The Dynamics of Speculative Bubbles under Learning

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ABSTRACT

I analyze the dynamics of a speculative bubble in a simple general equilibrium model in which a single asset is traded by two types of risk neutral agents facing a short sale constraint. The types have heterogeneous prior beliefs about the dividend process and they learn from common observations of the dividend. The speculative bubble is defined as the difference between the equilibrium price and the maximum buy-and-hold-forever valuation of the asset. I use a novel recursive methodology to obtain non-generic conditions for the data generating process so that the bubble is persistent with probability one. I also provide generic conditions under which the bubble can persist arbitrarily long in expected terms. Even in the cases where the bubble is persistent, I show that it appears highly infrequently on typical sample paths.

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1 Introduction

In many financial markets we observe periods of price behavior referred to as speculative bubbles. This term usually means that the price of an asset is significantly different from what is believed to be the asset's fundamental value. Furthermore, these events are often not isolated incidents involving a single asset, but rather entire regions or sectors of the economy. We observed this during the Japanese asset price bubble in the late 80's and during the "dot-com" bubble in the US in the late 90's. Other notable examples include the recent behavior of crude oil or real estate prices.

Harrison and Kreps (1972) provide one way to rationalize these events. In their paper, risk-neutral investors have heterogeneous prior beliefs about the persistence of the dividend process and face short-selling constraints. In some periods, the asset is not held by the agents for whom the expected discounted stream of dividends is higher, but by those who expect a higher dividend only in the following period and are hence hoping for profitable resale. This generates speculative trade and a price bubble, where the asset serves not only as a generator of dividends, but also as a period-to-period betting device. The bubble they obtain is not time-varying, in the sense that it depends on the history of dividends only through the current realization. This occurs because agents do not learn from the data. It seems that allowing for learning in this setting would quickly kill the bubble by removing heterogeneity from agents' posteriors. The goal of this paper is to address this objection by proposing a model in which rational traders fail to agree, despite learning from a common signal, resulting in a perpetual speculative bubble.

I develop a general equilibrium model of speculative bubbles in which agents differ in their prior beliefs. This model is based on Harrison and Kreps (1978), with an important addition, that I allow for learning. There is a market for a single asset that pays a stochastic dividend. The dividend follows a two-state Markov process, and can be either zero or one each period. There are two types of risk neutral agents, who face a short sale constraint, and both start with different prior beliefs concerning the transition matrix of the dividend process. The beliefs of both types are concentrated on two available matrices. This can be interpreted as a situation in which there are two theories or models being used by market

practitioners with matrices representing the theories.

A bubble is defined as that portion of the equilibrium price over and above the market fundamental. The market fundamental is the maximum buy-and-hold-forever valuation of the asset. I define the fundamental value process independently of a given equilibrium. This statistic reflects how much agents value the asset in the absence of a market. Considering the fundamental value separately from the market can be justified by the assumed risk neutrality of the agents. This way the agents do not need the market for insurance purposes. The only reason for trading is speculation. This model naturally extends to utility functions with risk aversion, but in that case the fundamental value needs to account for the insurance motive.

I use a recursive methodology borrowed from Slawski (2008) to show that if agents are sufficiently impatient, there exists a linear lower bound on a current bubble magnitude as a function of current belief differentials. This result means that as long as belief differentials stay bounded away from zero so does the bubble, and applies regardless of the true data-generating process of the dividend. The main theorem provides non-generic conditions for the true data-generating process so that the differences in the posteriors persist forever on almost all sample paths. Under these conditions a bubble of high magnitude appears infinitely often with probability one. I provide the following generic complement to the main theorem: the data-generating process can be chosen so that, in expected terms, the belief difference and the high magnitude bubble can persist arbitrarily long.

For these results to apply, it is necessary that agents' beliefs put zero mass on the true data-generating process. Here, I take as a given that even though market traders might be testing for all existing models, it is never the case that one of the available models coincides exactly with reality.

To get an idea of how the lack of convergence in the posteriors is obtained, consider the following thought experiment: there are two agents who observe a coin being flipped. The agents know the experiment produces an iid binomially distributed sequence of outcomes ($d = (d_1, d_2, \ldots)$) which are in the set {heads,tails}. However, the agents do not know the probability of heads, $\theta \in [0, 1]$. They assume some prior beliefs about θ, which are concentrated on the set $\{0.4, 0.6\}$. Using Bayes' rule we get that the posteriors are equal to the priors each time the number of heads equals the number of tails in a current history. For

each history, d^t, let $h_t(d^t)$ and $\tau_t(d^t)$ denote the number of heads and the number of tails, respectively. If the true data-generating process has $\theta = 0.5$, then it is clear that the process $h - \tau$ is a standard symmetric random walk and is thus persistent (i.e. $\Pr(\exists t > 0 \text{ s.th. } d_t = \tau_t) = 1$). Therefore agents' posterior beliefs fail to converge to each other on almost all sample paths.

One can see that the lack of convergence is achieved only for the true data-generating process with $\theta = 0.5$, and it is therefore not generic. However, using the weak law of large numbers we get the following robustness result: as θ approaches 0.5, the expected time that the process $h - \tau$ spends around the origin tends to infinity. This means that the initial disagreement can last arbitrarily long if θ is sufficiently close to 0.5. The generic result in this paper follows the above logic.

This example illustrates the way the model should be interpreted. The fact that the true probability lies in-between the two existing models reflects the view that people's knowledge is not complete. Models simplify reality so much that none of them can be exactly right. This paper highlights that making such simplifying assumptions can potentially lead to perpetual disagreement, which is reflected in the existence of speculative bubbles. The primary reason that I cannot directly use this example is that this sort of disagreement about an iid process does not lead to speculation. As such, I must extend this technique to the Markov environment in which bubbles can emerge.

My paper naturally divides into two parts: Section 2 develops a general equilibrium model of speculative trading and provides a numerical link between the current difference between agents' posteriors and the size of the speculative bubble, while Section 3 focuses on showing how to generate the data so that agents's posteriors fail to converge, in the spirit of the coin flip example presented earlier. This section also delivers the main result.

I close the introduction with a brief literature review. I focus on those models in which bubbles are driven by heterogeneous beliefs and a short sales constraint. This approach was first taken by Harrison and Kreps (1978). The main objection to this model is that agents concentrate their priors on separate matrices, so they do not learn over time and the speculative bubble stays on the same level for every history. Morris (1996) extends the example of Harrison and Kreps by accounting for learning. He uses an iid dividend process

and obtains a speculative bubble that vanishes over time at an exponential rate. Allen, Morris, and Postlewaite (1993) model bubbles by allowing for heterogeneous priors as well as private signals. Furthermore, there is a finite horizon, with an asset paying the dividend only once after the sequence of trading. This setup enables them to get a very strong notion of a bubble in which it is common knowledge that at some point the asset price is higher than the final dividend. The most recent contribution to this literature is the paper by Scheinkman and Xiong (2003). Their model can be treated as a fully developed continuous time version of Harrison and Kreps, in which (cumulative) dividend follows a diffusion process. Agents are getting two noisy signals and wrongfully interpreting one of them (each agent a different one) as carrying some information about the innovation part of the dividend process. In reality, neither of these processes is correlated with the innovation part. The dynamics of speculative trade are such that whenever the signal of an agent becomes more optimistic, the agent buy the asset. Even though this model predicts very interesting dynamics (high trading volumes and price volatility during bubbles), it still does not take into account the process of learning.

It is important to note that in the above models — including mine — having an infinite time horizon is not crucial to generate a bubble. The infinite time horizon is only used for analytical simplicity.

2 Speculative Equilibria

The main goal of this section is to provide a direct link between the current posteriors' asymmetry and the magnitude of the equilibrium bubble. I start with a detailed description of the market setup and the definition of sequential market equilibria. In Subsection 2.2, I introduce the notion of recursive equilibrium, which is a crucial tool in analyzing equilibria. I show that a recursive equilibrium generates a sequential market equilibrium with the lowest possible prices. I also introduce a functional operator, T, which characterizes the bounded recursive equilibria as its fixed point and serves as the main tool for analyzing them. Subsection 2.3 contains an illustrative benchmark case of no-learning, which can be explicitly solved. Subsection 2.4 contains the main result of this section, which is the linear lower bound for the speculative bubble.

2.1 Sequential Market Equilibria and Bubbles

I consider a market for a single stock with trading taking place at discrete time periods ($t = 0, 1, 2, \ldots$). The stock is in unit supply and the dividends are paid to the current holder at each period immediately prior to trading. Agents face a short sale constraint. I denote the random dividend process by $d = (d_0, d_1, d_2, \ldots)$. The dividends take values in the set $D \equiv \{0, 1\}$, and are observed up to the current period by all traders. The current owner has no control over the operation of the firm and the dividend process is the only (exogenous) source of uncertainty. The dividend process follows a two-state Markov chain with the transition matrix $Q = (q_{ij})_{0 \leq i,j \leq 1}$, which is not known to the traders.

There are two classes of investors, $i \in \{1, 2\}$. Each class is of measure one and is characterized by the initial beliefs about Q, $\pi_0^i \in \Delta(\{Q | Q \text{ is } 2 \times 2 \text{ probabilistic matrix}\})$. Each π_0^i determines the probability distribution over the realizations of process d. This probability distribution is denoted by $\Pr^{\pi_0^i} \in \Delta(D^\infty)$. The expected value operator with respect to that measure is denoted by $E^{\pi_0^i}$.

For the purpose of this paper, I consider beliefs of a particular form. Specifically, I assume that the beliefs of both agents are concentrated on two transition matrices, $Q^1 = (q^1_{dd'})_{0 \leq d,d' \leq 1}$ and $Q^2 = (q^2_{dd'})_{0 \leq d,d' \leq 1}$ (i.e., $\pi_0^i \in \Delta(\{Q^1, Q^2\})$). Belief π_0^i can be represented by one number, say $\pi_0^i(\{Q^1\}) \in [0, 1]$. In what follows I abuse the notation by writing π_0^i instead of $\pi_0^i(\{Q^1\})$.

One can think of Q^1 and Q^2 as the conceivable theories or models of the market. If both agents put positive probabilities on both, Q^1 and Q^2, then they have common support and I will refer to this situation as "learning." Agents may differ in the initial probabilities they assign to different theories, but they agree on what they consider possible.

The instances in which agents' beliefs are concentrated on single but different matrices (i.e., $\pi_0^i = \delta_{Q^i}$) are equivalent to the example of Harrison and Kreps, and I will refer to this situation as "no learning." In this case, none of the agents consider the theory used by the other type as possible, and therefore they do not update.

Both types' consumption sets are $C = \ell^\infty(\bigcup_t D^t) \equiv \bigcup_{\bar{c}} \{(c_t^i)_{t=0}^\infty | c_t^i : D^t \to [-\bar{c}, \bar{c}]\}$.

They are risk neutral with utility functions $U^i : C \to \mathbb{R}$, given by

$$U^i(c^i) = E^{\pi_0^i} \sum_{t=0}^{\infty} \beta^t c_t^i,$$

for each $c^i \in C$. The assumed boundedness of the set C ensures the limit always exists.

A sequential market equilibrium consists of the following components:

$$\begin{aligned}
&c \in C^2 \\
&\gamma = ((\gamma_t^i)_{i=1}^2)_{t=1}^{\infty}, \gamma_t^i : D^{t-1} \to \mathbb{R}_+ \\
&p = (p_t)_{t=0}^{\infty}, p_t : D^t \to \mathbb{R}.
\end{aligned}$$

The first component is the consumption allocation, the second one is the allocation of the asset across agents, and the third one is the contingent price.

A triplet (c, γ, p) constitutes an equilibrium if for each $i = 1, 2$, c^i, and γ^i solves:

$$\begin{aligned}
&\max_{(c^i,\gamma^i)} U(c^i) \\
&\text{s.t. } c_t^i(d^t) + p_t(d^t)\gamma_{t+1}^i(d^t) \leq p_t(d^t)\gamma_t^i(d^{t-1}) + \gamma_t^i(d^{t-1})d_t \\
&\text{s.t. } \gamma_0^i = 0, \gamma_{t+1}^i(d^t) \geq 0
\end{aligned} \quad \text{for each } t \text{ and } d^t \tag{1}$$

and the market-clearing conditions hold:

$$\begin{aligned}
&c_t^1(d^t) + c_t^2(d^t) = d_t, \text{ for each } t \text{ and } d^t \\
&\gamma_t^1(d^t) + \gamma_t^2(d^t) = 1, \text{ for each } t \text{ and } d^t.
\end{aligned}$$

I will now define the speculative bubble process for a given equilibrium. As mentioned before, the bubble process will relate the equilibrium price to the market fundamental. For each $i = 1, 2$ and for any t and d^t, define the process v^i by $v_t^i(d^t) \equiv E^{\pi_0^i}\left\{\sum_{\tau > t} \beta^t d_t | d^t\right\}$. This will be called the fundamental value of the asset for agent i, at time t, given the history d^t. It is the highest price agent i would be willing to pay for the asset if no future re-trade was possible. The market fundamental value of the asset at time t, given the history d^t, is the

process v, defined by $v_t(d^t) \equiv \max_{i=1,2} v_t^i(d^t)$. This gives the market price of the asset if the current period was the last day of trade. Now, for a given equilibrium price process p, I define the speculative bubble process s as $s_t \equiv p_t - v_t$. This statistic measures how much agents value the possibility of future asset trading, purely as a betting device.

2.2 Recursive Equilibrium

In this subsection I will introduce an auxiliary concept of recursive equilibrium. The concept, which is a helpful tool in solving for equilibria, will also be used to see directly how the current difference in posterior beliefs translates into the size of the speculative bubble. Having the link between posteriors and the bubble is possible because in the stationary equilibrium, the price is a function of the current posterior belief profile.

To simplify the notation for the law-of-motion of the posterior beliefs, I need an additional piece of notation. Each history of length 2, $(d, d') \in D^2$, defines the function $\lambda^{dd'} : [0,1] \to [0,1]$ by

$$\lambda^{dd'}(\pi^i) \equiv \frac{\pi^i q^1_{d_{t-1}d'}}{\pi^i q^1_{d_{t-1}d'} + (1-\pi^i) q^2_{dd'}}. \tag{2}$$

The above takes as an argument the previous beliefs and the type of transition that occurred in the last period, (d, d'). It gives as its value the new beliefs, updated using Bayes' rule. For notational convenience I denote the whole belief profile as $\pi = (\pi^1, \pi^2)$, and in what follows I will abuse notation by writing $\lambda^{dd'}(\pi)$, rather than $\left(\lambda^{dd'}(\pi^1), \lambda^{dd'}(\pi^2)\right)$.

A recursive equilibrium consists of:

$$\begin{aligned} W &: \mathbb{R}_+ \times D \times [0,1]^2 \to \mathbb{R}^2 \\ \gamma^* &: \mathbb{R}_+ \times D \times [0,1]^2 \to \mathbb{R}^2_+ \\ p^* &: D \times [0,1]^2 \to \mathbb{R}. \end{aligned}$$

The first component represents agents' value functions, and the second one their decision functions specifying agents' new asset holdings — as functions of the current asset holdings, the current dividend, and the current belief profile. The last component represents the price,

which is assumed not to depend on the current asset holdings but only on the state variables.

A triplet (W, γ^*, p^*) is a recursive equilibrium if for each $i =, 1, 2$, $\gamma = (\gamma^1, \gamma^2) \in \mathbb{R}^2_+$, $d \in D$, and $\pi \in [0,1]^2$, the following three conditions are satisfied.

1. Agent i's Bellman Equation holds

$$W^i(\gamma^i, d, \pi) = \max_{\gamma'^i \geq 0} \Big\{ \big[(\gamma^i - \gamma'^i)p(d) + \gamma^i d\big] + \sum_{d'} \big[\pi^i q^1_{dd'} + (1-\pi^i) q^2_{dd'}\big] W\big((\gamma'^i, \gamma^{*-i}), d', \lambda^{dd'}(\pi)\big) \Big\}. \tag{3}$$

2. $\gamma^{*i}(\gamma^i, d, \pi)$ is a solution to agent i's problem above.

3. Market clearing for the asset holds: if $\gamma^1 + \gamma^2 = 1$, then

$$\gamma^{*1}(\gamma^1, d, \pi) + \gamma^{*2}(\gamma^2, d, \pi) = 1.$$

The definitions of the fundamental values and the speculative bubble naturally extend to the recursive setup. The fundamental value of the asset for agent i, with beliefs π^i in state d, is $V^i(d, \pi^i) \equiv E^{\pi^i} \sum \beta^t d_t$. The market fundamental value is $V(d, \pi^1, \pi^2) \equiv \max_{i=1,2} V^i(d, \pi^i)$, and for a given recursive equilibrium price p^*, the speculative bubble is $S(d, \pi) \equiv p^*(d, \pi) - V(d, \pi)$.

Before introducing the tools to solve for a recursive equilibrium, I will provide a link between recursive equilibria and sequential market equilibria. If I specify the initial belief profile $\pi_0 = (\pi^1_0, \pi^2_0) \in [0,1]^2$, then any recursive equilibrium induces a sequential form price and allocation in a natural way. Namely, given a recursive equilibrium (W, γ^*, p^*), define a sequential allocation and price (c, γ, p) together with the implied process of the posterior

belief profiles π, recursively by:

$$\pi_{t+1}(d^{t+1}) = \lambda^{d_t d_{t+1}}(\pi_t(d^t))$$
$$p_t(d^t) = p^*(d_t, \pi_t(d^t))$$
$$\gamma_{t+2}(d^{t+1}) = \gamma^*(\gamma_{t+1}(d^t), d_{t+1}, \pi_{t+1})$$
$$c_t(d^t) = (\gamma_t - \gamma_{t+1})p_t + \gamma_t d_t.$$

In order to use recursive equilibrium techniques, we need to know that these sequences constitute a sequential market equilibrium. From Slawski (2008), we have the following:

Proposition 2.1. *Assume (W, γ^*, p^*) is a recursive equilibrium satisfying $W^1, W^2 \geq 0$. For some given $\pi_0 = (\pi_0^1, \pi_0^2) \in [0,1]^2$ let (c, γ, p) be the sequential allocation induced by it. If the following (transversality) condition is satisfied:*

$$\lim_{t\to\infty} \beta^t E^{\pi_0^i} W^i(\gamma_{t+1}^{*i}, d_t, \pi_t) = 0.$$

then (c, γ, p) is an equilibrium.

In particular, this means that any bounded recursive equilibrium naturally generates a sequential market equilibrium for any initial belief profile.

Another important property of bounded recursive equilibria is that they generate sequential market equilibria with the lowest possible prices in the class of all sequential market equilibria.

Proposition 2.2. *For any initial beliefs π_0, if p is a sequential equilibrium price system, and p^* is the bounded recursive equilibrium price, then for every history, d^t, we have $p(d^t) \geq p^*(d_t, \pi_t(d^t))$.*

Note that this implies that the same equality holds for the bubble: $s_t(d^t) \geq S(d_t, \pi_t(d^t))$.

Proof. If $(p_t(d^t))$ is an equilibrium price system, then it has to satisfy the first order conditions of (1), which, taking into account the fact that the market clearing condition must hold (i.e., $\gamma_t^i > 0$ for at least one agent), leads to:

$$p_t(d^t) = \max_{i=1,2} \beta E^{\pi_0^i}\big[p_{t+1}(d^{t+1}) + d_{t+1}|d^t\big]. \tag{4}$$

Define inductively a sequence of functions, $p_n : \{0,1\} \times [0,1]^2 \to \mathbb{R}$, by

$$\begin{aligned}
&p_0(d,\pi) \equiv 0 \\
&p_{n+1}(d,\pi) \equiv \max_{i=1,2} \beta \sum_{d'} \big[\pi^i q^1_{dd'} + (1-\pi^i) q^2_{dd'}\big]\big[d' + p_n(d', \lambda^{dd'}(\pi))\big].
\end{aligned}$$

Then by Proposition 2.5, we have $p^* = \lim_{n\to\infty} p_n$. I will show by induction that $p_t(d_t) \geq p_n(d_t, \pi_t^1(d_t), \pi_t^2(d^t))$, for each n, d^t.

For $n = 0$ this is obvious. Suppose that $p_t(d^t) \geq p_n(d^t, \pi_t(d^t))$ for some n and all d^t. Then, using (2) and (4),

$$\begin{aligned}
p_t(d^t) &= \max_{i=1,2} \beta E^{\pi_0^i}\big[d_{t+1} + p_{t+1}\big(d^{t+1}\big)|d^t\big] \\
&\geq \max_{i=1,2} \beta E^{\pi_0^i}\big[d_{t+1} + p_n\big(d_{t+1}, \pi_{t+1}(d^{t+1})\big)|d^t\big] \\
&= \max_{i=1,2} \beta \sum_{d_{t+1}\in\{0,1\}} \big[\pi^i q^1_{d_t d_{t+1}} + (1-\pi^i) q^2_{d_t d_{t+1}}\big]\big[d_{t+1} + p_n\left(d_{t+1}, \lambda^{d_t d_{t+1}}(\pi_t(d^t))\right)\big] \\
&= p_{n+1}(d_t, \pi_t(d^t)).
\end{aligned}$$

∎

In light of the last proposition, the significance of analyzing bounded recursive equilibria is that they provide the (highest) lower bound for the speculative bubble in sequential market equilibria. From now on I will focus solely on bounded recursive equilibria.

Next, I provide tools that allow me to solve for the bounded recursive equilibrium prices. These are then used to obtain the main results of this section. Consider the functional

operator $T : B(D \times [0,1]^2) \to B(D \times [0,1]^2)$ defined by[1]

$$Tp(d,\pi) \equiv \max_{i=1,2} \beta \sum_{d'} \left[\pi^i q^1_{dd'} + (1-\pi^i) q^2_{dd'}\right]\left[d' + p^*(d', \lambda^{d0}(\pi))\right]. \tag{5}$$

Proposition 2.3. *$p^* : D \times [0,1]^2 \to \mathbb{R}$ is the price of some bounded recursive equilibrium if and only if p^* is a fixed point of T.*

Proof. To show necessity take the first order conditions for (3) and use the envelope theorem to get $p^* = Tp^*$. For sufficiency, consider $W^i(\gamma, d, \pi) = \gamma^i(p^*(d,\pi) + d)$ and

$$\gamma^{*i}(\gamma, d, \pi) = \begin{cases} 0 & \text{if max in (5) is not achieved by } i \\ 1 & \text{if max in (5) is not achieved by } (-i) \\ 1/2 & \text{otherwise} \end{cases}$$

It is straightforward to verify this is a stationary equilibrium. ∎

Now I will briefly go over the properties of T.

Lemma 2.4. *T has the following properties:*

1. *It is a β-contraction with respect to the* sup-*norm on $B(D \times [0,1]^2)$*
2. *$T\big(C^0(D \times [0,1]^2)\big) \subseteq C^0(D \times [0,1]^2)$*
3. *It is monotone, i.e. for any $p, p' \in B\big(D \times [0,1]\big)$, if $p \geq p'$ then $Tp \geq p'$*

Proof. To get 1. apply Blackwell's sufficient conditions. 2. and 3. are obtained by a direct check. ∎

This lemma implies the following.

Proposition 2.5. *There exists a unique $p^* \in B(D \times [0,1]^2)$, such that $p^* = Tp^*$. Moreover, $p^* \in C^0(D \times [0,1]^2)$ and for any $p_0 \in B(D \times [0,1]^2)$, we have $p^* = \lim_{n\to\infty} T^n p_0$.*

[1] For a given topological space X, $B(X)$ denotes the Banach space of real-valued bounded functions defined on X, and $C^0(X)$ denotes the Banach space of all real-valued bounded continuous functions defined on X

The immediate consequence of monotonicity is a very useful lower/upper bound for prices, which will be the primary tool for proving the main result of this section.

Proposition 2.6. *If for some $p : D \times [0,1]^2 \to \mathbb{R}$ we have $Tp \geq p$ ($Tp \leq p$), then $p^* \geq Tp$ ($p^* \geq Tp$).*

2.3 No-Learning Benchmark

It is useful to go over the special case of no learning (i.e., $\pi^1 = 1$, $\pi^2 = 0$) in which I can explicitly solve for the fixed point of T. This is also an important benchmark for the general setup.

As there is no learning, the belief profile π stays the same for any history and the only relevant state variable is the dividend d. This allows me to omit beliefs from the argument of the recursive equilibrium objects throughout this subsection. For instance, $p(d)$ will stand for $p(d, 1, 0)$, and so on. First, I compute the fundamental values. I will use the notation $V_d^{Q^i}$ to denote the fundamental value corresponding to the transition matrix Q^i and the current dividend state, d. Standard recursive considerations lead to:

$$V^{Q^i} \equiv \begin{bmatrix} V_0^{Q^i} \\ V_1^{Q^i} \end{bmatrix} = \frac{\beta}{(1-\beta)(1+\beta(1-q_{00}^i - q_{11}^i))} \begin{bmatrix} q_{01}^i \\ q_{11}^i + \beta(1 - q_{00}^i - q_{11}^i) \end{bmatrix}. \tag{6}$$

The operator T can be redefined as $T : \mathbb{R}^2 \to \mathbb{R}^2$ with:

$$Tp = T \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} \equiv \begin{bmatrix} \max_{i=1,2} \beta \left(q_{00}^i p(0) + q_{01}^i (1 + p(1)) \right) \\ \max_{i=1,2} \beta \left(q_{10}^i p(0) + q_{11}^i (1 + p(1)) \right) \end{bmatrix},$$

where $p : D \to \mathbb{R}$ is a recursive price system, depending only on the dividend state, d. In this 2-dimensional case, the fixed point of T can be easily computed.

Proposition 2.7. *If $q_{00}^1 < q_{00}^2$ and $q_{11}^1 < q_{11}^2$, then the recursive equilibrium price system is given by*

$$p^* \equiv \begin{bmatrix} p^*(0) \\ p^*(1) \end{bmatrix} = \frac{\beta}{(1-\beta)(1+\beta(1-q_{00}^1 - q_{11}^2))} \begin{bmatrix} q_{01}^1 \\ q_{11}^2 + \beta(1 - q_{00}^1 - q_{11}^2) \end{bmatrix},$$

and in those circumstances the speculative bubble is strictly positive. Analogous result holds for the symmetric case (i.e., $q^2_{00} < q^1_{00}$ and $q^2_{11} < q^1_{11}$). Otherwise, the recursive equilibrium price is equal to the market fundamental.

One should note that the pricing above is equal to the fundamental valuation of a hypothetical "hybrid agent" with beliefs concentrated on the transition matrix the first row of which is taken from Q^1 and the second from Q^2.

Proof. First consider the case $q^1_{00} < q^2_{00}$ and $q^1_{11} < q^2_{11}$. p^*, above, is the solution to

$$p^*(0) = q^1_{00}p^*(0) + q^1_{01}(1 + p^*(1))$$
$$p^*(1) = q^2_{10}p^*(0) + q^2_{11}(1 + p^*(1)).$$

The only thing which has to be shown is that the RHSs above constitute the operator T for p^*. This is equivalent to $p^*(0) \leq p^*(1) + 1$. Indeed, this can be easily verified. To check that $S(d) > 0$ is a straightforward matter. The proof for the symmetric case $q^2_{00} < q^1_{00}$ and $q^2_{11} < q^1_{11}$ is the same.

If neither $q^1_{00} < q^2_{00}$ and $q^1_{11} < q^2_{11}$ nor $q^2_{00} < q^1_{00}$ and $q^2_{11} < q^1_{11}$, I apply the operator T to the market fundamental V to get $TV = V$. Consequently, $p^* = V$. ∎

I will now illustrate how this technique works on a specific numerical example. These numbers are taken from Harrison and Kreps (1978). Agents of respective groups have the following transition matrices:

$$Q^1 = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix} \qquad Q^2 = \begin{bmatrix} 2/3 & 1/3 \\ 1/4 & 3/4 \end{bmatrix},$$

with the discount factor $\beta = 0.75$. An agent of type one considers the dividend process more volatile than an agent of type two. Therefore, by Proposition 2.7 there must be a speculative bubble in the equilibrium. First, compute the fundamental value for each pair of beliefs and

each state. Using (6), I obtain:

$$V_0^{Q^1} = 4/3 = 1.33 \qquad V_1^{Q^1} = 11/9 = 1.22$$
$$V_0^{Q^2} = 16/11 = 1.45 \qquad V_1^{Q^2} = 21/11 = 1.91$$

This reveals that in both states the asset is valued more by agent 2. The market fundamental is therefore $V = V^{Q^2}$.

Using Proposition 2.7, the recursive equilibrium price is $p_0^* = 24/13 = 1.85$, $p_1^* = 27/13 = 2.08$ with the bubble

$$S(0) = 1.85 - 1.45 = 0.40$$
$$S(1) = 2.08 - 1.91 = 0.17$$

Note that for any history the speculative bubble is on the same level (depending only on the current dividend).

In order to intuitively see why there is a positive speculative bubble, let us apply the operator T to the fundamental value V:

$$T(V) = T\left(\begin{bmatrix} 1.45 \\ 1.91 \end{bmatrix}\right) = 0.75 \begin{bmatrix} \frac{1}{2}1.45 + \frac{1}{2}(1 + 1.91) \\ \frac{1}{4}1.45 + \frac{3}{4}(1 + 1.91) \end{bmatrix} = \begin{bmatrix} 1.64 \\ 1.91 \end{bmatrix}.$$

This means that if the price next period was known to be the fundamental value then, in state $d = 0$, an agent of type one would find the asset worth 1.64, which is strictly more than the fundamental value of 1.45. This occurs because he would subjectively perceive a higher probability of transition to state 1 next period (than type 2), wherein not only is the fundamental price much higher but he also gains the dividend. Consequently, because of risk neutrality and infinite financial resources, he would show an infinite demand, which is not possible in equilibrium. Attempting to set 1.64 as the price in state $d = 0$, would also not work because now agents of type two in state $d = 1$ would show infinite demand. This corresponds to the second iteration of the operator T. Those considerations not only show why the market fundamental is not an equilibrium price but also illustrate why the

equilibrium price has to be the limit of iterating the operator T.

2.4 Lower Bound for the Speculative Bubble

This subsection investigates under which conditions the recursive equilibrium bubble, S, is strictly positive for each $\pi^1 \neq \pi^2$. Additionally, it will be important to know if the magnitude of the bubble remains significant for a small belief asymmetry (i.e., $\pi^1 \approx \pi^2$).

In the previous subsection I characterized the situations in which there is a strictly positive bubble under the extreme belief differential (i.e., $\pi^1 = 1, \pi^2 = 0$). Specifically, by Proposition 2.7, if $\beta > 0$, $q^i_{00} < q^{-i}_{00}$ and $q^i_{11} < q^{-i}_{11}$ for some $i = 1, 2$, then $S(d, 1, 0) > 0$. By Proposition 2.5, p^* is continuous in beliefs and so is the bubble, S. This means that the speculative bubble is strictly positive in some neighborhood of $\pi = (0, 1)$. This, however, does not say how big the speculative bubble is when the beliefs are getting close to each other. It is not even clear if the bubble is strictly positive for all $\pi^1 \neq \pi^2$.

The following proposition provides conditions for the parameters of the model, ensuring that the bubble is not only strictly positive for any asymmetric beliefs ($\pi^1 \neq \pi^2$), but it is also bounded below by a strictly increasing linear function of $|\pi^1 - \pi^2|$. This means that any posterior belief asymmetry results in a speculative bubble of a significant magnitude and allows me to focus on the dynamics of the posteriors rather than the bubble dynamics, which I do in the next section.

Proposition 2.8. *If $q^1_{00} < q^2_{00}$, $q^1_{11} < q^2_{11}$ and one of the following holds:*

1. $V^{Q^1} < V^{Q^2}$ *and* $\beta \leq \frac{q^1_{00}}{q^2_{00}}$
2. $V^{Q^1} > V^{Q^2}$ *and* $\beta \leq \frac{q^1_{11}}{q^2_{11}}$
3. $V^{Q^1}_0 \geq V^{Q^2}_0$ *and* $V^{Q^2}_1 \geq V^{Q^1}_1$,

then there exist constants $B_0 > 0$ and $B_1 > 0$ such that for each $d \in \{0, 1\}$,

$$S(d, \pi^1, \pi^2) \geq B_d |\pi^1 - \pi^2|.$$

The proposition comes as a direct corollary to the more general Lemma A1, which provides much bigger range of β for which the conclusion still holds. However, due to a complex statement, Lemma A1 is presented in the Appendix. The idea of the proof is to consider a pricing function, which is linear in beliefs in the following form:

$$p^A(d, \pi^1, \pi^2) \equiv (1 - \pi^1)V_d^{Q^2} + \pi^2 V_d^{Q^1} + (\pi^1 - \pi^2)A_d,$$

where $A = (A_0, A_1)$. Using the identity

$$\begin{bmatrix} V^i(0, \pi^i) \\ V^i(1, \pi^i) \end{bmatrix} = \pi^i V^{Q^1} + (1 - \pi^i)V^{Q^2},$$

it can be seen that for $A_d = \max\{V_d^{Q^1}, V_d^{Q^2}\}$, the function p^A is equal to the market fundamental, and any higher values of A_d will generate a proportional bubble. For each value of the parameters, one can explicitly compute the highest values of the constants A_0 and A_1 so that $Tp^A(d, \pi^1, \pi^2) \geq p^A(d, \pi^1, \pi^2)$ for any $\pi \in [0, 1]^2$ and $d \in D$. Using the monotonicity of T (Proposition 2.6), for such constants the equilibrium price p^* satisfies $p^* \geq p^A$. In order to obtain Proposition 2.8, I need to make sure the constants A_d, obtained in this way, satisfy $A_d = \max\{V_d^{Q^1}, V_d^{Q^2}\}$. Then I just put $B_d \equiv A_d - \max\{V_d^{Q^1}, V_d^{Q^2}\}$. The following proposition specifies the highest values of Bs this technique allows us to achieve.

Proposition 2.9. *B_0 and B_1 in Proposition 2.8 can be taken as follows:*

In case 1: $B_0/R_2 = B_1 = \dfrac{V_1^{Q^2} - V_1^{Q^1} - R_1(V_0^{Q^2} - V_0^{Q^2})}{R_1R_2 - 1}$

In case 2: $B_0 = B_1/R_1 = \dfrac{V_0^{Q^1} - V_0^{Q^2} - R_2(V_1^{Q^1} - V_1^{Q^2})}{R_1R_2 - 1}$

In case 3: $B_0 = \dfrac{V_0^{Q^1} - V_0^{Q^2} - R_2(V_1^{Q^1} - V_1^{Q^2})}{R_1R_2 - 1}$ *and* $B_1 = \dfrac{V_1^{Q^2} - V_1^{Q^1} - R_1(V_0^{Q^2} - V_0^{Q^2})}{R_1R_2 - 1}$,

where the constants R_1, R_2 are given by

$$R_1 = \begin{cases} \frac{\left(1-\sqrt{\beta q^1_{00} q^2_{00}}\right)^2}{\beta q^1_{01} q^2_{01}} & \text{if } q^1_{00} \leq \beta q^2_{00} \\ \frac{1-\beta q^2_{00}}{\beta q^2_{01}} & \text{otherwise} \end{cases}$$

$$R_2 = \begin{cases} \frac{\left(1-\sqrt{\beta q^1_{11} q^2_{11}}\right)^2}{\beta q^1_{10} q^2_{10}} & \text{if } q^1_{11} \leq \beta q^2_{11} \\ \frac{1-\beta q^2_{11}}{\beta q^2_{10}} & \text{otherwise} \end{cases}$$

This proposition also follows directly from Lemma A1.

Now I will show how the values for Bs compare with the exact solution of the numerical example from the previous subsection. Since I allow for learning, the matrices Q^1 and Q^2 constitute the support of agents' beliefs. Without loss of generality I can assume their prior beliefs satisfy $\pi^1_0 > \pi^2_0$. In particular this assumption implies that $\pi^1_t(d^t) > \pi^2_t(d^t)$ for any t and any history d^t.

Since $V^{Q^2} > V^{Q^1}$ and $\beta = 3/4 = \frac{1/2}{2/3} = \frac{q^1_{00}}{q^2_{00}}$, the assumption of Proposition 2.8 is satisfied. Using the formulas from Proposiiton 2.9, I get $B_0 = .28$ and $B_1 = 0.11$. The linear lower bound for the equilibrium price bubble is then

$$S(0, \pi^1, \pi^2) \geq 0.28 \cdot |\pi^1 - \pi^2|$$
$$S(0, \pi^1, \pi^2) \geq 0.11 \cdot |\pi^1 - \pi^2|.$$

This compares to the exact values computed in the previous section for the no-learning environment (i.e., $\pi^1 = 1, \pi^2 = 0$). We had $S(0, 1, 0) = 0.40$ and $S(1, 1, 0) = 0.17$, suggesting that the linear lower bound captures a decent amount of speculation.

I close this section with a brief comment on high values of β, for which the assumption of Proposition 2.8 is not satisfied. Lemma A1 provides a much bigger range of βs but still does not work for $\beta \approx 1$. The question is: what is happening if β is high? Is it that the bubble cannot be bounded away from zero by a linear function of beliefs, or is it just a deficiency of the specific technique? Furthermore, is it true that for high values of β the bubble becomes insignificant for some small or intermediate values of $|\pi^1 - \pi^2|$? These questions are not addressed in this paper but some numerical experiments suggest that for $0 \ll \pi^1 \approx \pi^2 \ll 1$,

and high values of β the bubble becomes very small, relative to its highest value under no learning. Moreover, these numerical results do not even elucidate that for some small values of $|\pi^1 - \pi^2|$, the bubble is strictly positive.

3 The Dynamics of the Speculative Bubble under the True Transition Matrix

In this section I analyze the dynamics of posterior beliefs for a given true data-generating process. According to Proposition 2.8, this translates into the dynamics of the speculative bubble in the circumstances specified therein. In Section 3.1, I state the main result of the paper, which gives the condition for the true data-generating process to achieve a perpetual disagreement with probability one. I then discuss the idea of the proof. Subsection 3.2 provides some further results concerning the posterior dynamics, most notably the generic supplement to the main theorem.

3.1 The Main Result

I will now apply the technique associated with the coin flip example from the introduction into the Markov environment of my model.

Let $Q = (q_{dd'})_{0 \leq d,d' \leq 1}$ be the transition matrix of the true data generating process.

Proposition 3.1. *If $Q = (q_{ij})$ satisfies*

$$\frac{q_{00}}{q_{01}} \log\left(\frac{q^2_{00}}{q^1_{00}}\right) + \log\left(\frac{q^2_{01}}{q^1_{01}}\right) + \log\left(\frac{q^2_{10}}{q^1_{10}}\right) + \frac{q_{11}}{q_{10}} \log\left(\frac{q^2_{11}}{q^1_{11}}\right) = 0, \tag{7}$$

then for any $\alpha < 1$ the process of the posterior belief profile π satisfies $|\pi^1_t - \pi^2_t| > \alpha|\pi^1_0 - \pi^2_0|$ infinitely often with Q-probability one.

This proposition, together with Propositions 2.2 and 2.8, leads to the following theorem as an immediate corollary:

Theorem 1 (Main Theorem)**.** *Suppose the assumptions of Proposition 2.8 are satisfied and B_0 and B_1 are as in Proposition 2.9. Let s be a sequential market equilibrium bubble process*

and $\alpha < 1$. If the true data-generating process transition matrix Q satisfies (7), then $s_t > \alpha \cdot \max\{B_0, B_1\} \cdot |\pi_0^1 - \pi_0^2|$ infinitely often with Q-probability one:

$$\Pr^Q(s_t > \alpha \cdot \max\{B_0, B_1\} \cdot |\pi_0^1 - \pi_0^2| \textit{ infinitely often}) = 1.$$

This theorem says that if Q^1, Q^2, and β satisfy the assumptions of Proposition 2.8, then the condition for the true data generating matrix Q to generate the bubble of high magnitude infinitely often with probability one is given by (7). This condition is not generic. It defines a 1-dimensional manifold in the set of 2×2 probabilistic matrices, which is of dimension 2. Hence, the Lebesgue measure of the set of Q's satisfying (7) is zero. In the next subsection I will provide a generic complement for the main theorem.

Throughout the rest of this subsection I discuss the derivation of Proposition 3.1, leaving some technical details for the Appendix. I also introduce notation that is needed later.

Using Bayes' rule, upon observing the history d^t, the current beliefs at time t are given by $\pi_t^i(d^t) = \left(1 + \frac{1-\pi_0^i}{\pi_0^i} L_t(d^t)\right)^{-1}$, where $L_t(d^t)$ denotes the current likelihood ratio of the history d^t for matrices Q^2 vs Q^1 (i.e., $L_t(d^t) \equiv \frac{\Pr^{Q^2}\{d^t\}}{\Pr^{Q^1}\{d^t\}}$). The current belief difference becomes:

$$\pi_t^1 - \pi_t^2 = (\pi_0^1 - \pi_0^2) \cdot (L_t^{-1}\pi_0^1 + 1 - \pi_0^1)^{-1} \cdot (L_t(1 - \pi_0^2) + \pi_0^2)^{-1},$$

and can be fully analyzed in terms of the process $L = (L_t)_t$. Note that the original beliefs are re-achieved whenever $L_t = 1$. The reason for this is that in those periods the current history does not favor Q^1 or Q^2. Similarly, each time, $L_t > 1$ it means that the current history favors the theory Q^2 vs. the theory Q^1 and both agents move towards theory Q^2 in their current posteriors. If $L_t < 1$, the situation is reversed.

Without loss of generality, I assume that the initial beliefs satisfy $1 - \pi_0^1 = \pi_0^2 = \epsilon < 1/2$. This assumption is purely for convenience. Otherwise, it could happen that for some future histories the belief difference is bigger than the original one. In that situation, the initial difference would not be a natural benchmark and the conclusion of Lemma 3.1 would

be true even for some $\alpha > 1$.[2] We have the following:

$$|\pi_t^1 - \pi_t^2| = \pi_t^1 - \pi_t^2 = (1-2\epsilon)\cdot(L_t^{-1}(1-\epsilon)+\epsilon)^{-1}\cdot(L_t(1-\epsilon)+\epsilon)^{-1}$$

We can algebraically verify that for any $0 < \alpha < 1$, $\pi_t^1 - \pi_t^2 > \alpha \cdot (\pi_0^1 - \pi_0^2)$ if and only if

$$L_t \in I_\alpha \equiv \left(1 - \frac{2}{\sqrt{1+\frac{4\alpha\epsilon(1-\epsilon)}{1-\alpha}}+1}, 1 + \frac{2}{\sqrt{1+\frac{4\alpha\epsilon(1-\epsilon)}{1-\alpha}}-1}\right).$$

Clearly, $\pi_t^1 - \pi_t^2 = \pi_0^1 - \pi_0^2$ if and only if $L_t = 1$. It is therefore consistent to define $L_0 \equiv 1$. In order to analyze the dynamics of speculation for some given Q, it is enough to see how much time the process L_t spends in the interval I_α, for $\alpha > 0$ under that particular Q.

I will now analyze the dynamics of the process L. It is easy to see that for a given history d^t, $L_t(d^t)$ is given by

$$L_t(d^t) = \prod_{\tau=0}^{t-1}\left(\frac{q^1_{d_\tau d_{\tau+1}}}{q^2_{d_\tau d_{\tau+1}}}\right).$$

For algebraic convenience, I will consider the process L in log terms:

$$l_t \equiv \log L_t = \sum_{\tau=0}^{t-1}\log\left(\frac{q^1_{d_\tau d_{\tau+1}}}{q^2_{d_\tau d_{\tau+1}}}\right). \tag{8}$$

Defining a process Δ by $\Delta_t \equiv (d_{t-1}, d_t) \in D^2$, $l = (l_t)_t$ can be expressed as $l_t = \sum_{\tau=0}^{t-1} f(\Delta_\tau)$, where function $f : D^2 \to \mathbb{R}$ is defined via (8).

[2] If $1 - \pi_0^1 \neq \pi_0^2$, then for any $\delta > 0$ π_0^1, π_0^2 can be shown to be posterior beliefs of some priors $\tilde{\pi}_0^1, \tilde{\pi}_0^2$ such that $|1 - \tilde{\pi}_0^1 - \tilde{\pi}_0^2| < \delta$.

The process Δ is a 4-state Markov chain with the transition matrix

$$\bar{Q} \equiv \begin{bmatrix} q_{00} & q_{01} & 0 & 0 \\ 0 & 0 & q_{10} & q_{11} \\ q_{00} & q_{01} & 0 & 0 \\ 0 & 0 & q_{10} & q_{11} \end{bmatrix},$$

$\bar{Q}$ is ergodic if and only if Q is ergodic, which occurs if and only if $0 < q_{01}q_{10} < 1$. Hence, in such situations, $\bar{Q}$ has the unique ergodic distribution ν^Q. Simple algebra reveals the following:

$$\nu^Q = \frac{1}{\frac{q_{00}}{q_{01}} + 2 + \frac{q_{11}}{q_{10}}} \left[\frac{q_{00}}{q_{01}}, 1, 1, \frac{q_{11}}{q_{10}} \right].$$

Now, the intuition associated with the ergodic theorem would be as follows: The process l should be recurrent if the increment $f(\Delta_t)$ has the expected value of zero on the stationary (long run) distribution ν^Q. In other words, Q should be such that

$$E^{\nu^Q} f(d_t, d_{t+1}) \equiv \nu^Q_{00} f(0,0) + \nu^Q_{01} f(0,1) + \nu^Q_{10} f(1,0) + \nu^Q_{11} f(1,1) = 0.$$

This intuition is confirmed by the following lemma:

Lemma 3.2. *Let $I \subseteq \mathbb{R}$ be an interval that can be reached by the process l with positive probability, then $\Pr^Q(l_t \in I$ infinitely often$) = 1$ if and only if Q satisfies (7).*

This result, together with all the previous considerations, completes the proof of Proposition 3.1 and subsequently the main theorem: since the process l visits any interval infinitely often, so it does the interval $\log(I_\alpha)$. Therefore, the process L visits I_α infinitely often with probability one.

The idea with regard to the proof of Lemma 3.2 is to transform the process l to a standard independent random walk $\bar{l}$, with mean zero using the appropriate stopping times. The detailed proof can be found in the Appendix. In what follows I briefly explain the way the process l is transformed into the iid increments process $\bar{l}$. This will be of use in the final subsection, wherein I discuss some dynamic properties of the process of the posterior beliefs,

and I will do it in terms of the process $\bar{l}$.

Define a sequence of optional stopping times $\tau = (\tau_n)_{n\in\mathbb{N}}$:

$$\tau_1 \equiv \min\{t > 0 | \Delta_t = (0,0)\}$$
$$\tau_{n+1} \equiv \min\{t > \tau_n | \Delta_t = (0,0)\},$$

The realization of this random sequence gives us the number of periods in which the process Δ hits the state $(0,0)$. Now, I define the process $\bar{l}$ by

$$\bar{l}_t = l_{\tau_t}.$$

The process $\bar{l}$ shows the values of the process l only in periods in which $\Delta = (0,0)$. The strong Markov property for the process Δ roughly says that at any given random period, the future and the past of the process are independent, and also that the distribution of the future is solely determined by the current state. Now, since for all periods τ_t, the current state is $(0,0)$, it necessarily follows that the distribution of the future is the same for all periods τ_t. This means the increments of the process $\bar{l}$ are iid. It can also be argued that under the condition (7) they have zero expected value, so the process $\bar{l}$ has no drift and by standard properties of iid random walks it is recurrent. In other words, any interval that can be visited is visited infinitely often with probability one.

3.2 Further Insight into Bubble Dynamics

In this subsection I present several further results concerning bubble dynamics. The first result is a generic complement of the main theorem, and the second is that bubbles of high magnitude must appear very infrequently on typical sample paths. Finally, I present the so-called arc-sine law, which leads to a rather surprising conclusion. Even if none of the agents' models is a priori favored by the data generating process, on a typical sample path one of the models will be doing better most of the time. The results in this section are expressed in terms of the process $\bar{l}$ defined in the end of the last subsection. These results automatically translate into the log-likelihood process l and hence the bubble process s, but this transition

requires some notational complications caused by the fact that the process $\bar{l}$ samples the process l at random times.

The problem with the main result is that it is not generic. In order to have persistent speculation we need to assume that the data generating matrix, Q, satisfies condition (7). As mentioned before, the set of Qs, satisfying (7) is a 1-dimensional manifold in the set of 2×2-probabilistic matrices. I will now show that if Q is approaching this manifold, then the persistency of the speculative bubble is going to infinity.

Lemma 3.3. *Let*

$$\mu = \frac{q_{00}}{q_{01}} \log\left(\frac{q_{00}^2}{q_{00}^1}\right) + \log\left(\frac{q_{01}^2}{q_{01}^1}\right) + \log\left(\frac{q_{10}^2}{q_{10}^1}\right) + \frac{q_{11}}{q_{10}} \log\left(\frac{q_{11}^2}{q_{11}^1}\right),$$

then for any interval I we have $E^Q \sum_{t=1}^{\infty} 1_{\{\bar{l}_t \in I\}} \sim \frac{1}{\mu}$. In particular, $E^Q \sum_{t=1}^{\infty} 1_{\{\bar{l}_t \in I\}} \to \infty$ as $\mu \to 0$.

The proof, being an easy application of the weak law of large numbers, can be found in the Appendix. Taking $\alpha < 1$ and $I = I_\alpha$ in Lemma 3.3, we get the generic complement to the main result.

Theorem 2. *If $0 < \alpha < 1$ and μ as in Lemma 3.3, then $E^Q \sum_{t=1}^{\infty} 1_{\{s_t > \alpha \cdot |\pi_0^1 - \pi_0^2|\}} \to \infty$ as $\mu \to 0$.*

As stated above, the expected total number of future periods in which the bubble exceeds the level $\alpha \cdot |\pi_0^1 - \pi_0^2|$ goes to infinity as μ goes to zero.

I will now examine the question of how often, on average, high bubbles should arise in this model. Since the process $\bar{l}$ is a standard random walk, we have the following result (cf. Theorem 2 in XII.2 of Feller, 1966):

Proposition 3.4. *For any data generating process Q and any interval $I \subseteq \mathbb{R}$, such that $\bar{l}_0 \notin I$*

$$E^Q \inf\{t > 0 | \bar{l}_t \in I\} = \infty,$$

This means that the expected waiting time for the process $\bar{l}$ to reach any given level different than the initial one is infinity. After translating the above to the process s_t, one can

conclude that the expected waiting time for the bubble to reach any given magnitude higher than the current one is infinity. This suggests that even though under the conditions of the main theorem the bubbles are infinitely persistent, on average the periods of high magnitude rarely prevails. In practice, this infinite expected waiting time shows up in simulations in the form of having many runs with almost no bubbles. On some runs, however, we still observe a relatively high frequency of bubbles of a significant magnitude.

Another interesting property of the dynamics of $\bar{l}$ is provided by the arc sine law, which is a standard result in the theory of the iid random walk, and it extends naturally to my setup. For this to hold, it is required that $E\Delta_0^2 < \infty$, which can be easily verified. Necessarily, Theorems 1a and 2 in XII.8 of Feller (1966) imply the following:

Proposition 3.5. *Let* $\Pi_n \equiv \#\{t > 0 | \bar{l}_t > 0\}$, *then*

$$\frac{\Pi_n}{n} \to^d Beta(1/2, 1/2),$$

This means that the distribution of the fraction of strictly positive terms in $\bar{l}$, which can be thought of as an approximation of the fraction of time when the theory Q^2 has higher likelihood than theory Q^1, is asymptotically the Beta(1/2,1/2)-distribution with density $\frac{1}{\pi\sqrt{\alpha(1-\alpha)}}$.[3] It is unbounded at the endpoints 0 and 1 and has its minimum at $\frac{1}{2}$. This means that for long samples we are more likely to see either theory Q^1 or Q^2 explaining the data better rather than seeing close to equal lengths of one theory dominating. Since the speculative bubble is at its highest whenever the data is not conclusive, this suggests that, even though the periods of high bubble magnitude appear infinitely often, they will be observed relatively rarely and will typically be surrounded by much longer periods of low bubble magnitude. What we should generally see is a long period where theory Q^1 for example, dominates and both agents assign most of the beliefs to that theory. Indeed, during this time the speculative bubble is very small. At some point, the data will start to favor theory Q^2 which is associated with the beliefs being moved towards Q^2. During that time of transition the bubble explodes because one agent is more reluctant than the other to accept the new theory. Once both types settle their beliefs close to Q^2, speculative bubble vanishes

[3]The cdf of this distribution is given by the scaled arc-sine function, which justifies the name of the law.

again. This pattern is continued forever with probability one.

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Appendix

Lemma A1. *Suppose $q^1_{00} < q^2_{00}$ and $q^1_{11} < q^2_{11}$. Define the following constants:*

$$R_1 = \begin{cases} \frac{\left(1-\sqrt{\beta q^1_{00} q^2_{00}}\right)^2}{\beta q^1_{01} q^2_{01}} & \text{if } q^1_{00} \leq \beta q^2_{00} \\ \frac{1-\beta q^2_{00}}{\beta q^2_{01}} & \text{otherwise} \end{cases}$$

$$R_2 = \begin{cases} \frac{\left(1-\sqrt{\beta q^1_{11} q^2_{11}}\right)^2}{\beta q^1_{10} q^2_{10}} & \text{if } q^1_{11} \leq \beta q^2_{11} \\ \frac{1-\beta q^2_{11}}{\beta q^2_{10}} & \text{otherwise} \end{cases}$$

If one of the following conditions hold:

1. $V^{Q^1} > V^{Q^2}$ *and* $V_0^{Q^1} - V_0^{Q^2} > R_2\left(V_1^{Q^1} - V_1^{Q^2}\right)$
2. $V^{Q^1} > V^{Q^2}$ *and* $V_1^{Q^2} - V_1^{Q^1} > R_1\left(V_0^{Q^2} - V_0^{Q^1}\right)$
3. $V_0^{Q^1} \geq V_0^{Q^2}$ *and* $V_1^{Q^1} \leq V_1^{Q^2}$

then the recursive equilibrium price bubble, S, satisfies:

$$S(d, \pi^1, \pi^2) \geq (A_d - \max_i\{V_d^{Q^i}\})|\pi^1 - \pi^2| \tag{9}$$

where the constants A_0, A_1 are given by

$$A_0 = \frac{R_1 R_2 V_0^{Q^1} - R_2(V_1^{Q^1} - V_1^{Q^2}) - V_0^{Q^2}}{R_1 R_2 - 1}$$

$$A_1 = \frac{R_1 R_2 V_1^{Q^2} - R_1(V_0^{Q^2} - V_0^{Q^1}) - V_1^{Q^1}}{R_1 R_2 - 1}$$

and satisfy

$$A_0 > \max_i\{V_0^{Q^i}\} \text{ and } A_1 > \max_i\{V_1^{Q^i}\} \tag{10}$$

Proof. First I prove (10). Using the formula for A_0 and A_1 we get:

$$\begin{aligned} A_0 - V_0^1 &= \tfrac{A_1 - V_1^1}{R_1} = \tfrac{R_2(V_1^{Q^2} - V_1^{Q^1}) - (V_0^{Q^2} - V_0^{Q^1})}{R_1 R_2 - 1} \\ \tfrac{A_0 - V_0^{Q^2}}{R_2} &= A_1 - V_1^{Q^2} = \tfrac{R_1(V_0^{Q^1} - V_0^{Q^2}) - (V_1^{Q^1} - V_1^{Q^2})}{R_1 R_2 - 1} \end{aligned} \tag{11}$$

Simple algebra reveals $R_1, R_2 > 1$. Using (11) it is easy to verify, that any of the assumptions, 1 or 2, imply (10). To get that assumption 3 also implies (10) use the formula (6) to get

$$\begin{aligned} V_0^1 - V_0^2 &= \frac{\beta(q_0^2 - q_0^1 - \beta(q_1^2 - q_1^1) + \beta(q_0^1 q_1^2 - q_0^2 q_1^1))}{(1-\beta)(1+\beta(1-q_0^1-q_1^1))(1+\beta(1-q_0^2-q_1^2))} \\ V_1^1 - V_1^2 &= \frac{\beta(-(q_1^2 - q_1^1) + \beta(q_0^2 - q_0^1) + \beta(q_0^1 q_1^2 - q_0^2 q_1^1))}{(1-\beta)(1+\beta(1-q_0^1-q_1^1))(1+\beta(1-q_0^2-q_1^2))} \end{aligned}$$

which implies that we always have: $V_0^{Q^1} - V_0^{Q^2} \geq V_1^{Q^1} - V_1^{Q^2}$. This means (11) implies (10) also under assumption 3.

Now I move on to prove (9). I will also need the following identities, which are easy to verify,

$$\begin{aligned} R_1 &= \tfrac{A_1 - V_1^{Q^1}}{A_0 - V_0^{Q^1}} \\ R_2 &= \tfrac{A_0 - V_0^{Q^2}}{A_1 - V_1^{Q^2}} \end{aligned} \tag{12}$$

Without loss of generality we may assume $\pi^1 \geq \pi^2$ and in that case for any d, d' we have $\lambda^{dd'}(\pi^1) \geq \lambda^{dd'}(\pi^2)$. This means by induction, that whenever agents start with beliefs satisfying $\pi^1 > pi^2$ than no matter what history of dividends they observe, their updated beliefs will also satisfy this inequality, $\pi_t^1(d^t) \geq \pi_t^2(d^t)$.

Let

$$p^A(d, \pi^1, \pi^2) \equiv (1-\pi^1)V_d^{Q^2} + \pi^2 V_d^{Q^1} + (\pi^1 - \pi^2)A_d$$

Note that

$$p^A(d,\pi^1,\pi^2) - V(d,\pi^1,\pi^2) = (A_d - \max_i\{V^{Q^i}(0)\})|\pi^1 - \pi^2|$$

Hence by Proposition 2.6 in order to get (9) it is enough to show $Tp^A \geq p^A$. We will do it for each state, d, separately.

First take $d = 0$. We have

$$\begin{aligned} Tp^A(0,\pi^1,\pi^2) &\geq T^1p^A(0,\pi^1,\pi^2) \qquad (13)\\ &\equiv \beta(\pi^1 q^1_{00} + (1-\pi^1)q^2_{00})\left[(1-\lambda^{00}(\pi^1))V_0^{Q^2} + \lambda^{00}(\pi^2)V_0^{Q^1} + (\lambda^{00}(\pi^1) - \lambda^{00}(\pi^2))A_0\right]\\ &+ \beta(\pi^1 q^1_{01} + (1-\pi^1)q^2_{01})\left[1 + (1-\lambda^{01}(\pi^1))V_1^{Q^2} + \lambda^{01}(\pi^2)V_1^{Q^1} + (\lambda^{01}(\pi^1) - \lambda^{01}(\pi^2))A_1\right] \end{aligned}$$

Using

$$\begin{aligned} \lambda^{00}(\pi) &= \frac{\pi q^1_{00}}{\pi q^1_{00} + (1-\pi)q^2_{00}} = 1 - \frac{(1-\pi)q^2_{00}}{\pi q^1_{00} + (1-\pi)q^2_{00}}\\ \lambda^{01}(\pi) &= \frac{\pi q^1_{01}}{\pi q^1_{01} + (1-\pi)q^2_{01}} = 1 - \frac{(1-\pi)q^2_{01}}{\pi q^1_{01} + (1-\pi)q^2_{01}} \end{aligned}$$

we can see that both p^A and T^1p^A are linear in π^1. Also note that $T^1p^A = p^A$ for $\pi^1 = \pi^2$. In order to conclude $Tp^A(0,\pi^1,\pi^2) \geq p^A(0,\pi^1,\pi^2)$ it is enough then to prove that $T^1p^A \geq p^A$ for $pi^1 = 1$. Incidentally, note that we made use of the notation T^1p^A which was introduced when we defined operator T in (5).

We have:

$$\begin{aligned} &T^1p^A(0,1,\pi^2) - p^A(0,1,\pi^2) =\\ &=\beta q^1_{00}\left[\left(1 - \frac{(1-\pi^2)q^2_{00}}{\pi^2 q^1_{00} + (1-\pi^2)q^2_{00}}\right)V_0^{Q^1} + \frac{(1-\pi^2)q^2_{00}}{\pi^2 q^1_{00} + (1-\pi^2)q^2_{00}}\cdot A_0\right]\\ &\quad + \beta q^1_{01}\left[1 + (1 - \frac{(1-\pi^2)q^2_{01}}{\pi^2 q^1_{01} + (1-\pi^2)q^2_{01}})V_1^{Q^1} + \frac{(1-\pi^2)q^2_{01}}{\pi^2 q^1_{01} + (1-\pi^2)q^2_{01}}\cdot A_1\right]\\ &\quad - (1-\pi^2)(A_0 - V_0^{Q^1}) - V_0^{Q^1}\\ &=\beta(1-\pi^2)\left[\frac{q^1_{00}q^2_{00}}{\pi^2 q^1_{00} + (1-\pi^2)q^2_{00}}(A_0 - V_0^{Q^1}) + \frac{q^1_{01}q^2_{01}}{\pi^2 q^1_{01} + (1-\pi^2)q^2_{01}}(A_1 - V_1^{Q^1}) - \beta^{-1}(A_0 - V_0^{Q^1})\right] \end{aligned}$$

To show $T^1 p^A(0,1,\pi^2) \geq p^A(0,1,\pi^2)$ for all $\pi^2 \in [0,1]$ it is enough to prove:

$$\frac{q^1_{00}q^2_{00}}{\pi^2 q^1_{00} + (1-\pi^2)q^2_{00}}(A_0 - V_0^{Q^1}) + \frac{q^1_{01}q^2_{01}}{\pi^2 q^1_{01} + (1-\pi^2)q^2_{01}}(A_1 - V_1^{Q^1}) - \beta^{-1}(A_0 - V_0^{Q^1}) \geq 0$$

for all $\pi^2 \in [0,1]$. Thanks to (10), which I already proved, I can divide both sides by $A_0 - V_0^{Q^1}$. By (12), this gives

$$\frac{q^1_{00}q^2_{00}}{\pi^2 q^1_{00} + (1-\pi^2)q^2_{00}} + \frac{q^1_{01}q^2_{01}}{\pi^2 q^1_{01} + (1-\pi^2)q^2_{01}}R_1 - \beta^{-1} \geq 0$$

Denoting $\tilde{\pi} \equiv (1-\pi^2)$, and $q^i \equiv q^i_{00} = 1 - q^i_{01}$ this is equivalent to

$$\begin{aligned}&\tilde{\pi}^2 \cdot \beta^{-1}(q^2 - q^1)^2 + \tilde{\pi}(q^2 - q^1)[R_1(1-q^1)(1-q^2) - q^1 q^2 + \beta^{-1}q^1 - \beta^{-1}(1-q^1)] \\ &+ q^1(1-q^1)[q^2 + (1-q^2)R_1 - \beta^{-1}] \geq 0 \end{aligned} \tag{14}$$

for all $\tilde{\pi} \in [0,1]$. We have to consider two cases, which determine R_1 in the proposition. The first is $q^1 > \beta q^2$, then $R_1 = \frac{1-\beta q^2_{00}}{\beta q^2_{01}}$. Plugging this back into (14) we can see that the constant term becomes zero. It is also easy to see that the linear term must me positive under our assumptions (remember we assumed $q^1_{00} < q^2_{00}$). This, together with the fact that the quadratic term coefficient is positive means that the quadratic expression must be nonnegative for $\tilde{\pi} \geq 0$. Hence we are done in this case.

Now assume $q^1 > \beta q^2$. Then $R_1 = \frac{(1-\sqrt{\beta q^1_{00} q^2_{00}})^2}{\beta q^1_{01} q^2_{01}}$. In this case it is a matter of doing a simple algebra to check that the determinant of the quadratic is zero, which, together with the fact that the quadratic term coefficient is zero implies that (14) holds for all $\tilde{\pi}$. So we are done with the case $d = 0$.

Now consider $d = 1$. We have

$$\begin{aligned} Tp^A(1,\pi^1,\pi^2) &\geq T^2 p^A(1,\pi^1,\pi^2) \\ &\equiv \beta(\pi^2 q^1_{10} + (1-\pi^2)q^2_{10})\left[(1-\lambda^{10}(\pi^1))V_0^{Q^2} + \lambda^{10}(\pi^2)V_0^{Q^1} + (\lambda^{10}(\pi^1) - \lambda^{10}(\pi^2))A_0\right] \\ &+ \beta(\pi^2 q^1_{11} + (1-\pi^2)q^2_{11})\left[1 + (1-\lambda^{11}(\pi^1))V_1^{Q^2} + \lambda^{11}(\pi^2)V_1^{Q^1} + (\lambda^{11}(\pi^1) - \lambda^{11}(\pi^2))A_1\right] \end{aligned} \tag{15}$$

Using

$$\lambda^{10}(\pi) = \frac{\pi q^1_{10}}{\pi q^1_{10} + (1-\pi) q^2_{10}} = 1 - \frac{(1-\pi) q^2_{10}}{\pi q^1_{10} + (1-\pi) q^2_{10}}$$
$$\lambda^{11}(\pi) = \frac{\pi q^1_{11}}{\pi q^1_{11} + (1-\pi) q^2_{11}} = 1 - \frac{(1-\pi) q^2_{11}}{\pi q^1_{11} + (1-\pi) q^2_{11}}$$

we can see that both $p^A(1, \pi^1, \pi^2)$ and $T^2 p^A(1, \pi^1, \pi^2)$ are linear in π^2. Also note that $T^2 p^A = p^A$ for $\pi^1 = \pi^2$. In order to conclude $T p^A(0, \pi^1, \pi^2) \geq p^A(0, \pi^1, \pi^2)$ it is enough then to prove that $T^1 p^A(1, \pi^1, \pi^2) \geq p^A(1, \pi^1, \pi^2)$ for $\pi^2 = 0$. Note that I introduced auxiliary notation T^2 ("2" refers to agent 2 not to iterating T twice).

We have:

$$\begin{aligned}
&T^2 p^A(1, \pi^1, 0) - p^A(1, \pi^1, 0) = \\
&\quad = \beta q^2_{10} \left[\left(1 - \frac{\pi^1 q^1_{10}}{\pi^1 q^1_{10} + (1-\pi^1) q^2_{10}} \right) V_0^{Q^2} + \frac{\pi^1 q^1_{10}}{\pi^1 q^1_{10} + (1-\pi^1) q^2_{10}} \cdot A_0 \right] \\
&\qquad + \beta q^2_{11} \left[1 + (1 - \frac{\pi^1 q^1_{11}}{\pi^1 q^1_{11} + (1-\pi^1) q^2_{11}}) V_1^{Q^2} + \frac{\pi^1 q^1_{11}}{\pi^1 q^1_{11} + (1-\pi^1) q^2_{11}} \cdot A_1 \right] \\
&\qquad - \pi^1 (A_1 - V_1^{Q^2}) - V_1^{Q^2} \\
&\quad = \beta \pi^1 \left[\frac{q^1_{10} q^2_{10}}{\pi^1 q^1_{10} + (1-\pi^1) q^2_{10}} (A_0 - V_0^{Q^2}) + \frac{q^1_{11} q^2_{11}}{\pi^1 q^1_{11} + (1-\pi^1) q^2_{11}} (A_1 - V_1^{Q^2}) - \beta^{-1} (A_1 - V_1^{Q^2}) \right]
\end{aligned}$$

To show $T^2 p^A(1, \pi^1, 0) \geq p^A(1, \pi^1, 0)$ for all $\pi^1 \in [0, 1]$ it is enough to prove:

$$\frac{q^1_{10} q^2_{10}}{\pi^1 q^1_{10} + (1-\pi^1) q^2_{10}} (A_0 - V_0^{Q^2}) + \frac{q^1_{11} q^2_{11}}{\pi^1 q^1_{11} + (1-\pi^1) q^2_{11}} (A_1 - V_1^{Q^2}) - \beta^{-1} (A_1 - V_1^{Q^2}) \geq 0$$

for all $\pi^1 \in [0, 1]$. The already proved inequality (10) allows me to divide both sides by $A_1 - V_1^{Q^2}$. I get (using (12)) that it is enough to prove:

$$\frac{q^1_{10} q^2_{10}}{\pi^1 q^1_{10} + (1-\pi^1) q^2_{10}} R_2 + \frac{q^1_{11} q^2_{11}}{\pi^1 q^1_{11} + (1-\pi^1) q^2_{11}} - \beta^{-1} \geq 0$$

Denoting $\bar{\pi} \equiv 1 - \pi^1$, and $q^i \equiv q^i_{11} = 1 - q^i_{10}$ this is equivalent to

$$\bar{\pi}^2 \cdot \beta^{-1}(q^2 - q^1)^2 + \bar{\pi}(q^2 - q^1)[R_2(1-q^1)(1-q^2) - q^1 q^2 + \beta^{-1} q^1 - \beta^{-1}(1-q^1)]$$
$$+ q^1(1-q^1)[q^2 + (1-q^2)R_2 - \beta^{-1}] \geq 0 \tag{16}$$

for all $\bar{\pi} \in [0,1]$.

This looks the same as (14) and the proof goes exactly the same. We have to consider two cases, which determine R_2 in the proposition. The first is $q^1 > \beta q^2$, then $R_2 = \frac{1-\beta q^2_{11}}{\beta q^2_{10}}$. Plugging this back into (16) we can see that the constant term becomes zero. It is also easy to see that the linear term must me positive under our assumptions (remember we assumed $q^1_{11} < q^2_{11}$). This, together with the fact that the quadratic term coefficient is positive means that the quadratic expression must be nonnegative for $\bar{\pi} \geq 0$. Hence we are done in this case.

Now assume $q^1 \geq \beta q^2$. Then $R_2 = \frac{(1-\sqrt{\beta q^1_{11} q^2_{11}})^2}{\beta q^1_{10} q^2_{10}}$. In this case it is an easy calculation to check that the determinant of the quadratic is zero, which, together with the fact that the quadratic term coefficient is zero implies that (16) holds for all $\bar{\pi}$. So we are done with the case $d = 1$. ∎

Proof of Lemma 3.2. Consider the Markov family of distributions associated with $\bar{Q}$, with $P^{\bar{Q}}_{\bar{\Delta}}$ denoting the probability distribution associated with the Markov chain Δ starting from $\Delta_0 = \bar{\Delta}$. Obviously the distribution of interest is the one with $\bar{\Delta} = (0, d_0)$ but we need the whole Markov family to use recursive techniques.

Define a sequence of optional stopping times:

$$\tau_1 \equiv \min\{t > 0 | \Delta_t = \Delta_0\}$$
$$\tau_{n+1} \equiv \min\{t > \tau_n | \Delta_t = \Delta_0\}$$

Ergodicity of the process Δ_t implies that each state is recurrent (cf. Billingsley (1986) Sec. 8). This mean the stopping times defined above are finite. Using Lemma 8.3 from Bilingsley (1986) we get that also their first moments are finite:

$$E\tau_n = \frac{1}{\nu^Q_1},$$

for each $n \in \mathbb{N}$. Now we consider a selection of process l, given by stopping times τ_n i.e. process $\bar{l}_n \equiv l_{\tau_n}$. Now we will show that this process has iid increments. It is known that any time-homogeneous Markov chain has the strong Markov property (cf. Paragraph 8.6.3 in Wentzel (1980)), which in the case of process Δ can be expressed as:

$$P^Q_{D\bar{e}lta}(\theta_\tau^{-1}B|\mathcal{F}_\tau) = P^Q_{\Delta_\tau}(B) \qquad P_s\text{-almost surely on } \{\tau < \infty\}$$

for any $\bar{\Delta} \in D^2$, stopping time τ and event B measurable w.r.t the process Δ. $\theta_\tau : \{D^2\}^\infty \to \{D^2\}^\infty$ denotes the shift operator, i.e. $\theta_\tau(\Delta)_t \equiv \Delta_{t-\tau}$.

For any $m < n$, consider nth and mth increments of $\bar{l}$, $\Delta^{\bar{l}}_n \equiv \sum_{t=\tau_n}^{\tau_{n+1}-1} f(\Delta_t)$ and $\Delta^{\bar{l}}_m \equiv \sum_{t=\tau_m}^{\tau_{m+1}-1} f(\Delta_t)$, respectively. For any Borel set $A \subseteq \mathbb{R}$ we have: $\{\Delta^{\bar{l}}_n \in A\} = \theta^{-1}_{\tau_n}\{\Delta^{\bar{l}}_0 \in A\}$, and $\{\Delta^{\bar{l}}_m \in A\} = \theta^{-1}_{\tau_m}\{\Delta^{\bar{l}}_0 \in A\}$, hence the strong Markov property implies:

$$P^Q_{(0,d_0)}(\{\sum_{t=\tau_n}^{\tau_{n+1}-1} f(\Delta_t) \in A\}|\mathcal{F}_{\tau_n}) = P^Q_{(0,d_0)}(\{\sum_{t=\tau_m}^{\tau_{m+1}-1} f(\Delta_t) \in A\}|\mathcal{F}_{\tau_m}) = P^Q_{(0,d_0)}(\{\sum_{t=0}^{\tau_1-1} f(\Delta_t) \in A\})$$

The rightmost probability is unconditional hence it is a constant then also two other probabilities are constant. This implies that the nth increment is independent of $\mathcal{F}_{\tau_n}$ hence of all previous increments. Also the distribution is exactly the same as all the previous increments. By induction this lets us conclude that all the increments are iid.

This proves, that the process $\bar{l}$, which is some strictly increasing random selection from the process l is a random walk with iid increments. In order to prove it is recurrent we will use some standard results from the random walk theory. In order to apply those results we need to know that the increments have zero expected value. Let for each $D\bar{e}lta \in D^2$, $e_{\bar{\Delta}} = E^Q_{\bar{\Delta}}(\Delta^{\bar{l}}_0)$. The strong Markov property implies that e must satisfy the following recursive equation:

$$e = \begin{bmatrix} f(0,0) \\ f(0,1) \\ f(1,0) \\ f(1,1) \end{bmatrix} + \bar{Q} \begin{bmatrix} 0 \\ e_{(0,1)} \\ e_{(1,0)} \\ e_{(1,1)} \end{bmatrix}$$

Pre-multiplying both sides by the stationary distribution ν^Q we get:

$$\nu^Q e = \nu^Q \begin{bmatrix} f(0,0) \\ f(0,1) \\ f(1,0) \\ f(1,1) \end{bmatrix} + \nu^Q \begin{bmatrix} 0 \\ e_{(0,1)} \\ e_{(1,0)} \\ e_{(1,1)} \end{bmatrix} = \nu^Q \begin{bmatrix} 0 \\ e_{(0,1)} \\ e_{(1,0)} \\ e_{(1,1)} \end{bmatrix}$$

so $e_{(0,0)}$=0.

Now we can apply Theorem 3 and Theorem 4 from VI.10 of Feller (1966) to conclude that for any interval I the process $\bar{l}$ visits this interval infinitely often with probability 1, i.e. $P^Q(\bar{l}_n \in I \text{ infinitely often}) = 1$. ∎

Proof of Theorem 3.3. Let F denote the distribution of Δ_0. For any interval $I \subseteq \mathbb{R}$ denote

$$U(I) = E^Q \sum_{t=1}^{\infty} 1_I(\bar{l}_t) = \sum_{k=0}^{\infty} F^{k\bigstar}(\{I\})$$

Using the weak law of large numbers we have $P(|\bar{l}_n - \mu n| < \epsilon n) > \frac{1}{2}$ for $n > n_\epsilon$. It follows that $F^{k\bigstar}([-a,a]) > 1/2$ for $n_\epsilon < k < \frac{a}{\epsilon+\mu}$ so $a^{-1}U([-a,a]) > \frac{1}{2}(\frac{1}{\epsilon+\mu} - n_\epsilon a^{-1})$. It is a standard fact from the renewal theory (cf. Theorem 1 in VI.10 of Feller (1966)) that for any $a > 1$, $U([-a,a]) \leq (2a+1)U[-1,1])$. Using this we get

$$U([-1,1]) \geq \frac{1}{2a+1} U([-a,a]) \geq \frac{1}{3} a^{-1} U([-a,a]) > \frac{1}{6}(\frac{1}{\epsilon+\mu} - n_\epsilon a^{-1})$$

for any $a > 1$ and any $\epsilon > 0$, hence taking the limit $a \to \infty$, and $\epsilon \to 0$ we get

$$U([-1,1]) > \frac{1}{6\mu}$$

∎