisions of the signals $y$ and $x$ as

$$\tau_v \equiv \frac{\sigma_u^2}{\sigma_v^2} \text{ and } \tau_\varepsilon \equiv \frac{\sigma_u^2}{\sigma_\varepsilon^2}.$$

For future convenience, we also define the function

$$\lambda(s) \equiv \frac{1}{2}\left[\frac{1}{\rho}s + \left(\frac{1}{\rho} + \rho\right) \pm \sqrt{\left(\frac{1}{\rho}s + \left(\frac{1}{\rho} + \rho\right)\right)^2 - 4}\right], \text{ s.t. } |\lambda(s)| < 1.$$

We can now state two very useful results.

**Proposition 1** *The canonical factorization of the variance-covariance generating function is*

$$B^*(z) = \frac{1}{1-\rho z}\begin{pmatrix} 1 - \frac{\lambda\tau_v + \rho\tau_\varepsilon}{\tau_v + \tau_\varepsilon} z & \frac{\tau_\varepsilon(\rho-\lambda)}{\tau_v+\tau_\varepsilon} z \\ \frac{\tau_v(\rho-\lambda)}{\tau_v+\tau_\varepsilon} z & 1 - \frac{\lambda\tau_\varepsilon + \rho\tau_v}{\tau_v+\tau_\varepsilon} z \end{pmatrix}$$

$$V = \frac{1}{\tau_v + \tau_\varepsilon}\begin{pmatrix} \left(\frac{\tau_\varepsilon}{\tau_v} + \frac{\rho}{\lambda}\right) & (\rho - \lambda) \\ (\rho - \lambda) & \left(\frac{\tau_v}{\tau_\varepsilon} + \frac{\rho}{\lambda}\right) \end{pmatrix},$$

where

$$\lambda \equiv \lambda(\tau_v + \tau_\varepsilon).$$

**Corollary 1** *The optimal prediction of $a_t$ conditional on the current and past history of signals $y^t = \{y_t, y_{t-1}, ..\}$ and $x^t = \{x_t, x_{t-1}, ..\}$ is*

$$\mathbb{E}\left(a_t | y^t, x^t\right) = \frac{1}{\rho(1-\rho\lambda)}\frac{1}{1-\lambda L}(\tau_v y_t + \tau_\varepsilon x_t).$$

Proposition 1 and Corollary 1 show how the methods in analytic function theory are able to deliver simple closed form solutions for the prediction formulas in contexts of signal extraction from dynamic stochastic processes. Notice that Corollary 1 holds at the limit for $\rho \to 1$, i.e. it applies also when the process $a_t$ is a random walk. In section 4 we will see that the function $\lambda(s)$ plays a pivotal role in the learning dynamics in equilibrium. It essentially measures the extra-persistence of the innovation to the state variable that is embedded into the prediction formula because of the

imprecision of the signals on which the formula is based. As one can see from Corollary 1, the parameter $\lambda$ measures the persistence in the prediction term due to the uncertainty of whether an observed innovation in the signals $y$ or $x$ belongs to $a_t$ or to noise. It is easy to show that $|\lambda(s)|$ is inversely related to the precisions of the signals, formally $|\lambda(s)|' < 0$. In addition, any extra-persistence due to learning cannot possibly be higher than the persistence already present in the state variable, i.e. $|\lambda(s)| < |\rho|$ for $s > 0$. The term $\lambda$ in Corollary 1 also measures the effect of innovations to signals on the prediction of $a_t$ at impact. Since $sign(\lambda) = sign(\rho)$, the effect at impact has the same sign as $\rho$ and it is positively related to the precision of the signals since $|\lambda(s)|' < 0$. Summarizing, the more precise are the signals, the stronger is the reaction of $\mathbb{E}\left(a_t|y^t, x^t\right)$ at impact and the smaller is the extra-persistence due to learning. The properties of $\lambda(s)$ are showed more in detail in the appendix.

### 3.2 Fixed Point

Once the prediction formula has been derived as a function of the generic guess for the equilibrium variable, the next step involves the solution of a fixed point problem. We argue here that the analytic function approach is very powerful in solving for the set of coefficients of the linear equilibrium function. Let $\mathbf{p}_t$ be a sub-vector of $\mathbf{x}_t$ and let the equilibrium process be of the form

$$\mathbf{p}_t = \sum_{j=0}^{\infty} P_j \mathbf{u}_{t-j} = P(L)\,\mathbf{u}_t.$$

$P(L)$ is said to belong to the space of square-summable sequences $\ell^2$ if $\sum_{i=0}^{\infty} P_j^2 < \infty$. The space $\ell^2$ is normally referred to as the *time domain*. A powerful result in representation theory (the Riesz-Fischer theorem, see Sargent (1987) chapter XI) shows that the space of square summable sequences and the space of squared integrable analytic functions $L^2$ are *isometrically isomorphic*. This means that one can work in whichever space is more convenient since there always exists a mapping that allows one to recover a representation in $\ell^2$ from an analogous representation in $L^2$ and viceversa. The space $L^2$ is also known as the *Hardy* space. When the analytic functions are evaluate *on* the unit circle (i.e. at $z = e^{-i\omega}$ for $\omega \in [-\pi, \pi]$), the Hardy space is normally referred to as the *frequency domain*.

Suppose that $\mathbf{p}_t$ is endogenously defined by the linear equilibrium condition

$$\mathbf{p}_t = \gamma a(L)\, u_t + \gamma \bar{\mathbb{E}}(\mathbf{p}_t). \tag{16}$$

Solving for an equilibrium in the time domain in a linear stationary environment means solving a fixed point problem in the space of sequences of real numbers. This is normally achieved through the

method of undetermined coefficients which starts with a specific guess (a finite set of coefficients) of the equilibrium equation and whose solution usually consists of a system of non-linear equations in the equilibrium coefficients. In contrast, casting the problem in the Hardy space allows one to solve a fixed point problem in the space of functions. In particular, in problems of incomplete information where the signal extraction from endogenous variables is allowed in equilibrium, a specific guess might overlook the existence of equilibria that require a higher dimensional state space (possibly infinite). As a consequence, analytic function methods are more powerful than time domain methods in such models[12].

Despite this, the characterization of the space of equilibria in models of incomplete information with signal extraction from endogenous variable remains difficult. If one could show that an equilibrium under a given endogenous information set is always equal to an equilibrium under an exogenous information set (whose form possibly depend on the form of the endogenous information set assumed), then it would be enough to solve for the equilibrium under the exogenous information set and claim the equivalence. The analysis of the next section aims at establishing such an equivalence by extending the results in Futia (1981). An additional advantage of working with analytic functions is that the equivalence between spaces of equilibrium functions can be verified by studying the points where the equilbrium functions intersect with the complex plane, i.e. where $|z| \leq 1 : \det P(z) = 0$.

Let the equilibrium process in (16) be the analytic function $P(z) \in L^2$ and denote by $s$ the number of zeros of $P(z)$ for $|z| \leq 1$. Then the fixed point condition is

$$P(z) = \gamma a(z) + \gamma T(P(z)), \tag{17}$$

where $T(.)$ combines the mapping between the structural representation, the canonical representation and the optimal prediction formula. Because of this mapping, the function $P(z)$ has to satisfy condition (17) not only at the dynamic level but also at the informational level. The information conveyed by $\mathbf{p}_t$ in equilibrium should be consistent with the information used to compute the expectations for $\mathbf{p}_t$ in the first place. To reiterate on the key point of this section, this consistency requirement can be easily checked by studying the zeros of the analytic function $P(z)$.

# 4 Incomplete Information

In this section we apply the methods of section 3 to derive explicit forms for the expectational terms at the firms' level for the model economy outlined in section 2 under different assumptions

[12] See the discussion in Kasa(2000) on time domain versus frequency domain methods.

on the information sets of agents. We first solve for a price process under information sets that are exogenously given. We then show in section 5 how these solutions relate to a more interesting notion of equilibrium, one where endogenous variables are themselves conveyers of information.

In terms of our pricing model of section 2, it is useful to assume that the money supply is kept constant so that, $m_t = 0\ \forall t$. At the moment, it is convenient to abstract from nominal uncertainty in order to isolate the independent effect of uncertainty on real disturbances[13]. We report the key equations of the pricing model. The optimal individual pricing strategy for an arbitrary firm $i$ is

$$p_t(i) = (1-\gamma)\,\mathbb{E}\left(p_t|\Omega_{it}\right) - \gamma a_{it}. \tag{IP}$$

The equilibrium aggregate price is

$$p_t = \int_0^1 p_t(i)\,di. \tag{AP}$$

Individual productivity is given by (5). In addition, we assume that a *publicly* observable signal for aggregate productivity is available and it is equal to

$$x_t = a_t + \varepsilon_t, \tag{18}$$

where $\varepsilon_t$ is i.i.d. with variance $\sigma_\varepsilon^2$. We denote the precision of the private and public signals by $\tau_v$ and $\tau_\varepsilon$ as defined in section 3. The higher $\tau_v(\tau_\varepsilon)$, the more information about $u_t$ is contained in the signal $a_{it}(x_t)$.

### 4.1 Information Sets and Equilbrium Definitions

In section 2 we solved for an equilibrium under perfect common knowledge, which is the same equilibrium that would emerge if dispersed information was pooled so that $\Omega_{it} = \bigoplus_{i\in[0,1]} \mathbb{H}_t(a_i)$ for every $i$. Here we consider two information sets that allow for imperfect common knowledge.

(MS) *Morris-Shin Information:*

$$\Omega_{it}^{MS} = \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x).$$

(L) *Lucas Information:*

$$\Omega_{it}^{L} = \Omega_{it}^{MS} \oplus \mathbb{H}_{t-1}(a).$$

The two information sets are clearly related according to $\Omega_{it}^{MS} \subseteq \Omega_{it}^{L}$. We call (MS) the Morris-Shin information because it is the type of information assumption in Morris and Shin (2002). Under this

[13] Because the level of nominal activity is assumed to be known and constant, one could rewrite the entire model in real terms and normalize aggregate price to $p_t = 1$ in every period. Firm's individual prices would then express the distance from the aggregate price and aggregate information would be exctracted from realized aggregate output $y_t$.

information set there is a tension between the reaction to individual specific information ($\mathbb{H}_t(a_i)$) and information that is common knowledge ($\mathbb{H}_t(x)$). A special case of (MS) is when the public signal $x_t$ is completely uninformative about productivity, in other words its precision is $\tau_\varepsilon = 0$. Under this information set each agent is essentially solving a signal extraction problem which is completely isolated from the rest of the economy. In other words, there is no common component in agents' information sets besides the aggregate process $a_t$ driving $a_{it}$, which, however, is not common knowledge. This type of information set is the one assumed by Woodford (2003), therefore we will sometime refer to it as the Woodford information set (W). We call (L) the Lucas information set because it corresponds to the information set in Lucas (1973, 1975). Although at time $t$ there exists incomplete information about the current aggregate state of the economy, at the beginning of the next period the current state will be perfectly observed. This information set essentially assumes that the relevant state will be exogenously provided to agents with one period lag. Using the results of section 3 we show how to solve for the pricing model (IP)-(AP) under information (MS) and (L) and we compare the impulse response of both solutions to aggregate shocks in productivity.

With this benchmark in hand we then turn to a more intricated issue, that of characterizing an equilibrium when part of the information set is endogenously determined by the equilibrium behavior itself. It seems in fact reasonable to assume that an arbitrary firm $i$ can observe its own productivity together with the history of the aggregate price. More importantly, any firm can observe its own prediction error with respect to the aggregate price as the prediction error contains crucial information for the updating of the firm's beliefs about aggregate productivity. We work with the information set $\Omega^*_{it}$,

(*) *Endogenous Information*:

$$\Omega^*_{it} = \Omega^I_{it} \oplus \mathbb{H}_{t-1}(p, e_i), \quad I \in \{MS, L\}.$$

*Where the process for the prediction error is defined as*

$$e_{it} = p_t - \mathbb{E}(p_t|\Omega^*_{it}). \tag{PE}$$

Following Futia (1981) we define an important property of any information set. Let the process of interest be defined as $p_t = P(L)\, w_t$, then

*(I) An information set $\Omega_{it}$ is said to be informative when $H_{t-1}(w) \subseteq \Omega_{it}$.*

Notice that, when two information sets $\Omega_{it}$,$\Omega_{jt}$ are informative, then

$$\mathbb{E}[(P(L) - P(0))\, w_t|\Omega_{it}] = \mathbb{E}[(P(L) - P(0))\, w_t|\Omega_{jt}].$$

In other words, the prediction of that part of the process that depends on the information that is common knowledge, are symmetric across agents.

We are now ready to formally introduce the three notions of equilibria central to our results.

**Definition 1** ***Morris-Shin Equilibrium (MSE).*** *A Morris-Shin equilibrium is a stationary price process* $p_t \in \mathbb{H}_t(a, \varepsilon)$ *such that (IP)-(AP) hold with* $\Omega_{it} = \Omega_{it}^{MS}$.

**Definition 2** ***Lucas Equilibrium (LE).*** *A Lucas Equilibrium is a stationary price process* $p_t \in \mathbb{H}_t(a, \varepsilon)$ *such that (IP)-(AP) hold with* $\Omega_{it} = \Omega_{it}^{L}$.

**Definition 3** ***Information Equilibrium (IE)****. An information equilibrium is a stationary price process* $p_t \in \mathbb{H}_t(a, \varepsilon)$ *and a stationary prediction error process* $e_{it} \in \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(a, \varepsilon)$ *such that (IP)-(AP) hold with* $\Omega_{it} = \Omega_{it}^{*}$ *and* $e_{it}$ *defined by (PE).*

The definition (IE) might seem redundant in the sense that, once the price process has been defined, from equation ($PE$) one can immediately derive the prediction error, which is just a function of the aggregate price. However, the endogeneity of the information set $\Omega_{it}^{*}$ requires the definition of an equilibrium not only in terms of the variable to be predicted but also in terms of the prediction errors. In fact, we will show below that there are situations in which $\mathbb{H}_{t-1}(p) \subset \mathbb{H}_{t-1}(p, e_i)$, so that the prediction errors play a fundamental role to inform agent's predictions.

It is important to note that, unlike models of incomplete information in competitive markets, our assumption that current equilibrium prices do not enter agents' current information set makes a complete knowledge of the current state of the economy impossible, unless the variance of the idiosyncratic component of productivity $\sigma_v^2$ is driven to zero. In the equilibrium of our model, prices are set before markets are cleared and only subsequently convey information which, as a consequence, affects only future equilibrium prices[14].

Finally, since our emphasis is on the Hayekian idea of information transmission through the aggregate price process, we define the following property

[14] The choice of letting agents observe the last period equilibrium prices while natural in a setting where agents are price-setters it represents a non-standard assumption in models of competitive markets. However, Hellwig (1982) analyzes a model of competitive capital markets where agents can condition only on past prices. He shows that the impossibility of an informational efficient market (Grossman and Stiglitz, 1980) in that case tends to disappear as the current private information maintains a positive value in equilibrium and it makes investing in collecting information worthwhile.

(H) *A stationary price process* $p_t \in H_t(u, \varepsilon)$ *is said to be informative when* $H_t(u, \varepsilon) \subseteq H_t(p, x)$.

While property (I) applies to any potential information set, property (H) restricts the information set to be the one generated by the history of the aggregate price and the publicly available information.

### 4.2 Morris-Shin Equilibrium

The price equilibrium under the Morris-Shin information set is a function of current and past innovations to aggregate productivity and to the noise component of the public information signal $x_t$. Therefore, we want to solve for the two polynomials $P_u(L)$ and $P_\varepsilon(L)$ in

$$p_t^{MS} = P_u(L)\, u_t + P_\varepsilon(L)\, \varepsilon_t. \tag{19}$$

Using Theorem 1, Proposition 1 one can show the following.

**Proposition 2** *Let*

$$a_t = \rho a_{t-1} + u_t.$$

*The Morris-Shin price equilibrium exists, is unique and is given by (19) with*

$$P_u(L) = -\left(1 - (1-\gamma)\frac{\theta}{\rho}\right) \frac{\left(1 - \frac{\theta\gamma}{1-(1-\gamma)\frac{\theta}{\rho}} L\right)}{(1-\theta L)(1-\rho L)} \tag{20}$$

$$P_\varepsilon(L) = -(1-\gamma)\,\tau_\varepsilon \frac{\theta}{\rho(1-\rho\theta)} \frac{1}{(1-\theta L)} \tag{21}$$

*with*

$$\theta \equiv \lambda(\tau_\varepsilon + \gamma\tau_v). \tag{22}$$

When public information is not informative ($\tau_\varepsilon = 0$), the price equilibrium is $p_t^W = P_u(L)\, u_t$, where $P_u(L)$ is as in (20) and $\theta \equiv \lambda(\gamma\tau_v)$. We call $p_t^W$ the Woodford Equilibrium. We report the key steps in the derivation of the Woodford Equilibrium to show the power of the analytic function approach to deal with fixed point conditions in dynamic contexts. The proof for the more general case ($\tau_\varepsilon > 0$) follows the same lines. Using Theorem 1 one can show that firm $i$'s optimal prediction for the current aggregate price level when $\tau_\varepsilon = 0$ is

$$\mathbb{E}\left(p_t | \Omega_{it}^{MS}\right) = \tau_v \frac{\alpha(1-\rho L)}{\rho(1-\alpha L)} \left[\frac{L P_u(L)}{(L-\alpha)} - \frac{\alpha P_u(\alpha)}{(L-\alpha)}\right] a_{it}, \tag{23}$$

where $\alpha \equiv \lambda(\tau_v)$. Substituting (23) into the optimal pricing function (IP) and aggregating across firms gives the following condition for the equilibrium price function,

$$P_u(z) = -\frac{\gamma\rho\frac{1}{\tau_v}(1-\alpha z)(z-\alpha) - (1-\gamma)(1-\rho z)\alpha^2 P_u(\alpha)}{\left(\rho\frac{1}{\tau_v}(1-\alpha z)(z-\alpha) - (1-\gamma)\alpha z\right)(1-\rho z)}. \tag{24}$$

This expression is not a closed form solution for the price process because of the term $P(\alpha)$ on the right hand side. At $z = \alpha$, both sides are equal to $P_u(\alpha)$, therefore the constant is not determined. Nevertheless, the requirement that the solution has to be stationary provides the additional condition for the constant term (see Whiteman (1983)). If one of the roots in the denominator of (24) is inside the unit circle (which corresponds to an unstable eigenvalue), then the constant $P_u(\alpha)$ can be set so to cancel the root and obtain a unique stationary solution[15]. It can be shown that the denominator in (24) always satisfies this condition as it is proportional to $(1-\theta z)\left(1-\frac{1}{\theta}z\right)$where $\theta = \lambda(\gamma\tau_v)$. The condition to solve for the constant, $P_u(\alpha)$, can be then written as

$$\gamma\rho\frac{1}{\tau_v}(1-\alpha\theta)(\theta-\alpha) - (1-\gamma)(1-\rho\theta)\alpha^2 P_u(\alpha) = 0$$

which leads to $P_u(\alpha) = -\frac{\gamma\theta}{\alpha(1-\rho\theta)}$. Substituting back into (24) gives the equilibrium price in closed form.

**Corollary 2** *The price process for the Morris-Shin equilibrium is always informative.*

**Proof.** The public information available at the beginning of period $t+1$ is given by the entire history of $p_t$ and $x_t$. This information set can be represented as

$$\begin{pmatrix} p_t \\ x_t \end{pmatrix} = \begin{pmatrix} P_u(L) & P_\varepsilon(L) \\ a(L) & 1 \end{pmatrix} \begin{pmatrix} u_t \\ \varepsilon_t \end{pmatrix} \tag{25}$$

[15] In stationary equilibria under rational expectations non-uniqueness arises because, in general, a process for the forecast errors satisfying the orthogonality condition and being consistent with the law of motion for the system can be arbitrarily specified. Stationarity implies a unique solution if the the law of motion of the system contains an explosive autoregressive component, which is the case in (24). Under these conditions the process for the forecast errors cannot be arbitrary but it must be a function of the driving process such that the explosive autoregressive component is "succesfully contrasted". The constant $P(\alpha)$ is chosen in order to accomplish this task and deliver a stationary equilibrium that is unique. In models with forward looking rational expectations, uniqueness of a solution is usually ensured by a similar condition (see Beyer and Farmer (2006) and Cochrane (2006)). This is particularly interesting as no forward lookingness per-se is present in firms' pricing equations.

where $P_u(L)$ and $P_\varepsilon(L)$ are as in (20) and (21). If this polynomial matrix is invertible, then there exists a square summable linear combination of current and past $p_t$ and $x_t$ that returns the current and past history of $u_t$ and $\varepsilon_t$. In other words, the inverse of the polynomial matrix defines a bounded mapping from $\mathbb{H}_t(p,x)$ to $\mathbb{H}_t(u,\varepsilon)$ such that $\mathbb{H}_t(u,\varepsilon) \subseteq \mathbb{H}_t(p,x)$ and so property (H) is satisfied. We want to study the zeros of the determinant $P_u(L) - a(L)P_\varepsilon(L)$. It can be showed that this polynomial is proportional to

$$1 - \frac{\rho}{\frac{\tau_v+\tau_\varepsilon}{1-\rho\theta}+1}L.$$

Since $0 \leq 1-\rho\theta \leq 1$ for $|\rho| \leq 1$, then the root of the above polynomial is always smaller than one in absolute value, therefore the matrix (25) has a square summable inverse in positive powers of $L$.

■

It is instructive to look closely to the solution reported in Proposition 2. Let us first consider the effect of the innovation on productivity on the aggregate price. One can immediately notice that there are both a static wedge, $\left(1-(1-\gamma)\frac{\theta}{\rho}\right)$, and a dynamic wedge, $\left(1-\frac{\theta\gamma}{1-(1-\gamma)\frac{\theta}{\rho}}L\right)(1-\theta L)^{-1}$, that drive the difference with respect to the full information benchmark $p_t^* = -a_t$. The static wedge depends negatively on $1-\gamma$: the stronger the complementarities in price setting, the lower the reaction to the innovation in productivity at all horizons. Everything else equal, the overall uncertainty about the value of the innovation weighs more on the aggregate price. The static wedge depends also negatively on $\theta$: the higher $\theta$ the smaller the effect of an innovation in productivity at all horizons. From (22) we know that $\theta$ depends negatively on the degree of real rigidities $\gamma$ through the modified precision, $\tau_\varepsilon + \gamma\tau_v$. Even when the precision of the private signal is strong, a high $\tau_v$, if the real rigidities are substantial, $\theta$ will still be subtantially away from zero. The dynamic wedge can be written as

$$\left(1-\frac{\theta\gamma}{1-(1-\gamma)\frac{\theta}{\rho}}L\right)(1-\theta L)^{-1} = 1 + \frac{1}{1+\frac{\gamma\rho}{(1-\gamma)(\rho-\theta)}}\sum_{j=1}^{\infty}\theta^j L^j.$$

The term in front of the summation sign depends negatively on $\theta$. However, if the increase(decrease) in $\theta$ is due to an increase(decrease) in $1-\gamma$, then the same term is increased and the extra-persistence due to the perpetual learning is increased as well.

We now turn to the effect of the noise in the public signal on the aggregate price $\varepsilon_t$. In presence of full information this term is zero since $\theta \to 0$. It is also zero if the precision of the public signal $\tau_\varepsilon$ is zero as nobody will use such a signal to predict aggregate productivity. As the degree of real rigidity is increased, the incentive to price similarly is increased as well and firms would tend to rely more on variables that are common knowledge. It follows that, as $\gamma$ is decreased, the

scale factor $(1-\gamma)\frac{\theta}{1-\rho\theta}$ is increased together with the extra-persistence $\theta$. The difference with respect to the full information equilibrium and the private information equilibrium (W) due to the noise in the public signal increases when $\gamma$ is reduced. At the same time, the difference with respect of the full information equilibrium due to the underreaction to an aggregate productivity innovation increases for both the (W) equilibrium and the MS equilibrium. When $\gamma$ is large, the latter difference is minor, but as $\gamma$ gets smaller the difference is substantial. However, because of the increased difference due to the noise as $\gamma$ gets smaller, if the public signal is very imprecise, it might overall increase the difference with respect to the full information benchmark. This is the key tradeoff in the use of public information in presence of private information outlined in Morris and Shin (2002) in a static framework. Here the same type of tradeoff arises in a dynamic setting. The evaluation of the tradeoff in terms of the welfare for the model economy is outside the scope of this paper and it will be studied in future work.

**Higher Order Expectations.** The relevance of the private signal $a_{it}$ to the prediction of the price process depends on the degree of strategic complementarity $\gamma$. Equation (12) shows that $\gamma$ affects the relative importance of expectations of the higher order in the determination of the price level. As $\gamma$ becomes smaller the higher order expectations of the aggregate productivity receive a higher weight. It can be shown that agents rationally downplay the relevance of their private information as the degree of the expectations increases while increasing the importance of public information. Consider for example the direct expectation $E_t\left(a_t|\Omega_{it}^W\right)$, using Corollary 1 with $y_t = a_t + v_{it}$ we can write

$$\mathbb{E}\left(a_t|\Omega_{it}^{MS}\right) = \frac{\lambda}{\rho\left(1-\rho\lambda\right)}\frac{1}{\left(1-\lambda L\right)}\left(\tau_v a_{it} + \tau_\varepsilon x_t\right). \tag{26}$$

Average expectations are then

$$\bar{\mathbb{E}}_t\left(a_t\right) = \frac{\lambda}{\rho\left(1-\rho\lambda\right)\left(1-\lambda L\right)}\left(\tau_v a_t + \tau_\varepsilon x_t\right). \tag{27}$$

This term is not directly observed by agents, they can form an optimal prediction about it. Using Theorem 1 one can show that

$$\mathbb{E}\left(\bar{\mathbb{E}}_t\left(a_t\right)|\Omega_{it}^{MS}\right) = \left(\frac{\lambda\tau_v}{\rho\left(1-\rho\lambda\right)}\right)^2\frac{\left(1-\rho\lambda^2 L\right)}{\left(1-\lambda^2\right)\left(1-\lambda L\right)}a_{it} + C\left(L\right)x_t. \tag{28}$$

Comparing (26) to (28) the downplaying of private information is evident. Each firm knows that all the other firms are facing a similar signal extraction problem which leads to expectations of

the form (26). At the aggregate level (26) results into (27), which now becomes the object of prediction for each firm. As aggregate productivity is downplayed in the first order average market expectations, firms' estimates of the market expectations will downplay the private prediction of aggregate productivity. This will result in a further downplay of aggregate productivity in the market expectations of the market expectations. Proceeding in this fashion one can show that the higher order expectations terms depend less and less on private information $a_{it}$

$$\mathbb{E}\left(\bar{\mathbb{E}}_t^{(j-1)}(a_t)\,|\Omega_{it}^{MS}\right) = \left(\frac{\lambda\tau_v}{\rho(1-\rho\lambda)}\right)^j \frac{(a(j)-b(j)L)}{(1-\lambda^2)^{j-1}(1-\lambda L)} a_t + \tilde{C}(L)\,x_t. \tag{29}$$

From (29) the higher order expectation terms converge to a function of the public information $x_t$, the piece of information that is common knowledge across firms. This is consistent with a result in Morris and Shin (2002) in a static setting.

**Numerical Example 1.** We illustrate by a numerical example the properties of the aggregate price in a MS equilibrium. We fix the parameter values at $\rho = 0.9, \sigma_u = 1, \sigma_v = 1.5$. We consider three alternative values for the strategic complementarity parameter $\gamma \in \{.4, .2, .06\}$[16] and three alternative values for the precision of the public signal, $\tau_\varepsilon \in \{0, 1/16, 1\}$. The impulse response functions for a one-standard deviation negative shock to aggregate productivity are reported in Figures (1A-C). As strategic complementarities increase, the aggregate price tends to react less to aggregate innovation at impact and the deviation from the full information benchmark increases in persistence. The importane of the lack of perfect information is small when complementarities are weak and tends to increase markedly as complementarities are increased. It is especially interesting to notice that the "gain" from the public signal increases as complementarities are stronger compared to the private information case. Figures (2A-C) report the impulse response of the aggregate price to a standard deviation negative shock to the noise of the public signal. Both the full information and the Woodford aggregate prices are not affected by such a shock. On the other hand, the (MS) equilibrium price is affected because of the use of public information in the prediction formula. The innovation in the noise tends to be larger for the high precision MS equilibrium at impact and then the decay is substantially faster than the low precision MS equilibrium, especially for high values of complementarities. The inefficiency related to the use of a very imprecise public signal when complementarities are strong is evident. For low complementarities (Figure 2A), the

[16] Given our assumption on specific markets for factors, Table 3.1 at page 172 in Woodford (2003b) suggests a range of values from 0.17 to 0.06. See Woodford (2003b, pp 171-173) for additional discussion on the appropriate value of $\gamma$.

reaction at impact is smaller for the imprecise signal compared to the precise signal. This small reaction, combined with the closer reaction to the full information benchmark of Figure 1A justifies the use of public information even if imprecise. For high complementarities, the strong reaction at impact even for the imprecise signal case combined with a subtantial deviation from the full information benchmark (Figure 2C) shows that the use of noisy public information might be a bad idea when complementarities are strong.

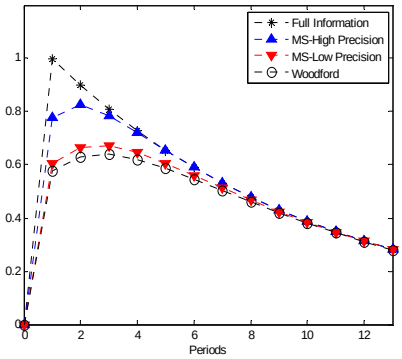


Figure 1A $\gamma = .4$

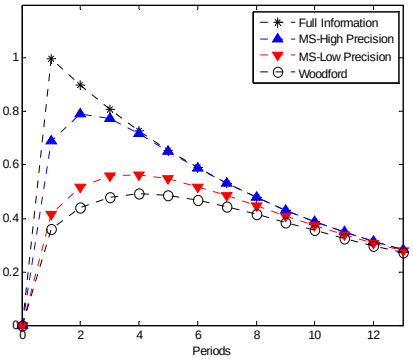


Figure 1B $\gamma = .2$

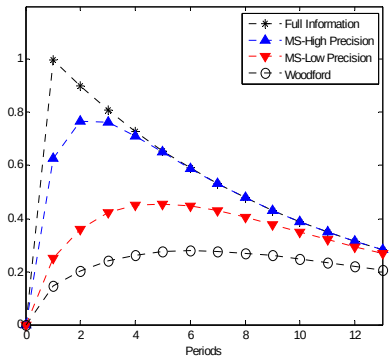


Figure 1C $\gamma = .06$

**Aggregate Price Impulse Response Functions** ($\Delta u = \sigma_u = 1$)

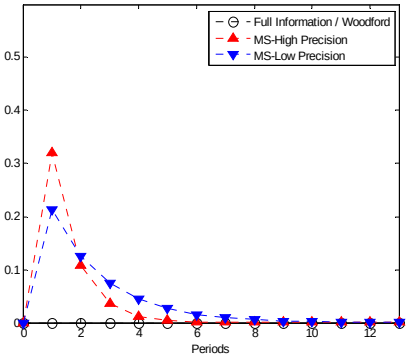


Figure 2A $\gamma = .4$

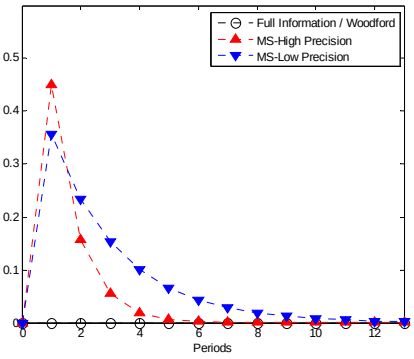


Figure 2B $\gamma = .2$

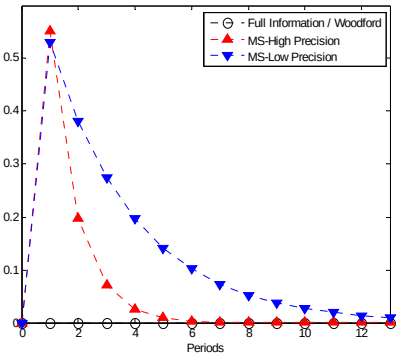


Figure 2C $\gamma = .06$

**Aggregate Price Impulse Response Functions** ($\Delta\varepsilon = \sigma_\varepsilon$)

If one interprets the aggregate price index as the Producer Price Index (PPI) released monthly by the US Bureau of Labor Statistics, the periods in the plots should be considered as months.

### 4.3 Lucas Equilibrium

In this section we solve for a Lucas Equilibrium. The stochastic processes in the information set $\Omega_{it}^{L}$ can be represented as

$$\begin{pmatrix} a_{it} \\ a_{t-1} \\ x_t \end{pmatrix} = \begin{pmatrix} a(L) & 1 & 0 \\ a(L)L & 0 & 0 \\ a(L) & 0 & 1 \end{pmatrix} \begin{pmatrix} u_t \\ v_{it} \\ \varepsilon_t \end{pmatrix}.$$

The form of a Lucas equilibrium is the usual

$$p_t^L = P(L)\, w_t,$$

with $P(L) = (P_u(L) \quad P_\varepsilon(L))$, and $w_t = (u_t \ \varepsilon_t)'$. Under the information set $\Omega_{it}^{L}$, using Theorem 1 one can show that

$$\mathbb{E}_t\left(p_t|\Omega_{it}^{L}\right) = (P(L) - P(0))\, w_t + \frac{P(0)}{\frac{\tau_\varepsilon}{\tau_v} + 1 + \frac{1}{\tau_v}} \begin{pmatrix} 1 & \frac{\tau_\varepsilon}{\tau_v} \\ -1 & 1 + \frac{1}{\tau_v} \end{pmatrix} \begin{pmatrix} u_t + v_{it} \\ u_t + \varepsilon_t \end{pmatrix}. \tag{30}$$

The individual forecast equation can be divided into two components. The first component represents the part of the state of the economy that is common across agents. In fact, any agent, by observing the past realization of aggregate productivity, can recover the structural disturbances $w = (u \ \varepsilon)'$ up to time $t-1$. This results in the first part of the forecast equation $(P(L) - P(0))\, w_t$. Next, agents need to optimally forecast the part of the process given by $P(0)\, w_t$. To understand the information used in this part of the forecast, note that the recovery of the structural disturbances up to time $t-1$ allow agents to recover the aggregate productivity up to the last period, $a_{t-1}$. It follows that the signal they rely upon to forecast $w_t$ is $a_{it} - a_{t-1} = u_t + v_{it}$ and $x_t - a_{t-1} = u_t + \varepsilon_t$. Aggregating (30) across agents, the agent's specific components $v_{it}$ average out to zero and the market expectation of the price level becomes

$$\bar{\mathbb{E}}_t(p_t) = P(L)\, w_t - \frac{P(0)}{\frac{\tau_\varepsilon}{\tau_v} + 1 + \frac{1}{\tau_v}} \begin{pmatrix} \frac{\tau_\varepsilon}{\tau_v} + \frac{1}{\tau_v} & -\frac{\tau_\varepsilon}{\tau_v} \\ 1 & \frac{\tau_\varepsilon}{\tau_v} \end{pmatrix} \begin{pmatrix} u_t \\ u_t + \varepsilon_t \end{pmatrix}.$$

This expression shows the effect of the individual signal extraction problem as expectations are aggregated. The lack of perfect correlation between firm-specific productivity and aggregate productivity (which results because $\sigma_v, \sigma_\varepsilon > 0$), prevents the market forecast from perfectly predicting the equilibrium price.

**Higher Order Expectations.** Elaborating on this expression for the Lucas equilibrium when $\tau_\varepsilon = 0$ one can show that the law of iterated expectations fails when applied to market expectations. Using our prediction formulas on the average price expectations one gets

$$\mathbb{E}\left(\bar{\mathbb{E}}_t\left(p_t\right)|\Omega_{it}^L\right) = \left(P\left(L\right) - P\left(0\right)\right)u_t + P\left(0\right)\left(\frac{1}{1+\frac{1}{\tau_v}}\right)^2\left(u_t + v_{it}\right).$$

Comparing the individual direct expectations (30) to the individual expectation of the market expectation one can see that agent $i$'s private information $(u_t + v_{it})$ is rationally given less importance since $\frac{1}{1+\frac{1}{\tau_v}} < 1$. Aggregating across agents one has

$$\bar{\mathbb{E}}_t\left(\bar{\mathbb{E}}_t\left(p_t\right)\right) = \left(P\left(L\right) - P_0\right)u_t + \left(\frac{1}{1+\frac{1}{\tau_v}}\right)^2 P_0 u_t,$$

which is different from $\bar{\mathbb{E}}_t\left(p_t\right)$ exactly because agents tend to give less importance to their private information in guessing the average market expectations. Repeating the procedure one can see that the $j$-th higher order market expectation is

$$\bar{\mathbb{E}}_t^{(j)}\left(p_t\right) = \left(P\left(L\right) - P_0\right)u_t + \left(\frac{1}{1+\frac{1}{\tau_v}}\right)^j P_0 u_t.$$

Taking the limit of this expression one immediately sees that $\lim_{j\to\infty}\bar{\mathbb{E}}_t^{(j)}\left(p_t\right) = \left(P\left(L\right) - P_0\right)u_t$. In the limit, higher order expectational terms are function solely of information that is common knowledge. For the information set (L) this is $\mathbb{H}_{t-1}\left(a\right) = \mathbb{H}_{t-1}\left(u\right)$. This result parallels a key result in Morris and Shin (2002).

Prediction formula (30) and market clearing lead to the following proposition.

**Proposition 3** *There exists a unique Lucas Equilibrium price process* $p_t = P_u^L\left(L\right)u_t + P_\varepsilon^L\left(L\right)\varepsilon_t$ *given by*

$$P_u^L\left(L\right) = -\left(a\left(L\right) - \frac{\left(1-\gamma\right)}{1+\tau_v\gamma+\tau_\varepsilon}\right), \tag{31}$$

$$P_\varepsilon^L\left(L\right) = -\frac{\left(1-\gamma\right)\tau_\varepsilon}{1+\tau_v\gamma+\tau_\varepsilon}. \tag{32}$$

**Corollary 3** *The price process from a Lucas Equilibrium is informative when the polynomial*

$$\left(1-\gamma\right) - \left(1+\gamma\left(\tau_v+\tau_\varepsilon\right)\right)a\left(L\right)$$

*does not have roots inside the unit circle.*

**Proof.** The proof follows the line of the proof of Corollary 2. ■

Using proposition 4 we can solve for a Lucas Equilibrium when $a_t = \rho a_{t-1} + u_t$ with $|\rho| \leq 1$. The unique Lucas Equilibrium in this case is

$$p_t^L = -\frac{\gamma\left(1+\tau_v\right)+\tau_\varepsilon}{1+\gamma\tau_v+\tau_\varepsilon}\left(\frac{1+\rho\frac{(1-\gamma)}{\gamma(1+\tau_v)+\tau_\varepsilon}L}{1-\rho L}\right)u_t - \frac{\left(1-\gamma\right)\tau_\varepsilon}{1+\tau_v\gamma+\tau_\varepsilon}\varepsilon_t.$$

The impulse responses of a Lucas equilibrium to innovations in aggregate productivity and in the noise component of the public information are reported in Figure 3 and Figure 4. Similarly to the Morris-Shin equilibrium, at impact there is an under-reaction of the aggregate price to aggregate productivity and an overreaction to public noise. Differently from the Morris-Shin equilibrium, any deviations from the full information benchmark completely dissipate in subsequent periods.

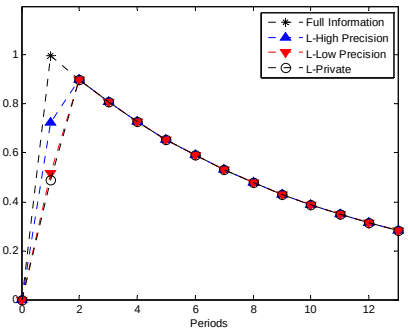


Figure 3A $\gamma = .4$

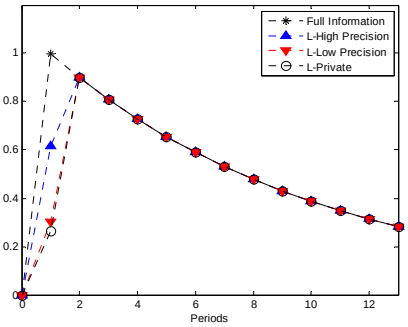


Figure 3B $\gamma = .2$

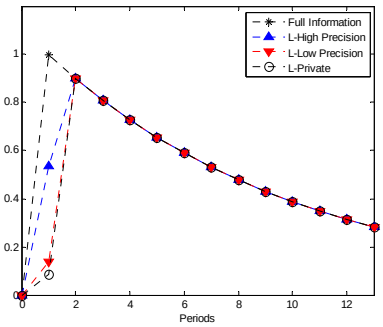


Figure 3C $\gamma = .06$

**Aggregate Price Impulse Response Functions** ($\Delta u = \sigma_u = 1$)

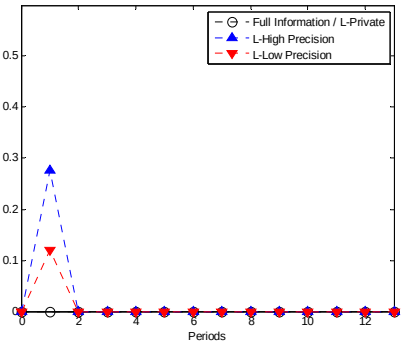


Figure 4A $\gamma = .4$

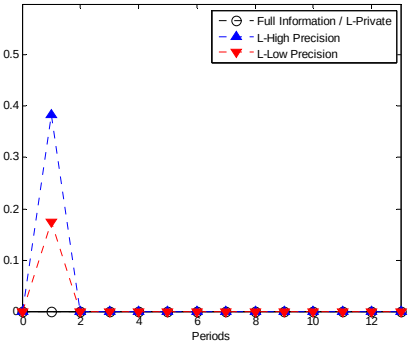


Figure 4B $\gamma = .2$

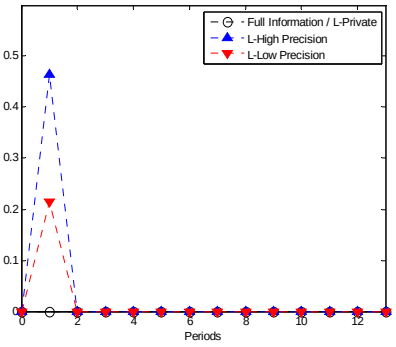


Figure 4C $\gamma = .06$

**Aggregate Price Impulse Response Functions** ($\Delta\varepsilon = \sigma_\varepsilon$)

# 5 Information Equilibrium

Studying equilibria that arise from an information set that is generated by the equilibrium process itself is a difficult task. For the two families of equilibria studied in section 4 we conjectured a price function and then we imposed the usual fixed point condition. Because the information sets (MS) and (L) contain only exogenous variables, the equilibrium conjecture did not involve a conjecture about the information set in equilibrium. In an Information Equilibrium this is no longer the case, the conjectured price must clear the fixed point condition at the strategy level and at the information level. This makes the task of characterizing a IE complicated. A fruitful way to proceed is to identify some equivalence properties between equilibria in spaces that are easily characterized and equilibria in the IE space.

From the definition of $\Omega_{it}^{*}$ and of an IE, the stochastic processes generating the information set $\Omega_{it}^{*}$ can be represented as

$$\begin{pmatrix} a_{it} \\ x_t \\ p_{t-1} \\ e_{it-1} \end{pmatrix} = \begin{pmatrix} a(L) & 1 & 0 \\ a(L) & 0 & 1 \\ P_u(L)L & 0 & P_\varepsilon(L)L \\ e_u(L)L & e_v(L)L & e_\varepsilon(L)L \end{pmatrix} \begin{pmatrix} u_t \\ v_{it} \\ \varepsilon_t \end{pmatrix}, \tag{33}$$

where the process for the prediction error has been specified, without loss of generality, as

$$e_{it} = e_u(L)u_t + e_v(L)v_{it} + e_\varepsilon(L)\varepsilon_t.$$

We are now ready to state two key results.

**Proposition 4** *All Lucas Equilibria are Information Equilibria.*

**Proof.** To prove this claim we need to show that the price functions $P_u^L(L)$ and $P_\varepsilon^L(L)$ are consistent with $\Omega_{it}^{*} = \Omega_{it}^{L}$. This last equality is formally given by

$$\mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}(p, e_i) = \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}(a, \varepsilon).$$

A sufficient condition for this to hold is that $\mathbb{H}_t(p) \oplus \mathbb{H}_t(x) = \mathbb{H}_t(a, \varepsilon)$, i.e. the (LE) price process $p_t$ is informative. However, corollary 2 tells us that this is not always the case. To ensure the claim we just need to prove that $\mathbb{H}_{t-1}(a_i) \oplus \mathbb{H}_{t-1}(e_i) = \mathbb{H}_{t-1}(a_i) \oplus \mathbb{H}_{t-1}(a)$. This equivalence can be

checked by studying the determinant of the system

$$\begin{pmatrix} a_{is} \\ x_s \\ e_{is} \end{pmatrix} = \begin{pmatrix} a(L) & 1 & 0 \\ a(L) & 0 & 1 \\ e_u(L) & e_v(L) & e_\varepsilon(L) \end{pmatrix} \begin{pmatrix} u_s \\ v_{is} \\ \varepsilon_s \end{pmatrix} \tag{34}$$

where

$$e_{it} = P(L) w_t - \mathbb{E}_t \left( P(L) w_t | \Omega_{it}^L \right).$$

From (30) and Proposition 3 one can show that

$$e_u(L) = -\frac{\gamma}{1+\gamma\tau_v+\tau_\varepsilon}, \quad e_v(L) = \frac{\gamma\tau_v}{1+\gamma\tau_v+\tau_\varepsilon}, \; e_\varepsilon(L) = \frac{\gamma\tau_\varepsilon}{1+\gamma\tau_v+\tau_\varepsilon}$$

The determinant of (34) is then proportional to

$$\left(1 + (\tau_v + \tau_\varepsilon) a(L)\right).$$

If $a(L)$ is rational and it does not vanish inside the unit circle, the above polynomial cannot possibly vanish inside the unit circle. For us this translates into the system (34) being enough to recover the structural innovations to productivity and thus aggregate productivity itself. To formally conclude the proof we just need to set $s = t-1$. The equilibrium price process is thus consistent with the initial information set. ■

**Proposition 5** *A Morris-Shin Equilibrium is not an Information Equilibrium, unless $a_t$ is i.i.d or either the public or the private signals are perfectly informative.*

**Proof.** We need to show that the form of a MS equilibrium derived in Proposition 2 is such that the information contained in the aggregate price and the prediction error is more than the information on which agents formulate their predictions. Formally, we want to show that $\Omega_{it}^{MS} \subset \Omega_{it}^*$ unless $a_t$ is i.i.d or $\tau_v \to \infty$ or $\tau_\varepsilon \to \infty$. Let $p_t^{MS}$ be a Morris-Shin equilibrium price, then $\mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \subset \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}\left(p^{MS}, e_i\right)$, which follows from the stationarity of $p_t^{MS}$. When $a_t$ is i.i.d. then $p_t^{MS} = p_t^L$ and, from proposition 4, the equivalence with the information equilibrium follows. On the other hand, when $\tau_v \to \infty$ or $\tau_\varepsilon \to \infty$, $\Omega_{it}^{MS} = \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) = \mathbb{H}_t(a, \varepsilon)$ and so $\mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}\left(p^{MS}, e_i\right) = \mathbb{H}_t(a, \varepsilon)$ which means that $\Omega_{it}^{MS} = \Omega_{it}^*$ and the equivalence follows from the definition of a MSE and a IE. For the more general case we want to show that

$$\mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \subset \mathbb{H}_t(a_i) \oplus \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}\left(p^{MS}, e_i\right). \tag{35}$$

From Corollary 2 we know that a MS equilibrium is always informative, i.e.

$$\mathbb{H}_t(x) \oplus \mathbb{H}_t(a) \subset \mathbb{H}_t(x) \oplus \mathbb{H}_{t-1}\left(p^{MS}\right).$$

Substituting the left hand side in (35) completes the proof. ■

Ideally, we would like to prove whether the converse of Proposition 4 holds, that is to say, whether any Information Equilibrium is a Lucas Equilibrium. However, it is not immediately clear how to proceed to prove such a result. In fact, one can consider a candidate IE price process where $P_L(L)$, $e_u(L)$, $e_v(L)$ and $e_\varepsilon(L)$ all vanish inside the unit circle. While proposition 4 shows that it would be inconsistent with a LE, it would not be inconsistent with the definition of an IE. Nevertheless, because of the generality of the existence result for the LE equilibrium and because of proposition 4, if an IE that is not LE exists, for the same set of parameters a LE must exist as well, and therefore an IE with different informational properties must exist too. Such a scenario suggests a multiplicity property of information equilibria that would be novel to rational expectations studies.

**Information Equilibrium and Identification of the Dynamic Responses** We conclude our analysis by interpreting the above results in terms of the issue of econometric identification of the dynamic response of aggregate price to innovations in productivity. The issues we point out are of the same nature as those pointed out by Hansen and Sargent (1991) and Lippi and Reichlin (1994). While we do not claim that the specific univariate setting presented in this paper is appropriate for estimation, we do believe that it offers a basis for a word of caution with respect to the ability to identify dynamic responses of aggregate variables from VAR studies when those aggregate variables are equilibrium outcome of an economy with heterogeneous dispersed information. Since we have proved that a Lucas equilibrium is an Information Equilibrium, we ask the question of whether the dynamic of the aggregate price in response to aggregate productivity can be measured if the economy is assumed to be in a Lucas equilibrium. It turns out that the answer depends on whether the Lucas equilibrium is informative (Corollary 3). Since a Lucas equilibrium becomes non-informative when $\gamma$ gets smaller, it follows that the correct identificatin of the impulse response function depends on the degree of real rigidity in the economy. For simplicity we look at an example where public information is uninformative ($\tau_\varepsilon = 0$). We let $\rho = 0.9, \sigma_u = 1, \sigma_v = 1.5$. For $\gamma = .4$. the IE price process is $p_t = -0.49(1 + 0.93L)a_t$. The price function vanishes outside the unit circle at $1/.93$, therefore the equilibrium price is informative. When $\gamma = .2$. on the other hand, the aggregate price follows the process $p_t = -0.26(1 + 2.49L)a_t$. In this case the price function vanishes inside the unit circle at $1/2.49$ and so the price process is not informative. In the first case, the impulse

response function is measured correctly (Figure 5A). In the second case, the measured impulse response function completely underestimates the dynamic response, except at impact. We conclude by noticing that the underestimation of the propagation effect can be quite severe, as Figure 5C shows, when complementarities in pricing are strong.

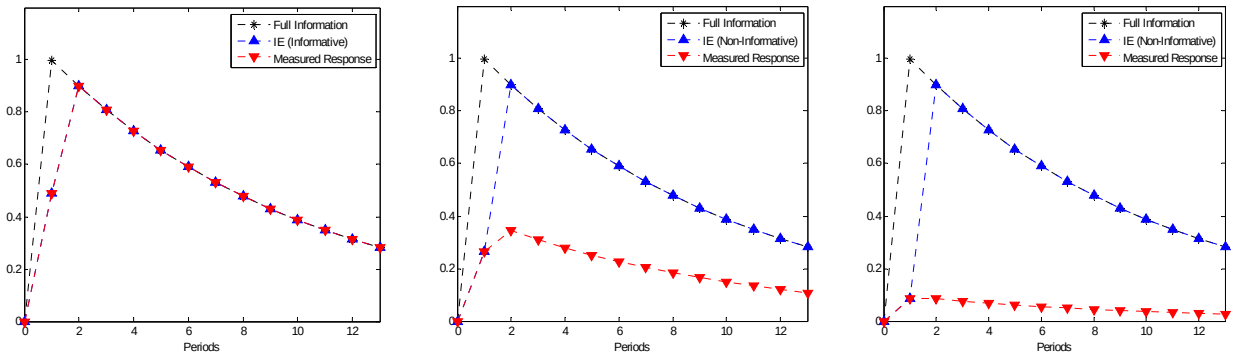


Figure 5A $\gamma = 0.4$ Figure 5B $\gamma = 0.2$ Figure 5C $\gamma = 0.06$

**Aggregate Price Impulse Response Functions: Actual (Blue) and Measured (Red)**

# 6 Conclusion

This paper studied the equilibrium aggregate price of a simple pricing economy under different assumptions about the information set of firms. We first characterized the space of equilibria for two exogenously specified information sets: one where agents observe noisy private and noisy public information (Morris-Shin equilibrium), and one where agents observe noisy private information and perfectly informative public information about the past state of the economy (Lucas equilibrium). Using methods in the space of complex-valued functions we derived prediction formulas in closed form as opposed to recursive form. We also showed that complex valued functions are especially powerful in solving for fixed point problems in linear rational expectations models. Using the closed form solutions for equilibria under exogenous information, we partially characterized the space of Information Equilibria, which are equilibria where endogenous variables convey information to imperfectly informed agents. We showed that a Lucas Equilibrium is always an Information Equilibrium. However, we could not rule out the possibility of Information Equilibria that are not Lucas Equilibria. We have also showed that a Morris and Shin equilibrium cannot possibly be an Information Equilibrium. We see this is as a very important result given the recent interest on the welfare properties of equilibria in presence of imperfect public and private information (see Angeletos and Pavan (2007)).

The results have been derived in a pricing model that is essentially static in nature. The only dynamics comes from the change in the exogenous state of the economy and the accumulation of information signals. In subsequent work the tools and the equilibrium notions employed in the present paper will be applied to contexts were agents are forward looking in the sense that their actions today depend on what they expect the average actions would be in the future. This additional dynamic component has the potential to substantially change the space of Information Equilibria.

# 7 Appendix

**Roots of the Characteristic Polynomial in Proposition 1** A sufficient condition for the existence and uniqueness of an equilibrium in the W and MS case is for the characteristic polynomial of the system to have one roots inside and one root outside the unit circle. The stable root of the polynomial is of particular importance because it describes the additional persistence of the equilibrium price inherent to the imperfect observation of the state of the economy. In both case the characteristic polynomial takes the form

$$1-\left(\frac{1}{\rho}s+\left(\frac{1}{\rho}+\rho\right)\right)z+z^2=(1-\lambda z)\left(1-\frac{1}{\lambda}z\right).$$

Under the assumptions of this paper, $s \geq 0$ and $|\rho| \leq 1$. Any root of this polynomial corresponds to the intersection point of a quadratic and a linear term, specifically

$$1+z^2=\left(\frac{1}{\rho}s+\left(\frac{1}{\rho}+\rho\right)\right)z.$$

The intersect of the linear term is always at the origin, while its slope has a minimum in absolute value at $\left|\frac{1}{\rho}+\rho\right| \geq 2$. When $s=0$ and $|\rho|=1$, the linear term is tangent to the quadratic term and the roots coincide and are equal to 1. For $s>0$, the slope of the linear term is bigger and the intersections with the

quadratic term happen one at the left and one at the right of unity.

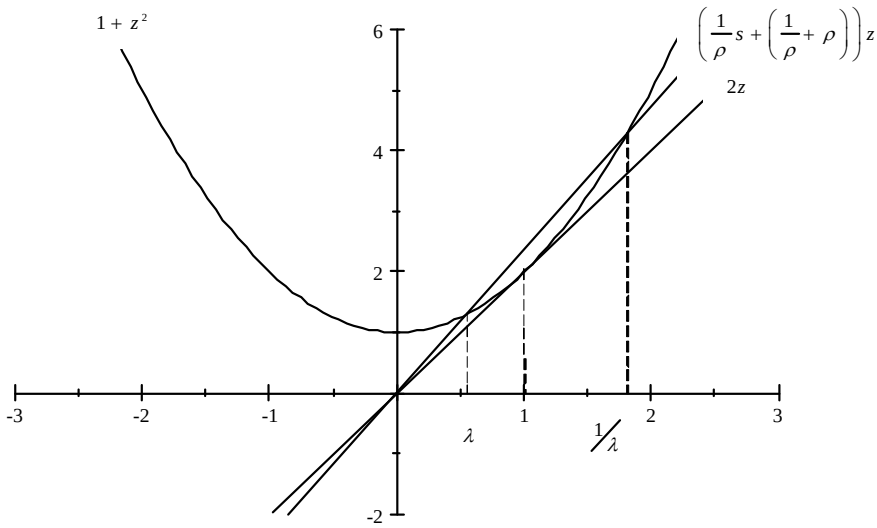


Figure A.1

Formally we have $\frac{\partial\lambda(s)}{\partial\rho} > 0$, $\frac{\partial\lambda(s)}{\partial s} < 0$. In addition, if we consider the simple polynomial

$$1 - \left(\frac{1}{\rho} + \rho\right) z + z^2,$$

with roots $\rho$ and $1/\rho$, it is easy to see that $|\lambda| < |\rho|$ and $|1/\lambda| > |1/\rho|$, it also follows that $sig(\lambda) = sig(\rho)$. Combining the results we have that $0 < \lambda/\rho < 1$.

# References


[1] Angeletos G. and A. Pavan, (2007) "Efficient Use of Information and Social Value of Information", Econometrica, forthcoming.

[2] Angeletos G. and I. Werning, (2006), "Crises and Prices: Information Aggregation, Multiplicity, and Volatility", American Economic Review, 95(6).

[3] Amato J. and H.S. Shin, (2006) "Imperfect Common Knowledge and the Information Value of Prices", Economic Theory, January 27(1).

[4] Adam, Klaus, "Optimal Monetary Policy with Imperfect Common Knowledge", Journal of Monetary Economics, forthcoming.

[5] Bacchetta P. and E. van Wincoop, (2008), "Higher Order Expectations and Asset Prices", "Journal of Money, Credit and Banking", *forthcoming*.

[6] Ball, L. and D. Romer (1990), "Real Rigidities and the Non-Neutrality of Money", Review of Economic Studies, 57, 183-203.

[7] Beyer, A. and R.E.A. Farmer, (2006), "On the Indeterminacy of Determinacy and Indeterminacy: Comments on the Paper "Testing for Indeterminacy" by Thomas Lubik and Frank Schorfheide", forthcoming American Economic Review.

[8] Blanchard, O.J. and N. Kiyotaki, (1987), "Monopolistic Competition and the Effects of Aggregate Demand", American Economic Review, 77(4), 647-666.

[9] Cochrane, John H., (2007), "Identification and Price Determination with Taylor Rules: A Critical Review", mimeo GSB University of Chicago.

[10] Futia, Carl (1981), "Rational Expectations in Stationary Linear Models", Econometrica, 49, 171-192.

[11] Fuhrer, Jeffrey C. (2005), "Intrinsic and Inherited Inflation Persistence", Federal Reserve Bank of Boston Working Paper No. 05-8.

[12] Galati, G. and W. Melick (2006), "The Evolving Inflation Process: an Overview", BIS Working Paper No 196, Basel.

[13] Hayek, Friedrich A. (1945), "The Use of Knowledge in Society", American Economic Review, 35, 519-530.

[14] Hellwig, Christian (2005), "Heterogeneous Information and the Welfare Effects of Public Information Disclosure", mimeo UCLA.

[15] Hellwig, Christian (2006), "Monetary Business Cycle Models: Imperfect Information", review article for The New Palgrave Dictionary of Economics, 2nd Edition, Durlauf S.N. and L.E. Blume.

[16] Hellwig, Martin F. (1982), "Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example", Journal of Economic Theory, 26(2), 279-312.

[17] Kailath, Thomas, (1981), Lectures on Wiener and Kalman Filtering, Springer-Werlag, New York.

[18] Kasa Kenneth (2000) Forecasting the Forecast of Others in the Frequency Domain, Review of Economic Dynamics, 3, 726-756.

[19] Kasa, K., T.B. Walker and C.H. Whiteman, (2006), "Asset Prices in a Time Series Model with Perpetually Disparately Informed, Competitive Traders", mimeo SFU.

[20] Kawamoto, Takuji, (2004), "Technology Shocks and the Role of Monetary Policy in the Beauty Contest Monetarist Model, IMES Discussion Paper no. 2004-E-10, Bank of Japan.

[21] Keynes, J.M. The General Theory of Employment, Interest and Money, MacMillan, London, 1936

[22] Lippi, M. and L. Reichlin, (1994), "VAR Analysis, Nonfundamental Representations, Blaschke Matrices", Journal of Econometrics, 63(1), 307-325.

[23] Lorenzoni, Guido, (2007a), "A Theory of Demand Shocks", mimeo MIT.

[24] Lorenzoni, Guido, (2007b), "Demand Shocks and Monetary Policy", mimeo MIT.

[25] Lucas, Robert E. (1972) "Expectations and the Neutrality of Money", Journal of Economic Theory, 4, 103-124.

[26] Lucas, Robert E. (1973) "Some International Evidence on Output-Inflation Tradeoff" American Economic Review, 63, 326-334.

[27] Lucas, Robert E. (1975) "An Equilibrium Model of the Business Cycle", Journal of Political Economy, 83(6) 1113-1144.

[28] Morris, S. and H.S. Shin (2002) "The Social Value of Public Information" American Economic Review, 92, 1521-1534.

[29] Morris, S. and H.S. Shin (2005), "Central Bank Transparency and the Signal Value of Prices", Brookings Papers in Economic Activity, W.C. Brainard and G.L. Perry (eds), 2.

[30] Muth, John (1961) "Rational Expectations and the Theory of Price Movements" Econometrica, 29(3) 315-335.

[31] Pearlman, J.G. and T.J. Sargent (2005), "Knowing the Forecasts of Others", Review of Economic Dynamics, 8, 480-497.

[32] Phelps, Edmund S., (1967), "Phillips Curves, Expectations of Inflation and Optimal Unemployment over Time," Economica, Vol. 34.

[33] Rozanov, Y.A. (1967), Stationary Random Processes, Holden-Day, San Francisco.

[34] Sargent, Thomas J., (1987), Macroeconomic Theory, Second Edition, in Economic Theory, Econometrics and Mathematical Economics series, Karl Shell (ed), Academic Press.

[35] Sargent, Thomas J., (1991), "Equilibrium with Signal Extraction from Endogenous Variables", Journal of Economic Dynamics and Control, 15(2), 245-273.

[36] Taub, Bart (1989), "Aggregate Fluctuations as an Information Transmission Mechanism", Journal of Economic Dynamics and Control, 13, 113-150.

[37] Townsend, Robert M. (1983), "Forecasting the Forecasts of Others", Journal of Political Economy, 91, 546-588.

[38] Walker, Todd (2007), "How Equilibrium Prices Reveal Information in Time Series Models with Disaparately Informed, Competitive Traders", forthcoming Journal of Economic Theory.

[39] Whittle, Peter (1983), Prediction and Regulation by Linear Least-Square Methods, Second Edition Revisited, University of Minnesota Press, Minneapolis.

[40] Woodford, Micheal (2003a), "Imperfect Common Knowledge and the Effects of Monetary Policy" in P. Aghion, R. Frydman, J. Stiglitz and M. Woodford (eds) Knowledge, Information and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps, Princeton University Press, Princeton, 25-58.

[41] Woodford, Micheal (2003b), Interest and Prices: Foundations of a Theory of Monetary Policy, Princeton University Press, Princeton.