Belief Disagreements and Collateral Constraints

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Abstract

Belief disagreements have been suggested as a major contributing factor to the recent financial crisis. This paper theoretically evaluates this hypothesis. I assume that optimists have limited wealth and take on leverage in order to take positions in line with their beliefs. To have a significant effect on asset prices, they need to borrow from traders with pessimistic beliefs using loans collateralized by the asset itself. Since pessimists do not value the collateral as much as optimists do, they are reluctant to lend, which provides an endogenous constraint on optimists' ability to borrow and to influence asset prices. I demonstrate that the tightness of this constraint depends on the nature of belief disagreements. Optimism concerning the probability of downside states has no or little effect on asset prices because these types of optimism are disciplined by this constraint. Instead, optimism concerning the relative probability of upside states could have significant effects on asset prices. This *asymmetric disciplining* effect is robust to allowing for short selling because pessimists that borrow the asset face a similar endogenous constraint. These results emphasize that what investors disagree about matters for asset prices, to a greater extent than the level of disagreements. When richer contracts are available, insurance contracts (similar to credit default swaps) endogenously emerge to facilitate betting. Richer contracts moderate the effect of belief disagreements on asset prices because the medium of betting shifts from buying (or shorting) the asset to trading alternative contracts.

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1 Introduction

The recent financial crisis highlighted the importance of investors' belief disagreements for asset prices. A number of commentators, e.g., Shiller (2005) and Reinhart and Rogoff (2008), have identified the optimism of a fraction of investors as a potential cause for the increase in asset prices in the run-up to the crisis in the markets for housing and mortgage backed securities. The optimistic buyers in these markets have often financed their asset purchases by borrowing from other, potentially less optimistic, investors. Much of this borrowing, e.g., mortgages, margin loans, or REPO loans, has been collateralized by the asset itself. In these transactions, the fraction of the asset price buyers pay is referred to as the *margin*. Fostel and Geanakoplos (2010) report that the margin to purchase a portfolio of AAA-rated mortgage backed securities was less than 2% in 2006, while it dramatically increased to 60% by the end of 2008. Why was it easy for optimists to borrow before the crisis? Why was it difficult during the crisis?

The situation was quite the opposite for pessimists: those investors who had a negative view of the housing market. To short sell a housing-related asset, these pessimists had to borrow the asset from other, potentially more optimistic, investors that owned them. Lewis (2010) reports that mortgage backed securities were impossible to borrow in the run-up to the crisis. An alternative was to borrow and short sell shares of companies that originated subprime mortgages. However, the fees that pessimists had to pay to borrow these shares were much larger than usual.[1] In the end, pessimists bet on their negative with the help of financial innovation: The introduction of credit default swaps into the mortgage market (in 2005) enabled pessimists to buy insurance on mortgage backed securities. Why was it difficult for pessimists to borrow? How did financial innovation help?

To understand optimists' and pessimists' borrowing constraints, this paper presents a model of credit markets with belief disagreements. Two types of traders with heterogeneous prior beliefs (optimists and pessimists) invest in a risky asset or (riskless) cash. Traders can also borrow (the asset or cash) from each other. The model feature a standard *collateral constraint*: All borrowing contracts must be secured by collateral (cash or the asset) which the borrower owns. If the borrower's promised payment exceeds the value of collateral, then she defaults on the contract and the lender receives the collateral. Different borrowing contracts are available for trade at competitive prices. The contracts that are actually traded, along with the price of the asset, are determined in general equilibrium.

The model reveals an *endogenous borrowing constraint* which has the potential to shed some light on borrowing contracts observed before and during the crisis. The constraint stems from borrowers' and lenders' belief disagreements. To fix ideas, consider optimists that purchase the asset by borrowing cash from pessimists. Pessimists might be reluctant to lend because they do not value optimists' collateral (the asset) as much as optimists do. This represents an endogenous constraint on optimists' ability to borrow and to influence asset prices. A symmetric argument

[1] Lewis (2010, p.92) reports that the fee for shorting New Century stock was 12% per year. In contrast, D'Avolio (2002) reports that the short fee for the vast majority of the US stocks in his sample was less than 1% per year.

shows that belief disagreements also constrain pessimists' ability to borrow the asset. In this paper, I systematically analyze the implications of this constraint. In my baseline setting, borrowing the asset (short selling) is not allowed and borrowing cash is restricted to simple debt contracts that promise a fixed payment independent of the future state. These assumptions not only provide a good starting point,[2] but they are also useful to isolate optimists' borrowing constraints. I gradually relax the assumptions first by allowing for short selling, which is useful to analyze pessimists' borrowing constraints; and then by considering richer contracts, which are useful to analyze the role of financial innovation.

Asymmetric Disciplining of Optimism. My main result, formalized in Theorems 2 and 3, concerns the baseline setting and shows that optimists face different endogenous constraints depending on the type of their optimism. To clarify, consider a simple example in which a single risky asset is traded. There are three states, good, normal and bad, in which the asset's future price will respectively be high, average and low. Pessimists assign an equal probability, 1/3, to each state. In contrast, optimists have a greater expected valuation of the asset. In this example, one can imagine two different types of optimism. For the first case, suppose optimists assign a probability less than 1/3 to the bad state, and equal probabilities to the normal and the good states. That is, optimism is on the downside in the sense that optimists think bad states are unlikely. An example of downside optimism was offered during the recent crisis (in the Fall of 2008), when a main dimension of disagreement was whether the upcoming recession would be a depression or a garden variety recession. In this case, I show that optimists borrow by using loans with relatively high margins and the current price is relatively low (in particular, relatively close to pessimists' valuation).

For the second case, suppose optimists agree with pessimists about the probability of the bad state, but they assign a greater probability to the good state than the normal state. That is, optimism is on the upside in the sense that optimists think good events are likely. An example of upside optimism was offered in the run-up to the crisis when a main dimension of disagreement was whether the house prices would continue to increase or not. One can construct this case such that optimists' valuation of the asset is the same as in the first case, so that optimism differs in type but not level. In this case, I show that optimists borrow by using loans with lower margins and the price is higher (in particular, closer to optimists' valuation).

One way to summarize this result is to note that optimism is *asymmetrically disciplined* by the endogenous borrowing constraint: Downside optimism is disciplined while upside optimism is not. The intuition for asymmetric disciplining is related to the asymmetry in the shape of the payoffs of collateralized loans. These loans make the same full payment in upside states, but they default and make losses in downside states. Consequently, any disagreement about the probability of downside states translates into a disagreement about how to value the loans, which in turn tightens the endogenous borrowing constraint. In contrast, disagreements about the relative

[2] This is because many collateralized loans (e.g., mortgages, REPOs) are relatively simple (i.e., they do not have many contingencies) and short selling of many assets other than stocks (and some stocks) is difficult and costly.

likelihood of upside states do not tighten the endogenous borrowing constraint.

More specifically, in the above example consider a loan with a sufficiently low margin that it will default in the bad state, but not in the normal or the good states. This loan always trades at an interest rate with a *spread* over the riskless rate, which compensates the lenders for expected losses in case of default. Moreover, in a competitive loan market, the spread on the loan is just enough to compensate the lenders for their expected losses according to their pessimistic belief. Nonetheless, in the first case of the example, this spread appears too high to optimists. This is because optimists assign a lower probability to the bad state, and thus they believe it is more likely that they will pay the loan in full. Therefore, optimists believe they will pay a higher expected interest rate than the riskless rate. As a result, optimists are induced to borrow using loans with higher margins on which there is relatively less disagreement. This decreases optimists' demand for the asset, leading to an equilibrium price closer to pessimists' valuation. In contrast, in the second case of the example, the spread on the loan with low margin appears fair to optimists because traders agree about the probability of the bad state. This leads to a lower margin and a higher price.

The asymmetric disciplining result lends itself to a number of comparative statics results regarding the effect of a change in the type and the level of belief disagreements. A literature initiated by Miller (1977) has argued that an increase in belief disagreements tends to increase the overvaluation of the asset (relative to the average valuation in the population) because the asset is held by the most optimistic investors. In contrast to this literature, the level of belief disagreements in my baseline setting has ambiguous effects on the asset price. This is because, while an increase in buyers' optimism tends to increase the price, an increase in lenders' pessimism tends to decrease the price through the tightening of the endogenous borrowing constraint. Theorems 3 and 4 qualify the Miller mechanism for environments in which optimists finance their asset purchases by borrowing from less optimistic lenders. They show that an increase in belief disagreements increases the asset price (and decreases margins) if the additional disagreements concern good states, but they have the opposite effect if the disagreements are about bad states. Put differently, what investors disagree about matters, to a greater extent than the level of their disagreements.

Asymmetric Disciplining of Pessimism. To analyze pessimists' borrowing constraints, it is necessary to consider an extension of the baseline setting with simple short contracts. Pessimists use these contracts to borrow the asset from optimists, which they then sell in the market. Pessimists use their cash holdings to collateralize their promise to return the asset. Moreover, they also pay a short fee to optimists in exchange for the loan. Both the cash-collateral pessimists pledge and the short fee they pay is determined in general equilibrium. Theorem 5 shows that pessimists also face different endogenous borrowing constraints depending on the type of their pessimism.

The argument is symmetric to the earlier argument for optimists. If the future asset price exceeds the value of cash-collateral, then pessimists default on their promise to return the asset and lenders receive the cash-collateral. The short fee that lenders charge compensates them for

expected losses according to their optimistic belief. When the pessimism is about the probability of upside states, this fee appears too high to pessimists because they find default unlikely. This induces pessimists to take asset-loans with higher cash-collateral on which there is relatively less disagreement, thereby decreasing their demand for short selling and leading to a higher asset price. In contrast, when the pessimism is about the relative probability of downside states, the short fee appears fair to pessimists, leading to a lower cash-collateral and a lower price.

Note that this result, which can be dubbed asymmetric disciplining of pessimism, is complementary to the asymmetric disciplining of optimism. When the belief disagreements are on the upside, the endogenous borrowing constraint is loose for optimists but tight for pessimists, which leads to a price closer to optimists' valuation. When the belief disagreements are on the downside, the situation is the opposite and the price is closer to pessimists' valuation. In particular, the asset price is typically different than a (wealth weighted) average of different beliefs. Belief disagreements systematically affect asset prices despite the fact that optimists and pessimists can borrow with an "equally rich" set of contracts (i.e., simple debt contracts and simple short contracts).

Richer Contracts. While simple debt and simple short contracts are common in financial markets, it is important to consider more general contracts especially because they introduce new economic forces. To this end, I consider the baseline setting (with no short selling) with the difference that optimists can borrow with unrestricted contracts subject only to the collateral constraint. Optimists' optimal contract takes a threshold form: Optimists promise zero dollars if the future asset price is above a threshold while promising as much as possible (subject to the collateral constraint) if the future price is below the threshold. Theorem 6 shows that the type of the collateral optimists choose to hold depends on the asset price, which leads to two cases of interest.

First, if the asset price (which is determined in equilibrium) is sufficiently low, then optimists collateralize their bets using the asset. In this case, optimists' investment strategy is similar to the baseline setting: They buy the asset, which they partly finance by borrowing cash from pessimists. Theorem 6 shows that, while the optimal loan is different than a simple debt contract, its shape is sufficiently similar that a version of asymmetric disciplining continues to hold. This result shows that asymmetric disciplining of optimism continues to apply with richer contracts.

Second and more interestingly, if the asset price is higher, then optimists collateralize their bets using cash instead of the asset. In this case, optimists' investment strategy looks different than the baseline setting: They sell an insurance contract (collateralized by cash) that promises a fixed payment if the future state is below a threshold. This result shows that insurance contracts (that resemble credit default swaps) endogenously emerge in this setting to facilitate betting.

Given that optimists are allowed to borrow with unrestricted contracts, one could conjecture that they would bid up the asset price higher in this setting than the baseline setting (since they are less constrained). I show that this conjecture is incorrect. Quite the opposite, the asset price is always lower than the baseline setting. This is because, while optimists' only betting option in the baseline setting is to buy the asset, richer contracts offer the alternative of selling insurance

contracts collateralized by cash. The availability of the alternative method reduces optimists' demand for the asset, which leads to a lower equilibrium price. A symmetric set of results holds for the setting in which pessimists are allowed to sell unrestricted contracts (instead of simple short contracts). Taken together, these results suggest that the availability of richer financial contracts moderates the effect of belief disagreements on the asset price.

Application: Asymmetric Disciplining of Speculative Bubbles. While the results described so far concern a static setting, the asymmetric disciplining mechanism naturally interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). I consider a dynamic extension of the baseline setting to analyze this interaction. In a dynamic economy in which the identity of optimists changes over time, a speculative phenomenon obtains as the current optimists purchase the asset not only because they believe it will yield greater dividend returns, but also because they expect to make capital gains by selling the asset to future optimists. The asset price exceeds the present discounted valuation of the asset with respect to the beliefs of any trader because of the resale option value introduced by the speculative trading motive. As Scheinkman and Xiong (2003) note, this resale option value may be reasonably called a "speculative bubble." This setup is the starting point of the dynamic extension, which introduces the additional element of the endogenous borrowing constraint for optimists. The dynamic model reveals that, when optimists need to purchase the asset by borrowing from pessimists, belief disagreements can lead to speculative asset price bubbles, but only if they concern upside states. When this is the case, however, the resale option value can increase the size of the speculative component considerably because large positions can be financed by loans collateralized by the speculative asset. This is because pessimists' valuation, as well as optimists' valuation, features a speculative component. Put differently, in a speculative episode, pessimists agree to finance optimists' purchase of the asset by extending large loans because they think, should the optimist default on the loan, they can sell the collateral (the asset) to another optimist.

The analysis of the dynamic model suggests that certain economic environments that generate uncertainty (and thus belief disagreements) about upside states are conducive to asset price bubbles financed by credit. This prediction is in line with Kindleberger (1978), who has argued that speculative episodes typically follow a novel event (which arguably generates upside uncertainty), and that the easy availability of credit plays an important role in these episodes.

Outline. The organization of the rest of this paper is as follows. The next subsection discusses the related literature. Section 2 describes the model and defines the equilibrium corresponding to a general set of borrowing contracts. Section 3 considers the baseline setting with simple debt contracts and characterizes the equilibrium. This section also presents the main result about the asymmetric disciplining of optimism. Section 4 analyzes the comparative statics of equilibrium with respect to the type and the level of belief disagreements. Sections 5 and 6 respectively present the extensions with simple short contracts and richer (unrestricted) contracts. Section 7 introduces the dynamic extension and derives the implications for speculative bubbles. Section 8

concludes. The paper ends with several appendices that present omitted proofs and extensions.

1.1 Related Literature

My paper is closely related to the work of Geanakoplos (2003, 2009), who considers asset prices and borrowing contracts in a model with two continuation states and investors with a continuum of belief types. In contrast, I consider a model with a continuum of continuation states and investors with two belief types (optimists and pessimists). My assumptions and results are relevant for understanding a number of economic issues. First, while Geanakoplos (2003) illustrates that an increase in belief disagreements can increase margins and decrease asset prices considerably, my paper qualifies this result and emphasizes the type of belief disagreements rather than the level. In the Geanakoplos' model, the increase in belief disagreements decreases the asset price because the disagreements are concentrated on downside states. My results show that an increase in belief disagreements in that model would actually increase the asset price (and decrease margins) if the additional disagreements were about upside states. Second, in the Geanakoplos' model loans that are traded in equilibrium are riskless (they are collateralized with respect to the worst case scenario). This feature does not generally hold when there are more than two continuation states. With a continuum of states, risky loans are also traded in equilibrium, which enables me to analyze the riskiness of lending. Third, I generalize the Geanakoplos' model by allowing for short selling which is not uncommon in financial markets. Fourth, I also generalize the model by allowing for unrestricted contracts. Geanakoplos (2009) analyzes CDS contracts and shows that their introduction reduces the asset price. In related work, Che and Sethi (2010) show that the availability of the CDS also reduces the asset supply (and thus, the lending to the real sector). My analysis complements these results by showing that insurance contracts (that resemble CDS) endogenously emerge as the optimal contract under appropriate assumptions.

My paper is part of a large literature that concerns the effect of borrowing constraints on asset prices, e.g., Shleifer and Vishny (1992, 1997), Kiyotaki and Moore (1997), Gromb and Vayanos (2002), Fostel and Geanakoplos (2008), Adrian and Shin (2010), Garleanu and Pedersen (2011), Brunnermeier and Pedersen (2009), Acharya, Gale and Yorulmazer (2009), Brunnermeier and Sannikov (2011). In addition, my results on loan margins are related to the corporate finance literature that concerns the determinants of leverage, e.g., Townsend (1979), Myers and Majluf (1984), Gale and Hellwig (1985), Hart and Moore (1994). The main difference from these literatures is the focus on belief disagreements as a friction that constrains borrowing, as opposed to asymmetric information, lack of commitment, or exogenously specified borrowing (or margin) constraints. In related work, He and Xiong (2011) analyze the implications of belief disagreements for loan maturity and asset prices.

My results on short selling contribute to a literature that analyzes the frictions that constrain asset lending. D'Avolio (2001) emphasizes that not all investors can participate in the short market (for legal and institutional reasons), and argues that the participation constraint of potential asset lenders is crucial to sustain positive short fees. Duffie, Garleanu and Pedersen (2002)

analyze the role of search frictions in generating large short fees. My paper analyzes the role of collateral constraints in restricting asset lending. This analysis reveals that the short fees (and short margins) are also affected by the nature of belief disagreements between asset lenders and borrowers.

The relationship of my paper to the literatures initiated by Miller (1977) and Harrison and Kreps (1978) has already been discussed.[3] A related literature concerns the plausibility of assuming heterogeneous (prior) beliefs in financial markets. The market selection hypothesis, which goes back to Alchian (1950) and Friedman (1953), posits that investors with incorrect beliefs should be driven out of the market as they would consistently lose money. Recent research has emphasized that the market selection hypothesis does not apply for incomplete markets, that is, traders with inaccurate (and heterogeneous) beliefs may have a permanent presence when asset markets are incomplete.[4] Of particular interest for my paper is the work by Cao (2010), who considers a similar economy in which markets are endogenously incomplete because of collateral constraints. Cao (2010) shows that belief disagreements in this economy remain in the long run, thus providing theoretical support for my central assumptions. Another strand of literature concerns whether investors' Bayesian learning dynamics would eventually lead to common beliefs. Recent work (e.g., by Acemoglu, Chernozhukov and Yildiz, 2009) has emphasized the limitations of Bayesian learning in generating long run agreement.[5]

2 Basic environment and borrowing constraints

Consider an economy with two dates, denoted by $t \in \{0, 1\}$, and a single consumption good which will be referred to as a dollar. The economy has a continuum of risk neutral traders who have endowments date 0, but who only consume at date 1. Traders can transfer their endowments to date 1 by investing in one of two ways. First, traders can keep their dollars in cash which yields one dollar at date 1 for each dollar invested at date 0. Cash is supplied elastically and its role is to fix the riskless interest rate for this economy, which is normalized to zero. Second, traders can also invest in a risky asset which is supplied inelastically at date 0 at a normalized supply of one unit. The asset yields dollars only at date 1, and it is traded at date 0 at a price p which will be endogenously determined.

There is a continuum of possible states at date 1, denoted by $s \in \mathcal{S} = \left[s^{\min}, s^{\max}\right]$. The asset pays s dollars if state s is realized, so the state captures the uncertainty in the asset's payoff. Traders have heterogeneous priors about the state. In particular, there are two types of traders, *optimists* and *pessimists*, respectively denoted by subscript $i \in \{1, 0\}$. Type i traders' prior

[3] There is a large literature that analyzes the asset pricing implications of heterogeneous beliefs and short selling constraints. In addition to the mentioned papers, an incomplete list includes Jones and Lamont (2001), Chen, Hong and Stein (2002), Diether, Malloy and Scherbina (2002), Ofek and Richardson (2003), Lamont and Stein (2004). Papers that analyze speculative bubbles include Allen, Morris, and Postlewaite (1993), Morris (1996), Hong and Sraer (2011), Burnside, Eichenbaum and Rebelo (2011).

[4] See, for example, Blume and Easley (1992, 2006), Sandroni (2000), Kogan, Ross, Wang, and Westerfield (2006).

[5] For further discussion on the merits of the common prior assumption in economic theory, see Bernheim (1986), Aumann (1986,1998), Morris (1995), and Gul (1998).

beliefs about the state is given by the probability distribution F_i over S. Traders know each others' priors, that is, optimists and pessimists agree to disagree.[6] The probability distributions F_1 and F_0 have density functions f_1, f_0 which are continuous and positive over $\mathcal{S}$. Let $E_i[\cdot]$ denote the expectation operator corresponding to the belief of a type i trader. Optimists are optimistic about the asset in the sense that:

$$E_1[s] > E_0[s]. \tag{1}$$

In subsequent sections, this assumption will be strengthened by various regularity conditions.

I normalize the population measure of each type traders to 1. Traders are initially endowed with n_i dollars and zero units of the asset. All units of the asset are initially endowed to unmodeled agents who sell their asset holdings at date 0 and consume.[7] An economy is denoted by the tuple $\mathcal{E} = (\mathcal{S}; \{F_i\}_i; \{n_i\}_i)$.

In view of assumption (1), optimists are the natural buyers of the asset. In addition to investing their endowments, optimists might want to borrow cash from pessimists to increase their investments in the asset. Relatedly, pessimists might want to borrow the asset from optimists to short sell. The common feature in these arrangements is borrowing (either cash or the asset). I next describe the frictions that constrain borrowing in this economy.

2.1 Borrowing Constraints in General Equilibrium

In this economy, all borrowing is subject to a collateral constraint. That is, promises made by borrowers must be collateralized by either the asset or the cash that they own. If the borrower does not pay the promised amount, then the lender receives the collateral. While the collateral constraint is commonly assumed in the literature, in this model it naturally emerges as a consequence of limited liability. In particular, since borrowers receive no additional endowments at date 1, limited liability implies that their promises cannot exceed the date 1 value of their durable assets. It is important to note that limited liability alone typically does not lead to a borrowing constraint.[8] However, limited liability combined with belief disagreements between borrowers and lenders generates a borrowing constraint. For example, pessimists might be reluctant to lend cash to optimists because they do not value optimists' collateral (the asset) as much as optimists do. Thus, in this economy belief disagreements represent an endogenous constraint on borrowing.

I model this endogenous constraint using a general equilibrium approach similar to Geanakoplos (2003, 2009). In this approach, borrowing contracts are not determined by a negotiation process between borrowers and lenders. Instead, borrowing contracts are treated as commodities

[6] This assumption is a simple way of capturing trade based on belief differences. To the extent that prices do not fully reveal information, e.g. because of liquidity or noise traders, this assumption could be viewed as a reduced form for a model in which belief differences are driven by differences in information rather than priors.

[7] The only role of this assumption is to simplify the analysis by eliminating the feedback effect from asset prices to traders' net worth. All results generalize to the setting without this assumption.

[8] For this reason, much of the corporate literature on borrowing constraints concerns an additional friction such as asymmetric information (e.g., Myers and Majluf, 1984) or lack of commitment (e.g., Hart and Moore, 1994).

that are traded in anonymous competitive markets. Traders choose their positions in all available contracts taking the prices of contracts as given. The contracts that are traded in non-zero quantities are determined in general equilibrium.[9]

Formally, a borrowing contract, $\beta \equiv \left([\boldsymbol{\varphi}(s)]_{s\in S}, \alpha, \gamma\right)$, is a promise of $\boldsymbol{\varphi}(s) \geq 0$ dollars in state s, collateralized by $\alpha \geq 0$ units of the asset and the $\gamma \geq 0$ units of cash. The contract is traded in an anonymous market at a competitive price $q(\beta)$ that will be endogenously determined. A trader who sells contract β, borrows $q(\beta)$ dollars at date 0. The borrower sets aside α units of the asset and γ units of cash that she owns as collateral. A trader who buys this contract, lends $q(\beta)$ dollars at date 0. The lender becomes entitled to a payment of $\boldsymbol{\varphi}(s)$ dollars in state s of date 1. However, the lender may not receive $\boldsymbol{\varphi}(s)$ dollars in full because the payment is only enforced by collateral. More specifically, if the future asset value s is such that $\boldsymbol{\varphi}(s) > \alpha s + \gamma$, then the contract defaults and the lender only receives the value of the collateral, $\alpha s + \gamma$. Combining the default and the non-default events, the payoff of contract β can be written as:

$$\min\left(\alpha s + \gamma, \boldsymbol{\varphi}(s)\right). \tag{2}$$

This framework can account for various different forms of collateralized borrowing arrangements, as illustrated by the following examples.

1. Simple debt contracts. Consider a contract that satisfies $\boldsymbol{\varphi}(s) \equiv \varphi$ for some $\varphi \in \mathbb{R}_+$. This corresponds to a debt contract in which the borrower promises a fixed payment of φ dollars at date 1. The debt contract is simple in the sense that the borrower's payment does not depend on the future state. Simple debt contracts provide a model of some common collateralized loans (e.g., REPOs, mortgages, asset purchases on margin) which do not have many contingencies. Sections 3.3 and 4 (the baseline setting) analyze the equilibrium corresponding to these types of contracts.

2. Simple short contracts. Consider a contract that satisfies $\boldsymbol{\varphi}(s) \equiv \varphi s$ for some $\varphi \in \mathbb{R}_+$. This corresponds to a short contract in which the borrower promises to return φ units of the asset at date 1. The short contract is simple in the sense that the promised number of assets does not depend on the future state. Simple short contracts provide a model of short selling the asset. Section 5 analyzes the equilibrium corresponding to this set of contracts.

3. Insurance contracts. Consider a contract that satisfies $\boldsymbol{\varphi}(s) = \begin{cases} \varphi, & \text{if } s < \bar{s}, \\ 0, & \text{if } s > \bar{s}. \end{cases}$ for some $\varphi \in \mathbb{R}_+$ and $\bar{s} \in S$. This corresponds to an insurance contract in which the borrower (i.e., the insurance provider) promises to pay φ dollars if the future state, s, is below a threshold, $\bar{s}$. The price of the contract, $q(\beta)$, corresponds to the insurance premium. Section 6 considers the equilibrium with an unrestricted contract space and shows that, under appropriate assumptions, insurance

[9] The general approach of treating contracts as commodities has been pioneered by Prescott and Townsend (1984a, 1984b). The equilibrium notion in this section has been first developed by Geanakoplos and Zame (1997, 2009).

contracts are traded in positive quantities. In this sense, insurance contracts endogenously emerge in this economy.

Let

$$\mathcal{B} = \left\{ \left([\boldsymbol{\varphi}(s)]_{s \in S}, \alpha, \gamma \right) \mid \boldsymbol{\varphi}(s) \in \mathbb{R}_+ \text{ for each } s \in S,\ \alpha \in \mathbb{R}_+,\ \gamma \in \mathbb{R}_+ \right\}$$

denote the set of all borrowing contracts. The rest of this section defines the general equilibrium corresponding to an exogenously specified subset of traded contracts, $\mathcal{B}^T \subset \mathcal{B}$. Sections 3.3-6 characterize this equilibrium for different specifications for $\mathcal{B}^T$.

Suppose $\mathcal{B}^T$ is a Borel set, and the price function $q(\cdot)$ is Borel measurable over $\mathcal{B}^T$. I model traders' positions in contracts as Borel measures over $\mathcal{B}^T$. In particular, type i traders choose two measures, μ_i^+, μ_i^-. The measure , μ_i^+, represents traders' positive positions on contracts: that is, it captures the contracts through which type i traders lend. The measure, μ_i^-, represents traders' negative positions in contracts: that is, it captures the contracts through which type i traders borrow. In addition, traders also choose their asset demands, $a_i \in \mathbb{R}_+$, and their cash holdings, $c_i \in \mathbb{R}_+$. Type i traders' budget constraint is given by:

$$pa_i + c_i + \int_{\beta \in \mathcal{B}^T} q(\beta)\, d\mu_i^+ - \int_{\beta \in \mathcal{B}^T} q(\beta)\, d\mu_i^- \leq n_i. \tag{3}$$

Note that the negative positions, μ_i^-, enable traders to borrow and expand their budget so that they can invest more in the asset or cash. However, borrowing is subject to the collateral constraints:

$$\begin{aligned} \int_{\left([\boldsymbol{\varphi}(s)]_{s \in S}, \alpha, \gamma\right) \in \mathcal{B}^T} \alpha d\mu_i^- &\leq a_i, \text{ and} \\ \int_{\left([\boldsymbol{\varphi}(s)]_{s \in S}, \alpha, \gamma\right) \in \mathcal{B}^T} \gamma d\mu_i^- &\leq c_i. \end{aligned} \tag{4}$$

In particular, traders must own sufficiently many assets and cash to pledge as collateral for the contracts they sell. Note that there is no analogous condition for traders' positive positions, μ_i^+, because lending does not require collateral. Type i traders choose their positions to solve:

$$\max_{(a_i, c_i) \in \mathbb{R}_+^2;\ \mu_i^+, \mu_i^-} \begin{array}{c} a_i E_i[s] + c_i + \int_{\mathcal{B}^T} E_i[\min(\boldsymbol{\varphi}(s), \alpha s + \gamma)]\, d\mu_i^+ \\ - \int_{\mathcal{B}^T} E_i[\min(\boldsymbol{\varphi}(s), \alpha s + \gamma)]\, d\mu_i^- \end{array}, \tag{5}$$

$$\text{subject to } (3) \text{ and } (4).$$

Note that traders calculate their expected payoffs (and payments) on assets and contracts according to their own beliefs.

The market for contracts is competitive. In particular, the price function $q(\cdot)$ is determined by debt market clearing, which can be written as:

$$\sum_{i\in\{1,0\}} \int_B d\mu_i^+ = \sum_{i\in\{1,0\}} \int_B d\mu_i^- \text{ for each Borel set } B \subset \mathcal{B}^T. \tag{6}$$

In words, the measure of positive positions on "each" contract must be equal to the measure of negative positions.

Definition 1. *A general equilibrium is a collection of prices* $(p, [q(\cdot)])$ *and allocations* $\left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)_{i\in\{1,0\}}$ *such that: traders' positions,* $\left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)$*, solve Problem* (5) *for each* $i \in \{0,1\}$*, the asset market clears,* $\sum_{i\in\{1,0\}} \hat{a}_i = 1$*, and debt markets clear [cf. Eq.* (6)*].*

3 Equilibrium with Simple Debt Contracts

This section characterizes the general equilibrium in the baseline setting with simple debt contracts. It also presents the main result which shows that optimism is asymmetrically disciplined by borrowing constraints.

Recall that a simple debt contract is denoted by $\left([\boldsymbol{\varphi}(s) = \varphi]_{s\in S}, \alpha, \gamma\right)$ for some $\varphi \in \mathbb{R}_+$. In addition, I assume the cash-collateral is zero, $\gamma = 0$, which is without loss of generality in this section.[10] I also normalize the contracts by taking the asset-collateral to be 1, i.e., $\alpha = 1$. Under these assumptions (and normalization), the set of traded contracts is given by:

$$\mathcal{B}^D \equiv \left\{\left([\boldsymbol{\varphi}(s) \equiv \varphi]_{s\in S}, 1, 0\right) \mid \varphi \in \mathbb{R}_+\right\}. \tag{7}$$

When there is no confusion, I denote a simple debt contract in $\mathcal{B}^D$ by its promised payment, φ.

Restricting attention to the contract set $\mathcal{B}^D$ represents two frictions in addition to the collateral constraint. The first friction is the absence of short selling. The second friction is the absence of more general debt contracts which may promise payments contingent on the asset's value s. It is natural to start with simple debt contracts, and defer the analysis of richer contracts to later sections, for a number of reasons. First, many assets other than stocks are difficult and costly to short sell, and many common collateralized loan arrangements (e.g., REPOs or asset purchases on margin) do not feature contingencies. Second, considering simple debt contracts conforms well with a strand of the collateral constraints literature that focuses on simple and riskless debt contracts (e.g., Kiyotaki and Moore, 1997, or more recently, Brunnermeier and Sannikov, 2011). The analysis in this section is more general than this strand of the literature because optimists are allowed to borrow with contracts that might default in some future states.

To characterize the equilibrium corresponding to set $\mathcal{B}^D$, I first consider an alternative principal-agent model of borrowing constraints which is more tractable than the general equilibrium model. I proceed by establishing the equivalence of the equilibria of the two models.

[10] To see this, consider a contract $(\varphi, \psi^a, \psi^c)$ with $\psi^c > 0$. The transfers generated by this contract can be equivalently captured by the contract, $(\max(0, \varphi - \psi^c), \psi^a, 0)$.

I then characterize the principal-agent equilibrium. The equivalence result not only provides a method of solving the general equilibrium, but it also clarifies (in the context of economy $\mathcal{E}$) the relationship between the general equilibrium and the principal-agent models of borrowing constraints.

3.1 An Alternative Principal-Agent Model of Borrowing Constraints

As an alternative to the general equilibrium model, one could imagine that contracts are determined as the result of a contractual negotiation process between borrowers and lenders. In this case, it would be possible to characterize the set of constrained efficient contracts. However, the contract allocation within this set would depend on the allocation of bargaining power between traders (as well as the details of the bargaining process). To make progress, it is common in the financial frictions literature to focus on the special case in which borrowers have all the bargaining power (cf. Gale and Hellwig, 1985, Holmstrom and Tirole, 1997, and also Kiyotaki and Moore, 1997 in the renegotiation stage). This leads to a principal-agent approach: The borrower chooses the debt contract subject to a set of frictions and lenders' participation constraint.

I next consider this principal-agent approach to characterize the simple debt contracts that are traded in economy $\mathcal{E}$. In particular, optimists (who are the natural borrowers with simple debt contracts) choose their borrowing and outstanding debt subject to a participation constraint for pessimists (who are the natural lenders). To eliminate corner cases, I also make the following assumption:

Assumption (A1). $n_1 < E_1[s] - s^{\min}$ *and* $n_0 > E_1[s] - n_1$.

The first part of the assumption ensures that optimists (in equilibrium) cannot fund their asset purchases with riskless debt. The second part ensures that pessimists' endowment is sufficiently large to meet optimists' borrowing demand. Given assumption (A1), I will establish that the equilibrium price satisfies $p \in (E_0[s], E_1[s])$. Since the price is greater than the pessimistic valuation, pessimists have no interest in investing in the asset. They use their endowment to invest in cash and to lend to optimists. In contrast, optimists make leveraged investments in the asset. The remaining question is how much optimists borrow and how many assets they demand.

To address this question, let a_1 denote optimists' asset position and φ denote their outstanding debt per-asset at date 1. In view of the collateral constraint, optimists' actual payment on their debt is given by $\min(s, \varphi)$. Consequently, lenders' participation constraint implies that their lending per-asset at date 0 is given by:

$$E_0[\min(s, \varphi)]. \tag{8}$$

Optimists choose their asset position and outstanding debt to solve:

$$\begin{aligned} \max_{(a_1,\varphi)\in\mathbb{R}^2_+} \quad & a_1 E_1[s] - a_1 E_1[\min(s,\varphi)], \\ \text{s.t.} \quad & a_1 p = n_1 + a_1 E_0[\min(s,\varphi)]. \end{aligned} \tag{9}$$

The first line is optimists' expected payoff at date 1: They receive a payoff from the asset but they make a payment on their debt. The second line is optimists' budget constraint which incorporates lenders' participation constraint.

Definition 2. *Given assumption (A1), a* principal-agent equilibrium *is a pair of asset price p and optimists' allocation (a_1^*, φ^*), such that: optimists' allocation solves problem* (9) *and the asset market clears, that is, $a_1^* = 1$.*

The principal-agent equilibrium will be characterized in Section 3.3. The characterization (and the equivalence result in Section 3.2) requires the following regularity condition on beliefs, which ensures that problem (9) has a unique solution for each p.

Assumption (A2). *The probability distributions F_1 and F_0 satisfy the hazard-rate order, that is:*

$$\frac{f_1(s)}{1-F_1(s)} < \frac{f_0(s)}{1-F_0(s)} \text{ for each } s \in \left(s^{\min}, s^{\max}\right). \tag{10}$$

The hazard-rate order is equivalent to saying that $\frac{1-F_1(s)}{1-F_0(s)}$ is strictly increasing over $\mathcal{S}$. Intuitively, this notion of optimism concerns optimists' relative probability assessment for the *upper-threshold events* $[s, s^{\max}] \subset \mathcal{S}$. It posits that optimists are increasingly optimistic for these events as the threshold level s is increased. It captures the idea that, the "better" the event becomes, the greater the optimism is for the event. The hazard-rate order is related to some well known regularity conditions. It is stronger than the first order stochastic order, that is, the inequality in (10) implies that F_1 dominates F_0 in the first order stochastic sense. However, it is weaker than the monotone likelihood ratio property: that is, if $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over $\mathcal{S}$, then the inequality in (10) holds.

3.2 Equivalence of the Principal-Agent and the General Equilibrium

The principal-agent approach is useful for its simplicity and tractability. However, the assumption that optimists have all the bargaining power requires motivation. In contrast, the general equilibrium approach does not a priori take a stance on optimists' and pessimists' relative bargaining powers. It is perhaps fortunate that for this economy the two approaches are equivalent, as shown by the following result.

Theorem 1 (Existence and Equivalence of Equilibria). *Suppose the contract space is restricted to simple debt contracts, $\mathcal{B}^T = \mathcal{B}^D$, and assumptions (A1) and (A2) hold.*

(i) There exists a unique principal-agent equilibrium $[p^*, (a_1^*, \varphi^*)]$.

(ii) There exists a general equilibrium $\left[(p, [q(\cdot)]), \left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)_{i\in\{1,0\}}\right]$. In this equilibrium, optimists borrow and pessimists lend, i.e., $\hat{\mu}_1^+ = \hat{\mu}_0^- = 0$. Moreover, only a single contract is traded in non-zero quantities, that is, $\hat{\mu}_1^-$ (and thus, $\hat{\mu}_0^+$) is a Dirac measure that puts weight only at one contract $\hat{\varphi} \in \mathcal{B}^D$. The general equilibrium is equivalent to the unique principal-agent

equilibrium in the sense that:

$$p = p^*,\ \hat{a}_1 = a_1^* = 1,\ \hat{\varphi} = \varphi^*,\ \textit{and}\ \hat{q}(\hat{\varphi}) = E_0[\min(s, \varphi^*)]. \tag{11}$$

The first part establishes the existence and uniqueness of the principal-agent equilibrium. The second part establishes the existence of a general equilibrium which is equivalent to the principal-agent equilibrium (see the proof in Appendix A.2). In particular, general equilibrium takes the same form as the principal-agent equilibrium: pessimists invest in cash and debt contracts (i.e., they lend to optimists), while optimists make leveraged investments in the asset by selling debt contracts. Moreover, the equilibrium prices, allocations, and traded debt contracts are the same. More specifically, in either model optimists have the same outstanding debt per-asset (i.e., $\varphi^* = \hat{\varphi}$) and they borrow the same amount ($\hat{q}(\hat{\varphi}) = E_0[\min(s, \varphi^*)]$). Put differently, the general equilibrium approach results in contracts "as if" optimists have all the bargaining power. Intuitively, this is because short selling is not allowed and pessimists' endowment is sufficiently large to meet optimists' demand for borrowing. The absence of short selling ensures that pessimists' only relevant investment options are investing in the storage technology and lending to optimists. Pessimists' large endowment [i.e., assumption (A1)] ensures that pessimists compete to make loans to optimists, rather than the opposite.

3.3 Characterization of Equilibrium

I next turn to the characterization of the principal-agent equilibrium, which corresponds to a general equilibrium in view of Theorem 1. The next result, which is also the main result, characterizes optimists' contract choice for a given price p. The next subsection combines this analysis with asset market clearing to solve for the equilibrium asset price.

Recall that optimists' outstanding debt per-asset is denoted by φ. Thus, optimists default on their debt if and only if the future asset value, s, is lower than the threshold state, $\bar{s} \equiv \varphi$. Consequently, I will refer to this simple debt contract as a *loan with riskiness* $\bar{s}$. Note also that optimists' borrowing per-asset is given by $E_0[\min(s, \bar{s})]$ [cf. Eq. (8)], which I refer to as the size of the loan. The size of the loan is increasing in its riskiness: that is, larger loans are also riskier loans. Given these definitions, one interpretation of problem (9) is that optimists choose from a menu of loans with different sizes (and thus, riskiness levels) which are priced by pessimists. The next result characterizes the optimal loan.

Theorem 2 (Asymmetric Disciplining of Optimism). *Suppose the contract space is restricted to simple debt contracts, $\mathcal{B}^T = \mathcal{B}^D$, and assumptions (A1) and (A2) hold. Fix asset price p that satisfies $p \in (E_0[s], E_1[s])$, and consider optimists' problem* (9)*. The riskiness, $\bar{s}$, of the optimal loan is the unique solution to the following equation over $\mathcal{S}$:*

$$p = p^{opt}(\bar{s}) \equiv \int_{s^{\min}}^{\bar{s}} s dF_0 + (1 - F_0(\bar{s})) \int_{\bar{s}}^{s^{\max}} s \frac{dF_1}{1 - F_1(\bar{s})}. \tag{12}$$

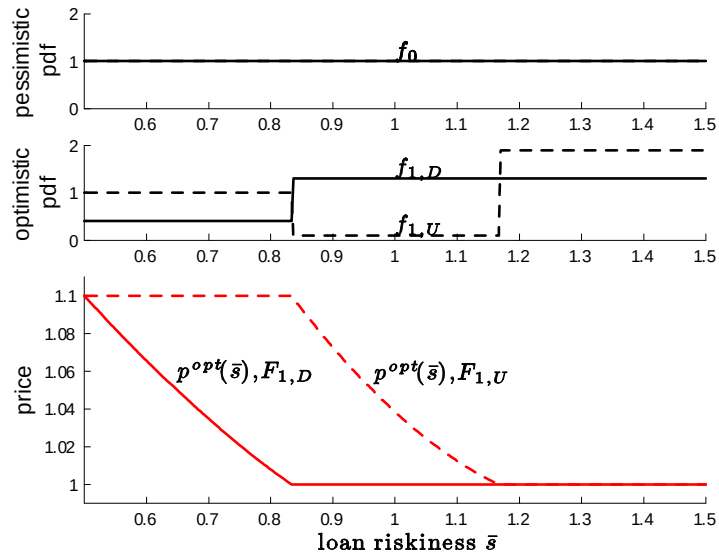

Figure 1: The top two panels display the probability density functions for traders' beliefs in the two scenarios of Example 1. The bottom panel displays the corresponding optimality curves, $p^{opt}(\bar{s})$, the inverse of which gives the optimal loan riskiness $\bar{s}$ for a given price level p.

I will shortly provide a sketch proof of this result along with an intuition. Before doing so, it is useful to note a few important aspects of the function, $p^{opt}(\cdot)$. First, this function is similar to an inverse demand function: Given the price on the y-axis, it describes the riskiness of the optimal loan on the x-axis. Second, assumption (A1) implies $p^{opt}(\cdot)$ is strictly decreasing and continuous (cf. Appendix A.1). Since $p^{opt}\left(s^{\min}\right) = E_1[s]$ and $p^{opt}(s^{\max}) = E_0[s]$, this further implies that there is a unique solution to Eq. (12).

Note also that $p^{opt}(\cdot)$ describes the equilibrium asset price conditional on the equilibrium loan riskiness $\bar{s}$. In particular, rewriting Eq. (12) yields the following asset pricing formula:

$$p^{opt}(\bar{s}) = F_0(\bar{s})\, E_0[s \mid s < \bar{s}] + (1 - F_0(\bar{s}))\, E_1[s \mid s \geq \bar{s}]. \tag{13}$$

This formula shows that optimism is asymmetrically disciplined in equilibrium. More specifically, the asset is priced with a mixture of optimistic and pessimistic beliefs. Pessimistic beliefs are used to assess the probability of default, $F_0(\bar{s})$, as well as the value of the asset conditional on default, $E_0[s \mid s < \bar{s}]$, while the optimistic beliefs are used to assess the value of the asset conditional on no default, $E_1[s \mid s \geq \bar{s}]$. Consequently, optimism about the probability of default states will not affect the asset price, while optimism about the relative probability of non-default states will increase the price. The following example further illustrates this asymmetric disciplining property.

Example 1 (Asymmetric Disciplining of Optimism). *Consider the state space* $\mathcal{S} = [1/2, 3/2]$. *Consider the following two cases that differ in the type of optimism.*

Case (i). *First suppose pessimists and optimists have the prior belief distributions* F_0 *and* $F_{1,D}$ *with density functions:*

$$f_0(s) = 1 \text{ for each } s \in \mathcal{S}, \qquad f_{1,D}(s) = \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \equiv [2/3 - 1/6, 2/3 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_N \equiv [1 - 1/6, 1 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_G \equiv [4/3 - 1/6, 4/3 + 1/6] \end{cases}.$$

Here, $\mathcal{S}_B$*,* $\mathcal{S}_N$*, and* $\mathcal{S}_G$ *capture "bad", "normal" and "good" events. Pessimists find all states equally likely while optimists have* downside optimism *in the sense that they think a bad even is unlikely.*[11]

Case (ii). *Next suppose pessimists have the same belief, but optimists' belief is changed to the distribution* $F_{1,U}$ *with density function*

$$f_{1,U} = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 0.1 & \text{if } s \in \mathcal{S}_N \\ 1.9 & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, optimists have upside optimism *in the sense that they think a good event is more likely than a normal event (while they agree with pessimists about the probability of the bad event). Note also that optimists are equally optimistic in both cases, that is,* $E_{1,U}[s] = E_{1,D}[s]$*.*

The bottom panel of Figure 1 displays the optimality curves, $p^{opt}(\cdot)$*, corresponding to the two cases. For any price* p*, optimists choose a larger and riskier loan in the second case (with upside optimism) than the first case (with downside optimism). Equivalently, for any level of loan riskiness* $\bar{s}$*, the asset price is higher in the second case than in the first case, illustrating the asymmetric disciplining of optimism.*

I next present a sketch proof of Theorem 2, which will be useful to provide the intuition. Optimists' problem (9), after substituting a_1 from the budget constraint, can be written as maximizing $n_1 R_1^L(\bar{s})$, where:

$$R_1^L(\bar{s}) \equiv \frac{E_1[s] - E_1[\min(s, \bar{s})]}{p - E_0[\min(s, \bar{s})]}. \tag{14}$$

This expression is the return of optimists who buy one unit of the asset and who finance part of the purchase using a loan with riskiness $\bar{s}$. The denominator is the downpayment optimists make: they borrow $E_0[\min(s, \bar{s})]$ from pessimists and they pay the rest of the purchase. The numerator is optimists' expected payoff: they expect to receive $E_1[s]$ from the asset and they also expect to pay $E_1[\min(s, \bar{s})]$ on their loan. Appendix A.1 shows that $R_1^L(\bar{s})$ has a unique maximum characterized by the first order condition. The first order condition is given by $p = p^{opt}(\bar{s})$, which

[11] Note that the belief distributions have discontinuous densities and they satisfy the hazard rate inequality, (10), only weakly. These distributions are used for illustration purposes because they provide a clear intuition. For analytical tractability, the formal results assume that beliefs have continuious densities and they satisfy the inequality in (10) strictly.

completes the sketch proof of Theorem 3.

For intuition, it is useful to break down $R_1^L(\bar{s})$ into two components. First consider the left hand side terms in the numerator and the denominator of (14), which constitute the *unleveraged return*:

$$R^U = \frac{E_1[s]}{p}.$$

This expression is the expected return of optimists if they buy the asset without borrowing. Since the theorem concerns prices that satisfy $p < E_1[s]$, the unleveraged return satisfies $R^U > 1$. That is, optimists perceive that investing in the asset yields a higher return than investing in the storage technology. This creates a force that pushes optimists towards taking larger and riskier loans to invest more in the asset. However, there is a second force that operates in the opposite direction. This force is related to the right hand side terms in the numerator and the denominator of (14), which constitute optimists' *perceived interest rate* on the loan:

$$1 + r_1^{per}(\bar{s}) \equiv \frac{E_1[\min(s, \bar{s})]}{E_0[\min(s, \bar{s})]}. \tag{15}$$

Optimists borrow $E_0[\min(s, \bar{s})]$ on the loan, but they expect to pay $E_1[\min(s, \bar{s})]$, which leads to the perceived interest rate $r_1^{per}(\bar{s})$. Assumption (A1) implies that $r_1^{per}(\bar{s})$ is always weakly greater than the riskless rate of 0, and that it is increasing in $\bar{s}$ [cf. Appendix A.1]. This in turn creates a force that pushes optimists towards taking smaller and safer loans. The intuition for the properties of $r_1^{per}(\cdot)$ relies on two observations. First, collateralized loans always trade at an interest rate with a spread over the riskless rate because lenders require compensation for their expected losses in case of default. Moreover, since the loan market is competitive, the spread on a loan is just enough to compensate lenders according to their pessimistic belief. Second, optimists believe that the loan will default less often than pessimists do. Hence optimists think they will end up paying the full loan amount more often. Consequently, optimists perceive that they will pay a greater interest rate than the riskless rate. Moreover, for greater levels of $\bar{s}$, the scope of disagreement for default is greater, which implies that $r_1^{per}(\bar{s})$ is increasing in $\bar{s}$.

It follows that, while a larger and riskier loan enables optimists to take larger positions on the asset, it also comes at a greater perceived interest rate, $r_1^{per}(\bar{s})$. Optimists' optimal loan choice balances these two forces. This breakdown of forces also provides an intuition for the observation that the optimality curve, $p^{opt}(\bar{s})$, is decreasing. When the price is lower, optimists' unleveraged return, $R^U = \frac{E_1[s]}{p}$, is greater. This induces optimists to borrow more by taking a larger and riskier loan, agreeing to pay a greater perceived interest rate at the margin.

To see the intuition for the asymmetric disciplining, fix a loan with riskiness $\bar{s}$, and consider how much the price should drop (from the optimistic valuation) to entice optimists to take this particular loan. Consider this question in the context of Example 1 for a riskiness level $\bar{s} = 0.8 \in \mathcal{S}_B$. In the first case of Example 1, optimists find the bad event $\mathcal{S}_B$ unlikely. Hence, given a loan with riskiness $\bar{s} \in \mathcal{S}_B$, there is disagreement about the probability of default, which implies $r_1^{per}(\bar{s}) > 0$. As this loan appears expensive to optimists, the asset price should drop considerably

to entice optimists to take this loan. Consider instead the second case of Example 1. In this case, for a loan with riskiness $\bar{s} \in S_B$, there is no disagreement about the probability of default, which implies $r_1^{per}(\bar{s}) = 0$. As the loan appears cheap to optimists, the asset price does not need to fall to entice them to take the loan (see Figure 1).

In other words, the asymmetric disciplining result operates through optimists' borrowing constraints. When the optimism is on the downside, optimists perceive tighter borrowing constraints [captured by a higher $r_1^{per}(\bar{s})$], which lowers their demand and leads to an asset price closer to pessimists' valuation. In contrast, upside optimism generates looser borrowing constraints and leads to an asset price closer to optimists' valuation.

3.4 Asset Market Clearing

I next consider asset market clearing and solve for the equilibrium. The budget constraint of problem (9) characterizes optimists' asset demand as:

$$a_1 = \frac{n_1}{p - E_0\left[\min\left(s, \bar{s}\right)\right]}. \tag{16}$$

The denominator of this expression is the downpayment optimists make to buy one unit of asset, using a loan with riskiness $\bar{s}$ to finance the rest of the purchase. The numerator is their endowment, all of which they spend to purchase assets. Market clearing requires equating optimists' asset demand in (16) with the asset supply of 1, which leads to:

$$p = p^{mc}(\bar{s}) \equiv n_1 + E_0\left[\min\left(s, \bar{s}\right)\right]. \tag{17}$$

Note that the market clearing curve, $p^{mc}(\bar{s})$, is increasing: When optimists take a larger and riskier loan, their demand for the asset is greater, which leads to a higher market clearing price. The equilibrium price and loan riskiness pair, $(p, \bar{s}^*)$, is determined as the unique intersection of the (decreasing) optimality curve, $p^{opt}(\bar{s})$ and the (increasing) market clearing curve, $p^{mc}(\cdot)$ (see Figure 2).

Figure 2 illustrates the equilibrium and shows the effect of a decline in optimists' net worth. As in Geanakoplos (2009), the price falls toward the pessimistic valuation. But this model features an additional effect: The equilibrium loan becomes larger and riskier. Intuitively, as the price falls, optimists see more of a bargain in the asset price which encourages them to invest more by taking larger and riskier loans. While the effects of optimists' net worth (and its dynamics) are important, the focus of this paper is on the effects of different types of belief heterogeneity, which I turn to next.

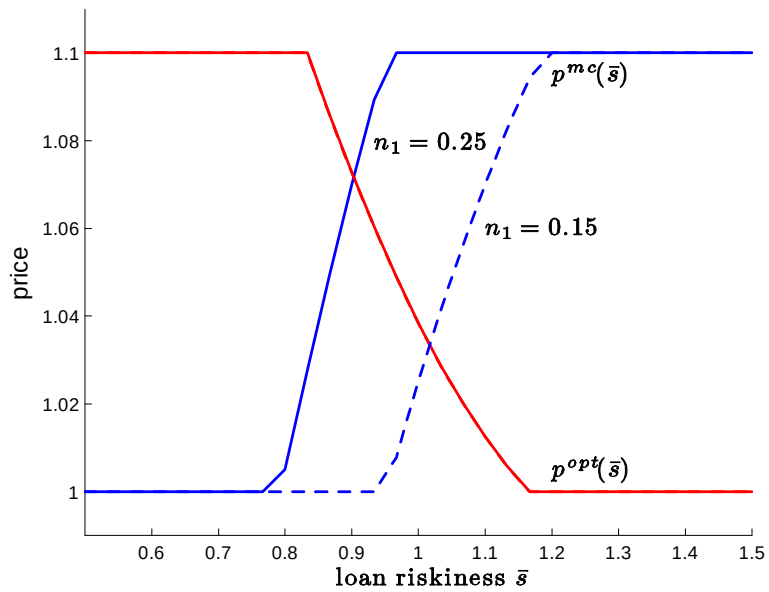

Figure 2: The figure displays the equilibrium, and the response of the equilibrium to a decline in optimists' initial endowment, n_1.

4 The Type and the Level of Belief Heterogeneity

This section establishes the comparative statics of equilibrium with respect to the type and the level of belief heterogeneity. In addition to the equilibrium loan riskiness, $\bar{s}^*$, and the asset price, p, a variable of interest is the *margin* on loans. The margin is defined as the fraction of the asset price optimists pay out of their own pocket,

$$m \equiv \frac{p - E_0\left[\min\left(s, \bar{s}^*\right)\right]}{p}, \tag{18}$$

while borrowing the rest using the collateralized loan. Note that, taking the prices as given, there is a one-to-one mapping between m and $\bar{s}^*$, with lower margins corresponding to higher riskiness levels. The comparative statics of the margin are important for a couple reasons. First, margins are readily observable for some common collateralized lending arrangements (e.g., REPOs, mortgages, or margin purchases). Second, recent studies, e.g., Garleanu and Pedersen (2011), develop general asset pricing models by taking margins as exogenous. The endogenous determination of the margin in this model might inform the choice of the exogenous margins when applying these theories.

To establish the comparative statics for the type of belief heterogeneity, it is necessary first to define the different types. The following definition introduces a notion of skewness of optimism, which intuitively captures the difference between optimists' beliefs in the two cases of Example 1.

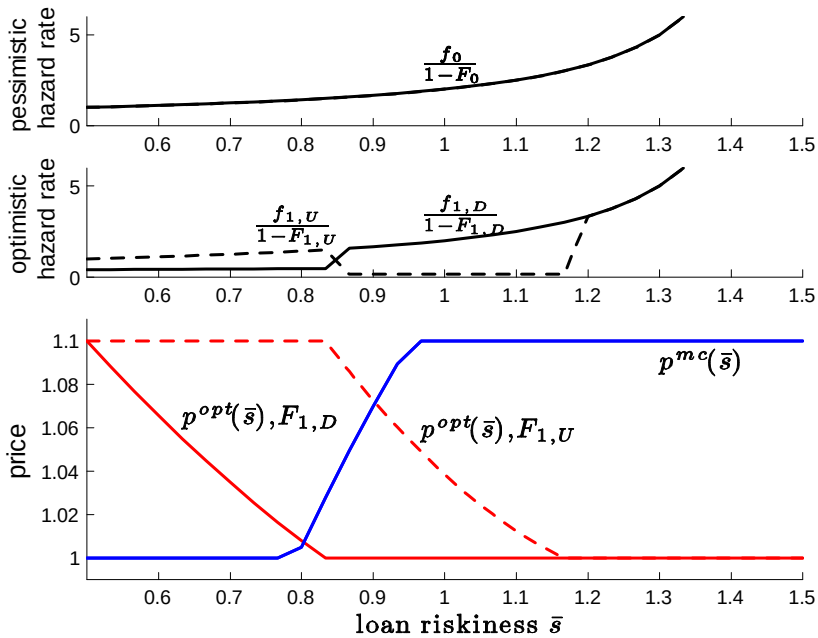

Figure 3: The top two panels display the hazard rates for traders' priors in the two cases analyzed in Example 1. The bottom panel plots the corresponding equilibria.

Definition 3 (Upside Skew of Optimism). *The optimism of distribution* $\tilde{F}_1$ *is* skewed more to the upside *than* F_1*, if and only if:*

(a) The distributions yield the same valuation of the asset, that is, $E\left[s\,;\tilde{F}_1\right] = E\left[s;\ F_1\right]$.

(b) The hazard rates of $\tilde{F}_1$ *and* F_1 *satisfy the (weak) single crossing condition:*

$$\begin{cases} \frac{\tilde{f}_1(s)}{1-\tilde{F}_1(s)} \geq \frac{f_1(s)}{1-F_1(s)} & \text{if } s < s^R, \\ \frac{\tilde{f}_1(s)}{1-\tilde{F}_1(s)} \leq \frac{f_1(s)}{1-F_1(s)} & \text{if } s > s^R, \end{cases} \quad \text{for some } s^R \in S. \tag{19}$$

To interpret this definition, note that the distributions $\tilde{F}_1$ and F_1 cannot be compared according to the hazard rate order of assumption (A2). In addition, these distributions lead to the same valuation of the asset, that is, they have the same "level" of optimism. Note also that $\tilde{F}_1$ has a lower hazard rate than F_1 over the region $\left(s^R, s^{\max}\right)$. Thus, conditional on states, $s \geq s^R$, $\tilde{F}_1$ is weakly more optimistic than F_1 in the sense of assumption (A2). In contrast, F_1 has a lower hazard rate than $\tilde{F}_1$ over the region $\left(s^{\min}, s^R\right)$, and thus its optimism is concentrated more on this region. Hence, the optimism of $\tilde{F}_1$ is skewed to the upside in the sense that it is concentrated more on good states.

The probability distributions, $F_{1,D}$ and $F_{1,U}$, for the two cases of Example 1 satisfy condition (19). In particular, the optimism of $F_{1,U}$ is more skewed to the upside, as illustrated in Figure 3. The same figure also plots the equilibrium for the two cases and illustrates that the equilib-

rium price and loan riskiness corresponding to $F_{1,U}$ are higher. The next result shows that this observation is generally true.

Theorem 3 (Type of Heterogeneity). *Consider the equilibrium characterized in Section 3. If optimism becomes more skewed to the upside, i.e., if F_1 is changed to $\tilde{F}_1$ that satisfies condition* (19) *[while still satisfying assumption (A2)], then: the asset price p and the loan riskiness $\bar{s}^*$ increase, and the margin m decreases.*

This result formalizes the sense in which optimism is asymmetrically disciplined. I provide a sketch proof which is completed in Appendix A.3. Eq. (13) shows that the optimistic belief affects the optimality curve, $p^{opt}(\bar{s})$, through optimists' conditional valuation, $E_1[s \mid s \geq \bar{s}]$. The analysis in the appendix shows that $\tilde{E}_1[s \mid s \geq \bar{s}] \geq E_1[s \mid s \geq \bar{s}]$ for each $\bar{s} \in \left(s^{\min}, s^{\max}\right)$. That is, an increase in the upside skewness of optimism increases the conditional valuation for each $\bar{s}$, even though it does not increase the unconditional valuation. It follows that the optimality curve shifts up pointwise. Since the market clearing curve, $p^{mc}(\cdot)$, is unchanged [cf. Eq. (17)], the comparative statics in the theorem statement follow. Intuitively, as optimism becomes more concentrated on upside states, optimists' borrowing constraints become looser and their demand for the asset becomes greater. This leads to a low margin and a high equilibrium price.

Theorem 3 shows that the type of the belief heterogeneity has an unambiguous effect on margins and asset prices. A natural question is whether the level of belief heterogeneity has similar robust predictions. The answer is no, as illustrated by the following example.

Example 2 (Ambiguous Effects of Increased Belief Heterogeneity). *Consider the following two cases each of which features an increase in the level of belief heterogeneity.*

Case (i). *Consider the first case of Example 1 with downside optimism. Suppose the beliefs are changed to $\tilde{F}_0$ and $\tilde{F}_1$ with density functions given by:*

$$\tilde{f}_0 = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 1+0.45 & \text{if } s \in \mathcal{S}_N \\ 1-0.45 & \text{if } s \in \mathcal{S}_G \end{cases}, \qquad \tilde{f}_1 = \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \\ 1.3-0.45 & \text{if } s \in \mathcal{S}_N \\ 1.3+0.45 & \text{if } s \in \mathcal{S}_G \end{cases}. \tag{20}$$

That is, pessimists' probability for the normal event increases and their probability for the good event decreases, while the opposite happens to optimists' prior. As the right panel of Figure 4 shows, in this case, the increase in belief heterogeneity leads to an increase in the asset price.

Case (ii). *Next consider the second case of Example 1 with upside optimism. Suppose the beliefs are changed to $\tilde{F}_0$ and $\tilde{F}_1$ with density functions given by:*

$$\tilde{f}_0 = \begin{cases} 1+2\eta_0 & \text{if } s \in \mathcal{S}_B \\ 1-\eta_0 & \text{if } s \in \mathcal{S}_N \\ 1-\eta_0 & \text{if } s \in \mathcal{S}_G \end{cases}, \qquad \tilde{f}_1 = \begin{cases} 1(1-2\eta_1) & \text{if } s \in \mathcal{S}_B \\ 0.1(1+\eta_1) & \text{if } s \in \mathcal{S}_N \\ 1.9(1+\eta_1) & \text{if } s \in \mathcal{S}_G \end{cases},$$

for some parameters η_0 and η_1. That is, pessimists' probability for the bad event increases (and

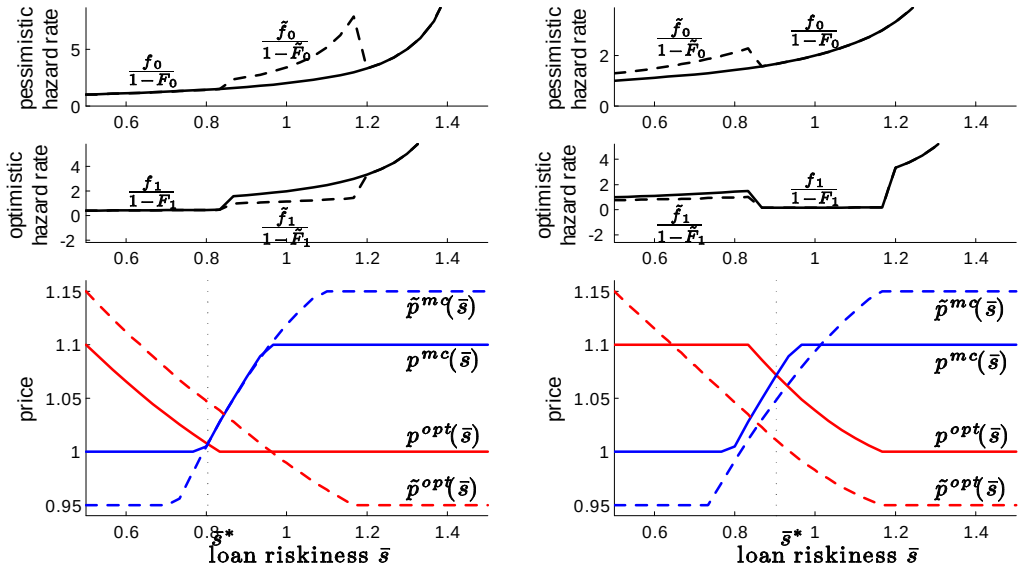

Figure 4: The left panel plots the equilibrium for the first case of Example 2: the increase in belief heterogeneity is to the right of $\bar{s}^*$ and it increases the asset price. The left panel plots the equilibrium for the second case of Example 2: the increase in belief heterogeneity is concentrated to the left of state $\bar{s}^*$ and it decreases the asset price.

their relative probability for good and the normal event remains constant), while optimists' probability for the bad event decreases. In addition, choose parameters η_0 and η_1 such that both pessimists' and optimists' valuation of the asset is the same as in the first case of this example. As Figure 4 shows, in this case, the increase in belief heterogeneity leads to a decrease in the asset price.

Miller (1977) had argued that an increase in traders' belief heterogeneity tends to increase the overvaluation of the asset (relative to the average valuation in the population) because the asset is held by the most optimistic traders. Example 2 illustrates that, in this model, the increase in belief heterogeneity [in the sense of assumption (A2)] has no robust predictions for the asset price. This is because both optimists' and pessimists' beliefs affect the price. While an increase in borrowers' optimism tends to increase the price, a decrease in lenders' pessimism tends to decrease it by tightening the borrowing constraints. This observation illustrates that the Miller mechanism might not apply in markets in which optimists finance their asset purchases by borrowing from less optimistic traders.

I next derive a qualified version of the Miller mechanism for these markets. In particular, the next result shows that increased belief heterogeneity has robust predictions if the type of the additional increase is taken into account. In the first case of Example 2, the increase in belief heterogeneity is concentrated on states below the default threshold $\bar{s}^*$, which leads to a

decrease in the asset price. In the second case of the example, the increase in belief heterogeneity is concentrated on states above the default threshold $\bar{s}^*$, which leads to an increase in the asset price increases (see also Figure 4). The next result establishes that these properties are general.

Theorem 4 (Level of Heterogeneity). *Consider the equilibrium characterized in Section 3. Consider an increase in belief heterogeneity, that is, suppose F_1, F_0 are changed to $\tilde{F}_1, \tilde{F}_0$ that satisfy:*

$$\frac{\tilde{f}_1(s)}{1-\tilde{F}_1(s)} \leq \frac{f_1(s)}{1-F_1(s)} \text{ and } \frac{\tilde{f}_0(s)}{1-\tilde{F}_0(s)} \geq \frac{f_0(s)}{1-F_0(s)} \text{ for } s \in S. \tag{21}$$

(i) Suppose the increase in belief heterogeneity is concentrated on states above $\bar{s}^*$ *in the sense that the hazard rate inequalities* (21) *are satisfied with equality for $s \in \left(s^{\min}, \bar{s}^*\right)$. Then, the asset price p and the loan riskiness $\bar{s}^*$ weakly increase, and the margin m weakly decreases.*

(ii) Suppose the increase in belief heterogeneity is concentrated on states below $\bar{s}^*$ *in the sense that the hazard rates inequalities* (21) *are satisfied with equality for $s \in (\bar{s}^*, s^{\max})$. Then, the asset price p weakly decreases and the margin m weakly decreases.*

Taken together, the results in this section suggest that the type of the belief heterogeneity is a more robust determinant of asset prices than the level of belief heterogeneity. In other words, what investors disagree about matters for asset prices, to a greater extent than the level of their disagreement.

5 Equilibrium with Short Selling

The baseline setting has focused on optimists' borrowing constraints. This section analyzes pessimists' borrowing constraints by considering short selling. The main result in this section shows that a version of the asymmetric disciplining result holds in this setting.

It is useful to start by reviewing the salient features of a typical short sale transaction in financial markets. In a typical short sale, a trader borrows the asset from a lender who owns the asset. The borrower raises p dollars from the short sale, which she leaves as cash-collateral with the lender. However, the lender requires additional protection, which induces the borrower to place an additional $m^S p$ dollars as cash-collateral. The *short margin*, m^S, is the analogue of the loan margin in the baseline setting since the borrower needs $m^S p$ dollars to short sell the asset. In this transaction, the borrower must also pay a *lending fee*, f, to the lender, which is subtracted from the cash-collateral.[12] Consequently, the net amount of collateral backing up the short contract is given by:

$$\left(1+m^S\right)p - f. \tag{22}$$

[12] More specifically, the lender rebates the borrower for the cash collateral at a *rebate rate*, r^{rebate}. This rate is lower than the riskless rate, r. The difference, $r - r^{rebate}$, corresponds to the lending fee paid by the borrower. In this model, the rebate rate is always negative since the riskless rate is taken to be zero (for simplicity). The rebate rate may also be negative in reality. The assets with low rebate rates (or high lending fees) are said to be "special." See D'Avolio (2002) or Lamont (2004) for excellent descriptions of the short market.

The lending fee, f, is the price through which the short market clears. If the trader does not return the asset, then the lender keeps the cash-collateral in (22).

The above short sale transaction can be captured in this model by the simple short contracts introduced in Section 2:[13]

$$\mathcal{B}^S \equiv \left\{ \left([\boldsymbol{\varphi}(s) \equiv s]_{s \in S}, 0, \gamma \right) \mid \gamma \in \mathbb{R}_+ \right\}. \tag{23}$$

In particular, a simple short contract promises a replica of the asset collateralized by γ units of cash. To map this short contract to a short sale, let m^S and f be given as the unique solutions to the following equations:

$$\gamma = \left(1 + m^S\right) p - f \text{ and } p - q(\gamma) = f. \tag{24}$$

In particular, the cash-collateral of the short contract corresponds to the net cash-collateral, (22), posted in the short sale transaction. The difference between the price of the asset and the price of its replica (i.e., the short contract) corresponds to the lending fee. In addition, the borrower defaults on the short contract if and only if the future asset price, s, exceeds the cash-collateral, γ. Thus, the actual payoff of the short contract is given by:

$$\min(\gamma, s). \tag{25}$$

In view of the mapping in (24), both the short margin, m^S, and the lending fee, f, will be endogenously determined in general equilibrium. I next characterize this general equilibrium with simple short contracts, $\mathcal{B}^S$. In particular, to isolate the effects of short selling, I rule out the simple debt contracts analyzed in the baseline setting. Appendix A.5 considers a model that features both simple debt and simple short contracts and shows that the economic insights of this section continue to apply in that setting.

The analysis of equilibrium follows closely the analysis in Section 3. The following is the analogue of assumption (A1) for this setting:

Assumption (A1^S). $n_1 > E_0[s] + \frac{n_0}{s^{\max} - E_0[s]}$.

This assumption ensures that optimists' endowment is sufficient to purchase the entire asset supply as well as the short contracts sold by pessimists. Given assumption (A1^S), I will establish that the equilibrium price satisfies $p \in (E_0[s], E_1[s])$. Since $p > E_0[s]$, pessimists invest only in cash and they sell short contracts. In contrast, optimists buy the asset and the short contracts sold by pessimists. In particular, the corresponding principal-agent equilibrium is one in which pessimists (who are the borrowers in this model) choose the short contract subject to optimists' participation constraint. Optimists are indifferent between buying the asset and the short contracts, which

[13] Recall that a simple short contract is generally denoted by $\beta = \left([\boldsymbol{\varphi}(s) = \varphi s]_{s \in S}, \psi^a, \psi^c \right)$ for some $\varphi \in \mathbb{R}_+$. The set, $\mathcal{B}^S$, features two restrictions which are without loss of generality. First, I assume $\psi^a = 0$ because the transfer implied by the contract, $(\varphi s, \psi^a, \psi^c)$, can be equivalently captured by the contract, $(\max(\varphi - \psi^a) s, 0, \psi^c)$. Second, I normalize the contracts by taking the number of assets short sold to be 1.

implies that the price of a short contract with cash-collateral γ is given by:

$$q(\gamma) = \frac{1}{\frac{E_1[s]}{p}} E_1 [\min(\gamma, s)]. \tag{26}$$

Note that the price of the short contract is increasing in its cash-collateral. Equivalently, keeping p fixed, the lending fee in (24) is decreasing in γ. Intuitively, optimists realize that the short contract will default in some future states, and they demand compensation for default in the form of a lending fee. When the cash-collateral is high, the short contract defaults less often and the lending fee is lower.[14]

Next note that, if pessimists sell x_0 units of this contract, then their cash position must be given by $c_0 = x_0\gamma$. Consequently, pessimists choose which short contract to sell, γ, and how many of these contracts to sell, x_0, to solve:

$$\begin{aligned} \max_{(x_0,\gamma)\in\mathbb{R}^2_+} \quad & x_0\gamma - x_0 E_0 [\min(\gamma, s)], \\ \text{s.t.} \quad & x_0\gamma = n_0 + x_0 \frac{1}{\frac{E_1[s]}{p}} E_1 [\min(\gamma, s)]. \end{aligned} \tag{27}$$

The first line is pessimists' expected return which consists of their return from cash net of their expected payments on short contracts. The second line is pessimists' budget constraint, which incorporates optimists' participation constraint. The budget constraint illustrates that choosing lower cash-collateral enables pessimists to sell a greater number of short contracts. On the other hand, recall that higher cash-collateral leads to a lower lending fee. Thus, pessimists face a trade-off between a greater short position and a lower lending fee.

The next result, which is the analogue of Theorem 2 for this setting, characterizes pessimists' optimal short contract. The result requires the following analogue of assumption (A2), which ensures that problem (27) has a unique solution for each p.

Assumption (A2^S). *The probability distributions F_1 and F_0 satisfy:*

$$\frac{f_1(s)}{\int_{s^{\min}}^{s} \tilde{s} dF_1} > \frac{f_0(s)}{\int_{s^{\min}}^{s} \tilde{s} dF_0} \text{ for each } s \in \left(s^{\min}, s^{\max}\right).$$

Assumptions (A2) and (A2^S) do not imply each other. However, Assumption (A2^S) [as well as assumption (A2)] is implied by the monotone likelihood ratio property (cf. Section 3.1).

Theorem 5 (Asymmetric Disciplining of Pessimism). *Suppose the contract space is restricted to simple short contracts, $\mathcal{B}^T = \mathcal{B}^S$, and assumptions (A1^S) and (A2^S) hold. Fix asset*

[14] For an example, note that $\gamma = s^{\max}$ leads to $q(\gamma) = p$ and a short fee $f = 0$. This is because this short contract is completely safe (i.e., it is collateralized according to the worst case scenario). From Eq. (24), note also that the margin on this contract is solved from $m^{sh}p = s^{\max} - p$. Hence, this contract corresponds to the short contract analyzed in Gromb and Vayanos (2002). The difference in this model is that riskier short contracts (with lower cash-collaterals) are also available for trade. The short contract that will be traded in equilibrium will be endogenously determined.

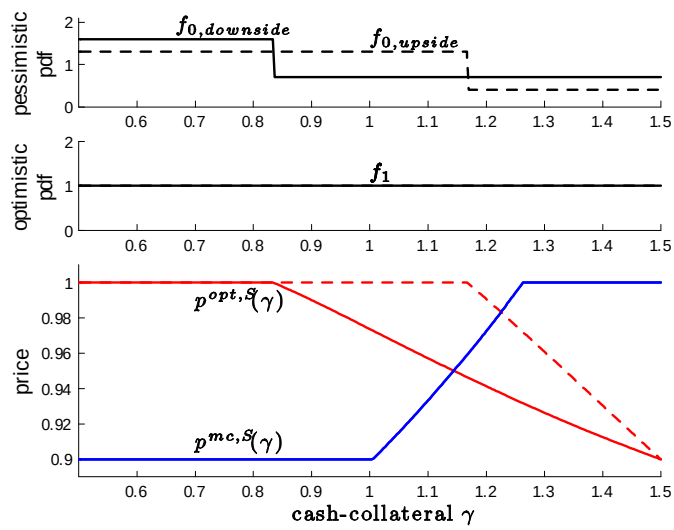

Figure 5: The top two panels display the pessimistic and the optimistic pdf's for two examples. The examples differ only in the type of pessimism: The bold lines correspond to pessimism on the downside while the dashed lines correspond to pessimism on the upside. The bottom panel plots the corresponding equilibria (the intersection of the optimality and market clearing curves).

price $p \in (E_0[s], E_1[s])$*, and consider pessimists' problem* (9)*. The cash-collateral,* γ*, of the optimal short contract is the unique solution to the following equation over the range* $\left(s^{\min}, s^{\max}\right)$*:*

$$p = p^{opt,S}(\gamma) \equiv \frac{E_1[s]}{F_0(\gamma)\frac{\int_{s^{\min}}^{\gamma} s dF_1}{\int_{s^{\min}}^{\gamma} s dF_0} + 1 - F_1(\gamma)}. \tag{28}$$

The pricing formula (28) shows that pessimism is asymmetrically disciplined in equilibrium. In particular, note that the pessimistic belief enters the pricing formula only through its effect on the following expression (which increases the price):

$$\int_{s^{\min}}^{\gamma} s \frac{dF_0}{F_0(\gamma)}.$$

This expression shows that pessimism about the relative likelihood of downside states, $s \leq \gamma$, decreases the asset price. More importantly, it also shows that any other type of pessimism does not decrease the asset price. For example, pessimism about the relative likelihood of upside states, $s \geq \gamma$, does not affect the price. Similarly, pessimism about the probability of the event $\{s \geq \gamma\}$ (while keeping the relative likelihood of states $s \leq \gamma$ fixed) does not decrease the price. Figure 5 illustrates this result by plotting the optimality curve, $p^{opt,S}(\gamma)$, for two examples that differ only in the type of pessimism. It shows that the asset price is greater when the pessimism on the upside than when it is on the downside.

The intuition for this result closely parallels the intuition for the asymmetric disciplining of

optimism (cf. Section 3.3). First consider why $p^{opt,S}(\gamma)$ is decreasing, that is, why pessimists choose lower cash-collateral when the price is higher. Given a higher p, pessimists have a greater incentive to bet. Consequently, they choose lower γ because this enables them to increase their short position [cf. problem (27)]. Next, to see the intuition for asymmetric disciplining, consider the cash-collateral, $\gamma = 1.1$, for the two examples plotted in Figure 5. Consider how much the price should increase (from the pessimistic valuation, $E_0[s]$) for pessimists to short sell with γ. The answer depends on the type of pessimism. When the pessimism is on the upside (for states $s \geq \gamma$), then optimists charge a positive short fee because they think the short contract is likely to default. Moreover, this short fee appears too high to pessimists because they think the contract is unlikely to default. Consequently, the price should increase considerably to entice pessimists to short sell with γ. In contrast, when the pessimism is concentrated on the downside, optimists and pessimists disagree less about the short fee corresponding to cash-collateral γ. Consequently, the price does not need to increase as much to entice pessimists to short sell with γ. Put differently, in this case pessimists face looser constraints for short selling, which leads to a lower price.

The equilibrium price is characterized by asset market clearing. Optimists' endowment is spent to purchase the asset and the short contracts sold by pessimists. Thus, asset market clearing implies: $n_1 = p + x_0 q(\gamma)$. Using Eqs. (24), (26) and (27), this condition can be rewritten as:

$$\frac{n_1}{p} = 1 + \frac{n_0}{p}\frac{1}{m^S}\frac{E_1[\min(\gamma, s)]}{E_1[s]}. \tag{29}$$

The left hand side of this expression corresponds to the demand for the asset. The right hand side corresponds to the effective supply, which is greater than the physical supply (one unit) in view of short selling. Note that a smaller short margin, m^S, enables pessimists to short sell a greater number of assets.[15] After substituting for m^S in terms of p and γ, Eq. (29) implicitly defines a market clearing relation between the price and the cash collateral, $p^{mc,S}(\gamma)$. Moreover, Appendix A.4 shows that $p^{mc,S}(\gamma)$ is increasing in γ. Intuitively, a smaller cash-collateral leads to a smaller short margin, which in turn leads to a greater short position and a lower market clearing price. The equilibrium price and cash-collateral pair, (p, γ), is determined as the unique intersection of the optimality curve, $p^{opt,S}(\gamma)$ and the market clearing curve, $p^{mc,S}(\gamma)$ (see Figure 5).

Note that the asymmetric disciplining of pessimism identified in this section is complementary to the asymmetric disciplining of optimism. When belief disagreements are on the upside, optimists will be less constrained while pessimists will be more constrained. In equilibrium, this will lead to lower loan margins and higher short margins, and an asset price closer to the optimistic valuation. In contrast, when belief disagreements are on the downside, loan margins will be higher, short margins will be lower, and the price will be closer to the pessimistic valuation. These points are further illustrated in Appendix A.5, which presents a model that features simple debt contracts together with the simple short contracts of this section.

[15] The last term in Eq. (29), $\frac{E_1[\min(\gamma,s)]}{E_1[s]}$, is a normalizing factor which emerges from the fact that the short contract is only an imperfect replica of the asset.

6 Equilibrium with Richer Contracts

While simple debt and simple short contracts are common in financial markets, it is important to consider richer contracts especially because they introduce new economic forces. This section considers the equilibrium with unrestricted contracts and obtains three results. First, it derives a version of the asymmetric disciplining result for this setting. Second, it shows that insurance or option-like contracts endogenously emerge to facilitate betting. Third, it establishes that these richer contracts moderate the effect of belief disagreements on the asset price.

Recall, from Section 2, that a borrowing contract is denoted by $\beta = \left([\boldsymbol{\varphi}(s)]_{s\in S}, \alpha, \gamma\right) \in \mathcal{B}$, with payoff (2). For expositional reasons, I start with the following trading restriction.

Assumption (PR). *Pessimists are restricted not to sell borrowing contracts, i.e.,* $\mu_0^- = 0$.

This assumption ensures that the corresponding principal-agent equilibrium is one in which optimists have all of the bargaining power. This equilibrium is useful because it isolates optimists' optimal contract. Pessimists' optimal contract, as well as the more general case without trading restrictions, will be discussed at the end of this section.

The analysis of equilibrium follows closely the analysis in Section 3. In particular, optimists choose their investment in the asset, a_1, and cash, c_1, along with one borrowing contract, $\left([\boldsymbol{\varphi}(s)]_{s\in S}, \alpha, \gamma\right)$, subject to pessimists' participation constraint. Moreover, optimists pledge all of their asset and cash holdings as collateral in the borrowing contract, i.e., $\alpha = a_1$ and $\gamma = c_1$, and they borrow as much as pessimists are willing to lend. Thus, optimists' problem can be written as:

$$\begin{aligned} \max_{(a_1,c_1)\in\mathbb{R}_+^2,\ [\boldsymbol{\varphi}(s)\in\mathbb{R}_+]_{s\in S}} \quad & a_1 E_1[s] + c_1 - E_1[\min(\boldsymbol{\varphi}(s), a_1 s + c_1)], \\ \text{s.t.} \quad & a_1 p + c_1 = n_1 + E_0[\min(\boldsymbol{\varphi}(s), a_1 s + c_1)]. \end{aligned} \tag{30}$$

Note that problem (30) is the analogue of problem (9), with two main differences. First, optimists' outstanding debt, $\boldsymbol{\varphi}(s)$, can be contingent on the future state. Second, optimists' cash investment is not a priori ruled out. While the assumption, $c_1 = 0$, was without loss of generality in Section 2, this is no longer the case in this section.

The next result, which is the analogue of Theorem 2 for this setting, characterizes optimists' optimal contract and portfolio choice. The result requires the following assumption, which is stronger than both assumptions (A2) and (A2^S), and which ensures that problem (27) has a unique solution for each p.

Assumption (MLRP). *The probability distributions F_1 and F_0 satisfy the monotone likelihood ratio property: that is, $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over S.*

Under assumption (MLRP), the optimal contract takes the form:

$$\boldsymbol{\varphi}(s) \equiv \begin{cases} a_1 s + c_1 & \text{if } s < \bar{s}, \\ 0 & 0 \text{ if } s > \bar{s}. \end{cases} \tag{31}$$

for a threshold state $\bar{s} \in \mathcal{S}$. That is, optimists make as large a promise as possible for states $s < \bar{s}$, while promising zero for states $s > \bar{s}$. Intuitively, optimists find bad states the least likely, and thus they concentrate all of their payments below a threshold state. The next result characterizes the optimal threshold, $\bar{s}$, of the optimal contract as well as optimists' investments, a_1 and c_1.

Theorem 6 (Optimists' Optimal Contract). *Suppose that the contract space is unrestricted, $\mathcal{B}^T = \mathcal{B}$, assumptions (PR) and (MLRP) hold, and that n_0 is sufficiently large [in particular, it satisfies condition* (*A*.36) *in Appendix A.6]. Fix asset price that satisfies $p \in (E_0[s], E_1[s])$, and consider optimists' problem* (30)*. The optimal contract takes the threshold form in* (31)*. There exists $\bar{p} < E_1[s]$ such that:*

(i) If $p < \bar{p}$, then optimists invest only in the asset, i.e., $c_1 = 0$. The threshold $\bar{s}$ of the optimal contract is the unique solution to the following equation over $\mathcal{S}$:

$$p = p^{opt,O}(\bar{s}) \equiv \int_{s^{\min}}^{\bar{s}} s dF_0 + \frac{f_0(\bar{s})}{f_1(\bar{s})} \int_{\bar{s}}^{s^{\max}} s dF_1. \tag{32}$$

(ii) If $p = \bar{p}$, then optimists are indifferent between investing in the asset and cash. The threshold $\bar{s}$ is the unique solution to $\bar{p} = p^{opt,O}(\bar{s})$.

(iii) If $p > \bar{p}$, then optimists invest only in cash, i.e., $a_1 = 0$. The threshold $\bar{s}$ is the unique solution to $\bar{p} = p^{opt,O}(\bar{s})$.

Part (i) of this theorem shows that, when the asset price is below a threshold $\bar{p}$ (defined in Eq. (*A*.35) in Appendix A.6), optimists make a leveraged investment in the asset as in the baseline setting. The only difference is that optimists borrow with a contingent loan which is not a simple debt contract. Nonetheless, the characterization of the optimal contingent contract shows that a version of the asymmetric disciplining result continues to apply in this setting. In particular, the pricing formula (32) shows that optimism about the relative likelihood of states above $\bar{s}$ increases the asset price, while optimism about the relative likelihood of states below $\bar{s}$ does not increase the price. The intuition for this result can be gleaned from the shape of the optimal loan [cf. Eq. (31)]. This loan makes the same payment (namely, zero) in all states $s > \bar{s}$. Hence, optimism about the relative likelihood of upside states does not lead to heterogeneity in the valuation for the optimal loan. Consequently, these types of optimism lead to looser borrowing constraints and a higher asset price. On the other hand, optimism about the relative likelihood of downside states, $s < \bar{s}$, leads to tighter borrowing constraints and a lower asset price.

Parts (ii) and (iii) consider the cases with a higher asset price. As the asset price increases, optimists' demand shifts from the asset to cash. Importantly, note that optimists stop buying the asset at a price, $\bar{p}$, which is lower than their valuation of the asset, $E_1[s]$. Using this result, Appendix A.6 closes the model and establishes that the equilibrium price satisfies $p \leq \bar{p}$. For intuition, consider the optimal contract (31) for part (iii) [and also part (ii)]. This corresponds to an insurance contract (or an option-like contract) that promises a fixed payment of c_1 dollars if the future state is below $\bar{s}$. Selling this contract provides optimists with an alternative method to

bet on their belief. This in turn lowers optimists' demand for the asset and leads to an equilibrium price bounded from above by $\bar{p} < E_1[s]$. In particular, the price always remains bounded away from the optimistic valuation, $E_1[s]$, regardless of optimists' endowment, n_1.

In line with this intuition, the following result establishes that the equilibrium price in this setting is always *lower* than the price in the baseline setting in which optimists are restricted to borrow with simple debt contracts.

Theorem 7 (Price Comparison Between Contingent and Simple Debt Contracts). *For the same beliefs, F_1 and F_0, and endowments, n_1 and n_0, the equilibrium price, p, in this section is strictly smaller than in Section 3. That is, allowing optimists to sell unrestricted contracts leads to a lower asset price than the case in which they are restricted to sell simple debt contracts.*

With richer borrowing contracts, optimists naturally face looser borrowing constraints. In view of this intuition, one could conjecture that allowing richer contracts would lead to a higher asset price. Theorem 7 shows that this conjecture is incorrect. Intuitively, richer contracts not only relax optimists' borrowing constraints, but they also provide optimists with alternative methods to bet on their belief. When the price is sufficiently high, the alternative methods (such as selling insurance contracts collateralized by cash) ensure that optimists do not demand the asset. In contrast, with simple contracts, optimists' only betting method is to make a leveraged investment in the asset. Consequently, the availability of richer contracts reduces optimists' demand for the asset and leads to a lower price.

Insurance contracts, options, and related derivative contracts are ubiquitous in financial markets. Moreover, these contracts arguably played an important role in facilitating speculation in the run-up to the recent crisis. For example, Lewis (2010) reports that the insurance company, AIG, sold large amounts of CDSs on mortgage backed securities to investors that were pessimistic about the housing market. In this model too, insurance contracts endogenously emerge to facilitate betting between optimists and pessimists. Moreover, as illustrated by Theorem 7, the availability of these contracts puts downward pressure on the asset price. This result creates a presumption that the trading of credit default swaps might have stopped the increase in house prices, and ultimately, might have expedited the subsequent decline.

In view of assumption (PR), the analysis in this section has focused on optimists' optimal contract. Appendix A.7 characterizes pessimists' optimal contract by considering the complementary case in which pessimists have all the bargaining power (and discusses the more general case without any trading restrictions). The analysis and the results parallel those of this section. In particular, pessimists sell a contract that promises a fixed payment if the future state is *above* a threshold state $\bar{s}$. Moreover, there exists a price level $\underline{p} > E_0[s]$ (and $\underline{p} < \bar{p}$) such that pessimists invest in cash if $p > \underline{p}$, and they invest in the asset $p \leq \underline{p}$. In either case, pessimists' investment strategy resembles selling a call option on the asset which pays if the future asset price is above a threshold level. Pessimists collateralize their promises either by holding cash or the asset depending on the asset price.

Appendix A.7 also shows that the equilibrium price always satisfies $p \geq \underline{p}$. In particular, the price always remains bounded away from the pessimistic valuation, $E_0[s]$, regardless of the pessimists' endowment, n_0. Furthermore, the equilibrium price is always *higher* than the price in Section 5 in which pessimists are only restricted to sell simple short contracts. Intuitively, richer contracts provide pessimists with alternative methods to bet on their belief. When the price is low, these alternative methods ensure that pessimists demand the asset (which they use as collateral for their promises in option-like contracts). In contrast, with simple short contracts, pessimists' only method to bet is to short sell the asset. Consequently, the availability of richer contracts increases pessimists' net demand for the asset and leads to a higher price.

A general picture emerges from the analysis in this section and Appendix A.7. These results show that insurance (or option-like) contracts endogenously emerge to facilitate betting among traders with different beliefs. Importantly, the availability of these contracts moderates the effect of belief heterogeneity on the asset price. More specifically, regardless of whether optimists or pessimists have the bargaining power, the equilibrium price with unrestricted contracts remains bounded in an interval $\left(\underline{p}, \overline{p}\right)$ which is a strict subset of the interval, $(E_0[s], E_1[s])$. With richer contracts, the medium of betting shifts to alternative contracts as opposed to the asset itself. This ensures that beliefs of the extreme optimists or extreme pessimists have a moderated influence on the asset price.

7 Dynamic Model: Financing Speculative Bubbles

The analysis so far has concerned a static economy. However, the asymmetric disciplining result also naturally interacts with the speculative component of asset prices identified by Harrison and Kreps (1978). This section considers a dynamic extension of the baseline setting with simple debt contracts and no short selling. It shows that "speculative bubbles" are also asymmetrically disciplined by the endogenous borrowing constraint. I first describe the basic environment without financial constraints and illustrate that the asset price features a speculative component. I then characterize the dynamic equilibrium with borrowing constraints.

7.1 Basic Dynamic Environment

Consider an infinite horizon overlapping generations economy with a single consumption good (dollar). The dates and the generations are denoted by $t \in \{0, 1, ...\}$. There is a continuum of traders in each generation t, who live at dates t and $t+1$. Consider the young traders at date t. These traders have endowments at date t, and they consume only at date $t+1$. They can transfer resources between dates by investing either in a bond or an asset. The bond is supplied elastically at a normalized price 1. Each unit of the bond yields $1+r$ dollars at the next date, and then fully depreciates. The bond is the analogue of cash in the earlier setting and its only role is to fix the riskless interest rate at some $r > 0$. The asset is in fixed supply, which is normalized to one.

The asset does not depreciate and yields y_t dollars at every date. The dividend yield follows:

$$y_{t+1} = y_t s_{t+1}. \tag{33}$$

Here, the dividend shock s_{t+1} is a random variable that takes values over a set, $\mathcal{S} = \left[s^{\min}, s^{\max}\right]$. Suppose that $S \in \mathbb{R}_{++}$ and that $1 \in S$. Suppose also that the shock, s_{t+1}, is independent of the past shocks, $\{s_1, .., s_t\}$.

Traders observe all past and current dividend yields, $\{y_1, .., y_t\}$. However, they have possibly heterogeneous prior beliefs about the next date's shock, s_{t+1}. In particular, there are two types of young traders, *pessimists* and *optimists*, respectively with prior beliefs F_0 and F_1 about s_{t+1}. Optimists are more optimistic in the hazard-rate sense, that is, beliefs satisfy assumption (A2) of Section 3. I also normalize the pessimistic belief to have mean 1, $E_{t,0}\left[s_{t+1}\right] = 1$. Assumption (A2) then ensures that $E_{t,1}\left[s_{t+1}\right] = 1 + \varepsilon$ for some $\varepsilon > 0$ which controls the level of optimism. Lastly, while traders disagree about the next date's shock, they agree about the shocks that are farther in the future. More specifically, they believe that the shock, s_{t+k} for $k \geq 2$, is distributed according to the pessimistic belief, F_0.

One interpretation of these assumptions is that traders normally believe that the dividend shock in (33) is a stationary random variable with mean 1. But at every date, a fraction of the traders (optimists) become optimistic regarding the next date's shock.[16] Under these assumptions, pessimists' present discounted value can be calculated as:

$$p_0^{pdv}\left(y_t\right) \equiv \sum_{k=1}^{\infty} \frac{E_{t,0}\left[y_{t+k}\right]}{\left(1+r\right)^k} = \frac{y_t}{r}. \tag{34}$$

On the other hand, optimists' present discounted value can be calculated as

$$p_1^{pdv}\left(y_t\right) \equiv \sum_{k=1}^{\infty} \frac{E_{t,1}\left[y_{t+k}\right]}{\left(1+r\right)^k} = \frac{y_t\left(1+\varepsilon\right)}{r}. \tag{35}$$

Intuitively, optimists expect the dividend yield to increase (on the average) to $y_t\left(1+\varepsilon\right)$. They then expect the yield to fluctuate around this higher level, which leads to the valuation in (35).

I normalize the population measure of young traders (of generation t) to 1. Moreover, I assume that their endowment is proportional to the current dividend yield: that is, type i traders are endowed with $n_i y_t$ dollars for some constant n_i.[17] This assumption, along with the earlier assumption on traders' beliefs, ensures that the economy has a recursive structure. In the rest of this section, I focus on recursive equilibria in which the asset price is a function of the current

[16] There could be a number of explanations for the source of this type of optimism. As in Scheinkman and Xiong (2003), optimists may be overconfident about a signal they receive about the next date's shock. Alternatively, optimists may be simply optimistic about the next date's shock, thinking that the current date is special. Reinhart and Rogoff (2008) refer to this type of optimism as "this time is different syndrome."

[17] This assumption is not necessary for the economic results, but it simplifies the analysis. I thank Ivan Werning for pointing out this simplification.

dividend yield, $p(y_t)$. I also drop t from the notation by denoting the current dividend yield by $y \equiv y_t$ and the future dividend shock by $s \equiv s_{t+1}$.

The main difference of this model than the earlier static economy is the fact that traders sell the asset to the next generation. In particular, the value of the asset is the sum of its dividend yield and its future price:

$$v(y,s) \equiv ys + p(ys) \text{ for each } s \in \mathcal{S}. \tag{36}$$

Note that the value of the asset is endogenous because it depends on the future asset price. The price is also endogenous because it depends on the future value of the asset. Consequently, the value function, $v(y,s)$, and the price function, $p(y)$, are jointly determined. The equilibrium price level naturally reflects the type of borrowing contracts available. Throughout this section, I assume that short contracts (i.e., short selling) are not allowed. Consequently, the analysis revolves around optimists' borrowing constraints. I first consider a benchmark setting in which optimists face no borrowing constraints. I then consider the main setting of this section in which optimists can only borrow with simple debt contracts.

7.2 Speculative Bubbles without Borrowing Constraints

As a benchmark, suppose optimists can borrow cash at rate r without any constraints (including limited liability).[18] In this case, optimists have an infinite demand for the asset whenever the asset price is below their valuation. Hence, the equilibrium asset price is equal to the optimistic valuation:

$$\begin{aligned} p(y) &= \frac{1}{1+r} E_1 [v(y,s)] \\ &= \frac{1}{1+r} \left(y(1+\varepsilon) + \int_{\mathcal{S}} p(ys)\, dF_1 \right), \text{ for all } y \in \mathbb{R}_{++}. \end{aligned} \tag{37}$$

Eq. (37) provides a recursive expression of the asset price, which can be solved as:

$$p(y) = \frac{y(1+\varepsilon)}{r-\varepsilon}. \tag{38}$$

Note that the asset price, $p(y)$, is higher than optimists' present discounted value in (35). The component of the asset price in excess of the present discounted value of the holder of the asset, $p(y) - p_1^{pdv}(y)$, is what Scheinkman and Xiong (2003) call a "speculative bubble." In addition, I define

$$\lambda = \frac{p(y) - p_1^{pdv}(y)}{p(y)} = \frac{\varepsilon}{r} \tag{39}$$

[18] This assumption is equivalent to a setting with borrowing constraints in which optimists' endowment is sufficiently large, i.e., $n_1 \to \infty$.

as *the share of the speculative component* in the asset price. The asset price features a speculative component because optimists hold the asset not only for the higher expected dividend yields, but also because they plan to sell the asset to future optimists (who will be even more optimistic then them). In view of these expected capital gains, optimists bid up the asset price higher than their present discounted value.

The expression in (39) also implies that the speculative component could represent a large fraction of the asset price, even for a relatively small level of belief heterogeneity, ε (especially when the interest rate is low). The rationale for this observation is related to a dynamic amplification effect. Note that future optimists also expect to make capital gains by selling the asset to yet more optimistic traders in the subsequent period, which increases the price in the next period. But this further increases the valuation of current optimists who are planning to sell to future optimists, increasing the current asset price further. In other words, a large speculative bubble forms through a dynamic accumulation of relatively small belief disagreements at each date.

7.3 Borrowing Constraints and Dynamic Equilibrium

Next consider optimists' borrowing constraints. In particular, suppose young traders at date t trade the simple debt contracts, $\mathcal{B}^D$ [cf. Eq. (7)], in addition to the asset. As in Section 3, the corresponding principal-agent equilibrium is one in which optimists choose their debt contract subject to pessimists' participation constraint. Let φ denote optimists' outstanding debt per-asset and a_1 denote their asset holdings. Then, given the current dividend yield y, optimists' solve the following analogue of problem (9):

$$\begin{aligned} \max_{(a_1,\varphi)\in\mathbb{R}^2_+} \quad & a_1 E_1\left[v\left(y,s\right)\right] - a_1 E_1\left[\min\left(v\left(y,s\right),\varphi\right)\right], \\ \text{s.t.} \quad & a_1 p = n_1 + a_1 \frac{E_0\left[\min\left(v\left(y,s\right),\varphi\right)\right]}{1+r}. \end{aligned} \tag{40}$$

Definition 4 (Dynamic Equilibrium). *A dynamic equilibrium is a price function $p(y)$ and allocations $\left(a_1(y), \varphi(y)\right)_{y\in\mathbb{R}_{++}}$ such that: for each dividend yield y, optimists' allocation solves problem* (40) *given the value function $v(y,s) = ys + p(ys)$ [cf. Eq.* (36)*] and the asset market clears [i.e., $a_1(y) = 1$].*

Since traders' endowments are proportional to the current dividend yield, I conjecture an equilibrium in which the price function is linear, i.e., $p(y) = p_d y$ for some $p_d > 0$. In this equilibrium, the value function is given by: $v(y,s) = ys(1+p_d)$. Note that the equilibrium debt contract $\varphi(y)$ defaults if and only if s is below $\bar{s}_d \equiv \frac{\varphi(y)}{y(1+p_d)}$. Hence, I refer to this contract as a loan with riskiness $\bar{s}_d$. The following result shows that the conjectured equilibrium exists and that the riskiness of the equilibrium loan does not depend on y.

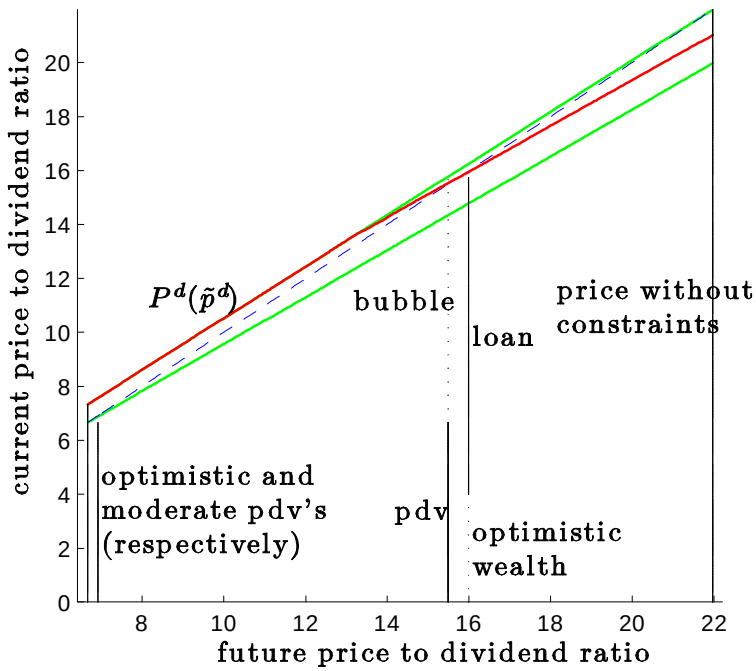

Figure 6: The x axis is the range of possible price to dividend ratios, $\left[p^{\min}, p^{\max}\right]$. The lowest and the highest solid curves respectively plot the pessimistic and the optimistic valuations when the future price to dividend ratio is given by the value at the x axis. The intermediate solid curve plots the price mapping, $P^d\left(\tilde{p}^d\right)$. The equilibrium is the intersection of this curve with the 45 degree line (the intermediate dashed curve).

Theorem 8 (Existence). *Suppose assumption (A2) holds and traders' net worths satisfy condition* (*A*.50) *in Appendix A.8. Then, there exists a dynamic equilibrium with price* $p(y) = p_d y$, *and riskiness* $\bar{s}_d^* \in \left(s^{\min}, s^{\max}\right)$, *for each* $y \in \mathbb{R}_{++}$. *The price to dividend ratio,* p_d, *lies in the interval,* $\left(\frac{1}{r}, \frac{1+\varepsilon}{r-\varepsilon}\right)$, *where the lower bound is the pessimistic pdv for the asset [cf. Eq.* (34)*] and the upper bound is the asset price in the benchmark without constraints [cf. Eq.* (38)*].*

To analyze this equilibrium, it is instructive to start with a partial equilibrium taking the future price function as given. In particular, let $P_d(\tilde{p}_d)$ denote the price to dividend ratio that would obtain today if the future price to dividend ratio is given by $\tilde{p}_d$. The proof of Theorem 8 in Appendix A.8 establishes that $P_d(\cdot)$ has a unique fixed point over the interval $\left[\frac{1}{r}, \frac{1+\varepsilon}{r-\varepsilon}\right]$, which corresponds to a dynamic equilibrium. Figure 6 plots $P_d(\cdot)$ for a particular example and illustrates the equilibrium. Note that the equilibrium price is higher than the pdv according to either the pessimistic or the optimistic beliefs. In particular, in this example, the price has a large speculative component despite optimists' borrowing constraints.

Figure 6 also illustrates optimists' balance sheet. In this example, optimists' downpayment is about 1/4 of the asset price. They finance the rest of the purchase by borrowing from pessimists

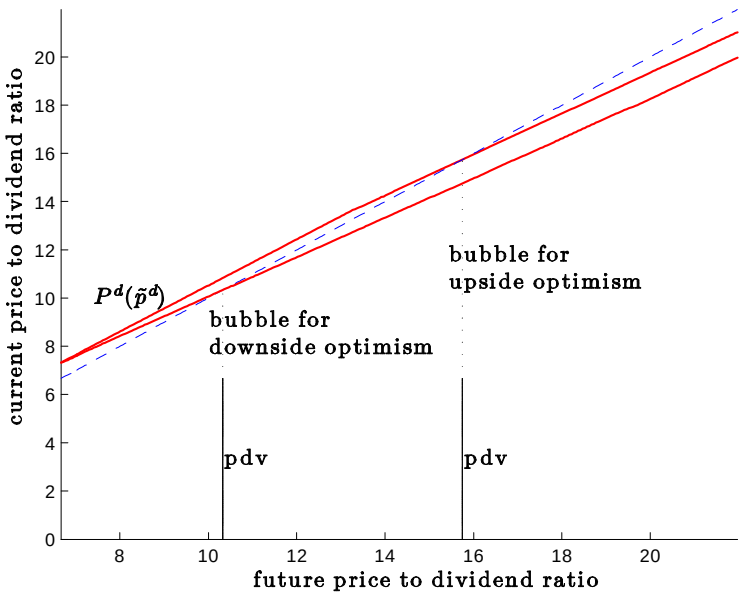

Figure 7: The lower (resp. the higher) solid curve plots the current price to dividend ratio as a function of the future price to dividend ratio for the belief distributions in the first case (resp. second case) of Example 1. The dynamic equilibrium is the intersection of this curve with the 45 degree line (dashed curve).

using the asset as collateral. In particular, pessimists agree to finance about 3/4 of the asset purchase despite the fact that they perceive the present discounted value of the asset to be less than half of its price. This feature illustrates how speculative bubbles can exist in environments in which optimists are borrowing constrained. In this example, the pessimistic lenders agree to extend large loans which are in part collateralized by the speculative component of the price. This is because lenders' valuation of the asset (the lower green line in Figure 6) also contains a speculative component. Intuitively, lenders agree to extend large loans because they think that, should the borrower default, they could always sell the collateral to a future optimist.

Put differently, an important feature of a speculative episode is that the bubble raises all boats: both optimists' and pessimists' valuations are greater than their respective present discounted values. Consequently, the difference between traders' valuations at any state (the difference between the two green lines in Figure 6) is relatively small. As in the unconstrained case, a large speculative bubble forms from a dynamic accumulation of small valuation differences. This is perhaps unfortunate, because a small valuation difference makes the financing of the asset relatively easy, opening the way for large speculative bubbles even when optimists are borrowing constrained.

Naturally, a small valuation difference does not always imply loose borrowing constraints. As in the static model, the tightness of borrowing constraints also depends on the type of belief

heterogeneity. To see this, consider the economy plotted in Figure 6 with the only difference that the optimistic priors are changed from $F_{1,U}$ to $F_{1,D}$ (cf. Example). That is, optimism is made more skewed to the downside (cf. Definition 3). Figure 7 shows that the speculative component shrinks by about half. Intuitively, when the optimism is on the downside, future optimists are unable to bid up the asset price because they face tighter borrowing constraints. This implies that the resale option value to future optimists is lower, which leads to a smaller speculative component.

The next result formalizes this intuition by showing that making optimism more skewed to the upside always increases the asset price and the share of the speculative component. To state the result, it is necessary to generalize the definition of the speculative component in (39) to the setting with borrowing constraints. To this end, I first define the overvaluation ratio $\theta_d \in (0,1)$ as the unique solution to

$$p(y) = (1-\theta_d)\frac{E_0[v(y,s)]}{1+r} + \theta_d \frac{E_1[v(y,s)]}{1+r}. \tag{41}$$

Intuitively, θ_d captures the fraction of the optimism in traders' beliefs that is reflected in the asset price. I next generalize the share of the speculative component as:[19]

$$\lambda_d = \frac{p(y) - p^{pdv}(y)}{p(y)}\text{ , where } p^{pdv}(y) = (1-\theta_d)\, p_0^{pdv}(y) + \theta_d p_1^{pdv}(y). \tag{42}$$

Unlike the unconstrained case, the marginal holder of the asset is not necessarily an optimist. Thus, the relevant present discounted value is defined by (41), which leads to the speculative component in (42).

Theorem 9 (Type of Heterogeneity and the Speculative Component). *Consider the dynamic equilibrium characterized in Theorem 8. If optimism becomes more skewed to the upside, i.e., if F_1 is changed to $\tilde{F}_1$ that satisfies condition* (19) *and $\tilde{F}_1 \succ_H F_0$ [so that assumption (A2) continues to hold], then: the price to dividend ratio p_d, the loan riskiness $\bar{s}_d^*$, and the share of the speculative component λ_d weakly increase.*

8 Conclusion

This paper theoretically analyzed the effect of belief disagreements on asset prices and financial contracts. The central feature of the model is that traders borrow by selling collateralized contracts to lenders that do not share the same beliefs. In particular, the lenders do not value the collateral as much as the borrowers do, which represents an endogenous borrowing constraint. I have considered the effect of this constraint in a number of settings that differ in the types of collateralized contracts that are available for trade.

[19] Note that both θ_d and λ_d are constants independent of y, because the price and the value functions are linear in the dividend yield.

In the baseline setting, I have restricted attention to simple debt contracts, which are useful to analyze optimists' borrowing constraints. I have also considered an extension with simple short contracts, which facilitate the analysis of pessimists' borrowing constraints. These analyses have established that both optimism and pessimism are *asymmetrically disciplined* by the endogenous borrowing constraint. In particular, the tightness of the constraint depends on the nature of belief disagreements. When belief disagreements are on the upside, optimists are less constrained while pessimists are more constrained, which leads to relatively low loan margins, relatively high short margins, and an asset price closer to the optimistic valuation. In contrast, when belief disagreements are on the downside, loan margins are relatively high, short margins are relatively low, and the asset price is closer to the pessimistic valuation. I have also considered a dynamic extension of the model which reveals that the speculative asset price bubbles, identified by Harrison and Kreps (1978), are also asymmetrically disciplined. These results suggest that what investors disagree about matters for asset prices. In particular, certain economic environments that generate uncertainty (and thus belief heterogeneity) about upside returns are conducive to asset price increases and speculative bubbles financed by credit.

An extension of the model with richer contracts has revealed that insurance contracts (that resemble credit default swaps) endogenously emerge to facilitate betting. Moreover, the availability of these contracts puts downward pressure on the asset price. These results provide one explanation for the introduction of the CDS contracts to the mortgage market in the run-up to the recent crisis. They are also consistent with the view that the CDS contracts might have played a role in the bursting of the house price bubble. More generally, this analysis has established that the availability of richer contracts moderates the effect of belief disagreements on asset prices.

This paper has opened up several avenues for future research. The first challenge is to develop a test of the main implications of the model; in particular, the effect of the *type* of belief disagreements on asset prices and borrowing contracts. An empirical literature in finance has analyzed the effect of the *level* of belief disagreements on asset prices (e.g., Chen, Hong and Stein, 2001, Diether, Malloy and Scherbina, 2002, and Ofek and Richardson, 2003). Following a strand of this literature, analysts' forecasts could be used as a proxy for investors' belief disagreements. However, measuring the type of belief disagreements requires finer information about the forecast of each analyst, e.g., a distribution rather than an average forecast. Once this level of information is compiled (or collected), the hypothesis of this paper can be tested.

Second, the analysis has been carried out with two belief types for tractability. Allowing for a greater number of belief types does not change the main implications, but it yields additional results. In particular, when there is a continuum of belief types, traders are endogenously divided into two groups such that those that are more optimistic than a threshold trader become natural buyers of the asset, while those that are more pessimistic become natural lenders. In addition, for certain types of belief disagreements, traders are assortatively matched through anonymous debt markets: Natural buyers that are relatively more optimistic borrow from natural lenders that are relatively more optimistic. Moreover, the relatively more optimistic pairs use loans with

relatively lower margins. Thus, this version of the model might be useful to study markets in which the same asset is simultaneously traded at heterogeneous margins. A model along these lines is analyzed by Fostel and Geanakoplos (2010).

Third, and more importantly, this model has not allowed debt contracts to be used as collateral. This is without loss of generality for the economy with two belief types considered in this paper, but not necessarily for economies with more belief types. For example, with three belief types, I conjecture that the equilibrium will feature a *pyramiding* arrangement (cf. Geanakoplos, 1997): The trader with intermediate beliefs will lend to optimists by buying their debt contracts, and it will borrow from extreme pessimists by using these debt contracts as collateral. This arrangement is interesting because it captures key features of housing-related credit markets. To give an example, senior tranches of subprime CDOs (collateralized debt obligations) are debt contracts backed by subprime mortgage backed securities, which are themselves debt contracts backed by subprime mortgages, which are themselves debt contracts backed by houses. Understanding the nature of pyramiding, and its impact on asset prices, is a fascinating topic which I leave for future work.

A Appendices: Omitted Proofs and Extensions

A.1 Principal-Agent Equilibrium with Simple Debt Contracts

This appendix completes the characterization of the principal-agent equilibrium analyzed in Section 3.3. The following lemma establishes a couple of properties implied by assumption (A2) which are used in the characterization. The rest of the appendix proves Theorem 2 and shows that the equilibrium price is interior, i.e., $p \in (E_0[s], E_1[s])$.

Lemma 1. *Suppose optimists' and pessimists' beliefs satisfy assumption (A2).*

(i) Optimists' perceived interest rate $1+r_1^{per}(\bar{s}) = \frac{E_1[\min(s,\bar{s})]}{E_0[\min(s,\bar{s})]}$ *[cf. Eq. (15)] is strictly increasing in* $\bar{s}$*. In particular,* $r_1^{per}(\bar{s}) > 0$ *for each* $\bar{s} > s^{\min}$*.*

(ii) $p^{opt}(\bar{s})$ *is continuously differentiable and strictly decreasing, i.e.,* $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$*. Moreover,* $p^{opt}(s^{\min}) = E_1[s]$ *and* $p^{opt}(s^{\max}) = E_0[s]$*.*

Proof of Lemma 1. Part (i). First note that the derivative of $E_i[\min(s,\bar{s})] = \int_{s^{\min}}^{\bar{s}} s dF_i(s) + \bar{s}(1-F_i(\bar{s}))$ is given by:

$$\frac{dE_i[\min(s,\bar{s})]}{d\bar{s}} = 1 - F_i(\bar{s}) > 0. \tag{A.1}$$

Using this expression, the derivative of $1 + r_1^{per}(\bar{s}) = \frac{E_1[\min(s,\bar{s})]}{E_0[\min(s,\bar{s})]}$ can be written as:

$$\frac{d(1+r_1^{per}(\bar{s}))}{d\bar{s}} = \frac{(1-F_1(\bar{s}))E_0[\min(s,\bar{s})] - E_1[\min(s,\bar{s})](1-F_0(\bar{s}))}{(E_0[\min(s,\bar{s})])^2}. \tag{A.2}$$

Thus, to prove that $1 + r_1^{per}(\bar{s})$ is increasing, it suffices to show that:

$$\frac{E_1[\min(s,\bar{s})]}{E_0[\min(s,\bar{s})]} < \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})} \text{ for each } \bar{s} \in \left(s^{\min}, s^{\max}\right). \tag{A.3}$$

To prove this, note that for each $\bar{s} \in \left(s^{\min}, s^{\max}\right)$,

$$\begin{aligned}
\frac{E_1[\min(s,\bar{s})]}{E_0[\min(s,\bar{s})]} &= \frac{\int_{s^{\min}}^{\bar{s}} s dF_1 + \bar{s}(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} s dF_0 + \bar{s}(1-F_0(\bar{s}))} \\
&< \frac{\int_{s^{\min}}^{\bar{s}} s\frac{1-F_1(s)}{1-F_0(s)} dF_0 + \bar{s}(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} s dF_0 + \bar{s}(1-F_0(\bar{s}))} \\
&< \frac{\int_{s^{\min}}^{\bar{s}} s dF_0 \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})} + \bar{s}(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} s dF_0 + \bar{s}(1-F_0(\bar{s}))} = \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})},
\end{aligned}$$

where the first inequality uses the hazard rate inequality (10) and the second inequality uses the fact that $\frac{1-F_1(s)}{1-F_0(s)}$ is strictly increasing.

Part (ii). Using Eq. (12), note that

$$\begin{aligned}
\frac{dp^{opt}(\bar{s})}{d\bar{s}} &\equiv \bar{s}f_0(\bar{s}) + \left(-f_0(\bar{s}) + f_1(\bar{s})\frac{1-F_0(\bar{s})}{1-F_1(\bar{s})}\right)\left(\int_{\bar{s}}^{s^{\max}} s\frac{dF_1}{1-F_1(\bar{s})}\right) - \frac{1-F_0(\bar{s})}{1-F_1(\bar{s})}\bar{s}f_1(\bar{s}) \\
&= -(1-F_0(\bar{s}))\left(\frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})}\right)(E_1[s \mid s \geq \bar{s}] - \bar{s}).
\end{aligned} \tag{A.4}$$

Here, the first line applies the chain rule while the second line substitutes $E_1[s \mid s \geq \bar{s}]$ and rearranges terms. The term, $\left(\frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})}\right)$, in Eq. $(A.4)$ is positive in view of the hazard rate inequality (10). Since the term, $E_1[s \mid s \geq \bar{s}] - \bar{s}$, is also positive, it follows that $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$. The second part of the statement follows by considering Eq. (12) for $\bar{s} = s^{\min}$ and $\bar{s} = s^{\max}$.■

Proof of Theorem 2. Most of the proof is provided in Section 3.3. The remaining steps are to show that $R_1^L(\bar{s})$ [cf. Eq. (14)] has a unique maximum characterized by $p = p^{opt}(\bar{s})$. To this end, consider the derivative of $R_1^L(\tilde{s})$, which can be calculated as:

$$\frac{dR_1^L(\tilde{s})}{d\tilde{s}} = \frac{1}{p - E_0[\min(s, \tilde{s})]}\left(R_1^L(\tilde{s})(1 - F_0(\tilde{s})) - (1 - F_1(\tilde{s}))\right). \tag{A.5}$$

Setting this expression to zero leads to the first order condition:

$$\frac{R_1^L(\tilde{s})}{1+r} = \frac{1 - F_1(\tilde{s})}{1 - F_0(\tilde{s})}.$$

Plugging this first order condition into (14) and rearranging terms yields $p = p^{opt}(\tilde{s})$. Next note by part (iii) of Lemma 1 that there exists exactly one $\bar{s} \in S$ that satisfies the first order condition.

It remains to show that the unique critical point characterized by $p = p^{opt}(\tilde{s})$ corresponds to a maximum of $R_1^L(\tilde{s})$. To this end, consider the value of the derivative, $\frac{dR_1^L(\tilde{s})}{d\tilde{s}}$, at the boundaries of region, $\tilde{s} = s^{\min}$ and $\tilde{s} = s^{\max}$. Note that $R_1^L\left(s^{\min}\right) = \frac{E_1[s] - s^{\min}}{p - s^{\min}} > 1$ because $p < E_1[s]$ (by assumption). Using this inequality, the derivative in $(A.5)$ satisfies $\left.\frac{dR_1^L(\bar{s})}{d\bar{s}}\right|_{\bar{s}=s^{\min}} > 0$. Similarly, note that $R_1^L(s^{\max}) = \frac{E_1[s] - E_1[s]}{p - E_0[s]} = 0$. Using this in inequality, the derivative in $(A.5)$ satisfies $\left.\frac{dR_1^L(\tilde{s})}{d\tilde{s}}\right|_{\tilde{s}=s^{\max}} \leq 0$. These boundary conditions imply that the unique critical point is a maximum of $R_1^L(\tilde{s})$ over S.■

Proof that the equilibrium price is interior. The analysis in Section 3.3 has characterized the equilibrium as the intersection of the optimality curve, $p^{opt}(\bar{s})$, and the market clearing curve, $p^{mc}(\bar{s})$. The remaining step is to verify the conjecture that the equilibrium price satisfies $p \in (E_0[s], E_1[s])$. Note that assumption (A1) implies:

$$p^{opt}\left(s^{\min}\right) = E_1[s] > p^{mc}\left(s^{\min}\right) = n_1 + s^{\min}.$$

Note also that:

$$p^{opt}(s^{\max}) = E_0[s] < p^{mc}(s^{\max}) = n_1 + E_0[s].$$

Thus, by continuity, $p^{opt}(\bar{s})$ and $p^{mc}(\bar{s})$ intersect for some interior $\bar{s} \in \left(s^{\min}, s^{\max}\right)$. From part (iii) of Lemma 1, it follows that the price at the intersection satisfies $p = p^{opt}(\bar{s}) \in (E_0[s], E_1[s])$.■

A.2 General Equilibrium and the Equivalence Result

This appendix provides the proof for Theorem 1 in Section 3.2.

Proof of Theorem 1. Part (i). Proof started in Section 3.3 and completed in Appendix A.1.
Part (ii).

For the proof, it is useful to define the notion of a *quasi-equilibrium*, which is a collection of prices $(p, [q(\cdot)])$ and portfolios $\left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)_{i\in\{1,0\}}$ such that markets clear and the portfolio of type $i \in \{1, 0\}$ traders solves Problem (5) with the additional requirement $\mu_1^+ = \mu_0^- = 0$.[20] That is, in a quasi-equilibrium, optimists are restricted not to buy debt contracts, and pessimists are restricted not to sell debt contracts.

The proof consists of two steps. I first prove part (ii) for a quasi-equilibrium. I then extend this result to general equilibrium.

Step 1: Existence of an Equivalent Quasi-Equilibrium

The proof is constructive. To construct a quasi-equilibrium, consider the following price function for debt contracts:

$$q(\varphi) = E_0[\min(s, \varphi)] \text{ for each } \varphi \in \mathbb{R}_+. \tag{A.6}$$

That is, suppose debt contracts are priced by pessimists. With this price function, characterization of pessimists' investment problem (5) is straightforward: they are indifferent between investing in cash and any simple debt contract. Moreover, since $p > E_0[s]$, they prefer these options to investing in the asset.

Next consider the investment problem (5) for optimists given the contract set $\mathcal{B}^D = \mathbb{R}_+$ and the restriction that $\mu_1^+ = 0$. Note that optimists choose $\hat{c}_1 = 0$ because $p < E_1[s]$. Using this observation and the contract prices in Eq. $(A.6)$, optimists' investment problem can be written as:

$$\begin{aligned} \max_{a_1\geq 0,\, \mu_1^-} \quad & a_1 E_1[s] - \int_{\varphi\in\mathbb{R}_+} E_1[\min(s,\varphi)]\, d\mu_1^-, \\ \text{s.t.} \quad & pa_1 - \textstyle\int_{\varphi\in\mathbb{R}_+} E_0[\min(s,\varphi)]\, d\mu_1^- \leq n_1 \quad \text{[budget constraint]}, \\ & \textstyle\int_{\varphi\in\mathbb{R}_+} d\mu_1^- \leq a_1. \quad \text{[collateral constraint]}. \end{aligned} \tag{A.7}$$

Since $p < E_1[s]$, the collateral constraint [as well as the budget constraint] binds. Let λ^B, λ^C respectively denote the Lagrange multipliers for the budget and collateral constraints. The first order condition for μ_1^- is given by:

$$\lambda^B E_0[\min(s,\varphi)] \leq E_1[\min(s,\varphi)] + \lambda^C, \text{ with equality if } \varphi \in supp\left(\mu_1^-\right).$$

The first order condition for a_1 is given by:

$$E_1[s] + \lambda^C = \lambda^B p.$$

Combining these first order conditions yields:

$$R_1^L(\varphi) = \frac{E_1[s] - E_1[\min(s,\varphi)]}{p - E_0[\min(s,\varphi)]} \leq \lambda^B, \text{ with equality only if } \varphi \in supp\left(\mu_1^-\right). \tag{A.8}$$

Recall from Appendix A.1 that $R_1^L(\varphi)$ has a unique maximum. It follows that there is a unique solution to the first order condition $(A.8)$. This shows that μ_1^- is a Dirac measure that puts weight only at one contract $\hat{\varphi} \in \mathbb{R}_+$.

[20] Here, $\mu_1^+ = 0$ (similarly $\mu_0^- = 0$) denotes the 0 measure, i.e., $\mu_i^+(B) = 0$ for each Borel set $B \subset \mathbb{R}_+$.

Next consider asset market clearing. Since the collateral constraint in $(A.7)$ binds, optimists sell $\hat{a}_1$ units of the traded contract, $\hat{\varphi}$. Plugging this into the budget constraint in $(A.7)$ implies:

$$p\hat{a}_1 - E_0\left[\min\left(s, \hat{\varphi}\right)\right]\hat{a}_1 = n_1.$$

Since only optimists demand the asset, market clearing requires $\hat{a}_1 = 1$. Plugging this into the previous equation gives the market clearing relation $p = p^{mc}\left(\hat{\varphi}\right)$ [cf. Eq. (17)].

It follows that the quasi-equilibrium contract and price pair, $(\hat{\varphi}, p)$, is such that $\hat{\varphi}$ maximizes $R_1^L\left(\varphi\right)$ and the market clearing relation holds. The analysis in Section 3.3 and Appendix A.1 show that there is a unique pair, $(\hat{\varphi}, p)$, that satisfies these conditions. Consequently, there exists a quasi-equilibrium. Moreover, since the same conditions [maximization of $R_1^L\left(\varphi\right)$ and the market clearing relation] characterize the principal-agent equilibrium, the two equilibria are equivalent. In particular, the equalities in (11) hold, which completes the proof of part (ii) for quasi-equilibrium.

Step 2: Existence of an Equivalent General Equilibrium

I next extend the proof in the previous step to general equilibrium. To this end, I claim that the constructed quasi-equilibrium corresponds to a general equilibrium with modified debt contract prices given by:

$$q\left(\varphi\right) = \max\left(E_0\left[\min\left(s, \varphi\right)\right], \frac{E_1\left[\min\left(s, \varphi\right)\right]}{R_1^L\left(\hat{\varphi}\right)}\right) \text{ for each } \varphi \in \mathbb{R}_+. \tag{A.9}$$

Note that $R_1^L\left(\hat{\varphi}\right)$ [cf. Eq. $(A.8)$] is optimists' expected gross rate of return in equilibrium. Thus, the expression $\frac{E_1[\min(s,\varphi)]}{R_1^L(\hat{\varphi})}$ is optimists' valuation of a debt contract φ in equilibrium. Unlike in a quasi-equilibrium, optimists can demand debt contracts in general equilibrium. Hence, the price of a debt contract is given by the upper-envelope of the pessimistic and optimistic valuations, as captured by $(A.9)$.

With the modified prices in $(A.9)$, the asset market continues to be in equilibrium. It remains to check that debt markets are also in equilibrium. To show this, I will establish a condition for equilibrium in debt markets in terms of traders' bid and ask prices, which I define next.

Bid prices for debt contracts: Given contract φ, consider the price that would make traders indifferent between buying this contract and investing in their equilibrium portfolios. Recall that optimists' gross rate of return is $R_1^L\left(\hat{\varphi}\right)$ while pessimists' gross rate of return is 1. Consequently, traders' bid prices are solved from:

$$\frac{E_0\left[\min\left(s, \varphi\right)\right]}{q_0^{bid}\left(\varphi\right)} = 1 \text{ and} \frac{E_1\left[\min\left(s, \varphi\right)\right]}{q_1^{bid}\left(\varphi\right)} = R_1^L\left(\hat{\varphi}\right). \tag{A.10}$$

Ask prices for debt contracts: Consider the price that would make traders indifferent between selling the debt contract φ and investing in their equilibrium portfolios. To be able to sell the contract φ, the trader must also hold 1 unit of the asset. Thus, traders' return from selling the contract is equal to their return from making a leveraged investment in the asset financed by contract φ. Consequently, traders' ask prices are solved from:

$$\frac{E_0\left[s\right] - E_0\left[\min\left(s, \varphi\right)\right]}{p - q_0^{ask}\left(\varphi\right)} = 1 \text{ and } \frac{E_1\left[s\right] - E_1\left[\min\left(s, \varphi\right)\right]}{p - q_1^{ask}\left(\varphi\right)} = R_1^L\left(\hat{\varphi}\right). \tag{A.11}$$

It can also be verified that [since $p \in (E_0[s], E_1[s])$] ask prices are strictly higher than bid prices:

$$q_i^{ask}(\varphi) > q_i^{bid}(\varphi) \text{ for each } \varphi \in \mathcal{S} \text{ and } i. \tag{A.12}$$

Intuitively, ask prices are higher because selling a contract requires collateral (which is scarce) but buying the same contract does not.

Conditions for equilibrium in debt markets: Define the aggregate bid and ask prices as:

$$q^{bid}(\varphi) = \max_i q_i^{bid}(\varphi) \text{ and } q^{ask}(\varphi) = \min_i q_i^{ask}(\varphi).$$

Note that, if $q(\varphi) < q^{bid}(\varphi)$, then there would be infinite demand for contract φ which would violate market clearing. Similarly, if $q(\varphi) > q^{ask}(\varphi)$, then there would be infinite negative demand for contract φ which would again violate market clearing. Thus, contract prices must satisfy:

$$q^{ask}(\varphi) \geq q(\varphi) \geq q^{bid}(\varphi) \text{ for each } \varphi. \tag{A.13}$$

Note also that positive trade in contract $\hat{\varphi}$ requires optimists to buy finite units of this contract while pessimists to sell finite units of it. Hence, the price of contract $\hat{\varphi}$ must satisfy:

$$q_1^{ask}(\hat{\varphi}) = q(\hat{\varphi}) = q_0^{bid}(\hat{\varphi}). \tag{A.14}$$

This analysis establishes that conditions $(A.13)$ and $(A.14)$ are necessary for equilibrium in debt markets. It can also be seen that these conditions are sufficient. More specifically, any debt contract prices satisfying these conditions (along with the quasi-equilibrium allocations) constitute a general equilibrium. Consequently, the remaining step is to check that the debt contract prices in $(A.9)$ satisfy conditions $(A.13)$ and $(A.14)$. Showing this requires some algebra which is carried out in the rest of the proof. Figure 8 graphically illustrates the proof by plotting bid and ask spreads in two examples.

The left panel of Figure 8 illustrates an example in which aggregate bid prices are always determined by pessimists while ask prices are always determined by optimists. The bid and ask prices are tangent to each other only at the equilibrium contract, $\hat{\varphi}$. The shaded region illustrates the set of possible general equilibrium prices. Note that the price of the traded contract is uniquely determined while the prices of non-traded contracts can take a range of values. The example in the right panel is similar except that, for sufficiently large φ, bid prices are determined by optimists rather than pessimists. However, this "crossing" happens in the no-trade region. In particular, the crossing does not change the price of the traded contract, $\hat{\varphi}$ (it only changes the range of possible prices for non-traded contracts). In either case, the contract prices in $(A.9)$ correspond to a general equilibrium because they constitute the lower bound of the shaded region [i.e., they correspond to aggregate bid prices].

Proof that the prices in $(A.9)$ **satisfy the debt market equilibrium conditions** $(A.13)$ **and** $(A.14)$**:** Note that the prices in $(A.6)$ imply $q(\varphi) = q^{bid}(\varphi)$ for each φ. Hence, checking conditions $(A.13)$ and $(A.14)$ amount to proving the claim:

$$q^{ask}(\varphi) \geq q^{bid}(\varphi) \text{ with equality iff } \varphi = \hat{\varphi}. \tag{A.15}$$

To prove this claim, first note $\hat{\varphi}$ is the unique maximum of $R_1^L(\varphi)$, which implies:

$$R_1^L(\hat{\varphi}) = \frac{E_1[s] - E_1[\min(s, \hat{\varphi})]}{p - E_0[\min(s, \hat{\varphi})]} > \frac{E_1[s] - E_1[\min(s, \varphi)]}{p - E_0[\min(s, \varphi)]} \text{ for each } \varphi \neq \hat{\varphi}.$$

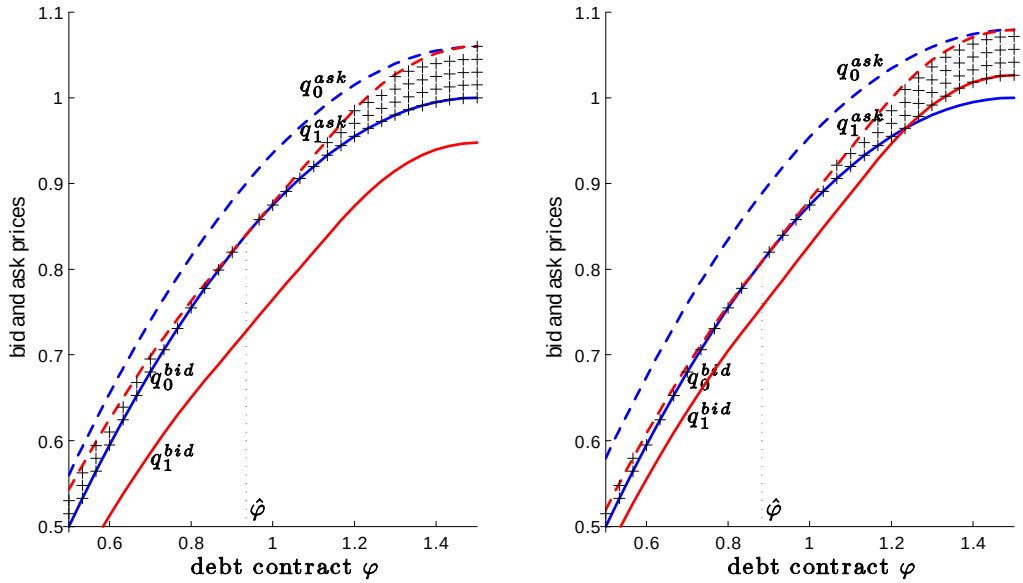

Figure 8: The left and right panels plot the bid and ask spreads in two examples which correspond to the two cases analyzed in the proof. The shaded areas display the set of possible general equilibrium debt contract prices in each case. The price of the traded debt contract, $\hat{\varphi}$, is uniquely determined, while the prices of non-traded contracts can take a range of values.

Using this inequality and the definition of $q_1^{ask}(\varphi)$ in $(A.11)$ shows

$$q_1^{ask}(\varphi) \geq E_0[\min(s,\varphi)] = q_0^{bid}(\varphi) \text{ with equality iff } \varphi = \hat{\varphi}. \tag{A.16}$$

Next, from the definition of bid prices in $(A.11)$, note that:

$$\frac{q_1^{bid}(\varphi)}{q_0^{bid}(\varphi)} = \frac{1}{R_L(\hat{\varphi})} \frac{E_1[\min(s,\varphi)]}{E_0[\min(s,\varphi)]}.$$

Thus, the ratio $\frac{q_1^{bid}(\varphi)}{q_0^{bid}(\varphi)}$ is less than 1 for $\varphi = s^{\min}$, and it is strictly increasing over $\varphi \in S$. In view of this observation, there are two cases to consider. For the first case, suppose $\frac{q_1^{bid}(\varphi)}{q_0^{bid}(\varphi)}$ never exceeds 1 over $\varphi \in S$ (as in the left hand side of Figure 8). This implies:

$$q_0^{ask}(\varphi) > q_0^{bid}(\varphi) > q_1^{bid}(\varphi) \text{ for each } \varphi \in \mathcal{S}, \tag{A.17}$$

where the left hand side inequality follows from Eq. $(A.12)$. In this case, combining $(A.16)$ and $(A.17)$ proves the claim in $(A.15)$.

For the second case, suppose $\frac{q_1^{bid}(\varphi)}{q_0^{bid}(\varphi)}$ exceeds 1 for sufficiently large φ (as in the left hand side

of Figure 8). In this case, it can be checked that $\frac{q_1^{bid}(\hat{\varphi})}{q_0^{bid}(\hat{\varphi})} < 1$,[21] which implies that the crossing takes place at some $\bar{\varphi} > \hat{\varphi}$. It can also be checked that $q_0^{ask}(\varphi) \geq q_1^{ask}(\varphi)$ for each $\varphi \geq \bar{\varphi}$.[22] Combining these observations, and using Eq. $(A.12)$, implies:

$$\begin{aligned} & q_0^{ask}(\varphi) > q_0^{bid}(\varphi) > q_1^{bid}(\varphi) \text{ for each } \varphi \in [s^{\min}, \bar{\varphi}), \\ q_1^{ask}(\varphi) \quad > \quad & q_0^{ask}(\varphi) > q_1^{bid}(\varphi) \geq q_0^{bid}(\varphi) \text{ for each } \varphi \in [\bar{\varphi}, s^{\max}]. \end{aligned} \tag{A.18}$$

Combining the inequalities in $(A.16)$ and $(A.18)$ proves claim $(A.15)$ also in this case.

This completes the proof that the quasi-equilibrium constructed in the first step corresponds to a general equilibrium with prices $(A.9)$. Note also that $a_1, p, \hat{\varphi}$ and $q(\hat{\varphi})$ are the same in the general equilibrium and the quasi-equilibrium. In view of step one, it follows that the equalities in (11) also hold for the general equilibrium. This shows that the constructed general equilibrium is equivalent to the principal-agent equilibrium, and completes the proof of the theorem.■

Remark 1 (Essential Uniqueness of General Equilibrium). *It can also be seen that in any general equilibrium, traders 'allocations, the asset price p, and the price of the traded debt contract, $q(\hat{\varphi})$, are uniquely determined. In other words, all equilibrium variables are uniquely determined except for the prices of non-traded debt contracts (as illustrated in Figure 8). This non-determinacy is inessential in the sense that it does not affect the real allocations in this economy.*

A.3 Comparative Statics with Simple Debt Contracts

This appendix provides proofs for the theorems in Section 4.

Proof of Theorem 3. Define the function $g : S \to R$ with:

$$g(\bar{s}) = \tilde{E}_1[s \mid s \geq \bar{s}] - E_1[s \mid s \geq \bar{s}] .$$

[21] To see this, note that the expression for the leveraged return $R_1^L(\hat{\varphi})$ [cf. Eq. (14)] can be rewritten as:

$$p - E_0[\min(s, \hat{\varphi})] = \frac{E_1[s]}{R_1^L(\hat{\varphi})} - \frac{E_1[\min(s, \hat{\varphi})]}{R_1^L(\hat{\varphi})}.$$

Next note that $p > \frac{E_1[s]}{R_1^L(\hat{\varphi})}$ (because $R_1^L(\hat{\varphi}) > \frac{E_1[s]}{p}$). Using this in the previous inequality implies:

$$E_0[\min(s, \hat{\varphi})] < \frac{E_1[\min(s, \hat{\varphi})]}{R_1^L(\hat{\varphi})}.$$

Rewriting this expresion implies $\frac{q_1^{bid}(\hat{\varphi})}{q_0^{bid}(\hat{\varphi})} < 1$.

[22] To see this, note that $\frac{E_1[s]}{E_0[s]} \geq \frac{E_1[\min(s,\varphi)]}{E_0[\min(s,\varphi)]}$, which implies:

$$\frac{E_1[s] - E_1[\min(s, \varphi)]}{E_0[s] - E_0[\min(s, \varphi)]} \geq \frac{E_1[\min(s, \varphi)]}{E_0[\min(s, \varphi)]}.$$

By definition of $\bar{\varphi}$, $\frac{E_1[\min(s,\varphi)]}{E_0[\min(s,\varphi)]} \geq R_1^L(\varphi)$ if and only if $\varphi \geq \bar{\varphi}$. Combining this with the previous displayed inequality, it follows that:

$$\frac{E_1[s] - E_1[\min(s, \varphi)]}{R_1^L(\hat{\varphi})} \geq E_0[\varphi] - E_0[\min(s, \varphi)] \text{ if } \varphi \geq \bar{\varphi}.$$

Using Eq. $(A.11)$, the previous displayed inequality implies $q_0^{ask}(\varphi) \geq q_1^{ask}(\varphi)$ for each $\varphi \geq \bar{\varphi}$.

Note that $g\left(s^{\min}\right)=0$ since $\tilde{E}_1[s]=E_1[s]$, and also that $g\left(s^{\max}\right)=0$. I claim that:

$$g(\bar{s})\geq 0 \text{ for all } \bar{s}\in\left(s^{\min},s^{\max}\right). \tag{A.19}$$

The comparative statics for p and $\bar{s}^*$ then follow by the argument provided after Theorem 3. For the comparative statics of the margin, substitute Eq. (17) into (18) to get

$$m=n_1/p. \tag{A.20}$$

It follows that the margin decreases because p increases and n_1 is unchanged.

The remaining step is the proof of claim $(A.19)$. I start by calculating $g'(\bar{s})$. Consider first the derivative of the conditional valuation:

$$\frac{d\tilde{E}_1[s\mid s\geq\bar{s}]}{d\bar{s}}=\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}\left(\tilde{E}_1[s\mid s\geq\bar{s}]-\bar{s}\right).$$

This expression further implies:

$$\begin{aligned} g'(\bar{s}) &= \frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}\left(\tilde{E}_1[s\mid s\geq\bar{s}]-\bar{s}\right)-\frac{f_1(\bar{s})}{1-F_1(\bar{s})}\left(E_1[s\mid s\geq\bar{s}]-\bar{s}\right) \\ &= \left(\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}-\frac{f_1(\bar{s})}{1-F_1(\bar{s})}\right)\left(\tilde{E}_1[s\mid s\geq\bar{s}]-\bar{s}\right)+\frac{f_1(\bar{s})}{1-F_1(\bar{s})}g(\bar{s}). \end{aligned} \tag{A.21}$$

I next prove the claim in $(A.19)$ in two steps. As the first step, consider $g(\bar{s})$ over the range $\bar{s}\in\left[s^{\min},s^R\right]$. Note that $\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}\geq\frac{f_1(\bar{s})}{1-F_1(\bar{s})}$ over this range. Thus, Eq. $(A.21)$ constitutes a differential equation of the form $g'(\bar{s})=A(\bar{s})+B(\bar{s})g(\bar{s})$ where $A(\bar{s})\geq 0$ and $B(\bar{s})>0$, with initial condition $g\left(s^{\min}\right)=0$. It follows that $g(\bar{s})\geq 0$ for each $\bar{s}\in\left[s^{\min},s^R\right]$, proving the claim over this range.

As the second step, consider $g(\bar{s})$ over the range $\bar{s}\in\left[s^R,s^{\max}\right]$. In this range, $\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}\leq\frac{f_1(\bar{s})}{1-F_1(\bar{s})}$. Note also that $\tilde{E}_1[s\mid s\geq\bar{s}]-\bar{s}>g(\bar{s})$. Using these inequalities, Eq. $(A.21)$ implies:

$$g'(\bar{s})\leq\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})}g(\bar{s}) \text{ for each } \bar{s}\in[s^R,s^{\max}). \tag{A.22}$$

Next suppose, to reach a contradiction, that there exists $\hat{s}<s^{\max}$ such that $g(\hat{s})<0$. Define $\hat{s}^{\max}$ with:

$$\hat{s}^{\max}=\sup\left\{s\in[\hat{s},s^{\max})\mid g(s)\leq g(\hat{s})\right\}.$$

Note that $g(\hat{s}^{\max})=g(\hat{s})<0$ by the continuity of $g(\cdot)$. Note also that $\hat{s}^{\max}<s^{\max}$ since $g(s^{\max})=0$. Then, the inequality in $(A.22)$ applies for $\hat{s}^{\max}$ and implies $g'(\hat{s}^{\max})\leq\frac{\tilde{f}_1(\hat{s})}{1-\tilde{F}_1(\hat{s})}g(\hat{s}^{\max})<0$. This further implies that there exists $s\in(\hat{s}^{\max},s^{\max})$ such that $g(s)<g(\hat{s}^{\max})=g(\hat{s})$, which contradicts the definition of $\hat{s}^{\max}$. It follows that $g(\bar{s})\geq 0$ for each $\bar{s}\in\left[s^R,s^{\max}\right]$, proving the claim also over this range.■

I next consider the proof of Theorem 4. The proof uses the following lemma.

Lemma 2. *Consider two probability distributions F_1 and F_0 that satisfy assumption (A2).*

(i) Suppose optimists become weakly more optimistic, i.e., consider their beliefs are changed to $\tilde{F}_1$ that satisfies the weak hazard rate inequality in (21)*. Then, the optimality curve shifts up pointwise, that is:*

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_1\right) \geq p^{opt}\left(\bar{s}\ ;\ F_1\right)\ \textit{for each}\ \bar{s} \in S.$$

The market clearing curve is unchanged.

(ii) Suppose pessimists become weakly more pessimistic, i.e., consider their beliefs are changed to $\tilde{F}_0$ that satisfies the weak hazard rate inequality in (21)*. Then, the optimality curve shifts down pointwise, that is:*

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0\right) \leq p^{opt}\left(\bar{s}\ ;\ F_0\right)\ \textit{for each}\ \bar{s} \in S. \tag{A.23}$$

Moreover, the market clearing curve also shifts down, that is:

$$p^{mc}\left(\bar{s}\ ;\ \tilde{F}_0\right) \leq p^{mc}\left(\bar{s}\ ;\ F_0\right)\ \textit{for each}\ \bar{s} \in S. \tag{A.24}$$

Proof of Lemma 2. Part (i). In view of Eq. (13), it suffices to show that

$$\tilde{E}_1\left[s \mid s \geq \bar{s}\right] \geq E_1\left[s \mid s \geq \bar{s}\right]\ \text{for each}\ \bar{s} \in S.$$

Note that $\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})} \leq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}$ for each $\bar{s} \in S$. Thus, the previous displayed inequality follows from the argument in (the second step of) the proof of Theorem 3.

Part (ii). To show $(A.23)$, define the function $h : S \to R$ with

$$h\left(\bar{s}\right) = p^{opt}\left(\bar{s}\ ;\ F_0\right) - p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0\right).$$

Note that $(A.23)$ is equivalent to showing that $h\left(\bar{s}\right) \geq 0$ for each $\bar{s} \in S$. I will prove the stronger claim that $h\left(\bar{s}\right)$ is weakly increasing over S. This claim implies that $h\left(\bar{s}\right) \geq 0$ because $h\left(s^{\min}\right) = 0$. To prove that $h\left(\bar{s}\right)$ is weakly increasing, consider the derivative of $h\left(\bar{s}\right)$. After using Eq. $(A.4)$ and rearranging terms, this derivative can be written as:

$$\frac{dh\left(\bar{s}\right)}{d\bar{s}} = \begin{pmatrix} \tilde{f}_0\left(\bar{s}\right) - f_0\left(\bar{s}\right) \\ -\frac{f_1(\bar{s})}{1-F_1(\bar{s})}\left(F_0\left(\bar{s}\right) - \tilde{F}_0\left(\bar{s}\right)\right) \end{pmatrix}\left(E_1\left[s \mid s \geq \bar{s}\right] - \bar{s}\right). \tag{A.25}$$

Next note that $\frac{\tilde{f}_0(\bar{s})}{1-\tilde{F}_0(\bar{s})} \geq \frac{f_0(\bar{s})}{1-F_0(\bar{s})}$, which also implies $\frac{\tilde{f}_0(\bar{s})-f_0(\bar{s})}{F_0(\bar{s})-\tilde{F}_0(\bar{s})} \geq \frac{f_0(\bar{s})}{1-F_0(\bar{s})}$. Combining this inequality with $\frac{f_0(\bar{s})}{1-F_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}$ implies:

$$\frac{\tilde{f}_0\left(\bar{s}\right) - f_0\left(\bar{s}\right)}{F_0\left(\bar{s}\right) - \tilde{F}_0\left(\bar{s}\right)} \geq \frac{f_1\left(\bar{s}\right)}{1 - F_1\left(\bar{s}\right)}.$$

Using this inequality in Eq. $(A.25)$ shows that $\frac{dh(\bar{s})}{d\bar{s}} \geq 0$, which completes the proof of $(A.23)$.

To show $(A.24)$, recall from Lemma 1 that $\frac{E_1[\min(s,\bar{s})]}{E_0[\min(s,\bar{s})]}$ is increasing in $\bar{s}$. Taking F_0 in place of F_1 and $\tilde{F}_0$ in place of F_0, this implies $\frac{E_0[\min(s,\bar{s})]}{\tilde{E}_0[\min(s,\bar{s})]}$ is increasing in $\bar{s}$. This in turn implies:

$$E_0\left[\min\left(s, \bar{s}\right)\right] \geq \tilde{E}_0\left[\min\left(s, \bar{s}\right)\right]\ \text{for each}\ \ \bar{s} \in \mathcal{S}.$$

Using this inequality in Eq. (17) proves $(A.24)$.■

Proof of Theorem 4. Part (i). The fact that the conditions in (21) are satisfied with equality over $\left(s^{\min}, \bar{s}^*\right)$ implies F_0 and F_1 are unchanged over this range. From Eq. (17), it follows that $p^{mc}(\bar{s})$ is also unchanged over this range. On the other hand, I claim that $p^{opt}(\bar{s})$ shifts up over this range. Since $p^{opt}(\cdot)$ is decreasing and $p^{mc}(\cdot)$ is increasing, this claim implies that the new intersection point is for a greater $\bar{s}^*$ and a greater p. It follows that p and $\bar{s}^*$ increase. This also implies that the margin $m = n_1/p$ decreases [cf. Eq. $(A.20)$].

Thus, it remains to prove the claim that $p^{opt}(\bar{s})$ shifts up over the range, $\left(s^{\min}, \bar{s}^*\right)$. To this end, first note that part (i) of Lemma 2 implies:

$$p^{opt}\left(\bar{s}\ ;\ F_0, \tilde{F}_1\right) \geq p^{opt}\left(\bar{s}\ ;\ F_0, F_1\right) \text{ for each } \bar{s} \in S. \tag{A.26}$$

That is, ignoring the change in F_0, the optimality curve shifts up. Next recall that F_0 is unchanged over $\left(s^{\min}, \bar{s}^*\right)$. By Eq. (12), this implies:

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, \tilde{F}_1\right) = p^{opt}\left(\bar{s}\ ;\ F_0, \tilde{F}_1\right) \text{ for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right). \tag{A.27}$$

That is, the change in F_0 [which is concentrated on the region $(\bar{s}^*, s^{\max})$] does not affect the optimality curve over the region $\left(s^{\min}, \bar{s}^*\right)$. Combining Eq. $(A.27)$ with the inequality in $(A.26)$ proves that $p^{opt}(\bar{s})$ shifts up for $\bar{s} \in \left(s^{\min}, \bar{s}^*\right)$.

Part (ii). Parts (i) and (ii) of Lemma 2 imply that $p^{mc}(\bar{s})$ shifts down over the entire range S. I also claim that $p^{opt}(\bar{s})$ shifts down over the range $(\bar{s}^*, s^{\max})$. Since $p^{opt}(\cdot)$ is decreasing and $p^{mc}(\cdot)$ is increasing, this claim implies that the new intersection point is for a lower price p. It also follows that the margin $m = n_1/p$ increases.

Thus, it remains to prove the claim that $p^{opt}(\bar{s})$ shifts down over the range $(\bar{s}^*, s^{\max})$. To this end, first note that part (ii) of Lemma 2 implies:

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, F_1\right) \leq p^{opt}\left(\bar{s}\ ;\ F_0, F_1\right) \text{ for each } \bar{s} \in S. \tag{A.28}$$

That is, ignoring the change in F_1, the optimality curve shifts down. Next recall that by assumption $\frac{f_0}{1-F_0}$ is unchanged over $(\bar{s}^*, s^{\max})$. By Eq. (12), this implies:

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, \tilde{F}_1\right) = p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, F_1\right) \text{ for each } \bar{s} \in \left(\bar{s}^*, s^{\max}\right). \tag{A.29}$$

That is, the change in F_1 [which is concentrated on the region $\left(s^{\min}, \bar{s}^*\right)$] does not affect the optimality curve over the region $(\bar{s}^*, s^{\max})$. Combining Eq. $(A.29)$ with the inequality in $(A.28)$ proves that $p^{opt}(\bar{s})$ shifts down for $\bar{s} \in (\bar{s}^*, s^{\max})$.■

A.4 Equilibrium with Short Contracts

This appendix completes the characterization of the equilibrium with short selling analyzed in Section 3.3.

Proof of Theorem 5. I first show that there is unique solution to $p = p^{opt,S}(\gamma)$. With some algebra, the derivative of $p^{opt,S}(\gamma)$ in Eq. (28) can be calculated as:

$$\frac{dp^{opt,S}(\gamma)}{d\gamma} = -\frac{E_1[s]\left(\int_{s^{\min}}^{\gamma} sdF_1\right)}{\left(1 - F_1(\gamma) + F_0(\gamma)\frac{\int_{s^{\min}}^{\gamma} sdF_1}{\int_{s^{\min}}^{\gamma} sdF_0}\right)^2}\left[\frac{f_1(\gamma)}{\int_{s^{\min}}^{\gamma} sdF_1} - \frac{f_0(\gamma)}{\int_{s^{\min}}^{\gamma} sdF_0}\right]\left[\frac{F_0(\gamma)\gamma}{\int_{s^{\min}}^{\gamma} sdF_0} - 1\right]$$

It can be checked that the last bracketed term is positive. The second to last bracketed term is also positive in view of assumption (A2^S). This implies $\frac{dp^{opt,S}(\gamma)}{d\gamma} < 0$. Note also that $p^{opt,S}(s^{\max}) = E_1[s]$ and $p^{opt,S}\left(s^{\min}\right) = E_0[s]$. Since $p \in (E_0[s], E_1[s])$, it follows that there is a unique solution to $p = p^{opt,S}(\gamma)$ over the range $\gamma \in \left[s^{\min}, s^{\max}\right]$.

I next show that the unique solution to $p = p^{opt,S}(\gamma)$ corresponds to the solution to problem (27). Substituting x_0 from the budget constraint, this problem can be written as:

$$\max_{\gamma} R_1^S(\gamma) \equiv \frac{\gamma - E_0[\min(\gamma, s)]}{\gamma - \frac{1}{\frac{E_1[s]}{p}} E_1[\min(\gamma, s)]}.$$

This expression is the expected return of pessimists who sell one unit of short contract with cash collateral γ. The derivative of $R_1^L(\tilde{s})$ can be calculated as:

$$\frac{dR_1^S(\gamma)}{d\gamma} = R_1^S(\gamma)\left(\frac{F_0(\gamma)}{\gamma - E_0[\min(\gamma, s)]} - \frac{1 - \frac{1}{\frac{E_1[s]}{p}}(1 - F_1(\gamma))}{\gamma - \frac{1}{\frac{E_1[s]}{p}} E_1[\min(\gamma, s)]}\right).$$

Setting this expression to zero and rearranging terms implies the first order condition $p = p^{opt,S}(\gamma)$. Since there is a unique solution to $p = p^{opt,S}(\gamma)$, it follows that there is a unique maximum $R_1^S(\gamma)$ characterized by the first order condition.■

Asset market clearing and characterization of equilibrium. Eq. (24) implies $m^S = \frac{\gamma - q(\gamma)}{p}$. Substituting for $q(\gamma)$ from Eq. (26) gives

$$m^S(\gamma, p) = \frac{1}{p}\left(\gamma - p\frac{E_1[\min(\gamma, s)]}{E_1[s]}\right). \tag{A.30}$$

Using this expression, Eq. (29) can be written as:

$$n_1 = p\left(1 + \frac{n_0}{\gamma\frac{E_1[s]}{E_1[\min(\gamma,s)]} - p}\right). \tag{A.31}$$

Note that the right hand side is strictly increasing in p. Consequently, for each $\gamma \in \left[s^{\min}, s^{\max}\right]$, there is a unique solution to this equation denoted by $p^{mc,S}(\gamma)$. Moreover, $p^{mc,S}(\gamma)$ is increasing because the right hand side is strictly decreasing in γ. It can also be checked that $p^{mc,S}\left(s^{\min}\right) < E_1[s]$ and that $p^{mc,S}(s^{\max}) > E_1[s]$ [in view of condition (A1)]. Consequently, $p^{opt,S}(\gamma)$ and $p^{mc,S}(\gamma)$ intersect at some $\gamma \in \left(s^{\min}, s^{\max}\right)$ and $p \in (E_0[s], E_1[s])$. This completes the characterization of equilibrium.■

A.5 Equilibrium with Simple Short and Debt Contracts

Sections 3 and 5 considered the equilibrium with either simple debt contracts or simple short contracts in isolation. This appendix presents a model that features both types of contracts, and shows that the asymmetric disciplining result continues to apply in this setting.

Consider a variant of the general equilibrium defined in Section 2 with the contract space $\mathcal{B}^T = \mathcal{B}^D \cup \mathcal{B}^S$, where $\mathcal{B}^D$ and $\mathcal{B}^S$ are respectively the set of simple debt and short contracts defined in Eqs. (7) and (23). To keep the analysis tractable, I also assume that only a fraction $\gamma^S \in [0,1]$ of traders can sell short contracts, while only a fraction $\gamma^D \in [0,1]$ can sell debt contracts. That is, only a fraction of traders can short sell the asset or leverage their investments in the asset. These assumptions are not unreasonable because short selling in financial markets (and to some extent, leverage) is confined to a small fraction of investors. Assume also that the parameters $\left(n_0, n_1, \gamma^S, \gamma^D\right)$ are such that the equilibrium price is interior, $p \in (E_0[s], E_1[s])$ [which will be verified after the characterization].

Under these assumptions, there exists an equilibrium of the following form. The fraction, γ^D, of optimists leverage their investments in the asset using a loan with riskiness $\bar{s} \in S$. The debt contracts they sell are bought by the fraction, $1-\gamma^S$, of pessimists who are unable to short sell. On the other hand, the fraction, γ^S, of pessimists short sell the asset using cash collateral $\gamma \in S$. The short contracts they sell are bought by the fraction, $1-\gamma^D$, of optimists who are unable to leverage. To see the intuition for this matching, note that pessimists that are able to short sell require a greater interest rate to part with their endowment than those who are unable to short sell. This is because they have a greater expected rate of return on their endowment (in view of their ability to short sell). Consequently, optimists borrow cash from the latter type of pessimists. A similar reasoning shows that pessimists borrow the asset from optimists who are unable to leverage.

An equilibrium can then be represented by a triple, $(p, \bar{s}, \gamma)$. Moreover, the optimal debt and short contracts are characterized by the same analysis in Sections 3 and 5. In particular, the riskiness of the optimal loan, $\bar{s}$, is characterized as the solution to $p = p^{opt}(\bar{s})$ [cf. Eq. (12)]. Similarly, the cash-collateral of the optimal short contract is characterized as the solution to $p = p^{mc}(\bar{s})$. Assume that beliefs satisfy assumption (MLRP) of Section 6, which ensures that the optimal debt and short contracts are uniquely determined. The remaining step is asset market clearing, which can be written as:

$$\frac{n_1}{p}\left[\gamma^D \frac{1}{m(\bar{s},p)} + 1 - \gamma^D\right] = 1 + \gamma^S \frac{n_0}{p}\left[\frac{1}{m^S(\gamma,p)}\right]\frac{E_1[\min(\gamma,s)]}{E_1[s]}. \tag{A.32}$$

Here, $m(\bar{s},p)$ denotes the loan margin [cf. Eq. (18)] and $m^S(\gamma,p)$ denotes the short margin [cf. Eq. $(A.30)$]. The left hand side of Eq. $(A.32)$ is the total demand for the asset which consists of demand by the fraction, γ^D, of optimists who leverage their investments and the fraction, $1-\gamma^D$, of optimists who are unable to leverage. The right hand side is the total supply of the asset which consists of the physical supply of 1 unit and the short contracts sold by the fraction, γ^S, of pessimists who are able to short sell.

Eq. $(A.32)$ defines a third (market clearing) equation between the asset price p and the contracts $(\bar{s}, \gamma)$. The equilibrium triple, $(p, \bar{s}, \gamma)$, is characterized by the solution to this equation along with the optimality conditions $p = p^{opt}(\bar{s}) = p^{opt,S}(\gamma)$. To illustrate the asymmetric disciplining result, suppose the belief heterogeneity shifts to upside states (while keeping $E_1[s]$ and $E_0[s]$ constant). By the analysis in Section 4, this induces optimists to leverage using loans with a smaller riskiness, $\bar{s}$, and a lower loan margin, m. By the analysis in Section 5, this also

induces pessimists to short sell using a larger cash-collateral, γ, and a higher short margin, m^S. In view of these observations, the market clearing relation $(A.32)$ implies that (keeping p constant) the demand for the asset increases while the effective supply of the asset decreases. This in turn leads to a higher equilibrium price. Conversely, if the belief heterogeneity shifts to downside states, then the loan margin increases, the short margin decreases, and the asset price decreases.

A.6 Equilibrium with Richer Contracts and Optimists' Bargaining Power

This appendix completes the characterization of the equilibrium analyzed in Section 6, in which the contract set is unrestricted and optimists have all of the bargaining power.

Proof of Theorem 6. Let λ^B denote the Lagrange multiplier for the budget constraint of problem (30). Consider the first order condition for $\boldsymbol{\varphi}(s)$, which implies:

$$\boldsymbol{\varphi}(s)\begin{cases} = a_1 s + c_1, & \text{if } -f_1(s) + \lambda^B f_0(s) > 0, \\ \text{intermediate}, & \text{if } -f_1(s) + \lambda^B f_0(s) = 0, \\ = 0, & \text{if } -f_1(s) + \lambda^B f_0(s) < 0. \end{cases}$$

Recall by assumption (MLRP) that $f_1(s)/f_0(s)$ is increasing over S. It follows that $\boldsymbol{\varphi}(s)$ takes the threshold form in (31) for some threshold state, $\bar{s}$, which is also a choice variable.

Next consider the first order conditions for a_1 and c_1, which are respectively given by:

$$\begin{aligned} E_1[s] - \int_{s^{\min}}^{\bar{s}} s dF_1 &\leq \lambda^B \left(p - \int_{s^{\min}}^{\bar{s}} s dF_0\right) \text{ with inequality only if } a_1 = 0, \\ 1 - \int_{s^{\min}}^{\bar{s}} dF_1 &\leq \lambda^B \left(1 - \int_{s^{\min}}^{\bar{s}} dF_0\right) \text{ with inequality only if } c_1 = 0. \end{aligned}$$

In view of the budget constraint, at least one of these first order conditions is satisfied with equality. Using this observation, problem (30) can be rewritten as:

$$\begin{aligned} \max_{\bar{s}, n_1^a \geq 0, n_1^c \geq 0} \quad & n_1^a R_1^a(\bar{s}, p) + n_1^c R_1^c(\bar{s}, p), \\ \text{s.t.} \quad & n_1^a + n_1^c = n_1. \\ \text{where } R_1^a(\bar{s}, p) \equiv & \frac{E_1[s] - \int_{s^{\min}}^{\bar{s}} s dF_1}{p - \int_{s^{\min}}^{\bar{s}} s dF_0} \text{ and } R_1^c(\bar{s}, p) = \frac{1 - F_1(\bar{s})}{1 - F_0(\bar{s})}. \end{aligned} \tag{A.33}$$

The expression, $R_1^a(\bar{s}, p)$, is optimists' return from selling the contingent debt contract (with threshold $\bar{s}$) collateralized by one unit of the asset. The expression, $R_1^I(\bar{s}, p)$, is optimists' return from selling the contingent debt contract collateralized by one unit of cash. The budget constraint of problem $(A.33)$ is obtained from the budget constraint of problem (30) after substituting $a_1 = \frac{n_1^a}{p - \int_{s^{\min}}^{\bar{s}} s dF_0}$ and $c_1 = \frac{n_1^c}{1 - \int_{s^{\min}}^{\bar{s}} F_0}$ [along with the contingent debt contract from Eq. (31)].

To solve problem $(A.33)$, first consider the maximization of $R_1^a(\bar{s}, p)$ over $\bar{s} \in S$. The first order condition for this problem is given by $p = p^{opt,O}(\bar{s})$ [cf. Eq. (32)]. Moreover, under assumption (MLRP), there is a unique solution to the first order condition. This implies that $R_1^a(\bar{s}, p)$ has a unique maximum, $\bar{s}^a(p) \in \left(s^{\min}, s^{\max}\right)$, characterized by the first order condition. Next consider the maximization of $R_1^c(\bar{s}, p)$ over $\bar{s} \in S$. The first order condition for this problem is given by

$$\frac{f_1(\bar{s})}{f_0(\bar{s})} = \frac{1 - F_1(\bar{s})}{1 - F_0(\bar{s})}. \tag{A.34}$$

Under assumption (MLRP), there is a unique solution to the first order condition. This implies that $R_1^c(\bar{s}, p)$ has a unique maximum, $\bar{s}^c \in \left(s^{\min}, s^{\max}\right)$, characterized by the first order condition.

Next define, $\bar{p}$, as the price for which the maxima for the two types of returns coincide, that is $\bar{s}^a(p) = \bar{s}^c$. In particular, define:

$$\bar{p} \equiv p^{opt,O}(\bar{s}^c) = \int_{s^{\min}}^{\bar{s}^c} s dF_0 + \frac{1 - F_0(\bar{s}^c)}{1 - F_1(\bar{s}^c)} \int_{\bar{s}^c}^{s^{\max}} s dF_1, \tag{A.35}$$

which ensures that $\bar{s}^a(\bar{p}) = \bar{s}^c$. The second equality in this expression uses Eq. (32) along with the definition of $\bar{s}^c$ in Eq. $(A.34)$. Plugging $\bar{p}$ into the definition of $R_1^a(\bar{s}^c, p)$ in $(A.33)$ implies:

$$R_1^a(\bar{s}^c, \bar{p}) = \frac{1 - F_1(\bar{s}^c)}{1 - F_0(\bar{s}^c)} = R_1^c(\bar{s}^c, \bar{p}).$$

Since $\bar{s}^a(\bar{p}) = \bar{s}^c$, it follows that $\max_{\bar{s}} R_1^a(\bar{s}, \bar{p}) = \max_{\bar{s}} R_1^c(\bar{s}, \bar{p})$.

Finally, note that $R_1^a(\bar{s}, p)$, is strictly decreasing in p, while $R_1^c(\bar{s}, p)$, is independent of p [cf. $(A.33)$]. It follows that $\max_{\bar{s}} R_1^a(\bar{s}, \bar{p}) > \max_{\bar{s}} R_1^c(\bar{s}, \bar{p})$ if $p < \bar{p}$, while the inequality is reversed if $p > \bar{p}$. I next use this observation to present the solution to problem $(A.33)$ along with the proofs for parts (i)-(iii).

Part (i). If $p < \bar{p}$, then optimists invest only in the asset, i.e., $n_1^a = n_1, n_1^c = 0$, and thus $c_1 = 0$. Optimists choose $\bar{s}$ to maximize $R_1^a(\bar{s}, p)$. The optimal threshold, $\bar{s}$, is characterized as the solution to $p = p^{opt,O}(\bar{s})$.

Part (ii). If $p = \bar{p}$, then optimists are indifferent between investing in the asset or cash, i.e., $n_1^a, n_1^c \in [0, n_1]$. The optimal threshold, $\bar{s} = \bar{s}^c$, is characterized as the solution to $\bar{p} = p^{opt,O}(\bar{s})$. Equivalently, $\bar{s}^c$ is also the solution to Eq. $(A.34)$.

Part (iii). If $p > \bar{p}$, then optimists invest only in cash, i.e., $n_1^a = 0, n_1^c = n_1$, and thus $a_1 = 0$. Optimists choose $\bar{s}$ to maximize $R_1^c(\bar{s}, p)$. The optimal threshold is $\bar{s} = \bar{s}^c$.

To complete the proof of the theorem, note that debt market clearing requires pessimists' endowment, n_0, to be sufficiently large to meet optimists' borrowing. If $p < \bar{p}$, so that part (i) applies, then neither optimists use their endowment and their borrowing to invest only in the asset. Consequently, the debt market clearing condition can be written as:

$$n_0 > p - n_1.$$

That is, optimists' and pessimists' total endowment must be sufficiently large to purchase the entire asset supply. If instead $p > \bar{p}$, so that part (iii) applies, then the debt market clearing condition requires:

$$n_0 > n_1 \frac{\int_{s^{\min}}^{\bar{s}^c} F_0}{1 - \int_{s^{\min}}^{\bar{s}^c} F_0}.$$

Combining the cases $p < \bar{p}$ or $p > \bar{p}$, a sufficient condition for debt market clearing is:

$$n_0 > \max\left(\bar{p} - n_1, n_1 \frac{\int_{s^{\min}}^{\bar{s}^c} F_0}{1 - \int_{s^{\min}}^{\bar{s}^c} F_0}\right), \tag{A.36}$$

where $\bar{s}^c$ is defined as the solution to Eq. $(A.34)$.■

Asset market clearing and characterization of equilibrium. When $p > E_0[s]$ (which will be the case in equilibrium), pessimists do not invest in the asset. Thus, asset market clearing

requires optimists to hold the entire asset supply, i.e.,, $a_1 = 1$. In view of the characterization in Theorem 6, optimists' demand for the asset is given by:

$$a_1 = \begin{cases} \frac{n_1}{p - \int_{s^{\min}}^{\bar{s}} s dF_0}, & \text{if } p < \bar{p}, \\ \text{intermediate}, & \text{if } p = \bar{p}, \\ = 0, & \text{if } p > \bar{p}. \end{cases}$$

This implies the following market clearing condition:

$$p^{mc,O}(\bar{s}) = \min\left(n_1 + \int_{s^{\min}}^{\bar{s}} s dF_0,\ \bar{p}\right). \tag{A.37}$$

The equilibrium price is characterized as the unique intersection of the decreasing optimality relation, $p^{opt,O}(\bar{s})$ and the weakly increasing market clearing relation $p^{mc,O}(\bar{s})$. These two curves always intersect at some $p \in (E_0[s], \bar{p}]$. Given the equilibrium price, the threshold of the optimal contract, $\bar{s}$, (along with optimists' portfolio) is characterized by Theorem 6.■

Proof of Theorem 7. Let $\left(p^{debt}, \bar{s}^{debt}\right)$ denote the equilibrium in Section 3, which is characterized as the intersection of $p^{opt}(\bar{s})$ and $p^{mc}(\bar{s})$ [cf. Eqs. (12) and (17)]. If $p^{debt} > \bar{p}$, then the result follows because the equilibrium price with unrestricted contracts is always weakly smaller than $\bar{p}$. Thus, suppose $p^{debt} \leq \bar{p}$. Comparing Eqs. (17) and $(A.37)$ illustrates that

$$p^{mc,O}\left(\bar{s}^{debt}\right) < p^{mc}\left(\bar{s}^{debt}\right) = p^{debt}.$$

I also claim that:

$$p^{opt,O}\left(\bar{s}^{debt}\right) \leq p^{opt}\left(\bar{s}^{debt}\right) = p^{debt}. \tag{A.38}$$

In view of the last two displayed equations, it follows that $p^{mc,O}(\bar{s})$ and $p^{opt,O}(\bar{s})$ intersect at some $p < p^{debt}$, proving the result.

The remaining step is to prove the claim in $(A.38)$. From Eq. $(A.35)$, recall that $\bar{p}$ is defined as the solution to $\bar{p} = p^{opt,O}(\bar{s}^c)$, where $\bar{s}^c$ is the solution to Eq. $(A.34)$. Using assumption (MLRP), Eq. $(A.34)$ implies that:

$$\frac{f_0(\bar{s})}{f_1(\bar{s})} \leq \frac{1 - F_0(\bar{s})}{1 - F_1(\bar{s})} \text{ for } \bar{s} \geq \bar{s}^c, \text{ with equality iff } \bar{s} = \bar{s}^c.$$

Comparing Eqs. (12) and (32), it follows that:

$$p^{opt,O}(\bar{s}) \leq p^{opt}(\bar{s}) \text{ for } \bar{s} \geq \bar{s}^c, \text{ with equality iff } \bar{s} = \bar{s}^c.$$

This expression has two implications. First, since $\bar{p} = p^{opt}(\bar{s}^c)$, the assumption $p^{debt} \leq \bar{p}$ implies $\bar{s}^{debt} \geq \bar{s}^c$. Second, since $\bar{s}^{debt} \geq \bar{s}^c$, the claim in $(A.38)$ follows.■

A.7 Equilibrium with Richer Contracts and Pessimists' Bargaining Power

Section 6 considered the equilibrium with richer contracts when optimists had all of the bargaining power. This appendix analyzes the equilibrium in the complementary case in which pessimists have all of the bargaining power. It also discusses the more general case without any trading restrictions.

To characterize pessimists' optimal contract, consider the following alternative to assumption (PR).

Assumption (OR). *Optimists are restricted not to sell borrowing contracts, i.e.,* $\mu_1^- = 0$.

With this assumption, pessimists choose their portfolio allocation subject to optimists' participation constraint, i.e., they solve:

$$\max_{(a_0,c_0)\in\mathbb{R}_+^2,\ [\boldsymbol{\varphi}(s)\in\mathbb{R}_+]_{s\in S}} \quad a_0 E_0[s] + c_0 - E_0[\min(\boldsymbol{\varphi}(s), a_0 s + c_0)], \tag{A.39}$$

$$\text{s.t.} \quad a_0 p + c_0 = n_0 + \frac{E_1[\min(\boldsymbol{\varphi}(s), a_0 s + c_0)]}{E_1[s]/p}.$$

Note that optimists' required rate of return is $E_1[s]/p$ because they are indifferent between investing in the asset and lending to optimists. The following result is the analogue of Theorem 6 for this setting.

Theorem 10 (Pessimists' Optimal Contract). *Suppose that the contract space is unrestricted,* $\mathcal{B}^T = \mathcal{B}$*, assumptions (OR) and (MLRP) hold, and that* n_1 *is sufficiently large [in particular, it satisfies condition* (A.46)*]. Fix asset price that satisfies* $p \in (E_0[s], E_1[s])$*, and consider pessimists' problem* (30)*. The optimal contract takes the threshold form*

$$\boldsymbol{\varphi}(s) \equiv \begin{cases} a_0 s + c_0 & \text{if } s > \bar{s}, \\ 0 & \text{if } s < \bar{s}. \end{cases} \tag{A.40}$$

for a threshold state $\bar{s} \in \mathcal{S}$*. There exists* $\underline{p} > E_0[s]$ *such that:*

(i) If $p > \underline{p}$*, then pessimists invest only in cash, i.e.,* $a_1 = 0$*. The threshold* $\bar{s}$ *of the optimal contract is the unique solution to the following equation over* $\mathcal{S}$*:*

$$p = p^{opt,P}(\bar{s}) \equiv \frac{E_1[s]}{F_0(\bar{s})\frac{f_1(\bar{s})}{f_0(\bar{s})} + 1 - F_1(\bar{s})}. \tag{A.41}$$

(ii) If $p = \underline{p}$*, then pessimists are indifferent between investing in the asset and cash. The threshold* $\bar{s}$ *is the unique solution to* $\bar{p} = p^{opt,P}(\bar{s})$*.*

(iii) If $p < \underline{p}$*, then pessimists invest only in the asset, i.e.,* $c_1 = 0$*. The threshold* $\bar{s}$ *is the unique solution to* $\bar{p} = p^{opt,P}(\bar{s})$*.*

Proof of Theorem 10. Let λ^B denote the Lagrange multiplier for the budget constraint of $(A.39)$. Consider the first order condition for $\boldsymbol{\varphi}(s)$, which implies:

$$\boldsymbol{\varphi}(s) \begin{cases} = a_0 s + c_0, & \text{if } -f_0(s) + \lambda^B f_1(s) > 0, \\ \text{intermediate}, & \text{if } -f_0(s) + \lambda^B f_1(s) = 0, \\ = 0, & \text{if } -f_0(s) + \lambda^B f_1(s) < 0. \end{cases}$$

Recall by assumption (MLRP) that $f_1(s)/f_0(s)$ is increasing over S. It follows that $\boldsymbol{\varphi}(s)$ takes the threshold form in $(A.40)$ for some threshold state, $\bar{s}$.

Next consider the first order conditions for c_0 and a_0, which are respectively given by:

$$\begin{aligned} 1-\int_{\bar{s}}^{s^{\max}} dF_0 &\leq \lambda^B\left(1-\frac{\int_{\bar{s}}^{s^{\max}} dF_1}{E_1[s]/p}\right) \text{ with inequality only if } c_0=0. \\ E_0[s]-\int_{\bar{s}}^{s^{\max}} sdF_0 &\leq \lambda^B\frac{p}{E_1[s]}\left(E_1[s]-\int_{\bar{s}}^{s^{\max}} sdF_1\right) \text{ with inequality only if } a_0=0, \end{aligned}$$

It follows that, as in the proof of Theorem 6, pessimists' optimal investment decision depends on the comparison between:

$$\max_{\bar{s}\in S} R_0^c(\bar{s},p) \equiv \frac{\int_{s^{\min}}^{\bar{s}} sdF_0}{1-\frac{p}{E_1[s]}\int_{\bar{s}}^{s^{\max}} dF_1} \text{ and } \max_{\bar{s}\in S} R_0^a(\bar{s},p) \equiv \frac{E_1[s]}{p}\frac{\int_{s^{\min}}^{\bar{s}} sdF_0}{\int_{s^{\min}}^{\bar{s}} sdF_1}. \tag{A.42}$$

The expression, $R_0^c(\bar{s},p)$, is pessimists' return from selling the contingent contract (with threshold $\bar{s}$) collateralized by cash. The expression, $R^c(\bar{s},p)$, is pessimists' return from selling the contingent contract collateralized by the asset.

Let $\bar{s}^c(p)$ denote the maximum of $R_0^c(\bar{s},p)$ over S. From the first order condition, $\bar{s}^c(p)$ is characterized as the solution to $p = p^{opt,P}(\bar{s})$ [cf. Eq. $(A.41)$]. Moreover, under assumption (MLRP), there is a unique solution to this equation [since $p^{opt,P}(\bar{s})$ is strictly decreasing]. Similarly, let $\bar{s}^a \in S$ denote the maximum of $R_0^a(\bar{s},p)$ over S. From the first order condition, $\bar{s}^a$ is characterized as the unique solution to

$$\frac{f_0(\bar{s})}{f_1(\bar{s})} = \frac{\int_{s^{\min}}^{\bar{s}} sdF_0}{\int_{s^{\min}}^{\bar{s}} sdF_1}. \tag{A.43}$$

Next define $\underline{p}$ as the price level for which the two optima, $\bar{s}^c(p)$ and $\bar{s}^a$, coincide:

$$\underline{p} = p^{opt,P}(\bar{s}^a) = \frac{E_1[s]}{F_0(\bar{s}^a)\frac{\int_{s^{\min}}^{\bar{s}^a} sdF_1}{\int_{s^{\min}}^{\bar{s}^a} sdF_0} + 1 - F_1(\bar{s}^a)}. \tag{A.44}$$

Here, the second equality uses Eq. $(A.41)$ and the definition of $\bar{s}^a$ in Eq. $(A.43)$. Plugging $\underline{p}$ into the definition of $R_0^c(\bar{s},p)$ in Eq. $(A.42)$, it follows that:

$$R_0^c\left(\bar{s}^a,\underline{p}\right) = \frac{E_1[s]}{p}\frac{\int_{s^{\min}}^{\bar{s}^a} sdF_0}{\int_{s^{\min}}^{\bar{s}^a} sdF_1} = R_0^a\left(\bar{s}^a,\underline{p}\right).$$

Since $\bar{s}^c\left(\underline{p}\right) = \bar{s}^a$, it follows that $\max_{\bar{s}} R_0^c\left(\bar{s},\underline{p}\right) = \max_{\bar{s}} R_0^a\left(\bar{s},\underline{p}\right)$.

Finally, note that $R_0^c(\bar{s},p)$, is strictly increasing in p, while $R_0^a(\bar{s},p)$ is strictly decreasing in p [cf. $(A.42)$]. It follows that $\max_{\bar{s}} R_0^c\left(\bar{s},\underline{p}\right) > \max_{\bar{s}} R_0^a\left(\bar{s},\underline{p}\right)$ if $p > \underline{p}$, while the inequality is reversed if $p < \underline{p}$. I next use this observation to characterize pessimists' optimal investment decision and to present the proofs for parts (i)-(iii).

Part (i). If $p > \underline{p}$, then pessimists only invest in cash, i.e., $a_0 = 0$. Pessimists choose $\bar{s}$ to maximize $R_0^c(\bar{s},p)$. The optimal threshold, $\bar{s}$, is characterized as the solution to $p = p^{opt,P}(\bar{s})$.

Part (ii). If $p = \underline{p}$, then pessimists are indifferent between investing in cash and the asset. The optimal threshold, $\bar{s} = \bar{s}^a$, is characterized as the solution to $\underline{p} = p^{opt,P}(\bar{s})$. Equivalently, $\bar{s}^a$ is also the solution to Eq. $(A.43)$.

Part (iii). If $p < \underline{p}$, then pessimists invest only in the asset, i.e., $c_0 = 0$. Pessimists choose $\bar{s}$ to

maximize $R_0^a(\bar{s}, p)$. The optimal threshold is $\bar{s} = \bar{s}^a$.

To complete the proof of the theorem, note that debt market clearing requires optimists' endowment, n_1, to be sufficiently large to meet pessimists' borrowing. If $p < \underline{p}$, so that part (iii) applies, then in equilibrium all of the aggregate endowment is invested in the asset. Consequently, the debt market clearing condition can be written as:

$$n_1 > p - n_0.$$

If instead $p > \underline{p}$, so that part (i) applies, then the debt market clearing condition requires:

$$\begin{aligned} n_1 &> n_0 \frac{\frac{1-F_1(\bar{s}^c(p))}{E_1[s]/p}}{1 - \frac{1-F_1(\bar{s}^c(p))}{E_1[s]/p}} \\ &= n_0 \frac{1}{\frac{E_1[s]}{p(1-F_1(\bar{s}^c(p)))} - 1} = n_0 \left[\frac{1 - F_1(\bar{s}^c(p))}{F_0(\bar{s}^c(p))} \frac{f_0(\bar{s}^c(p))}{f_1(\bar{s}^c(p))} \right]. \end{aligned} \tag{A.45}$$

Here, the last equality uses Eq. $(A.41)$. The expression in brackets is a decreasing function of $\bar{s}^c(p)$. Since $\bar{s}^c(p) \geq \bar{s}^a$ [in view of Eq. $(A.41)$ and $p > \underline{p}$], a sufficient condition for debt market clearing in this case is:

$$n_1 > n_0 \left[\frac{1 - F_1(\bar{s}^a)}{F_0(\bar{s}^a)} \frac{f_0(\bar{s}^a)}{f_1(\bar{s}^a)} \right].$$

Combining the cases $p < \underline{p}$ and $p > \underline{p}$, the following condition is sufficient for debt market clearing regardless of the equilibrium asset price:

$$n_1 > \max\left(\underline{p} - n_0, \frac{1 - F_1(\bar{s}^a)}{F_0(\bar{s}^a)} \frac{f_0(\bar{s}^a)}{f_1(\bar{s}^a)} \right), \tag{A.46}$$

where $\bar{s}^a$ is defined as the solution to Eq. $(A.43)$.■

Asset market clearing and characterization of equilibrium. Next, to characterize the equilibrium price, consider asset market clearing. First suppose $p > \underline{p}$, so that case (i) of Theorem 10 applies. In this case, only optimists purchase assets. Moreover, their endowment is split between purchasing the asset supply and lending to pessimists. Recall that their lending to pessimists is given by the expression in $(A.45)$. Thus, asset market clearing condition can be written as:

$$n_1 = p + n_0 \frac{p}{\frac{E_1[s]}{1-F_1(\bar{s})} - p}. \tag{A.47}$$

This expression implicitly defines an increasing market clearing relation, $p^{mc,\tilde{P}}(\bar{s})$. Next suppose $p = \underline{p}$, so that case (ii) of Theorem 10 applies. In this case, pessimists also purchase some assets. Thus, market clearing requires $(A.47)$ to hold with weak inequality, or equivalently, $p \geq p^{mc,\tilde{P}}(\bar{s})$. Finally, suppose $p < \underline{p}$, so that case (ii) of Theorem 10 applies. In this case, some of the asset is purchased by pessimists. Thus, the total amount of dollars spent on purchasing the asset is given by $n_1 + n_0$. The demand for the asset is given by $\frac{n_1+n_0}{p} > \frac{n_1+n_0}{\underline{p}} > 1$, where the last inequality follows by assumption $(A.46)$. It follows that asset market clearing is violated. Consequently, $p < \underline{p}$ does not arise in equilibrium. Combining the analysis for the three cases, the asset market

clearing condition is given by:

$$p^{mc,P}(\bar{s}) = \max\left(p^{mc,\tilde{P}}(\bar{s}), \underline{p}\right).$$

Note that $p^{mc,P}(\bar{s})$ is weakly increasing in $\bar{s}$. Consequently, the equilibrium price is characterized as the unique intersection of the decreasing optimality relation, $p^{opt,P}(\bar{s})$ and the weakly increasing market clearing relation $p^{mc,P}(\bar{s})$. These two curves always intersect at some $p \in [\underline{p}, E_1(s))$. Given the equilibrium price, the threshold of the optimal contract, $\bar{s}$, (along with optimists' portfolio) is characterized by Theorem 10.

Next consider the following analogue of Theorem 7, which shows that the asset price with richer contracts is higher than the price in Section 3 in which pessimists are restricted to sell simple short contracts.

Theorem 11 (Price Comparison Between Contingent and Simple Short Contracts). *For the same beliefs, F_1 and F_0, and endowments, n_1 and n_0, the equilibrium price, p, in this appendix is strictly greater than in Section 3. That is, allowing pessimists to sell unrestricted borrowing contracts leads to a higher asset price than the case in which they are restricted to sell simple short contracts.*

Proof of Theorem 11. Let $\left(p^{short}, \bar{s}^{short}\right)$ denote the equilibrium in Section 5, which is characterized as the intersection of $p^{opt,S}(\bar{s})$ and $p^{mc,S}(\bar{s})$ [cf. Eqs. (28) and $(A.31)$]. If $p^{short} < \underline{p}$, then the result follows because the equilibrium price with unrestricted contracts is always weakly greater than $\underline{p}$. Thus, suppose $p^{short} \geq \underline{p}$. Comparing Eqs. $(A.31)$ and $(A.47)$ illustrates that $p^{mc,S}(\bar{s}) > p^{mc,\tilde{P}}(\bar{s})$ for each $\bar{s} \in S$, which further implies:

$$p^{mc,P}\left(\bar{s}^{short}\right) > p^{mc,S}\left(\bar{s}^{short}\right) = p^{short}.$$

I also claim that:

$$p^{opt,P}\left(\bar{s}^{short}\right) \geq p^{opt,S}\left(\bar{s}^{short}\right) = p^{short}. \tag{A.48}$$

In view of the last two displayed equations, it follows that $p^{mc,P}(\bar{s})$ and $p^{opt,P}(\bar{s})$ intersect at some $p > p^{short}$, proving the result.

The remaining step is to prove the claim in $(A.48)$. From Eq. $(A.44)$, recall that $\underline{p}$ is defined as the solution to $\underline{p} = p^{opt,P}(\bar{s}^a)$, where $\bar{s}^a$ is the solution to Eq. $(A.43)$. Using Eq. $(A.43)$ and assumption (MLRP), it follows that:

$$\frac{f_1(\bar{s})}{f_0(\bar{s})} \leq \frac{\int_{s^{\min}}^{\bar{s}} s dF_1}{\int_{s^{\min}}^{\bar{s}} s dF_0} \text{ for } \bar{s} \leq \bar{s}^a, \text{ with equality iff } \bar{s} = \bar{s}^a.$$

Comparing Eqs. $(A.41)$ and (28), it follows that:

$$p^{opt,P}(\bar{s}) \geq p^{opt,S}(\bar{s}) \text{ for } \bar{s} \leq \bar{s}^a, \text{ with equality iff } \bar{s} = \bar{s}^a.$$

This expression has two implications. First, since $\underline{p} = p^{opt,P}(\bar{s}^a)$, the assumption $p^{short} \geq \underline{p}$ implies $\bar{s}^{short} \leq \bar{s}^a$. Second, since $\bar{s}^{short} \leq \bar{s}^a$, the claim in $(A.48)$ follows.■

Equilibrium without trading restrictions. The analyses in Section 6 and this appendix considered the equilibrium with a trading restriction [assumption (PR) or (OR)]. Consider the case without any trading restrictions. In this case, the equilibrium can be solved as a five-tuple: $(p, R_1, R_0, \beta_1, \beta_0)$. Here, $R_1 \geq \frac{E_1[s]}{p}$ and $R_0 \geq 1$ are respectively optimists' and pessimists' gross rates of return on their endowments in equilibrium. The contract, β_1, is optimists' optimal borrowing contract which solves problem (30) with the only difference that pessimists lend $\frac{E_0[\min(\boldsymbol{\varphi}(s), a_1 s + c_1)]}{R_0}$ instead of $E_0\left[\min\left(\boldsymbol{\varphi}\left(s\right), a_1 s + c_1\right)\right]$. The contract, β_0, is pessimists' optimal borrowing contract which solves problem $(A.39)$ with the only difference that optimists lend $\frac{E_1[\min(\boldsymbol{\varphi}(s), a_0 s + c_0)]}{R_1}$ instead of $\frac{E_1[\min(\boldsymbol{\varphi}(s), a_0 s + c_0)]}{E_1[s]/p}$. In equilibrium, optimists (resp. pessimists) are indifferent between selling contract β_1 (resp. contract β_0) and lending to pessimists (resp. optimists). The prices, p, R_1, R_0, are determined by market clearing conditions respectively for the asset, optimists' borrowing, and pessimists' borrowing.

A.8 Dynamic Equilibrium

This section completes the characterization of the dynamic equilibrium analyzed in Section 7.

Proof of Theorem 8. Fix some future price to dividend ratio $\tilde{p}_d$, i.e., suppose the price at the next date is given by $p\left(y\right) = \tilde{p}_d y$ for some

$$\tilde{p}_d \in \left[\frac{1}{r}, \frac{1+\varepsilon}{r-\varepsilon}\right]. \tag{A.49}$$

I first characterize the current price to dividend ratio, $P_d\left(\tilde{p}_d\right)$. I then show that the price mapping, $P_d\left(\cdot\right)$, has a unique fixed point, which corresponds to a dynamic equilibrium.

To characterize the current price, first note that the value function is given by $v\left(s, y\right) = y\left(1 + \tilde{p}_d\right) s$. Define an alternative state space as the interval of possible asset payoffs in current dollars: $\hat{S} = \left[\frac{y(1+p_d)}{1+r} s^{\min}, \frac{y(1+p_d)}{1+r} s^{\max}\right] \subset \mathbb{R}_{++}$. Note that the economy at date t is isomorphic to the static economy analyzed in Section 3 with state space $\hat{S}$ and endowments $n_1 y$ and $n_0 y$. I claim that this economy satisfies assumptions (A1) and (A2) of Section 3. To see this, note that assumption (A2) holds because F_1 and F_2 satisfy the same assumption for the original state space S. Note also that assumption (A1) can be written as:

$$n_1 y < \left(E_1\left[s\right] - s^{\min}\right) \frac{1+\tilde{p}_d}{1+r} y \text{ and } n_0 y > E_1\left[s\right] \frac{1+\tilde{p}_d}{1+r} y - n_1 y.$$

Suppose n_1 and n_0 are such that the previous inequality is satisfied for each $\tilde{p}_d$ in the interval $(A.49)$. After canceling the y terms, this condition can be written as:

$$n_1 < \frac{E_1\left[s\right] - s^{\min}}{r} \text{ and } n_0 > \frac{E_1\left[s\right]}{r-\varepsilon} - n_1. \tag{A.50}$$

It follows that, under condition $(A.50)$, the corresponding static economy satisfies assumptions (A1) and (A2). This in turn implies that the analysis in Section 3.3 applies to this economy. In particular, the equilibrium price, $p\left(y\right)$, and the threshold, $\bar{s} \in \hat{S}$, are characterized as the unique solution to Eqs. (12) and (17). After plugging in $p\left(y\right) = p_d y$ and $\bar{s} = \bar{s}_d \frac{y(1+p_d)}{1+r}$, Eqs. (12) and

(17) can be written as:

$$\begin{aligned} p_d &= \frac{1+\tilde{p}_d}{1+r}\left[\int_{s^{\min}}^{\bar{s}_d} s dF_0 + \left(1-F_0\left(\bar{s}_d\right)\right)\int_{\bar{s}_d}^{s^{\max}} s\frac{dF_1}{1-F_1\left(s\right)}\right], \\ p_d &= n_1 + \frac{1+\tilde{p}_d}{1+r}\int_{s^{\min}}^{\bar{s}_d} s dF_0. \end{aligned} \tag{A.51}$$

By the analysis in Section 3.3, for each $\tilde{p}_d$ these equations have a unique solution, $(p_d, \bar{s}_d)$. Denote the unique solution by $\left(P_d\left(\tilde{p}_d\right), S_d\left(\tilde{p}_d\right)\right)$ and note that:

$$P_d\left(\tilde{p}_d\right) \in \left[E_0\left[s\right]\frac{1+\tilde{p}_d}{1+r}, E_1\left[s\right]\frac{1+\tilde{p}_d}{1+r}\right] \text{ and } S_d\left(\tilde{p}_d\right) \in \left(s^{\min}, s^{\max}\right).$$

I next show that $P_d\left(\cdot\right)$ has a unique fixed point over the interval $(A.49)$. To this end, I first claim that the loan riskiness $S_d\left(\tilde{p}_d\right)$ is increasing in $\tilde{p}_d$. Combining the two equations in $(A.51)$, $S_d\left(\tilde{p}_d\right) \in \left(s^{\min}, s^{\max}\right)$ is determined as the unique solution to:

$$\frac{1-F_0\left(\bar{s}_d\right)}{1-F_1\left(\bar{s}_d\right)}\int_{\bar{s}_d}^{s^{\max}}\left(s-\bar{s}_d\right)dF_1 = n_1\frac{1+r}{1+\tilde{p}_d}.$$

By assumption (A2), the left hand side of this expression is a strictly decreasing function of $\bar{s}_d$. Since the right hand side is decreasing in $\tilde{p}_d$, it follows that $S_d\left(\tilde{p}_d\right)$ is increasing in $\tilde{p}_d$. From the second equation in $(A.51)$, this also implies that $P_d\left(\tilde{p}_d\right)$ is increasing in $\tilde{p}_d$. I next claim that $P_d\left(\cdot\right)$ satisfies the boundary conditions:

$$P_d\left(\frac{1}{r}\right) > \frac{1}{r} \text{ and } P_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right) < \frac{1+\varepsilon}{r-\varepsilon}. \tag{A.52}$$

Since $P_d\left(\cdot\right)$ is increasing and continuous, it follows that it has a unique fixed point over the interval $(A.49)$. This establishes the existence of a dynamic equilibrium: $p\left(y\right) = p_d y$ and $\bar{s}_d^* = S_d\left(p_d\right) \in \left(s^{\min}, s^{\max}\right)$, where p_d corresponds to the unique fixed point.

The remaining step is to show the boundary conditions in $(A.52)$. To this end, consider Eq. $(A.51)$ for $\tilde{p}_d = \frac{1}{r}$:

$$\begin{aligned} P_d\left(\frac{1}{r}\right) &= \frac{1+\frac{1}{r}}{1+r}\left(\int_{s^{\min}}^{S_d\left(\frac{1}{r}\right)} s dF_0 + \frac{1-F_0\left(S_d\left(\frac{1}{r}\right)\right)}{1-F_1\left(S_d\left(\frac{1}{r}\right)\right)}\int_{S_d\left(\frac{1}{r}\right)}^{s^{\max}} s dF_1\right) \\ &> \frac{1+\frac{1}{r}}{1+r}\left(\int_{s^{\min}}^{s^{\max}} s dF_0 + \frac{1-F_0\left(s^{\max}\right)}{1-F_1\left(s^{\max}\right)}\int_{s^{\max}}^{s^{\max}} s dF_1\right) = \frac{1+\frac{1}{r}}{1+r}E_0\left[s\right] = \frac{1}{r}. \end{aligned}$$

Here, the second line replaces $S_d\left(\frac{1}{r}\right)$ in the first line with $s^{\max} > S_d\left(\frac{1}{r}\right)$, and the inequality follows since the expression in the first line is a decreasing function of $S_d\left(\frac{1}{r}\right)$. This establishes the first boundary condition in $(A.52)$. Similarly, consider Eq. $(A.51)$ for $\tilde{p}_d = \frac{1+\varepsilon}{r-\varepsilon}$:

$$\begin{aligned} P_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right) &= \frac{1+\frac{1+\varepsilon}{r-\varepsilon}}{1+r}\left(\int_{s^{\min}}^{S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)} s dF_0 + \frac{1-F_0\left(S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)\right)}{1-F_1\left(S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)\right)}\int_{S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)}^{s^{\max}} s dF_1\right) \\ &\leq \frac{1}{r-\varepsilon}\left(\int_{s^{\min}}^{s^{\min}} s dF_0 + \frac{1-F_0\left(s^{\min}\right)}{1-F_1\left(s^{\min}\right)}\int_{s^{\min}}^{s^{\max}} s dF_1\right) = \frac{E_1\left[s\right]}{r-\varepsilon} = \frac{1+\varepsilon}{r-\varepsilon}. \end{aligned}$$

Here, the second line replaces $S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)$ in the first line with $s^{\min} \leq S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)$, and the last equality uses the definition of ε. This establishes the second boundary condition in $(A.52)$ and completes the proof of the theorem.■

Proof of Theorem 9. Recall that the equilibrium is a fixed point of the mapping $P_d(\cdot)$, where $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ is characterized as the solution to the equations in $(A.51)$. Suppose optimism becomes more skewed to the upside. Applying Theorem 3 to the corresponding static economy implies that $P_d(\tilde{p}_d)$ weakly increases for each $\tilde{p}_d$. In particular, the mapping $P_d(\cdot)$ shifts up pointwise. It follows that the fixed point, p_d, weakly increases. Applying Theorem 3 once more implies that $S_d(\tilde{p}_d)$ weakly increases for each $\tilde{p}_d$. Since p_d weakly increases, and since $S_d(\cdot)$ is an increasing function (cf. proof of Theorem 8), it follows that $\bar{s}_d^* = S_d(p_d)$ also weakly increases.

Next consider the share of the speculative component, λ_d. Plugging in the value function, $v(y,s) = s(1+p_d)y$, and using $E_0[s] = 1$ and $E_1[s] = 1+\varepsilon$, Eq. (41) can be rewritten as:

$$\frac{p_d}{1+p_d} = (1-\theta_d)\frac{1}{1+r} + \theta_d\frac{1+\varepsilon}{1+r}.$$

Note also that Eq. (42) can be written as

$$1-\lambda_d = \frac{p^{pdv}(y)}{p(y)} = \frac{1}{p_d}\left((1-\theta_d)\frac{1}{r} + \theta_d\frac{1+\varepsilon}{r}\right).$$

Combining the last two displayed equalities, the share of the speculative component is given by:

$$\lambda_d = 1 - \frac{1+1/r}{1+p_d}$$

Since p_d weakly increases, λ_d also weakly increases.■

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