State-Dependent Probability Distributions in Non Linear Rational Expectations Models*

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Abstract

In this paper, we solve a large class of non-linear rational expectations models with regime switching, i.e. recurring shifts in parameters. The regime-switches and the shocks may follow state-dependent probability distributions. Using a perturbation approach, we first prove that sufficient determinacy conditions - conditions ensuring the existence of a unique stable equilibrium - follow from the corresponding conditions in an exogenous regime-switching model. We provide easily verifiable and sufficient determinacy conditions and a first-order approximation of the solution for purely forward-looking models. Finally, we illustrate our results with a Fisherian model of inflation determination in which monetary policy switches endogenously between a less-active (dovish) and a more-active (hawkish) reaction against inflation. This example highlights that the state-dependent fluctuations of transition probabilities can substantially alter the equilibrium dynamics through economic agents' expectations.

Keywords: Perturbation methods, regime switching, DSGE, indeterminacy, monetary policy.

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1 Introduction

In this paper, we consider a class of non-linear discrete-time rational expectations models with discrete jump processes, describing regime switching, and continuous stochastic processes, the usual "shocks". Both processes can follow state-dependent probability distributions meaning that the distribution of shocks as well as the probabilities of regime switches may depend on the state of the economy. We find sufficient determinacy conditions, i.e. conditions ensuring the existence of a unique stable solution, and provide the first-order Taylor expansion of the solution. We apply our results to a Fisherian model of inflation determination with endogenous regime switching across different monetary policy regimes and we show that the state-dependent fluctuations of transition probabilities can substantially alter the equilibrium dynamics through economic agents' expectations.

A substantial line of empirical studies challenges the common assumption of economic agents' time-invariant behavior (Clarida et al., 2000). Recent papers (e.g., Svennson and Williams, 2009; Farmer et al., 2009b) thus relax the assumption of time-invariant behavior by introducing recurring shifts in structural parameters ; yet, a common approach remains that changes in policy behavior are exogenous.

However, there is a wide range of empirical (Kim et al., 2003) and theoretical (e.g. Mendoza and Yue, 2011, for defaults) reasons to believe that regimes switches may be influenced by macroeconomic fluctuations or, more generally, are endogenous to the model's economic outcomes. Thus, this paper addresses this issue and considers models where shocks and regime changes have state-dependent distributions.

To solve these models, we first extend the perturbation approach to state-dependent probability distributions. By using the perturbation approach, Woodford (1986) infers the stability properties of non-linear stochastic models from their linearized counterpart assuming "small" shocks. In the same way, we link the stability properties of the endogenous regime switching to the exogenous regime switching assuming "weak" endogeneity. Second, we find sufficient determinacy conditions for exogenous regime switching models by generalizing results of Farmer et al. (2009a) to multivariate exogenous regime switching models.

Our contributions are threefold.

First, by applying the Implicit Function Theorem, we show that the existence and the uniqueness of a bounded solution in this class of models rely both on the existence of a unique solution for a simplified, linear model with exogenous regime switching. Furthermore, we formulate the first-order Taylor expansion of the solution.

Second, we provide sufficient and easily verifiable determinacy conditions for linear multivariate exogenous Markov switching models when agents are purely forward-looking. To our knowledge, determinacy results for this type of model do not exist in the literature and we supply them by generalizing Farmer et al. (2009a). We base our conditions on eigenvalue computations that can be thought of as a generalization of Blanchard and Kahn (1980) conditions in a context of regime switching. These conditions also ensure determinacy for the non-linear endogenous regime switching model.

The introduction of endogenous transition probabilities do not alter the determinacy conditions, as we assume that these probabilities do not vary too much. However, the economic dynamics differ substantially. As a consequence, one can expect that estimating a model with endogenous regime switching may result in different point estimates and hence modify our understanding of the business cycle. Our third contribution is to apply our method to a Fisherian model of inflation determination in which the monetary policy rule may change across regimes according to a state-dependent jump process. We find determinacy conditions similar to Farmer et al. (2009a) which extend Davig and Leeper (2007) and we obtain the first-order Taylor expansion of the solution.

We highlight two effects peculiar to the endogenous nature of regime switching.

First, the dynamics in each regime is modified compared to the one that would prevail

if the transition probabilities were constant. Real interest rate shocks are less destabilizing, especially for large shocks. The more likely a switch becomes -the larger a shock is-, the more the expectations of future switches will affect agents' contemporaneous decisions. Thus, the standard inflationary effect of such a shock is weakened by the endogenous expectations formation effect, which tends to decrease the immediate response of inflation. When the real interest rate shock is considerable, the expectations formation effect is so large that it dominates and leads to a hump-shaped Impulse Response Function. This illustrates the importance of such mechanism and the potential bias introduced when assuming constant transition probabilities.

Second, apart from these state-dependent expectations formation effects, there is an additional mechanism since the endogenous fluctuations of probabilities alter the conditional average response of the economy to shocks. We call this second phenomenon the endogenous selection effect. This effect of state-dependent transition probabilities directly impacts the likelihood of each regime. It introduces differences in the average response of the economy to a shock, which could modify the forecasting performance of such models.

Related Literature Recent papers have challenged the empirical validity of models based on time-invariant behavior of economic agents. For instance, several papers analyze the sharp decreases in output and inflation volatility around the mid 80s, the so-called "Great Moderation", by allowing for time-varying economic behavior. Among the competing sources of parameter changes, some papers have allowed for breaks in the variance of structural shocks (Sims and Zha, 2006; Justiniano and Primiceri, 2008; Fernández-Villaverde et al., 2010; Liu et al., 2010), others for shifts in the parameters of monetary policy rules (Clarida et al., 2000; Lubik and Schorfheide, 2004).

Within the context of forward looking economic agents, the possibility of future regimes switches alters agents' current decision rules (Sims, 1982) through what Leeper and Zha (2003) call the *expectations formation effects*. In simple calibrated monetary models with Markov switching monetary policy rules, Davig and Leeper (2007) show that these effects can be relatively large and can substantially modify determinacy conditions (see also Farmer et al., 2009a).

In most of the literature, the expectations formation effects are state-invariant as regime switches are exogenous. However, allowing for endogenous regime switching appears useful in many contexts: monetary policy regime (Davig and Leeper, 2008), financial crises (Coe, 2002), defaults (Mendoza and Yue, 2011), policy regimes (Rothert, 2009), occasionally binding constraints (Christiano and Fisher, 2000), and sudden stops (Calvo, 1998) among many others. In such contexts, we expect the expectations formation effects to depend on the likelihood of future switches and, hence, on the current state of the economy. Davig and Leeper (2008) show that a monetary policy rule that changes when certain endogenous variables intersect specified thresholds leads to substantial and state-dependent expectations formation effects. However, Davig and Leeper's approach relies on numerical methods which imply both a high computational cost and the lack of analytical results.

Foerster et al. (2011) propose a solution technique to solve exogenous Markov switching models by applying a perturbation approach. Davig and Doh (2008) also solve a non-linear Markov switching rational expectations model by linearizing it and then using the undetermined coefficient approach. Nevertheless, none of these papers give explicit arguments to apply the Implicit Function Theorem and prove that there exists a unique "stable" solution of the original non-linear model.

As our method relies on solving linear exogenous Markov switching models, this paper is also closely related to the Markov switching literature. This literature distinguishes two main solution techniques: the undetermined coefficient approach (Blake and Zampolli, 2006; Davig and Leeper, 2007; Svennson and Williams, 2009) and a direct approach (Farmer et al., 2010b). Our approach is closer to the undetermined coefficient approach since both methods

are equivalent when models are linear.

In the context of exogenous Markov switching models, the characterization of the full class of solutions (the existence, the uniqueness and the form of the solution) is already a challenging task. Davig and Leeper (2007) propose a simple determinacy condition for forward-looking Markov switching models. However, Farmer et al. (2010a) have casted doubts on their methods. Because of this controversy, most of the literature has turned to the Mean Square Stability concept (see Farmer et al., 2009b) following the influential book by Costa et al. (2005). This definition of stability is, however, incompatible with the perturbation approach[1] and, hence, does not provide the adequate stability concept for solving non-linear Markov Switching DSGE models with perturbation theory.

Disregarding rational expectations, endogenous changes have been well studied in purely econometric frameworks. Following the seminal paper by Hamilton (1989), Filardo (1994) and Filardo and Gordon (1998) estimate Markov switching regressions with time-varying transition probabilities. More recently, Kim et al. (2003) propose a technique for estimating multivariate models with endogenous regime switching, i.e. where transition probabilities depend on endogenous variables. However, these significant progresses cannot be replicated yet to estimate rational expectations models with endogenous regime switching.

The remainder of the paper is organized as follows. Section 2 presents the class of models we study as well as some notations. Then, we provide our main theoretical results in Section 3. Finally, Section 4 illustrates our findings through an endogenous regime switching Fisherian model of inflation determination.

2 The setup

This section presents the class of models we consider and specifies some notations required for our analysis.

Most of recent rational expectations macroeconomic models with regime switching can be reduced to the following system:

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, z_{t-1}, \gamma v_t)] = 0 \tag{1}$$

when z is a vector of endogenous variables evolving in a bounded closed set F of $\mathbb{R}^n$, v is a multi-dimensional stochastic process evolving in a bounded domain V of $\mathbb{R}^p$ and γ is a scalar. The parameter γ will tend to zero when applying the perturbation approach. Let s_t denoting the regime of the economy at date t and taking values in $\{1, \cdots N\}$ where N is the number of possible regimes. For any i, f_i is a regular function and $\mathbb{E}_t$ is the expectation operator given the information available at time t, namely current and past shocks and regimes.

We denote by $u_t \in U$ the concatenation of the regime and the current shock (s_t, v_t). The set U^∞ represents the set of infinite sequences[2] $u^t = (u_t, u_{t-1}, \cdots)$. $\mathcal{M}(U)$ stands for the set of measures on U and Σ is the sigma field of U generated by the product of the singletons $\{i\}_{i\in\{1,\cdots,N\}}$ and the Borel set of V. We consider a measure μ_γ in $\mathcal{M}(U)$, depending on a policy function ϕ mapping U^∞ to $\mathbb{R}^n$ and a sequence $u^{t-1} \in U^\infty$:

$$\mu_\gamma(s, v, \phi, u^{t-1}) = \sum_{i=1}^{N} h^i(v, \phi, u^{t-1}) p^i_\gamma(\phi, u^{t-1}) \delta_i(s) \tag{2}$$

This measure is a combination of Lebesgue-continuous measures, h^i, and mass-point measures. δ_i denotes the Dirac distribution in i. We suppose that for any i, p^i_γ and h^i are smooth. We focus on this particular class of measures as they behave conveniently and encompass a large class of economic models. We present some useful properties of this class of measures

[1] Applying the Implicit Function Theorem requires Banach spaces

[2] For more details about this formalism, see Woodford (1986).

in appendix A. It is worth noticing that we allow the measure of v to vary across regimes as its probability measure depends on i in the sum. We assume that h^i is normalized:

$$\forall i, \quad \forall \phi, \quad \forall u^{t-1} \int_V h^i(v, \phi, u^{t-1}) dv = 1 \tag{3}$$

In addition, we assume that for all i, p_0^i does not depend on ϕ and we denote them by $p_0^i(u^{t-1})$.

We denote by Σ^∞ the infinite product of sigma field Σ and we consider elements u^t representing infinite histories of a stochastic process such that the conditional probability of u^t conditional on the history u^{t-1} is given by:

$$u^t | u^{t-1} \sim \mu_\gamma(\cdot, \phi, u^{t-1})$$

Then the process u^t can be represented by a probability measure $\pi_\gamma(\phi)$ acting on Σ^∞:

$$\pi_\gamma \quad : \quad \phi \quad \mapsto \quad \mu(\cdot, \phi) = \prod_{k=0}^{\infty} \mu_\gamma(u_{t-k}, \phi, u^{t-k-1}) \tag{4}$$

When γ is equal to 0, the model corresponds to an exogenous regime switching model without shocks. γ thus measures simultaneously the size of the shocks and the degree of endogeneity of regime switching[3] (the slope of the mass-point probabilities).

Now, we introduce the type of solutions we are looking for.

Definition 1. *A stationary rational expectations equilibrium (s.r.e.e.) of model (1) is a* ***continuous*** *function $\phi : U^\infty \to F$ such that:*

1. $||\phi||_\infty = \sup_{U^\infty} \|\phi(u^t)\| < \infty$
2. *If u is a U valued stochastic process associated with the probability measure $\pi_\gamma(\phi)$ then $z_t = \phi(u^t)$ is solution of (1).*

Furthermore, this solution is a steady state if ϕ is constant.

We restrict our analysis to continuous and bounded functions. Thus, we introduce $\mathcal{B}$, the set of functions ϕ on U^∞ such that, for all s, the map $v \mapsto \phi(s, v)$ is continuous and such that $(s, v) \mapsto \phi(s, v)$ is bounded. This definition is more restrictive than the mean-square stability required by Farmer et al. (2007) but necessary since our framework relies on a perturbation approach. In addition, our space of solutions is very close to, but slightly tighter than, the set of essentially bounded functions considered by Woodford (1986). This restriction arises from the state-dependence of the measure μ_γ.

We will prove the existence and the uniqueness of a *s.r.e.e.* when the continuous shocks are small enough and the regime switching is weakly endogenous by applying the Implicit Function Theorem (IFT) to an operator acting on the Banach space of bounded and continuous functions, $\mathcal{B}$.

3 Solving Rational Expectations Models with perturbation approach

In this section, we prove that there exists a unique *s.r.e.e.* of the model with small continuous shocks and weakly endogenous regime switching (i.e. γ small enough) if the underlying exogenous regime switching model admits a unique solution [Theorem 1]. In the absence of regime switching, this result extends Woodford (1986) result by allowing shocks' distribution

[3] We could have distinguished these two dimensions without any substantial modification

to be state-dependent [section 3.2]. Then, we give sufficient existence and uniqueness conditions for a non-linear endogenous regime switching model in a purely forward-looking context [Proposition 1]. Finally, we illustrate this Proposition in the context of linear endogenous regime-switching model and compute the first-order Taylor expansion [Proposition 2].

3.1 General result

Our first result is derived from the Implicit Function Theorem (IFT) in Banach Spaces (see Abraham et al., 1988, and section B.1 in the appendix). To apply this latter, we define an operator $\mathcal{N}$, whose zeros correspond to a *s.r.e.e* of model (1). The operator $\mathcal{N}$ is acting on $\mathcal{B} \times \mathbb{R}$ with values in $\mathcal{B}$ and is such that, for any u^t:

$$\mathcal{N}(\phi, \gamma)(u^t) = \int_U f_{s_t}(\phi(uu^t), \phi(u^t), \phi(u^{t-1}), \gamma v_t)\mu_\gamma(u, \phi, u^t)du \tag{5}$$

As explained in section 2, we assume that the mass-point probabilities do not depend on ϕ when γ is equal to 0. Thus:

$$\mathcal{N}(\phi, 0)(u^t) = \sum_{i=1}^{N} p_0^i(u^t) \int_V f_{s_t}(\phi(is^t, vv^t), \phi(u^t), \phi(u^{t-1}), 0)h^i(v, \phi, u^{t-1})dv, \qquad \forall u^t \in U^\infty \tag{6}$$

This operator corresponds to a model with purely exogenous regime switching and the shocks, v, are sunspot shocks (do not appear in the model, f_{s_t}). One may notice that transition probabilities can depend on past regime and shocks. This thus encompasses Markov Switching models. Besides, even if the shocks, v, are sunspots their probability distribution functions can depend on the equilibrium, ϕ.

We will apply the Implicit Function Theorem to $\mathcal{N}$ to prove the existence of a unique *s.r.e.e.* To apply this theorem, we have to know a particular solution when $\gamma = 0$. The invertibility of $D_\phi\mathcal{N}(\phi_0, 0)$ coincides with the existence of a unique bounded continuous function, h, satisfying (for any bounded ψ):

$$D_\phi\mathcal{N}(\phi_0, 0)(h) = \psi \tag{7}$$

In other words, the invertibility of $D_\phi\mathcal{N}(\phi_0, 0)$ corresponds to the existence of a unique *s.r.e.e*, for any bounded stochastic process, ψ_t, of the following "linearized" regime switching model:

$$\mathbb{E}_t[A(s^{t+1})z_{t+1}] + B(s^t)z_t + C(s^t)z_{t-1} = \psi_t \tag{8}$$

where the shocks u_t follow the probability distribution μ_0, i.e. the regime switches are exogenous while the shocks may follow state dependent probability distributions. The matrices $A(s^{t+1})$, $B(s^t)$ and $C(s^t)$ depend on the derivatives of f_{s_t} and on the regimes history. The coincidence between the invertibility of $D_\phi\mathcal{N}(\phi_0, 0)$ and the uniqueness of a solution of model 8 leads to the following theorem.

Theorem 1. *If there exists a continuous function $\phi_0 : \{1, \cdots, N\}^\infty \to F$ such that*

1. *ϕ_0 is a particular* s.r.e.e. *of the non-linear exogenous regime switching model without shocks ($\mathcal{N}(\phi_0, 0) = 0$)*
2. *The model (8) admits a unique* s.r.e.e. *for any bounded process ψ_t.*

Then there exists γ_0 small enough such that, for any γ smaller than γ_0, there exists a unique s.r.e.e. *of model (5) around ϕ_0. Furthermore, the first-order Taylor expansion of the solution, $\phi(\gamma)$ in γ is given by:*

$$\forall u^t \in U, \phi(\gamma)(u^t) = \phi_0(s^t) + \gamma D_\phi \mathcal{N}(\phi_0, 0)^{-1} D_\gamma \mathcal{N}(\phi_0, 0) + o(\gamma)$$

Proof. This theorem is a direct application of IFT in Banach Space. See appendix B.1 for more details. □

This result shows that the weakly endogenous regime switching model has the same properties as the exogenous one, and that the solutions are close. It derives very general conditions of determinacy from properties of the model with exogenous regime switching.

The first hypothesis can usually be checked by hand while the second hypothesis coincides with Blanchard and Kahn conditions of the underlying linearized model. However, contrary to Woodford (1986), this theorem only gives sufficient conditions of determinacy but is silent on the reciprocal. In addition, the solution, $\phi(\gamma)$, is not necessarily recursive contrary to Jin and Judd (2002).

In most cases, these conditions are difficult to verify. The remainder of this section gives examples for which we can explicitly derive condition 2. of Theorem 1.

3.2 Case I: state-dependent probability distribution in the absence of regime switching

We first consider the model (5) in the absence of regime switching:

$$\mathbb{E}_t[f(z_{t+1}, z_t, z_{t-1}, \gamma v_t)] = 0 \tag{9}$$

where v^t follows a continuous law, $h(v, \phi, v^{t-1})$ ($\mu_\gamma = h$). We assume that h is Lebesgue-continuous and C^1 according to its second component.

Let us assume that the model (9) admits a *steady-state* when $\gamma = 0$. We denote it by $\bar{z}$ and call it the *deterministic steady-state*. Thus, $\bar{z}$ satisfies $f(\bar{z}, \bar{z}, \bar{z}, 0) = 0$. Unexpectedly, the linearized model in $\phi_0 = \bar{z}$ is a well-studied linear rational expectations model with **exogenous shocks** (see Appendix B.3 for more details):

$$\underbrace{\begin{pmatrix} \partial_2 f(\bar{z}, \bar{z}, \bar{z}, 0) & \partial_1 f(\bar{z}, \bar{z}, \bar{z}, 0) \\ I_n & 0 \end{pmatrix}}_{A} Z_t = \underbrace{\begin{pmatrix} -\partial_3 f(\bar{z}, \bar{z}, \bar{z}, 0) & 0 \\ 0 & I_n \end{pmatrix}}_{B} Z_{t-1} + \begin{pmatrix} \Psi_t \\ 0 \end{pmatrix} \tag{10}$$

where the shocks follow a state-invariant distribution, $h(v, \bar{z}, v^{t-1})$ and $Z_t = [z_t' \ \mathbb{E}_t z_{t+1}']'$.

As a consequence, the second condition of Theorem 1 reduces to the determinacy conditions of model (10) for any ψ_t. Blanchard and Kahn (1980) give these determinacy conditions[4]. If all the generalized eigenvalues of the pencil $< A, B >$[5] lie inside the unit circle, there exists a unique *s.r.e.e.* of model (10).

Then, from a direct application of Theorem 1, one can show that if the Blanchard and Kahn conditions for the linearized model in $\bar{z}$ are satisfied, there exists $\gamma_0 > 0$ such that for γ smaller than γ_0 -i.e. for "small" shocks —, the model (9) admits a unique *s.r.e.e.* Furthermore, the first order expansion of this solution exactly coincides with the solution of the linearized model when $\psi_t = \gamma \partial_4 f(\bar{z}, \bar{z}, \bar{z}, 0) v_t$. We give some details in Appendix B.3.

This result generalizes Theorem 2 by Woodford (1986) to the case of shocks with state-dependent probability distributions. Neither the underlying linearized model nor the first-order Taylor expansion of the solution change compared to the exogenous case.

[4] Klein (2000) extends Blanchard and Kahn (1980) when the matrix $\partial_1 f(\bar{z}, \bar{z}, \bar{z}, 0)$ is singular.

[5] When A is invertible, the generalized eigenvalues of $< A, B >$ coincide with the standard eigenvalues of $A^{-1}B$

3.3 Case II: endogenous regime switching in a forward-looking environment

Let us turn to the regime switching model. We consider the following purely forward-looking model:

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, \gamma v_t)] = 0. \tag{11}$$

We assume that the transition probability from regime i to regime j only depends on the past value of endogenous variables, $\phi(u^{t-1})$. Thus, there exists a function p_{ij} with values in $[0, 1]$ such that:

$$\forall u^t \in U^\infty, \qquad p^j_\gamma(\phi, (is^{t-1}, v^t)) = p_{ij}(\gamma, \phi(u^{t-1})) \tag{12}$$

In addition, we assume that the probabilities p_{ij} are smooth (C^1) and constant for $\gamma = 0$ ($p_{ij}(0, .) = \bar{p}_{ij}$). We can check that the implied measure, μ_γ, has all the needed properties described in Section 2.

As in the absence of regime switching, we assume that there exists a solution of the model when there is no shock, ϕ_0. In addition, we suppose that this solution only depends on the current regime, i.e., $\phi_0(is^{t-1}) = \bar{z}_i$ where $(\bar{z}_1, \cdots, \bar{z}_N)$ is solution of the $N \times n$ following equations, for any $i \in \{1, \cdots, N\}$:

$$\sum_{j=1}^{N} \bar{p}_{ij} f_i(\bar{z}_j, \bar{z}_i, 0) = 0$$

In this context, the associated linearized model can be written as:

$$\mathbb{E}_t[A_{s_t s_{t+1}} x_{t+1}] + B_{s_t} x_t = \psi_t \tag{13}$$

where A_{ij} and B_i depend on the first derivatives of f_i in $(\bar{z}_j, \bar{z}_i, 0)$ and are explicitly defined in Equation (24) in Appendix B.4. Finally, we impose that B_i is invertible for any i.

We need to find conditions ensuring the existence of a unique *s.r.e.e.* of model (13) to apply Theorem 1. As in the fixed regime model, we can find such conditions by computing eigenvalues of certain matrices. For a fixed operator norm on $\mathcal{M}_n(\mathbb{R})$, $|||.|||$, we introduce the matrix S_p defined for $p > 1$ by:

$$S_p = \left(\sum_{(k_1, \cdots, k_{p-1}) \in \{1, \cdots, N\}^{p-1}} \bar{p}_{ik_1} \cdots \bar{p}_{k_{p-1}j} |||A_{ik_1} \beta_{k_1}^{-1} \cdots A_{k_{p-1}j} \beta_j^{-1}||| \right)_{ij} \tag{14}$$

and, by convention for $p = 1$,

$$S_1 = \left(\bar{p}_{ij} |||A_{ij} \beta_j^{-1}||| \right)_{ij}$$

We have the following sufficient condition for the existence of a unique *s.r.e.e.* of model (13):

Proposition 1. *If there exists an integer p such that all the eigenvalues of S_p lie inside the unit circle, then the model* (13) *admits a unique* s.r.e.e..

Proof. We construct the solution of model (13) (or the inverse of $D_\phi \mathcal{N}(\phi_0, 0)$) as the sum of an infinite operator series. Then, if this operator series is absolutely convergent, the linearized model admits a unique solution ($D_\phi \mathcal{N}(\phi_0, 0)$ is invertible). Finally, it remains to prove that if, for some $p > 0$, all the eigenvalues of S_p lie inside the unit circle, the series converges. See appendix B.4 for the details. □

This proposition combined with Theorem 1 gives the condition for the existence of a unique *s.r.e.e.* of the non-linear endogenous regime switching model, equations (11) and (12).

This Proposition leads to several remarks. First of all, in the absence of regime switching the condition that S_1 has no explosive eigenvalue exactly corresponds to Blanchard and Kahn (1980) conditions. Second, if the model is univariate ($n = 1$), then checking the eigenvalues of S_1 is enough as $S_p = S_1^p$. Third, the determinacy condition found by Farmer et al. (2009a) in the Fisherian model of inflation determination coincides with our condition when $p = 1$ (more details are provided in Section 4). Thus, this proposition is a generalization of conditions found by Farmer et al. (2009a) to a multivariate Markov switching model. Loosely speaking, we replace the absolute values they use in one-dimension by operator norms. Indeed, the sub-multiplicative property of any operator norms overcomes the non-commutativity of matrices products.

3.4 Case III: endogenous regime switching in a forward-looking and linear environment

When a forward-looking regime switching model is linear, one may solve it for any size of the shocks. Therefore, there is no necessity to assume small shocks. We thus present a refinement of Theorem 1 in case of linear model (f_i is linear for any $i \in \{1, \cdots N\}$). More explicitly, we consider the following model:

$$A_{s_t}\mathbb{E}_t(x_{t+1}) + B_{s_t}x_t + \sigma C_{s_t}v_t = 0 \tag{15}$$

where the probabilities of transitions from regime i to regime j are $p_{ij}(\gamma, \phi(u^{t-1}))$. For simplicity, we assume that shocks, v_t, follow a first order Vectorial Auto-Regressive process:

$$v_{t+1} = \Lambda v_t + \mu_t$$

Where μ follows a centered standardized truncated gaussian (whose p.d.f. is h). In this special case, we can perform a simple perturbation approach assuming weakly endogenous probabilities around the exogenous regime switching model **with** shocks.

Proposition 2. *If condition of Proposition 1 is satisfied, then, for γ small enough, the model (15) admits a unique* s.r.e.e.*, $\phi(\gamma)$, satisfying:*

$$\begin{aligned} \phi(\gamma)(s^t, v^t) &= B_{s_t}^{-1}R_{s_t}v_t + \dots \\ \dots B_{s_t}^{-1} \quad \mathcal{O} & \left[A_{s_t}\sum_{j=1}^{N}\int_V \gamma\partial_1 p_{s_t j}(0, \sigma B_{s_t}^{-1}R_{s_t}(\Lambda v_t + \mu))B_j^{-1}R_j(\Lambda v_t + \mu)h(\mu)d\mu\right] + o(\gamma) \end{aligned} \tag{16}$$

*where $\mathcal{O}$ is an **operator** acting on $\mathcal{B}$ defined in (31) and R_{s_t} is a matrix given by (32) (see Appendix B.5).*

Proof. This Proposition follows from Proposition 1. Proof is given in Appendix B.5. □

The determinacy condition is exactly the same as in Proposition 1. Equation (16) gives the first-order Taylor expansion of the unique *s.r.e.e.* of the model (15). The first term, $B_{s_t}^{-1}Rv_t$, is the exact solution of the exogenous regime switching model ($\gamma = 0$). The second term corresponds to the first-order wedge introduced by the state-dependence of transition probabilities.

This second term can be interpreted as the (first-order) additional expectations formation effect caused by expected fluctuations of transition probabilities. More precisely, for any function $\Psi \in \mathcal{B}$, $\mathcal{O}\Psi$ corresponds to the expected cumulative effect of Ψ from $t + 1$ to ∞

when (s^t, v^t) follows the measure μ_0. This operator is applied to the sum over the regimes of the product between the marginal deviation of the considered transition probability (due to its state-dependence) and the solution of the exogenous regime switching in the associated regime. Thus, the second term in (16) corresponds to the contemporaneous impact of the expected cumulative effect ($\mathcal{O}$) of the marginal effect of changes in probability in p periods, for $p \in \mathbb{N}$.

Practically, the second term can be either computed by hands in simple examples or numerically approximated when the probabilities are too complex (e.g. not polynomial). We compute explicitly this term in the application described in Section 4.

4 A Fisherian model of inflation determination

We apply results of Proposition 2 to study a Fisherian model of inflation determination with endogenous transition probabilities. In this model, the reaction to inflation in the monetary policy rule may change across regimes. When the probabilities are exogenous, Davig and Leeper (2007) and Farmer et al. (2009a) have established determinacy conditions for this model - called the *Long Run Taylor Principle* - saying that a combination between the transition probabilities and the central bank's reaction to inflation parameters has to be greater than one. Applying Proposition 2, we obtain that determinacy conditions are given by the Long Run Taylor Principle, but the dynamics of the solution is slightly more complex. We identify two mechanisms stemming from the state-dependence of transition probabilities: the endogenous expectations formation effect and the endogenous selection effect. First, the expectations formation effects due to the possibility of future switches are state contingent, whereas these effects are constant in an exogenous Markov switching model. Second, the endogenous fluctuations of transition probabilities lead to a state-dependent likelihood of each regime after a shock. This selection effect triggers substantial modifications of Impulse Response Functions compared to those obtained for exogenous regime switching.

4.1 The model

Consider a nominal bond that costs 1 at date t and pays off $1 + i_t$ at date $t + 1$. Then, the asset pricing equation for this bond can be written in log form as:

$$i_t = \mathbb{E}_t(\pi_{t+1}) + r_t \tag{17}$$

where r_t is the ex-ante equilibrium interest rate and evolves as

$$r_t = \rho r_{t-1} + v_t$$

where $\rho < 1$ and v_t is a zero-mean i.i.d bounded process. Monetary policy follows a simplified Taylor rule, adjusting the nominal interest rate in response to inflation, where the reaction to inflation evolves stochastically across regimes,

$$i_t = \alpha_{s_t} \pi_t \tag{18}$$

where s_t is the realized monetary policy regime and takes two values $1, 2$. We use the formalism introduced in section 3.4 and assume that the switching process follows a Markov chain with transition probabilities $p_{ij} = p(s_t = j | s_{t-1} = i)$ depending on past inflation, π_{t-1}. To simplify the solution and the exposition, we focus on probabilities satisfying:

$$p_{ij}(\pi_{t-1}) = \bar{p_{ij}} + \gamma \lambda_{ij} \pi_{t-1}^2 \tag{19}$$

Where λ_{ij} is a parameter reflecting the sensitivity of the probability, p_{ij}, to inflation and γ is the scale parameter. For consistency, $\sum_i \bar{p_{ij}} = 1$ and $\sum_i \lambda_{ij} = 0$. Furthermore, we

assume that γ is small enough to guarantee that the probabilities remain between 0 and 1 (this obviously requires that $\bar{p_{ij}} \in (0,1)$.).

As mentioned by Filardo (1994), endogenous regime switching as exemplified by equation (19) allows for state-dependent duration of each regime. In our example, if λ_{11} is negative, the average duration of regime 1 decreases when inflation deviates from its steady-state.

4.2 The solution

We assume that $|\alpha_1| > p_{11}$ and $|\alpha_2| > p_{22}$ and we apply Proposition 2 to find determinacy conditions of the Fisherian model and the first-order Taylor expansion of the solution.

Proposition 3. *If the policy parameters satisfy the following determinacy condition:*

$$|\alpha_1|.|\alpha_2| + p_{22}(1 - |\alpha_1|) + p_{11}(1 - |\alpha_2|) > 1 \tag{20}$$

Then there exists a unique s.r.e.e. for γ small enough and the solution satisfies:

$$\pi_t = -\frac{r_t \Lambda_{s_t}}{\alpha_t} + \rho\gamma[c_{s_t} r_t^3 + d_{s_t} \frac{var(v)}{1-\rho^2} r_t] + o(\gamma) \tag{21}$$

Where c_{s_t}, d_{s_t} and Λ_{s_t} are regime-dependent scalars (see Appendix C for their expressions)

Proof. This Proposition is an application of Propositions 1 and 2. See appendix C for the complete proof. □

As emphasized by Theorem 1, the determinacy conditions coincide with those of the exogenous Markov switching model. Thus, condition (20) is similar to determinacy conditions by Farmer et al. (2009a) and extends those obtained by Davig and Leeper (2007). The interpretation of equation (20) is similar to the *long-run Taylor principle* by Davig and Leeper (2007) : there may exist a unique *s.r.e.e.* even if policy deviates from the Taylor principle "substantially for brief periods or modestly for prolonged periods".

Equation (21) gives the first-order Taylor expansion of inflation with respect to the scale parameter. The first term exactly coincides with the solution of the model when probabilities are constant. The second term results from the expectations of the third raw moments of the real interest rate shock. Generically, the higher the variance of the real interest rate shock is, the larger the state-dependence of probabilities matters.

Let us consider a central bank that can switch between an active monetary policy regime ($\alpha_2 = 3$) and a passive monetary regime in which monetary authority responds less than one to one to inflation ($\alpha_1 = 0.9$). Furthermore, we assume the higher the deviation from the steady-state inflation is, the more likely the central bank is to choose the active regime ($\lambda_{11} < 0$). The parameters are calibrated for an illustrative purpose to deliver plausible Impulse Response Functions (see Table 1 in Appendix D). In this calibration, a deviation of 1% (−1%) from the steady-state level of inflation decreases the probability of remaining in regime 1 by −0.4%. In this case, the first-order Taylor expansion of inflation (in γ) is:

$$\text{When } s_t = 1, \qquad \tilde{\pi}_t = (6.15 - 0.15\gamma)\tilde{r}_t - 75 * 10^2 \gamma \tilde{r}_t^3 \tag{22}$$

$$\text{When } s_t = 2, \qquad \tilde{\pi}_t = (0.60 - 0.004\gamma)\tilde{r}_t - -1.4 * 10^2 \gamma \tilde{r}_t^3 \tag{23}$$

As we could expect, the responses to a real interest rate shock in both regimes are smaller than in the exogenous regime-switching ($\gamma = 0$) as the probability of switching toward the aggressive regime increasingly anchors the expectations. In what follows, we discuss two new features of the state-dependent regime switching equilibrium in comparison with the standard exogenous Markov switching model as studied in Davig and Leeper (2007): the endogenous expectations formation effects and the endogenous selection effect.

4.3 Endogenous Expectations Formation Effects

In this section, we analyze the impact of the state-dependence of transition probabilities on expectations. In exogenous Markov switching models, Leeper and Zha (2003) and Davig and Leeper (2007) show that the possibility of future switches alters the current decisions of economic agents following the suggestion by Sims (1982). This mechanism is known as the Expectations Formation Effects. Higher probabilities of switching from one regime to another amplify these effects. Therefore, one may expect that these effects become state-dependent in an endogenous regime switching framework.

To only focus on these endogenous expectations formation effects, we study Impulse Response Functions of inflation to a real interest rate shock in each regime. We plot these functions in figure 1. The left graph depicts the response to a one-standard deviation shock while the right graph displays the inflation response to a ten-standard deviation shock. The blue (red resp.) line represents the response when monetary policy is active (passive resp.). In dashed thin line, we display the response of r_t to a real interest shock v_t. We gather all the responses for different sizes of shock in figure 2. In the left graph, in blue (in the right graph, in red resp.), we represent the ratio between the contemporaneous response to a shock and the size of the shock in the active (passive resp.) regime. In dashed thick lines, we represent the maximal response divided by the size of the shock in each regime. Finally, in dashed thin line, we plot these multipliers when probabilities are constant. From figures 1 and 2, we draw three stylized facts.

First, whatever the size of the shock, the multipliers in both regime are strictly lower than those of the corresponding exogenous Markov switching model (see figure 2). This means that the threat of switching toward a more aggressive regime when the inflation is higher helps stabilizing the economy in both regimes.

Second, there is a wedge between the exogenous solution and the endogenous one due to the uncertainty around the future values of shocks. This wedge is illustrated by the little difference between the dashed line and the red line when shocks are close to zero and by the variance term in equation (21). This constant difference between the exogenous multiplier and the endogenous has a significant impact particularly when the shocks are small.

Third, the larger the shock (in absolute value), the lower the multiplier. In the exogenous case, the size of the shock proportionally shifts the response of inflation as the solution is linear (the multiplier is constant, see figure 2). In the endogenous regime switching, a larger shock leads to a relatively smaller response of inflation. The initial response to a shock divided by the size of the shock is quadratic in the size of the shock (figure 2). We notice that the symmetric shape is a consequence of the quadratic term[6] in (19), and stems from the cubic term in equation (21). In other words, the impact of the expectations is higher when the shock is larger. Indeed, a larger shock triggers larger expectations of future switches toward the active monetary policy regime, and this tends to stabilize the expectations. This endogenous expectations formation effect can be so huge that it also alters the inflation responses in the active monetary policy regime itself. Indeed, in such a situation, even if the probability of switching to the passive regime has not changed, economic agents internalize that if the economy switches to the passive monetary regime, the probability of staying in this regime has decreased. Consequently, the multiplier in the active regime also depends on the size of the shocks.

Fourth, when the size of the shock is very large, it generates hump-shaped inflation response (see figure 1). Actually, the cubic polynomial (22) admits a maximum, r_M. If $r_t > r_M$, the response to the shock first increases then decreases, leading to a hump-shape. The size of this maximal response does not depend on the size of the shocks and this feature differs from hump-shaped responses in linear rational expectations models where the maximum depends on the size of the shock.

[6] Assuming linear probabilities leads to an asymmetric behavior

The causes of the hump-shape and of the decreasing multiplier are the combination of two mechanisms. First, in an exogenous regime switching, a fall in r_t corresponds to an exogenous decrease in inflation (due to less demand). Second, a fall in the shock, r_t, leads also to increase the probability of remaining in the passive monetary policy regime and thus tends to increase inflationary pressures due to the expectations formation effects. While the first impact is linear, the second is not. For small shocks, the first direct mechanism is preeminent and explains the monotonic decrease of inflation in reaction to shocks. The larger the shock, the more important the second mechanism. Hence, when the size of the shock increases, their relative impact decreases. From a certain threshold, the second mechanism - the threat to switch toward a more aggressive monetary policy - becomes predominant explaining the hump-shape.

Finally, we argue that endogenous regime switches triggers endogenous expectations formation effects which are absent in exogenous regime switching models. This new mechanism seems particularly desirable in a large number of economic issues such as occasionally binding constraints, zero lower bound or defaults.

4.4 Endogenous Selection Effect

We now turn to the direct implication of the state-dependence of transition probabilities. In the exogenous Markov switching model, once computed the response in both regimes, the simulation of trajectories is straightforward as probabilities are constant. In such a context, the conditional average IRF is thus simply the weighted average of the IRF of each regime. This is no more the case when probabilities are state dependent.

We represent in figure 3 the expected impulse response function to a real interest rate shock (in red) for small and large shocks assuming that the initial regime is the passive one. We also plot the expected response (in blue) considering a model in which inflation can switch between the two values given by equations (22) and (23) according to a Markov process with constant transition probabilities ($\bar{p}_{11}$ and $\bar{p}_{22}$). The difference between the two curves is only a consequence of a selection effect due to endogenous fluctuations of transition probabilities. In addition, we plot the relative difference between the cumulated IRF for fixed probabilities and the one for endogenous probabilities (figure 4). Intuitively, this difference measures the gap between the two areas described in figure 3 for different sizes of shock. We will refer to this gap as the selection effect.

When shocks are small, the selection effect is negligible (figures 3 and 4). The larger the shock, the larger this selection effect. This finding is consistent with the fact that transition probabilities are quadratic and hence more sensitive to large inflation deviations and thus to large shocks. After a large shock (but not too large), inflation increases in both regimes, hence the likelihood of the active monetary policy regime increases and the expected response, is thus closer to the response of this active regime than to the other. If the shock is very large, since the maximal hump-shaped response does not depend on the size of the shock, the selection effect is not increasing anymore, leading to a reversal in the evolution of this effect. Figure 4 illustrates the cubic behavior of this effect.

Finally, in our application, we emphasize a direct implication of the state-dependence on the likelihood of each regime; this leads to complex behavior of expected IRF through what we call the selection effect. This mechanism is already observed in endogenous regime switching model without rational expectations.

5 Conclusion

In this paper, we study non-linear rational expectations models with regime switching. We allow both regime switches and shocks to follow state-dependent probability distributions. In such models, we show how to use the perturbation approach and prove that determinacy

conditions can be derived from the corresponding conditions in an exogenous regime switching model. In addition, we provide new determinacy conditions for forward-looking rational expectations models with exogenous Markov-switching. Combining these two results leads to sufficient determinacy conditions for forward-looking models with endogenous regime-switching.

We apply our findings to a model of inflation determination with monetary policy following a Markov-switching process between a dovish regime and a hawkish regime. Furthermore, we assume that the hawkish regime is more likely when inflation deviates from its targeted level. The endogeneity does not matter for the determinacy conditions as we assume small fluctuations of probabilities. However, as expectations of future switches vary across regimes, they change the economic dynamics. The effects of the expectations increase with the likelihood of a switch. This mechanism tends to stabilize the economy, particularly in case of large shocks.

Evidences on a state-dependent expectation effect highlight the theoretical importance of endogenous regime switching.

APPENDIX

A Preliminary: measures μ_γ

We consider a stochastic process, $u = [s, v]$ with the following conditional probability distribution:

$$\mu_\gamma(s, v, \phi, u^{t-1}) = \sum_{i=1}^{N} h^i(v, \phi, u^{t-1}) p^i_\gamma(\phi, u^{t-1}) \delta_i(s)$$

Where we assume that:

- p^i_γ is C^1 according to its first argument and $\gamma \mapsto p^i_\gamma$ is C^1.
- For any $\phi \in \mathcal{B}$ and $u^{t-1} \in U^\infty$, $h^i(\cdot, \phi, u^{t-1})$ is an integrable function on V. h^i is also C^1 according to its second argument.
- For any $u^t \in U^\infty$, $\phi \mapsto h^i(v, \phi, u^{t-1})$ is C^1.

The measure $\mu_\gamma(\cdot, \phi, u^{t-1})$ can be seen as a continuous linear form on $\mathcal{B}$ and we write abusively $\int_U \mu_\gamma(u, \phi, u^{t-1}) \Psi(u) du =< \Psi, \mu_\gamma >$ even if this measure is not Lebesgue-continuous in general.

Lemma 1. *1. For any $\Psi \in \mathcal{B}$, for any $\gamma \in [0, 1]$ and for any $u^{t-1} \in U^\infty$, $\phi \mapsto \int_U \Psi(u) \mu_\gamma(u, \phi, u^{t-1}) du$ is C^1.*
We abusively denote by $\int_U \Psi(u) D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) du$ its differential at ϕ applied to $\hat{\phi}$.

2. It exists $C > 0$, such that for any Ψ in $\mathcal{B}$ for any $\gamma \in [0, 1]$, for any $u^{t-1} \in U^\infty$, for any ϕ and $\hat{\phi}$ of norm equal to 1,

$$|\int_U \Psi(u) D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) du| \leq C \|\Psi\|_\infty$$

3. For any $\Psi \in \mathcal{B}$, for any ϕ and for any $u^{t-1} \in U^\infty$, $\gamma \mapsto \int_U \Psi(u) \mu_\gamma(u, \phi, u^{t-1}) du$ is $C^1([0, 1])$.

Proof. We first check that μ_γ satisfies 1. Fix $\Psi \in \mathcal{B}$, $\gamma \in [0, 1]$ and $u^{t-1} \in U^\infty$, we compute:

$$< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >= \gamma \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V h^i(v, \phi, u^{t-1}) \Psi([i \quad v]) dv$$

$\phi \mapsto< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >$ is derivable; indeed, this function is the sum of the product of derivable function and an integral which is C^1 by dominated convergence theorem.
Then, we check that 2. is satisfied.

$$< \Psi, D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) >= \gamma \sum_{i=1}^{N} D_\phi p^i_\gamma(\phi, u^{t-1})(\hat{\phi}) \int_V D_\phi h^i(v, \phi, u^{t-1})(\hat{\phi}) \Psi([i \quad v]) dv$$

and

$$| < \Psi, D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) > | \leq (\sum_{i=1}^{N} \sum_{i=1}^{N} \sup |||D_\phi p^i_\gamma(., .)||| \sup ||||D_\phi h_\gamma(., ., .)||||) \|\Psi\|_\infty$$

Finally, we verify 3.

$$< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >= \gamma \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V h^i(v, \phi, u^{t-1}) \Psi([i \quad v]) dv$$

As we suppose that for any $i \in [1, N]$, $\gamma \mapsto p^i_\gamma$ is C^1 then $\gamma \mapsto< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >$ is also C^1. □

These properties imply that the model as a whole is smooth enough to apply Implicit Function Theorem. Obviously, when measures are state-invariant, these two properties are immediate for any measures. We notice that point 2. can be interpreted as the fact that $D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi})$ is a distribution of order 0 uniformly bounded. Point 3. guarantees that the measures μ_γ are a C^1 path between μ_0 and μ_1 and hence we can apply perturbation method around $\gamma = 0$.

B Proof of Theorem 1 and Propositions 1 and 2

B.1 Proof of Theorem 1

In this part, we prove Theorem 1. The proof is a consequence of implicit function theorem applied to operator $\mathcal{N}$. First, we recall the Implicit Function Theorem(IFT).

Theorem 2. *[Abraham et al. (1988)] Let E, F, G be 3 Banach spaces, let $U \subset E, V \subset F$ be open and $f : U \times V \to G$ be C^r, $r \geq 1$. For some $x_0 \in U$, $y_0 \in V$ assume $D_y f(x_0, y_0) : F \to G$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique C^r map $g : U_0 \times W_0 \to V$ such that, for all $(x, w) \in U_0 \times W_0$*

$$f(x, g(x, w)) = w$$

Thus, we check that:

1. $\mathcal{B}$ with the norm $\|\|_\infty$, and $\mathbb{R}$ with $||$ are Banach spaces.
2. $\mathcal{N}$ is C^1 on $\mathcal{B} \times]-M, M[$.
3. ϕ_0 satisfies $\mathcal{N}(\phi_0, 0) = 0$.
4. $D_\phi \mathcal{N}(\phi_0, 0)$ is invertible.

The first point is immediate, $\mathcal{B}$ with the norm $\|\|_\infty$ is a Banach space as a product of Banach spaces. Point 2. results from the following lemma.

Lemma 2. *$(\phi, \gamma) \mapsto \mathcal{N}(\phi, \gamma)$ is C^1 for $\phi \in \mathcal{B}$ and $\gamma \in]-M, M[$*

Proof. For any $\phi \in \mathcal{B}$, the function $\gamma \mapsto \mathcal{N}(\phi, \gamma)$ is C^1 by regularity of p^i_γ and f. For the differentiability in ϕ, we check that $\Phi \mapsto \mathcal{N}(\phi, \gamma)$ is differentiable, with continuous differential:

$$\mathcal{N}(\phi, \gamma) = \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \phi) dv$$

$\phi \mapsto \int_V f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \phi) dv$ is differentiable by regularity of f, h_i, and Lebesgue's dominated convergence Theorem. It results from the differentiability of $\phi \mapsto p^i_\gamma(\phi, u^{t-1})$ that $\phi \mapsto \mathcal{N}(\phi, \gamma)$ is differentiable and moreover:

$$D_\phi \mathcal{N}(\phi, \gamma) H = \sum_{i=1}^{N} \partial_1 p^i_\gamma(\phi, u^{t-1}) H(s^{t-1}, v^{t-1}) \int_V f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \phi) dv$$

$$+ \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V \partial_1 f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(is^t, vv^t) h_i(v, \phi) dv$$

$$+ \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V \partial_2 f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^t, v^t) h_i(v, \phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\phi, u^{t-1}) \int_V \partial_3 f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^{t-1}, v^{t-1}) h_i(v, \phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\phi, u^{t-1}) \int_V f(\phi(is^t, vv^t), \phi(s^t, v^t), \phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^{t-1}, v^{t-1}) \partial_2 h_i(v, \phi) H(s^{t-1}, v^{t-1}) dv$$

Thus, $(\phi, \gamma) \mapsto D_\phi \mathcal{N}(\phi, \gamma) H$ is continuous on $\mathcal{B} \times]-M, M[$. □

Points 3. and 4. result from assumptions of Theorem 1. We end the section with the following result, showing that the regularity of μ_γ implies that the differential of the operator behaves as the differential of an operator where the probabilities are exogenous.

Lemma 3. *Under assumptions of Theorem 1, the differential $D_\phi \mathcal{N}(\phi_0, 0)$ satisfies:*

$$D_\phi \mathcal{N}(\phi_0, 0) H =< \partial_1 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(\cdot S^t, \cdot v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > +$$
$$< \partial_2 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^t, v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > +$$
$$< \partial_3 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^{t-1}, v^{t-1}), \mu_0(\cdot, \phi_0, u^{t-1}) >$$

Proof. We compute:

$$D_\phi \mathcal{N}(\phi_0, 0) H =< \partial_1 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(\cdot S^t, \cdot v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > +$$
$$< \partial_2 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^t, v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > +$$
$$< \partial_3 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^{t-1}, v^{t-1}), \mu_0(\cdot, \phi_0, u^{t-1}) >$$
$$+ < f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0), D_\phi \mu_0(u, \phi_0, u^{t-1}) H(s^{t-1}, v^{t-1}) >$$

Since ϕ_0 does not depend on v, the last term is zero. □

B.2 Notations

We introduce some notations, useful for the following. Fix a function $\phi_0 \in \mathcal{B}$, we define the operators $\mathcal{F}_i$, for $i \in \{1, \cdots N\}$ and $\mathcal{L}$ on $\mathcal{B}$.

$$\mathcal{F}_i : H \mapsto ((s^t, v^t) \mapsto \int_V H(is^t, vv^t) h_i(v, \phi_0, s^t, v^t) dv)$$

$$\mathcal{L} : H \mapsto ((s^t, v^t) \mapsto H(s^{t-1}, v^{t-1})$$

Equation (3) imply that F_i and L have the following straightforward properties.

1. $\mathcal{F}_i \mathcal{L} = \mathbf{1}$
2. $|||\mathcal{F}_i||| = 1$ and $|||\mathcal{L}||| = 1$

Point 1. and point 2. are classical results in theory of operators on sequences. The first result is obtained by straightforward computation. The second follows from the fact that:

$$\forall H \in \mathcal{B}, \qquad \|\mathcal{L} H\| = \|H\|, \qquad \|\mathcal{F}_i H\| \leq \|H\|$$

and the last inequality is an equality if H is constant.

B.3 State-dependent probability distribution in the absence of regime switching: some details

This part is devoted to the study of model (9). We show that $\mathcal{N}$ satisfies points 1. and 2. of Theorem 1. Here, there is no discrete part, thus we omit the dependence in s_t and the indexation in i. The function ϕ_0 is the constant $\phi_0(v^t) = \bar{z}$. By construction, ϕ_0 satisfies:

$$\mathcal{N}(\phi_0, 0) = f(\bar{z}, \bar{z}, \bar{z}, 0) = 0$$

We introduce the operator $\mathcal{F}$ associated to ϕ_0. Due to Lemma 2, $\mathcal{N}$ is differentiable and according to Lemma 3, we compute:

$$D_\phi \mathcal{N}(\phi_0, 0)h = \partial_1 f(\bar{z}, \bar{z}, \bar{z}, 0)\mathcal{F}h + \partial_2 f(\bar{z}, \bar{z}, \bar{z}, 0)h + \partial_3 f(\bar{z}, \bar{z}, \bar{z}, 0)\mathcal{L}h$$

Thus, the model is exactly similar to the models studied in Woodford (1986) and Klein (2000). Assume that Blanchard and Kahn conditions are satisfied for the linearized model (see section 3.2), then $D_\phi \mathcal{N}(\phi_0, 0)$ is invertible. Precisely, we consider the pencil (A, B) defined in equation (10) and introduce its real generalized Schur decomposition, following Klein (2000): there exist unitary matrices Q and Z, quasi triangular matrices T and S such that:

$$A = QTZ \quad \text{and} \quad B = QSZ$$

Furthermore, we rank the generalized eigenvalues such that $|T_{ii}| > |S_{ii}|$ for $i \in [1, n]$ and $|S_{ii}| > |T_{ii}|$ for $i \in [n+1, 2n]$ which is possible if and only if the number of explosive generalized eigenvalues is n (Blanchard and Kahn, 1980; Klein, 2000). In this case, $D_\phi \mathcal{N}(\phi_0, 0)$ is invertible and

$$D_\phi \mathcal{N}(\phi_0, 0)^{-1} = (1 + Z_{22}^{-1} Z_{21} L)^{-1} Z_{22}^{-1} (1 - S_{22}^{-1} T_{22} P)^{-1} S_{22}^{-1} Q'_{12}$$

where, for a matrix $M \in \mathcal{M}_{2n}(\mathbb{R})$, we write M by blocks of $\mathcal{M}_n(\mathbb{R})$:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

B.4 Proof of Proposition 1

In this section, we prove Proposition 1. We compute:

$$\mathcal{N}(\phi, 0)(s^t, v^t) = \sum_{j=1}^{N} \bar{p}_{s_t j} \int_V f_{s_t}(\phi(js^t, vv^t), \phi(s^t, v^t), 0) h_{s_t}(v, \phi, u^t) dv$$

We define the function ϕ_0 such that:

$$\forall i \in \{1, \cdots N\}, \qquad \phi_0(s^t, v^t) = \bar{z}_{s_t}$$

According to assumption 3., the function ϕ_0 satisfies :

$$\mathcal{N}(\phi_0, 0) = 0$$

We compute now $D_\phi \mathcal{N}(\phi_0, 0)$, using Lemma 3, we have:

$$D_\phi \mathcal{N}(\phi_0, 0)h(s^t) = \sum_{j=1}^{N} \bar{p}_{s_t j} \partial_1 f_{s_t}(\bar{z}_j, \bar{z}_{s_t}, 0) \mathcal{F}_j h$$

$$+(\sum_{j=1}^{N}\bar{p}_{s_tj}\partial_2 f_{s_t}(\bar{z}_j,\bar{z}_{s_t},0))h$$

Introducing A_{ij} and β_i defined for $(i,j)\in\{1,\cdots,N\}^2$ by:

$$\beta_i=\sum_{j=1}^{N}\bar{p}_{ij}\partial_2 f_i(\bar{z}_j,\bar{z}_i,0)\quad \texttt{and}\quad A_{ij}=\partial_1 f_i(\bar{z}_j,\bar{z}_i,0) \tag{24}$$

We get that:

$$D_\phi\mathcal{N}(\phi_0,0)h=\sum_{j=1}^{N}\bar{p}_{s_tj}A_{s_tj}\mathcal{F}_j+\beta_{s_t}$$

Let Ψ a function in $\mathcal{B}(U^\infty)$ and consider the equation $D_\phi\mathcal{N}(\phi_0,0)h=\Psi$. Then, for all $i\in\{1,\cdots N\}$,

$$D_\phi\mathcal{N}(\phi_0,0)h=\sum_{j=1}^{N}\bar{p}_{s_tj}A_{s_tj}\mathcal{F}_jh+\beta_{s_t}h=\Psi$$

This relation implies that for any $P\geq 2$

$$\begin{aligned} h &= \beta_{s_t}^{-1}\Psi-\beta_{s_t}^{-1}\sum_{p=2}^{P}(-1)^p\sum_{\{s_2,\cdots,s_p\}\in\{1,\cdots,N\}^{p-1}}\bar{p}_{s_ts_2}A_{s_ts_2}\beta_{s_2}^{-1}\cdots\bar{p}_{s_{p-1}s_p}A_{s_{p-1}s_p}\beta_{s_p}^{-1}\mathcal{F}_{s_2}\mathcal{F}_{s_3}\cdots\mathcal{F}_{s_p}\Psi \\ &+ (-1)^P\beta_{s_t}^{-1}\sum_{\{s_2,\cdots,s_{P+1}\}\in\{1,\cdots,N\}^P}\bar{p}_{s_ts_2}A_{s_ts_2}\beta_{s_2}^{-1}\cdots\bar{p}_{s_{p-1}s_p}A_{s_{p-1}s_p}\beta_{s_p}^{-1}\mathcal{F}_{s_2}\cdots\mathcal{F}_{s_P}\mathcal{F}_{s_{P+1}}h \end{aligned} \tag{25}$$

We then define the operator series of general term, $\mathcal{A}_p$:

$$\mathcal{A}_p\ :\phi\mapsto\left((s^t,v^t)\mapsto\sum_{s_2,\cdots,s_p}(-\bar{p}_{s_ts_1}A_{s_ts_1})\beta_{s_1}^{-1}\cdots(-\bar{p}_{s_{p-1}s_p}A_{s_{p-1}s_p})\beta_{s_p}^{-1}\mathcal{F}_{s_2}\cdots\mathcal{F}_{s_{p-1}}\mathcal{F}_{s_p}\phi\right) \tag{26}$$

and by convention $\mathcal{A}_1=\mathbf{1}$.

If $\sum\mathcal{A}_p$ converges, then, the third member of equation (25) tends to 0 when p tends to ∞ and the second member converges in $\mathcal{B}$. Thus, h is uniquely defined for any $\psi\in\mathcal{B}$ by:

$$h(s^t,v^t)\quad=\quad\beta_{s_t}^{-1}F_{s_t}\Psi-\beta_{s_t}^{-1}\sum_{p=2}^{\infty}\mathcal{A}_p\Psi=\beta_{s_t}^{-1}\sum_{p=1}\mathcal{A}_p\Psi \tag{27}$$

This results proves that for any $\Psi\in\mathcal{B}(U^\infty)$, we have found a unique solution h such that:

$$D_\phi\mathcal{N}(\phi_0,0)h=\Psi$$

Thus, $D_\phi\mathcal{N}(\phi_0,0)$ is invertible and moreover,

$$D_\phi\mathcal{N}(\phi_0,0)^{-1}\Psi=\beta_{s_t}^{-1}\sum_{p=1}\mathcal{A}_p\Psi \tag{28}$$

Now, we consider the matrix S_p defined by

$$S_p=\left(\sum_{(k_1,\cdots,k_{p-1})\in\{1,\cdots,N\}^{p-1}}|||\bar{p}_{ik_1}A_{ik_1}\beta_{k_1}^{-1}\cdots\bar{p}_{k_{p-1}j}A_{k_{p-1}j}\beta_j^{-1}|||\right)_{ij}$$

Fix p such that the eigenvalues of S_p lie inside the unit circle ; we will show that $\sum \mathcal{A}_p$ is absolutely convergent. We define :

$$\tilde{A}_{ij} = \bar{p}_{ij} A_{ij}$$

For any $(q,r) \in \mathbb{N} \times \{0, \cdots, p-1\}$, we use sub-multiplicativity of norm $|||\cdot|||$ and compute, for $n = pq + r$:

$$\sum_{\{s_2,\cdots,s_n\}\in\{1,\cdots,N\}^{n-1}} |||\tilde{A}_{ik_1}\beta_{k_1}^{-1}\cdots\tilde{A}_{k_{p-1}j}\beta_j^{-1}||| \leq$$

$$\sum_{\{s_2,\cdots,s_{pq}\}\in\{1,\cdots,N\}^{pq-1}} |||\tilde{A}_{is_2}\beta_{s_2}^{-1}\cdots\tilde{A}_{s_{p-1}s_p}\beta_{s_p}^{-1}|||\cdots|||\tilde{A}_{s_{p(q-1)}s_{p(q-1)+1}}\beta_{s_{p(q-1)}}^{-1}\cdots\tilde{A}_{s_{pq-1}s_{pq}}\beta_{s_{pq}}^{-1}|||\times$$

$$\sum_{\{s_2,\cdots,s_r\}\in\{1,\cdots,N\}^{r-1}} |||\tilde{A}_{s_{pq}s_2}\beta_{s_2}^{-1}\cdots\tilde{A}_{s_{r-1}s_r}\beta_{s_r}^{-1}|||$$

We find an upper bound for both terms of the previous inequality. Concerning the second term, there exists $C > 0$,such that for any $r \in \{0, \cdots, p-1\}$,

$$\sup_{i\in\{1,\cdots,N\}} \sum_{\{s_2,\cdots,s_r\}\in\{1,\cdots,N\}^{r-1}} |||\tilde{A}_{s_i s_2}\beta_{s_2}^{-1}\cdot\tilde{A}_{s_{r-1}s_r}\beta_{s_r}^{-1}||| < C \tag{29}$$

We rewrite the first term as:

$$\sum_{\{s_2,\cdots,s_{pq}\}\in\{1,\cdots,N\}^{pq-1}} |||\tilde{A}_{is_2}\beta_{s_2}^{-1}\cdots\tilde{A}_{s_{p-1}s_p}\beta_{s_p}^{-1}|||\cdots|||\tilde{A}_{s_{p(q-1)}s_{p(q-1)+1}}\beta_{s_{p(q-1)}}^{-1}\cdots\tilde{A}_{s_{pq-1}s_{pq}}\beta_{s_{pq}}^{-1}|||$$

$$= \sum_{s_{pq}\in\{1,\cdots,N\}} (S_p^q)_{is_{pq}} \tag{30}$$

Combining (29) and (30) leads to

$$|||\mathcal{A}_n||| < \sup_i \sum_{j=1}^{N} (S_p^q)_{ij}$$

Then, denoting by $[x]$ the integer part of a real number x, we obtain[7] :

$$|||\mathcal{A}_n||| \leq C|||(S_p)^{\left[\frac{n}{p}\right]}|||_\infty$$

Since all the eigenvalues of S_p lie inside the unit circle, due to the Gelfand's formula (Gelfand, 1941), for any matrix norm,

$$\lim_{q\to\infty} |||S_p^q|||^{1/q} = \rho < 1$$

This implies that $\lim_{n\to+\infty} |||S_p^{\left[\frac{n}{p}\right]}|||^{1/n} = \sqrt[p]{\rho} < 1$. Finally, using the Cauchy rule, the series $\sum |||(S_p)^{\left[\frac{n}{p}\right]}|||_\infty$ is convergent and thus $\sum \mathcal{A}_n$ is absolutely convergent.

[7] We recall that for a matrix $M \in \mathcal{M}_N(\mathbb{R})$, $|||M|||_\infty = \sup_{i\in\{1,\cdots,n\}} \sum_{j=1}^{N} |M_{ij}|$

B.5 Proof of Proposition 2

In this section, we prove Proposition 2. The proof relies on a refinement of the method used in Proposition 1. We first compute the solution when $\gamma = 0$. Since the model is completely linear, the solution is defined for any γ. Then we solve the model for a small γ by perturbation around the model with exogenous probabilities.

We consider the following model:

$$A_{s_t}\mathbb{E}_t(x_{t+1}) + B_{s_t}x_t + \sigma C_{s_t}v_t = 0$$

We assume that for any $i \in \{1, \cdots, N\}$, the matrices B_i are invertible.

In this case, the operators $\mathcal{A}_p$ defined in equation (26) satisfy, for $p > 0$:

$$\mathcal{A}_p \ : \phi \mapsto ((s^t, v^t) \mapsto \sum_{s_2,\cdots,s_p} (-\bar{p}_{s_t s_2} A_{s_t}) B_{s_t}^{-1} \cdots (-\bar{p}_{s_{p-1} s_p} A_{s_p}) B_{s_p}^{-1} \mathcal{F}_{s_2} \cdots \mathcal{F}_{s_{p-1}} \mathcal{F}_{s_p} \phi)$$

Defining $\Psi_0(s^t, v^t) = \sigma C_{s_t} v_t$ and using the computations in section B.4, we get that the solution of the model is given by :

$$\phi(s^t, v^t) = \mathcal{O}\Psi_0$$

where:

$$\mathcal{O} = \sum_{p=1}^{\infty} \mathcal{A}_p \tag{31}$$

First, we compute

$$\mathcal{A}_p \Psi_0 = \sigma \sum_{s_2,\cdots,s_p} (-\bar{p}_{s_t s_2} A_{s_t}) B_{s_t}^{-1} \cdots (-\bar{p}_{s_{p-1} s_p} A_{s_p}) B_{s_p}^{-1} C_{s_p} \Lambda^p v_t$$

Defining the matrix $P \in \mathcal{M}_{nN}(\mathbb{R})$ by blocks as:

$$P_{ij} = p_{ij} A_j (B_j)^{-1}$$

$$\mathcal{A}_p \phi_0 = (P^p)_{s_t} \times C \times \Lambda^p$$

where $C = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}$. This leads to:

$$\sum_{p=1}^{\infty} \mathcal{A}_p \phi_0 = \sigma (P^p)_{s_t} \times C \times \Lambda^p$$

Defining $R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix}$ such that $\mathcal{O}\phi_0(s^t, v^t) = R_{s_t} v_t$, then R satisfies:

$$\text{Vect}(R) = (I - (\Lambda \otimes P))^{-1} \text{Vect} \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix} \tag{32}$$

Consequently, the solution of this linear model when $\gamma = 0$ satisfies:

$$\phi(\sigma, 0)(s^t, v^t) = \sigma B_{s_t}^{-1} R_{s_t} v_t$$

We apply Theorem 1 and compute

$$D_\phi \mathcal{N}(\phi(\sigma,0),\sigma,0)^{-1}h(s^t,v^t) = B_{s_t}^{-1}(\mathcal{O}h)(s^t,v^t)$$

Moreover, we compute:

$$D_\gamma \mathcal{N}(\phi(\sigma,0),\sigma,0)(s^t,v^t) = A_{s_t}\sum_{j=1}^{N}\gamma\partial_1 p_{s_t j}(0,\sigma B_{s_t}^{-1}R_{s_t}v_t)\sigma B_j^{-1}R_j\Lambda v_t$$

and: This implies that:

$$\phi(\sigma,\gamma)(s^t,v^t) = \sigma B_{s_t}^{-1}R_{s_t}v_t + \sigma B_{s_t}^{-1}\mathcal{O}\left[A_{s_t}\sum_{j=1}^{N}\int_V \gamma\partial_1 p_{s_t j}(0,\sigma B_{s_t}^{-1}R_{s_t}(\Lambda v_t+\mu))B_j^{-1}R_j(\Lambda v_t+\mu)h(\mu)d\mu\right]$$

C Proof of Proposition 3

C.1 Existence and uniqueness of a s.r.e.e.

This results from Proposition 1.

$$S_1 = \begin{bmatrix} \frac{p_{11}}{|\alpha_1|} & \frac{p_{12}}{|\alpha_2|} \\ \frac{p_{21}}{|\alpha_1|} & \frac{p_{22}}{|\alpha_2|} \end{bmatrix}$$

Here, $S_p = S_1^p$ and we only need to check that all eigenvalues of S_1 are smaller than 1. This condition exactly coincides with Farmer et al. (2009a) determinacy condition.

Furthermore, when $|\alpha_1| > p_{11}$ and $|\alpha_2| > p_{22}$, Davig and Leeper (2007) proves that the eigenvalues of S_1 are smaller than one if and only if:

$$|\alpha_1|.|\alpha_2| + p_{22}(1-|\alpha_1|) + p_{11}(1-|\alpha_2|) > 1.$$

C.2 The solution when probabilities are exogenous

We directly apply Proposition 15 and with the notations of section B.5, $\Lambda = \rho$ and

$$P = \begin{bmatrix} \frac{p_{11}}{\alpha_1} & \frac{1-p_{11}}{\alpha_2} \\ \frac{1-p_{22}}{\alpha_1} & \frac{p_{22}}{\alpha_2} \end{bmatrix}$$

We know that:

$$\pi_t = \frac{\Lambda_t r_t}{\alpha_t}$$

where

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = (I-\rho P)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In particular, if $\rho = 0$, $\Lambda_1 = \Lambda_2 = 1$.

C.3 Computation of $D_\gamma \mathcal{N}(\phi_0, 0)$

The expression of $\mathcal{N}(\phi,\gamma)$ leads to:

$$D_\gamma \mathcal{N}(\phi_0,0) = \rho\frac{\Lambda_{s_t}}{\alpha_{s_t}}(\frac{\Lambda_1}{\alpha_1} - \frac{\Lambda_2}{\alpha_2})[\frac{\Lambda_{s_t}}{\alpha_{s_t}}\lambda_{s_t 1}^2 r_t]r_t^2$$

We thus define $\gamma_i = \frac{\Lambda_i^2}{\alpha_i^2}(\frac{\Lambda_1}{\alpha_1} - \frac{\Lambda_2}{\alpha_2})\lambda_{i1}^2$, to rewrite the differential as follows:

$$D_\gamma \mathcal{N}(\phi_0,0) = \rho\gamma_{s_t}r_t^3$$

C.4 Computation of $D_\phi \mathcal{N}(\phi_0, 0)^{-1} D_\gamma \mathcal{N}(\phi_0, 0)$

Computing :

$$D_\phi \mathcal{N}(\phi_0, 0)^{-1} D_\gamma \mathcal{N}(\phi_0, 0) = -\frac{r_t \Lambda_{s_t}}{\alpha_{s_t}} + \gamma\rho \sum_{k=0}^{\infty} \mathbb{E}_t \frac{\gamma_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}} r_{t+k}^3$$

As r_t is independent from the switching process,

$$D_\phi \mathcal{N}(\Phi_0, 0)^{-1} D_\gamma \mathcal{N}(\phi_0, 0) = -\frac{r_t \Lambda_{s_t}}{\alpha_{s_t}} + \gamma\rho \sum_{k=0}^{\infty} \mathbb{E}_t \frac{\gamma_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}} \mathbb{E}_t r_{t+k}^3$$

We notice that:

$$\mathbb{E}_t \frac{\gamma_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}} = \frac{1}{\alpha_{s_t}} \tilde{P}_{s_t}^k \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Now we compute first and second raw moments of r_t:

$$\mathbb{E}_t r_{t+k}^3 = 3\mathrm{var}(v) \mathbb{E}_t r_{t+k-1} + \rho^3 \mathbb{E}_t r_{t+k-1}^3$$

$$\mathbb{E}_t r_{t+k}^3 = 3\mathrm{var}(v) r_t \sum_{i=1}^{k} (\rho^3)^{i-1} \rho^{k+1-i} + (\rho^3)^k r_t^3$$

$$\mathbb{E}_t r_{t+k}^3 = 3\rho^k \frac{1-(\rho^2)^k}{1-\rho^2} \mathrm{var}(v) r_t + (\rho^3)^k r_t^3$$

$$\mathbb{E}_t r_{t+k}^3 = 3\frac{\rho^k}{1-\rho^2} \mathrm{var}(v) r_t + (\rho^3)^k [r_t^3 - \frac{\mathrm{var}(v) r_t}{1-\rho^2}]$$

Thus,

$$\gamma\rho \sum_{k=0}^{\infty} \mathbb{E}_t \frac{\gamma_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}} \mathbb{E}_t r_{t+k}^3 = \gamma \frac{\rho}{\alpha_{s_t}} \frac{\mathrm{var}(v) r_t}{1-\rho^2} \sum_{k=0}^{\infty} \rho^k \tilde{P}_{s_t}^k \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + (r_t^3 - \frac{\mathrm{var}(v) r_t}{1-\rho^2}) \sum_{k=0}^{\infty} \rho^{3k} \tilde{P}_{s_t}^k \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$\gamma\rho \sum_{k=0}^{\infty} E_t \frac{\gamma_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}} \mathbb{E}_t r_{t+k}^3 = \gamma \frac{\rho}{\alpha_{s_t}} \frac{\mathrm{var}(v) r_t}{1-\rho^2} (I - \rho\tilde{P})^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + (r_t^3 - \frac{\mathrm{var}(v) r_t}{1-\rho^2})(I - \rho^3 \tilde{P})^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Finally by applying Proposition 2 we find:

$$\pi_t = -\frac{r_t \Lambda_{s_t}}{\alpha_{s_t}} + \rho\gamma(c_{s_t} r_t^3 + d_{s_t} \frac{\mathrm{var}(v)}{1-\rho^2} r_t) + o(\gamma)$$

Where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1} (I - \rho^3 \tilde{P})^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = A^{-1} [(I - \rho\tilde{P})^{-1} - (I - \rho^3 \tilde{P})^{-1}] \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

D Calibration

Parameter	Calibration
p_{11}	0.8
p_{22}	0.95
ρ	0.9
$\sqrt{var(v)}$	0.25%
α_1	0.9
α_2	2
λ_{11}^2	-40
λ_{22}^2	0
γ	1

Table 1: Parameters calibration

E Figures

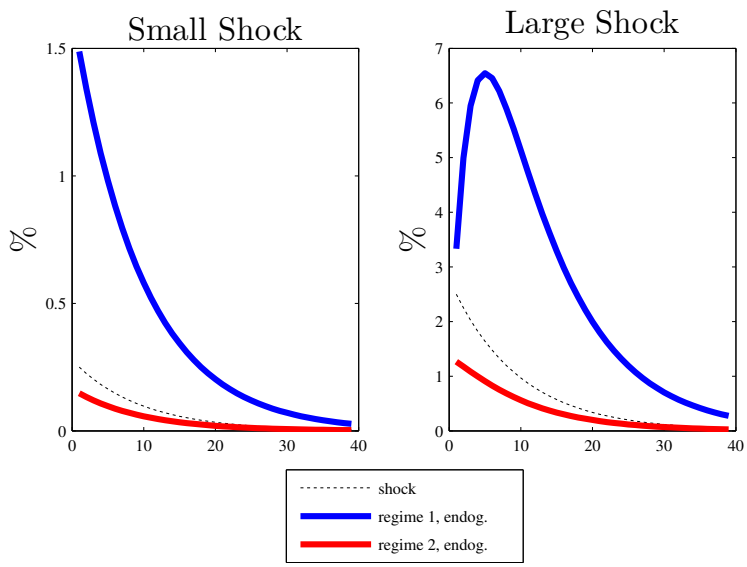

Figure 1: Impulse Response Function to a real interest rate shock

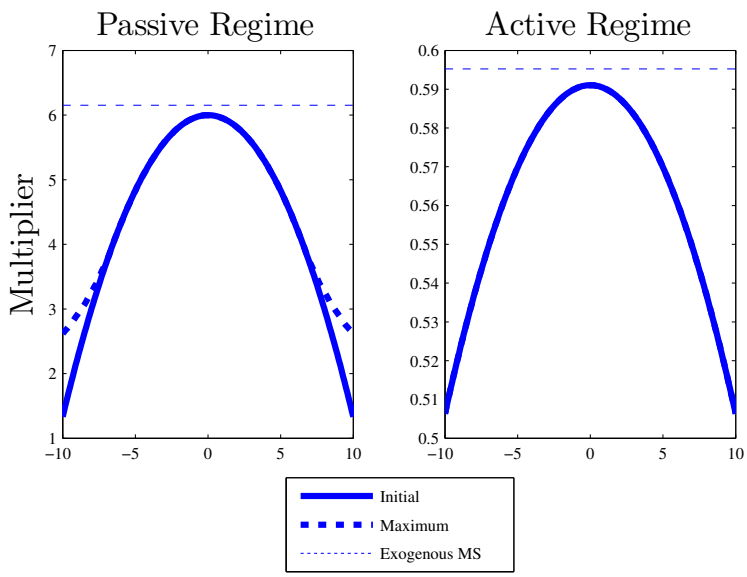

Figure 2: Multiplier with respect to the size of the shock

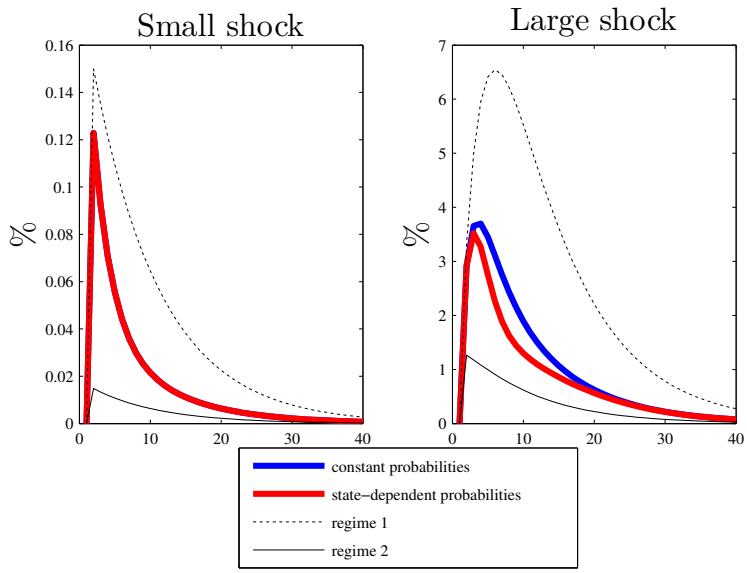

Figure 3: Impulse Response Function to a real interest rate shock

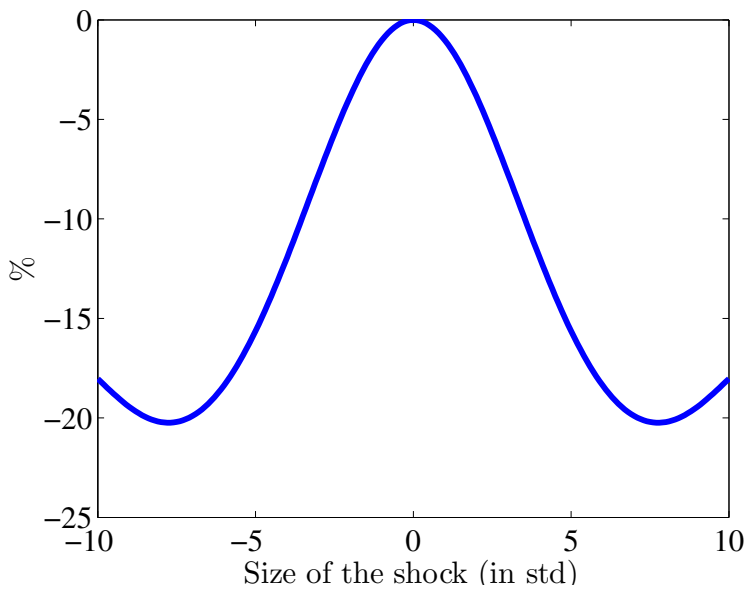

Figure 4: Selection effect

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