Entrepreneurs, Ambiguity and Investments*

Preliminary and Incomplete

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Abstract

Entrepreneurs are often considered engines of growth due to their ability and/or willingness to delve into uncharted economic terrain. We follow a literature that captures this idea with a multiprior approach to decision making. Entrepreneurs are endowed with investment opportunities whose outcomes are ambiguous. When self financed we show that ambiguity may cause entrepreneurs to be cautious in expansion but also reluctant to abandon prior investment decisions even when abandonment payoffs are relatively large. With external financing, the optimal contracts critically depends on differences in ambiguity as seen by financiers and entrepreneurs. By developing investment and financing implications we provide a base for future empirical work that will shed light on how new ambiguous opportunities are in fact handled.

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1 Introduction

Entrepreneurs have long been considered engines of growth due to their ability and/or willingness to delve into uncharted economic terrain. Knight (1921) introduced the idea that delving into uncharted economic terrain could be thought of as taking an action when one is unable to identify a unique distribution governing the action's outcomes. Such a situation is widely referred to as Knightian uncertainty or ambiguity. Ambiguity is distinct from risk where the decision maker feels an act being contemplated is governed by a unique prior distribution. Experimental evidence in several fields suggests that, all else equal, humans prefer not to take actions with ambiguous outcomes, i.e. they exhibit ambiguity aversion. Entrepreneurs are special because they identify ambiguous opportunities and/or they are more able and willing to deal with them. As Knight put it

> "...the facts upon which the working-out of the organization depends can no longer be objectively determined with accuracy by experiment; all the data in the case must be *estimated*, subject to a larger or smaller margin of error . . .The function of making these estimates and of 'guaranteeing' their value to the other participating members of the group falls to the responsible entrepreneur..." (his emphasis)

We examine a model of entrepreneurs who are assumed to have monopoly access to an ambiguous project. We follow a literature that has formalized the idea of ambiguity in terms of multiple prior distributions. In this setting we examine the allocation of savings to investment projects when ambiguity is present. We consider a canonical corporate finance model: an entrepreneur (E) has a potentially economically valuable idea but lacks the required investment resources while a financier (F) has the required resources but does not have direct access to investment opportunities. We examine how investment decisions are influenced by the existence of ambiguity aversion and how financing choices are affected by the extent to which E and F differ in their assessment of the ambiguity of the project.

We present a number of novel findings from this inquiry. For instance, we show that entrepreneurs may continue operating a project even when they believe the project may be worth less than the scrap value of the asset. We also show how financial securities can be designed to bring financiers and entrepreneurs who differ in their view of ambiguity together.

We contrast our analysis to standard finance theory that is built on the subjective expected utility (SEU) paradigm. The fundamental assumptions in the SEU approach are that individuals select actions that have risky outcomes by attaching a utility that reflects risk attitudes to each outcome and a unique probability to the likelihood that the outcome will obtain. The expected utility of each action is then evaluated and the decision maker is assumed to choose actions that maximize expected utility.

The ability of agents to assess and use probabilities in a way that is consistent with SEU has been widely challenged over the years. Particularly compelling evidence against the expected utility hypothesis comes from psychological experiments that have identified 'Ambiguity Aversion', often apparent in what has been called the Ellsberg Paradox (Ellsberg, 1961). The experiment has the following structure. Subjects are presented with two urns. In one urn, the unambiguous urn, they are able to verify that there are 50 white balls and 50 blue balls. In the second urn, the ambiguous urn, they are assured that there are 100 balls in total, some white and some blue but they are given no information about how many white or blue balls are in the urn. The subjects are then offered a gamble: they will be paid, say, \$10 if they draw a blue ball and they must choose which urn to draw from. The majority choose the unambiguous urn. According to the expected utility hypothesis, this action implies that the subject has concluded that the probability of drawing a blue ball in the ambiguous urn is less than 50%.

The subject is then offered the same bet but with the colors reversed, i.e. the \$10 prize is given for drawing a white ball. In this case the subjects again choose the unambiguous urn. According to the expected utility hypothesis, this choice is consistent with the subject believing the probability of drawing a blue ball from the ambiguous urn is more than 50%. So the same subject has taken actions that, according to the expected utility hypothesis, implies that they believe the probability of a specific outcome (drawing a blue ball from the ambiguous urn) is both less than and greater than 50%. This is clearly inconsistent with and therefore presents a significant challenge to the expected utility hypothesis.

An interesting interpretation of the experimental evidence is that agents base decisions on multiple prior distributions and follow a non-baysian approach to decision making. For instance, suppose that the subject believes that there were two possible distributions for the ambiguous urn: one that attaches a probability of 40% to drawing a blue ball and the other assigns a probability of 60% to this outcome. Further, assume that the subject acts to maximize the minimum possible payoff (maxmin expected utility or MEU), an approach axiomatized by Gilboa and Schmeidler (1989). Under these assumptions, regardless of the color selected, the unambiguous urn has a unique expected payoff of 5 while selecting the ambiguous urn has the set of expected payoffs $\{4,6\}$, with the elements of this set corresponding to each of the two possible distributions. Hence the maximum of the minimum payoffs from the two acts is 5 attached to the choice of the unambiguous urn.

An alternative decision rule based on multiple priors will be referred to as Inertia Based Expected Utility (IBEU) due to Bewley (2002). To implement this approach it is necessary to identify a status quo (for instance the status quo could be that you will be paid 10 if blue is drawn from the unambiguous urn) and an alternative (for instance you will be paid 10 if blue is drawn from the ambiguous urn). The decision rule is that you will only decide to accept the alternative if it is better under all possible distributions than the status quo. We will see that MEU and IBEU often give similar choices but in some cases imply different decisions.

Several other approaches to decision making have been examined in more general settings and Gilboa and Marinacci (2011) provide an excellent survey of these approaches. It is, however, beyond the scope of our paper to explore all of these approaches as we are primarily concerned with basic financing issues. As a result, we examine the entrepreneur's problem in the context of SEU, MEU and IBEU only.

Recent studies have applied the multiple prior approach to finance problems. For the most part, theses applications have been in the asset pricing and portfolio choice areas.

Epstein and Schneider (2010) provide an excellent survey. Fewer studies have considered how ambiguity aversion could affect corporate decisions and it is to this literature that we contribute. Nishimura and Ozaki (2007), Miao and Wang (2011), and Flor and Hessel (2011) examine the exercise decision in a real option setting but do so only in an MEU framework and do not consider financing issues.

The pioneering work of Rigotti (2004) is closest to our study in examining financing decisions in a IBEU setting. We complete and extend the analysis of Rigotti by considering both static and dynamic settings as well as real and financial decisions. Moreover, by drawing out investment and financing implications under both MEU and IBEU we open the possibility of using actual investment and financing decisions to settle the empirical question of which approach best describes managerial decision making.

In the next section we set out some preliminary elements of our analysis. We do this in the simple setting of a single decision maker selecting between a safe asset and an asset that is ambiguous. We also discuss our approach to modelling entrepreneurs. In section 3 we consider dynamic investment problem with expansion and contraction options. Section 4 characterizes the solution of the investment problem and show that different representations of ambiguity lead to stark difference in observed investment behavior. Section 5 introduces financing of a project in a static setting and Section 6 considers the problem of financing a multi-stage project. Section 7 concludes. Appendix A contains proofs for all propositions.

2 Ambiguity and Decision Making

We illustrate our approach to ambiguity and decision making through two decisions being contemplated by a decision maker (DM). Both decisions involve a certain cash amount I and a gamble $\tilde{C}$ that will produce an unknown future cash flow. The realization of $\tilde{C}$ will depend on the state of the world drawn from the set $\Omega = \{U, D\}$ and for now assume the cash flow is simply the state value, u or d. Both decision problems require that the DM

choose between a 'status quo' or current wealth level and an alternative that the status quo can be exchanged for. The decision problems are:

- **Investment:** The status quo is I and the alternative is $\tilde{C}$;
- **Contraction:** The status quo is $\tilde{C}$ and the alternative is I.

In a subsequent section we set out a two period model where an investment is made that can subsequently be expanded or contracted. Hence the decision problem and status quo for the same DM will change from an Investment to a Contraction/Expansion decision.

In evaluating a choice, the value the DM attaches to each outcome is described by a utility function. We assume all agents are risk neutral and that the utility of each outcome is the outcome itself.

As a benchmark we consider a SEU DM who believes that a unique subjective probability governs the outcome of the gamble $\tilde{C}$. Let p denote the probability attached to the outcome u so that the probability of d is $(1-p)$. Let $E_p(\tilde{C})$ denote the expected value of a gamble under the distribution p, i.e. $E_p(\tilde{C}) = D + p(U - D)$.

An *ambiguous* gamble is one where the DM feels that the outcome of a particular choice is governed by one distribution, denoted π, taken from a set of several possible distributions, denoted Π. The DM is, however, unable to quantify the likelihood that any particular distribution governs the outcome of the gamble. Specifically for our example, we assume that Π contains two distributions, $\Pi = \{p - \epsilon, p + \epsilon\}$ where $p - \epsilon$ represent the probability of U under the first distribution and $p + \epsilon$ the probability of U under the second distribution. Because the DM does not have a unique distribution in mind we cannot compute the expected utility or a net present value of the gamble unless $\epsilon = 0$. We will refer to a cash flow for which the DM feels that $\epsilon = 0$ as an unambiguous cash flow. We will also take ϵ to be a measure of the ambiguity of the cash flow. Since ambiguity is in the eye of the beholder, we will also refer to a DM who sees a cash flow as being unambiguous as ambiguity neutral.

Ambiguity aversion refers to the way in which the DM selects among ambiguous choices, with the general notion being that, all else equal, the DM prefers less ambiguous actions. In the Introduction we described two specific approaches to decision making; Inertia Based Expected Utility (IBEU) and Maxmin Expected Utility (MEU). To illustrate the SEU, IBEU, and MEU approaches to decision making, assume $p = .5, U = 10, D = 0$, and define I^* as the largest value of I at which the DM is willing to leave the status quo for the alternative. In what follows we use the symbol $\succ$ to denotes preferences over alternative actions available to the DM.

Subjective Expected Utility (SEU)

A SEU decision maker will use the following decision rule regardless of the status quo.

$$\tilde{C} \succ I \iff 5 > I \tag{1}$$

$$I \succ \tilde{C} \iff I > 5 \tag{2}$$

Hence $I^* = 5$. Suppose $I \leq 5$. If I is the status quo the DM will choose the alterntative of $\tilde{C}$ while if $\tilde{C}$ is the status quo then the DM will stay with the status quo rather than sell it for I. This illustrates that, when $I \leq I^*$ the DM selects I over $\tilde{C}$ regardless of the status quo. In general, the status quo plays no role in SEU decision making.

In the case of SEU, we can define the net present value of the investment to the decision maker as

$$NPV_{invest} = E_p\tilde{C}) - I.$$

The contraction NPV is a similar expression but with the signs reversed;

$$NPV_{contract} = -E_p(\tilde{C}) + I.$$

Hence either decision can be evaluated in terms of the familiar NPV rule.

In moving to an ambiguity averse decision maker we assume the DM feels that $\epsilon > 0$ and, to be concrete, assume $\epsilon = .3$. Since there are two distributions in Π, the DM feels there are two possible expected values of the gamble: $E_{p-\epsilon}(\tilde{C}) = 2$ and $E_{p+\epsilon}(\tilde{C}) = 8$.

Inertia Based Expected Utility (IBEU)

Bewley (2002) elegantly presents and motivates Inertia Based Decision Making. His approach to modelling ambiguity consists in starting from the traditional Savage (1954) SEU axioms but dropping the completeness one. Once this axiom is removed, he shows that the DM preferences cannot be represented via a unique probability distribution. In particular, the decisions among alternative gambles are characterized by a "unanimity rule", i.e., one lottery is preferred over another if its expected value is higher under *all* possible distributions. This decision rule leads naturally to an incomplete preference ordering and the model will not always specify what the DM will do. Most important, the model cannot be used to assign a "value" to a lottery as it is typically done under SEU. To resolve the indeterminacy in the presence of incomparable gambles, Bewley introduces the following *status quo* or *inertia* assumption:

Inertia Assumption A DM will only accept a gamble if the expected value of the gamble is strictly better than the status quo for each $\pi \in \Pi$.

As we will see, this assumption implies that the status quo or reference point is important in making a decision. We adopt the convention that, when present, the status quo is the gamble appearing on the RHS of the preference relation $\succ$. We refer to this representation of preference as Inertia Based Expected Utility (IBEU). Under IBEU we have that

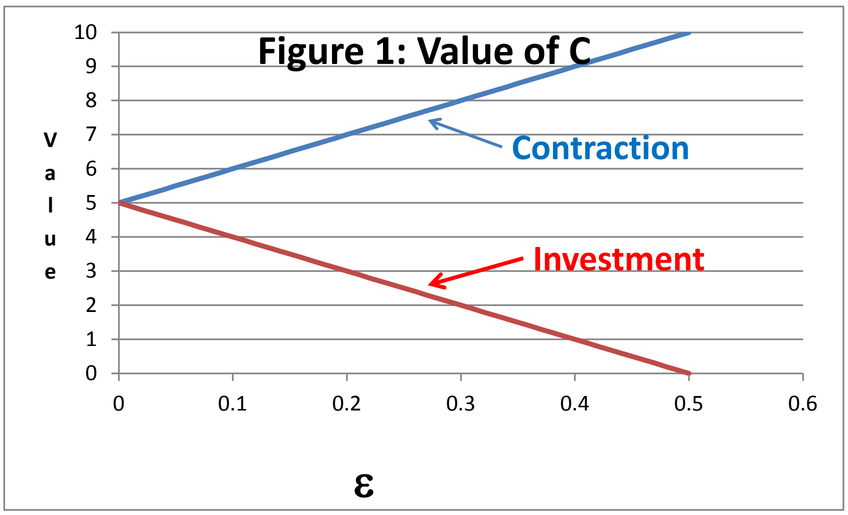
$$\tilde{A} \succ \tilde{B} \iff E_\pi(\tilde{A}) > E_\pi(\tilde{B}) \ \forall \pi \in \Pi \tag{3}$$

In the case of our simple example, the IBEU implies the following rules.

$$\tilde{C} \succ I \iff 2 > I \tag{4}$$

$$I \succ \tilde{C} \iff I > 8 \tag{5}$$

In words, (4) states that if the status quo is the certain amount of cash I, the DM will only leave the status quo if the expected value of the gamble under the most pessimistic prior in Π is greater than I. On the other hand, if the DM possesses the gamble $\tilde{C}$, she will only give up the gamble if the payment received, I, is larger than the expected value of the gamble under the most optimistic distribution in Π. Essentially, IBEU takes into account the opportunity cost or potential regret of an action.



The behavior implied by inertia displays interesting features. Figure 1 shows how the relative value of the gamble depends on the action being considered (i.e. the status quo) and the degree of ambiguity of the DM. For an investment the status quo is I, so the choice is one of *buying* the gamble. When buying, the value of the gamble decreases in the ambiguity measure ϵ. For a contraction the status quo is the gamble $\tilde{C}$ the DM is *selling* the gamble and the value of the gamble is increasing in the ambiguity measure ϵ.

Maxmin Expected Utility (MEU)

In general under MEU, the decision of which lottery to accept, $\tilde{A}$ or $\tilde{B}$, is given by

$$\tilde{A} \succ \tilde{B} \iff \min_{\pi} E_{\pi}(\tilde{A}) > \min_{\pi} E_{\pi}(\tilde{B}).$$

Note that there is no special role for the status quo in this framework.

For our simple example, a MEU decision maker will have the following ordering.

$$\tilde{C} \succ I \iff 2 > I \tag{6}$$

$$I \succ \tilde{C} \iff I > 2 \tag{7}$$

Hence, $I^* = 2$. In this case, as in the case of the SEU, we can determine the unique price at which the DM will leave the status quo for the alternative, regardless of which object is the status quo. The willingness to invest, represented by I^*, is lower under MEU than under SEU. It is worth noting, however, that I^* under MEU would be identical to I^* under SEU but with a more pessimistic but unique prior distribution. Moreover, this critical "price" is the same whether buying or selling the gamble.

As stated earlier, IBEU takes into account the opportunity cost or potential regret of an action. In contrast MEU decision making is "pessimistic" about any long position in a gamble, whether the gamble is currently owned or is to be acquired.

Interestingly, based on the choices made it is not possible to distinguish a MEU DM who sees a gamble as ambiguous from one who has a unique but pessimistic view of the gamble. They are reluctant to expand and quick to contract. On the other hand, IBEU implies choices that are distinct from a pessimist: the DM seems pessimistic and reluctant to invest but then appears to be optimistic in her reluctance to contract.

2.1 Entrepreneurs

The focus of our study is on decision makers that are referred to as Entrepreneurs. There seems to be two broad dimensions along which entrepreneurs are assumed to be different from other DMs.

1. **Technology:** Entrepreneurs are seen as individuals who create new productive opportunities in the economy. We will implement this view by assuming that the Entrepreneur has monopoly access to ambiguous investment opportunities.

2. **Ambiguity attitude:** Another possibility is that Entrepreneurs see a world that is different from others. Sometimes they are described as being more tolerant of ambiguity, more decisive (e.g., Rigotti (2004), Amarante, Ozgür, and Phelps (2011), and Amarante, Ghossoub, and Phelps (2011)). While this may be the case, we feel it is important to consider situations where Entrepreneurs can see either more ambiguity or less ambiguity than others in the economy.

3 A Model of Investment

We consider a simple model of a risk neutral Entrepreneur (E) with an investment opportunity that has payoffs over three dates, t_0, t_1, and t_2, defining two periods. The Entrepreneur has the ability to turn investment into future cash flows C_t through an ambiguous project. Details of the technology are presented below. The entrepreneur is endowed with access to the project and with the ability to affect output through an endowed productivity parameter θ_{S_t}, $t \in \{1, 2\}$ where S_t is the state.

We assume initially that the Entrepreneur has sufficient funds to self finance and chooses to do so. This setting gives us an opportunity to examine the real consequences of ambiguity without the added effects of financing. In Section 4 we relax this assumption and introduce the need for E to arrange external financing from a risk neutral financier F.

3.1 Technology and Ambiguity

The state space is the product

$$\Omega = \Omega_1 \times \Omega_2$$

where $\Omega_1 = \Omega_2 = \{u, d\}$ so that

$$\Omega = \{(u, u), (u, d), (d, u), (d, d))\},$$

where $(\cdot,\cdot)$ represents a "path" on the binomial tree. The space is endowed with the binomial tree filtration $\mathcal{F} = \{\mathcal{F}_0; \mathcal{F}_1, \mathcal{F}_2\}$.

We capture the ambiguity with which E views this project at each time t through the existence of a set Π_t of priors π_t over the possible state realizations. The set of priors on $\mathcal{F}_1$ can be described by the probability of the up node ($\pi_0 = \text{prob}\left(\{(u,u),(u,d)\}\right)$)

$$\Pi_0 = \{\pi_0 : \; p - \epsilon \leq \pi_0 \leq p + \epsilon\}$$

The conditional set of priors on $\mathcal{F}_2$ can be described by the probability $\pi_1 = \text{prob}\left((\bullet, u)\right) \mid \mathcal{F}_1)$

$$\Pi_1 = \{\pi_1 : \; p - \epsilon \leq \pi_1 \leq p + \epsilon\}.$$

For ease of exposition we consider the case in which $\Pi_0 = \Pi_1 = \Pi$. We denote the expected value of a gamble under the prior $\pi \in \Pi$ by $E_\pi(\tilde{C})$.

3.2 Technology and decisions

The entrepreneur must pay I_0 if he wants to launch the project. At time 1, he gets the state contingent cash flow $c_1(u)$ or $c_1(d)$.

In the up state (at time 1), the entrepreneur can keep the same project scale and get the cash flows θ_{uu} or θ_{ud} with the condition $\theta_{uu} > \theta_{ud}$. Alternatively, he can expand the firm by paying the amount I_1 at $t = 1$ and get the cash flows $\theta_{uu}(1+\lambda)$ or $\theta_{ud}(1+\lambda)$ at $t = 2$.

In the down state (at time 1), the entrepreneur can keep the same project scale and get the cash flows θ_{du} or θ_{dd} with the condition $\theta_{du} > \theta_{dd}$. Alternatively, he can shut dow the firm and get immediately the cash flow S and nothing at time $t = 2$.

We assume that the entrepreneur behave as in Beweley with linear utilities and a bias toward the status quo (inertia). For simplicity, we assume that their initial status quo is zero wealth.

The project requires that one unit of capital be installed at t_0 at a cost of I_0.This unit of capital will produce cash flows at both t_1 and t_2 as well as the ability to expand or contract capacity at t_1. One period later, at t_1, the capital generates a cash flow of C_1, where $C_1 \in \{\theta_u, \theta_d\}$. In addition to the first cash flow, at t_1 the entrepreneur must decide to either maintain the current size, expand or contract. To simplify we assume that the entrepreneur can only expand in state u and can only contract in state d.[1] In the second period C_2 will be realized. If E continues without expanding or contracting then $C_2 \in \{\theta_{uu}, \theta_{ud}, \theta_{du}, \theta_{dd}\}$. Expansion will scale C_2 up to λC_2 while contraction results in a state independent scrap value S.

4 Optimal expansion and contraction decisions

We follow a dynamic programming approach to the analysis, determining the expansion/contraction decisions and then the initial investment decision. To insure dynamic consistency in the presence of ambiguity, we invoke Strotz (1955)consistent planning principle.[2] We therefore first determine the t_1 expansion and contraction decision and take these as given at t_0.

4.1 The Expansion Decision

At t_1 in state u the entrepreneur is able to expand capacity by exchanging the current unit of capital for $\lambda > 1$ units of productive capital by paying I_1 . For simplicity, we assume that $\theta_u > I_1$ so that E does not have to contribute further once the initial investment of I_0 is made at t_0. Hence, in state u the status quo for E under a particular prior $\pi_E \in \Pi_E$ is the gamble:

$$Status\ \ Quo - Continue: \quad E_{\pi_E}(\theta_{S_2})$$

[1]The assumption that the technology can only expand in state u and contract in state d is for simplicity only and is without loss of generality.

[2]See Siniscalchi (2011) for an analysis of the consistent planning principle in dynamic choice under ambiguity.

while the alternative is to expand capacity providing

$$Alternate - Expand: \quad \lambda E_{\pi_E}(\theta_{S_2}) - I_1$$

4.1.1 SEU

If E is ambiguity neutral she will make decisions based on SEU implying

$$\begin{aligned} Expand \succ Continue &\iff \lambda E_p(\theta_{S_2}) - I_1 > E_p(\theta_{S_2}) \\ &\iff (\lambda - 1)E_p(\theta_{S_2}) > I_1 \end{aligned} \tag{8}$$

In this case (8) is the familiar NPV rule.

4.1.2 IBEU

If E is ambiguity averse so that $\epsilon > 0$ and makes decisions based on IBEU, then she will make the following decisions.

$$Expand \succ Continue \iff \lambda E_{\pi_E}(\theta_{S_2}) - I_1 > E_{\pi_E}(\theta_{S_2}),\ \forall \pi_E \in \Pi_E \tag{9}$$

$$\iff (\lambda - 1)E_{p-\epsilon}(\theta_{S_2}) > I_1. \tag{10}$$

In solving (9) we know that (10) is the binding constraint since (10) is increasing in π_E. In other words, for an expansion both the status quo and the alternate are long positions in the underlying asset so the alternate is evaluated with the most pessimistic priors.

4.1.3 MEU

If E is ambiguity averse so that $\epsilon > 0$ and makes decisions based on MEU, then she will make the following decisions.

$$Expand \succ Continue \iff \min_{\pi \in \Pi_E} (\lambda E_{\pi_E}(\theta_{S_2}) - I_1) > \min_{\pi \in \Pi_E} E_{\pi_E}(\theta_{S_2}) \tag{11}$$

$$\iff (\lambda - 1)E_{p-\epsilon}(\theta_{S_2})) > I_1 \tag{12}$$

Interestingly, although the criteria used in IBEU and MEU are different, the condition under which a firm will expand, as given by (10) and (12) are identical.

4.2 The Contraction Opportunity

At date 1 in state d the entrepreneur is able to convert capacity into an alternative, no ambiguous use that has an immediate risk free salvage value of S. Hence, in state u the status quo for E under a particular prior $\pi_E \in \Pi_E$ is the gamble:

$$Status\ \ Quo - Continue: \quad E_{\pi_E}(\theta_{S_2})$$

while the alternative is to contract capacity providing

$$Alternate - Contract: \quad S$$

4.2.1 SEU

If E is ambiguity neutral she will make decisions based on SEU implying

$$Contract \succ Continue \quad \Longleftrightarrow \quad S > E_p(\theta_{S_2}) \tag{13}$$

As with the expansion decision, in this case the decision criteria (13) is the familiar NPV rule.

4.2.2 IBEU

If E is ambiguity averse so that $\epsilon > 0$ and makes decisions based on IBEU, then she will make the following decisions.

$$Contract \succ Continue \quad \Longleftrightarrow \quad S > E_{\pi_E}(\theta_{S_2})\ \forall \pi_E \in \Pi_E \tag{14}$$

$$\Longleftrightarrow \quad S > E_{p+\epsilon}(\theta_{S_2}) \tag{15}$$

4.2.3 MEU

If E is ambiguity averse so that $\epsilon > 0$ and makes decisions based on MEU, then she will make the following decisions.

$$Contract \succ Continue \iff S > \min_{\pi \in \Pi_E} E_{\pi_E}(\theta_{S_2} \tag{16}$$

$$\iff S > E_{p-\epsilon}(\theta_{S_2}) \tag{17}$$

While it is difficult to distinguish SEU from MEU and IBEU for expansion decisions beyond relative pessimism, a contraction decision delivers very different results. Again SEU and MEU differ in pessimism, but IBEU is very differnt from the other two. A comparison of (17) with (15) shows that an MEU decision maker is pessimistic even in contraction while and IBEU decision maker is optimistic. Hence, even when E can sell a firm for S and may even consider S to be larger than her expected payoff from continuing, she will not contract because there is some chance that, following the contraction, economic conditions could improve and the asset value would be higher than S.

It has been observed that managers are reluctant to divest or shut down projects that have not done well, a result that has been explained by agency problems, reputation concerns and asymmetric information (see, for example, Boot (1992) and Weisbach (1995)). Our explanation is in terms of ambiguity aversion under symmetric information. If DMs are ambiguity averse and base decisions based on IBEU they will be reluctant to terminate a project because of the importance that potential regret plays in their decision making.

4.3 Initial investment

At t_0 the status quo for E is I_0 and the alternative is an investment that will lead to the expansion and contraction decisions given above. We take the expansion and contraction decisions as given and evaluate them as of t_0.

$$Status\ \ Quo - Do\ not\ invest: \quad I_0$$

while the alternative is to contract capacity providing

$$Alternate - Invest: \quad E_{\pi_E}(\theta_{S_1}) + E_{\pi_E}(X|a^*)$$

where, by the principle of consistent planning, a^* indicates optimal action (expansion, contraction or status quo) that will be chosen at time $t = 1$, and

$$X = \begin{cases} \theta_{S_2} & \text{if } u \text{ and continue} \\ \lambda\theta_{S_2} - I_1 & \text{if } u \text{ and investment} \\ \theta_{S_2} & \text{if } d \text{ and continue} \\ \text{S} & \text{if } d \text{ and contraction} \end{cases}$$

For example if E expands in u and contracts in d, then the entrepreneur invests if and only if

$$\pi_0\left(-I_1 + c_1(u) + \pi_1\theta_{uu}(1+\lambda) + (1-\pi_1)\theta_{ud}(1+\lambda)\right) + (1-\pi_0)(c_1(d) + S) \geq I_0$$

for all $\pi_0 \in \Pi_0$ and $\pi_1 \in \Pi_1$. This condition is satisfied if and only if

$$\pi_0\left(-I_1 + c_1(u) + (p-\epsilon)\theta_{uu}(1+\lambda) + (1-(p-\epsilon))\theta_{ud}(1+\lambda)\right) + (1-\pi_0)(c_1(d) + S) \geq I_0$$

where $\pi_0 \in \{p-\epsilon, p+\epsilon\}$ depending on the values of the model parameters. For example if S is large enough, then the above investment condition holds with $\pi_0 = p + \epsilon$. On the other hand if λ is large or if I_0 is small, then the above investment condition holds with $\pi_0 = p - \epsilon$.

5 External Financing

In this section we assume that E has no funds and must obtain financing from a risk neutral financier F. We consider two extreme cases: one where E is ambiguity averse ($\epsilon > 0$) and the financier is ambiguity neutral ($\epsilon = 0$) as well as the opposite. We consider first the case of a single period financing arrangement. The next section studies contracting in a dynamic setting.

5.1 F ambiguity averse, E ambiguity neutral

Assume that $\epsilon_E = 0$ and $\epsilon_F > 0$. The financier's belief is represented by the set $\Pi = [p - \varepsilon, p + \varepsilon]$ and E_π denotes the expectation under the probability π.

5.1.1 IBEU choices

We assume that F behaves like in the Bewley model with linear utility and inertia. Each time E sells a security to F, E is going to realize a negative NPV transaction. In fact, given the Bewley preferences of F and the fact that E's prior belong to Π_F, the maximum amount that F is willing to pay for a security is always smaller than the NPV of this security according to the E prior (p). As a result, if E has some cash, he will always prefer self financing to raising money. If E does not have cash, he must issue a security $d = (D_u, D_d)$ to finance the project and, as we show, must raise just the necessary amount I. The security has to be in the feasible set $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ D = (D_u, D_d) \in R^{2+} \mid D_u \leq \theta_u,\ D_d \leq \theta_d, E_\pi(D) \geq I \text{ for all } \pi \in \Pi_F \right\}$$

The entrepreneur is going to solve

$$\sup_{D=(D_u,D_d)\in\mathcal{D}} U_E(D) = -I + \inf_{\pi\in\Pi} E_\pi(D) + E_p(\theta) - E_p(D).$$

Because E's belief p is an element of the set Π_F, the inequality

$$U_E(D) \leq -I + E_p(\theta) \tag{18}$$

always holds. Notice that inequality (18) binds in two important cases. The first case is when the set Π is the singleton $\Pi = \{p\}$ and the second case is when d payoffs are state independent, that is $d_u = D_d$.

Proposition 1. *Self financing is always the best option. If the entrepreneur has no cash the optimal policy is to finance it with a security* (D_u^*, D_d^*) *that has the form*

Range of I:	$0 \leq I \leq \theta_d$	$\theta_d \leq I \leq E_{p-\varepsilon}(\theta)$	$E_{p-\varepsilon}(\theta) < I$
Financing:	$D^* = (I, I)$	$D^* = (\theta_d + \frac{1}{p-\varepsilon}(I - \theta_d), \theta_d)$	*None*

The intuition for the proposition is that each time E issues a security, he is going to lose money. When possible he wants to avoid issuing securities. If he must raise money, he will first issue a security with constant payoff because their NPV is insensitive to beliefs. In fact both E and F agree on the valuation of a constant payoff security (a riskless bond). If issuing riskless bonds does not allow to raise enough money to finance the investment, then E start issuing state contingent payoff securities (equity like) up to a point where the NPV of the whole firm under the worst belief is larger than I in which case, the project is too costly and E abandons it.

Notice that if E has some cash $W_0 < I$ then the above proposition applies by changing I to $I - W_0$. We therefore have a preference order for securities. First, using cash to finance a project is the best solution, then issuing bonds is the second choice and then issuing equities (or equity like) is the third choice.

5.1.2 MEU Choices

If F has MEU preferences, then given the set of priors Π_F the maximum price that they are willing to pay for a security d is

$$\inf_{\pi \in \Pi_F} E_\pi(D).$$

As a result, when the entrepreneur is choosing the security that he will issue, he faces the same feasible set $\mathcal{D}$ as the one that would prevail if F have Bewley preferences. This is due to the equivalence

$$(E_\pi(D) \geq I \text{ for all } \pi \in \Pi_F) \Leftrightarrow (\inf_{\pi \in \Pi_F} E_\pi(D) \geq I)$$

Therefore Proposition 1 will also hold when F has MEU preferences and the same order for securities will prevail. We conclude then that we will observe the same type of contracts when the financiers use MEU or IBEU to make decisions.

5.2 E ambiguity averse and F ambiguity neutral

We assume that E beliefs are represented by the set $\Pi_E = [p - \varepsilon, p + \varepsilon]$ where each element of the set Π_E represents the probability of the up state. We denote by E_π the expectation under the probability π. We also denote by $u_E^\pi(D)$ the utility that E derives from starting the project by issuing d under the belief that the probability distributuion is π. The financier's beliefs on the other hand are determined by fix prior defined by the probability of the "up" state and we assume that their beliefs are centered and equal to p.

5.2.1 IBEU Choices

When E makes IBEU choices and F makes SEU choices, the set of feasible contracts are:

$$\mathcal{G} = \left\{ D = (D_u, D_d) \in R^{2+} \mid D_u \leq \theta_u,\ D_d \leq \theta_d, E_p(D) \geq I \right\}$$

E finances the project with the security d if and only if

$$U_E^\pi(D) = -I + E_p(D) + E_\pi(\theta - D) \geq 0 \quad \text{for all } \pi \in \Pi_F. \tag{19}$$

This constraint defines the set of implementable securities. Because $\theta \geq D$, we see that any security in $\mathcal{G}$ satisfies constraint (19) and as a result any security in $\mathcal{G}$ can be implemented in the context of an entrepreneur using IBEU to make decisions. So the only restrictive constraint here is the financing constraint $E_p(D) \geq I$ and if it is satisfied then E is happy to start the project (the utility is positive under any prior π).

Proposition 2. *If $I \leq E_p(\theta)$, the entrepreneur is happy to start the project by issuing any security in the set $\mathcal{G}$. Due to incomplete preferences, the entrepreneur is unable to rank the different financing options.*

The set $\mathcal{G}$ can be easily described in the plane. The set $(\mathcal{G})$ always contains the point $\theta = (\theta_u, \theta_d)$. If $\theta_d < I$, then $\mathcal{G}$ is a simple triangle inside the square $\{0 \leq D \leq \theta\}$. If $\theta_d > I$, then $\mathcal{G}$ is the union of two triangles inside the square $\{0 \leq D \leq \theta\}$ and unlike the former case, the second triangle contains a region below the 45 degree line.

Notice that all the projects with $E_{p-\varepsilon}(\theta) < I < E_p(\theta)$ are going to be started when E is is ambiguity averse (and F is ambiguity neutral) but are abandoned when it is F who is ambiguity averse (and E is ambiguity neutral). In our context, it is the financier's attitude toward ambiguity which determine whether the project is started. Notice also that the optimal security from Proposition 1 is contained the set $\mathcal{G}$ is all subcases.

5.2.2 MEU Choices

When E makes MEU choices and F makes SEU choices, the set of feasible contracts is:

$$\mathcal{G} = \left\{ D = (D_u, D_d) \in R^{2+} \mid D_u \leq \theta_u,\ D_d \leq \theta_d, E_p(D) \geq I \right\}$$

If E starts the project, he will get the utility

$$U_E(D) = -I + E_p(D) + \inf_{\pi \in \Pi_E} E_\pi(\theta - D)$$

The optimization problem for the entrepreneur is then

$$\sup_{D \in G} U_E(D)$$

and he will start the project whenever this quantity is positive.

Proposition 3. *Under MEU preferences, E will start the project if and only if*

$$I \leq E_p(\theta).$$

If this condition holds, then it is optimal to sell the whole firm ($D = \theta$).

Notice that there are multiple optima because the set $\mathcal{G}$ intersects the set of contracts leaving E with flat payoff $(\theta_u - D_u = \theta_d - D_u)$ then any element D of this intersection gives the utility

$$U_E(D) = -I + E_p(D) + \inf_{\pi} E_\pi(\theta - D) = -I + E_p(D) + E_p(\theta - D) = -I + E_p(\theta)$$

and therefore the security D is also an optimal choice.

5.3 Financing when both E and F are ambiguity averse

In this section we assume that E has a multiple prior set Π_E and F has a multiple prior set Π_F and we make the assumption that

$$\Pi_E \subseteq \Pi_F$$

5.3.1 IBEU Choices

We assume that both F and E make IBEU choices. The set of contracts satisfying the financing constraints is

$$\mathcal{H} = \left\{ D \mid D \leq \theta \text{ and } \inf_{\pi \in \Pi_F} E_\pi(D) \geq I \right\}$$

If E finances the project with $d \in \mathcal{H}$, he gets the utility The utility

$$U_E^\pi = -I + \inf_{\pi \in \Pi_F} E_\pi D + E_\pi(\theta - D)$$

for any given prior $\pi \in \Pi_E$. Because $d \leq \theta$ any contract from $\mathcal{H}$ is implementable and E is happy to start the firm and finance it with any $d \in \mathcal{H}$. Note that the structure of the set Π_E is irrelevant for this result provided that $\Pi_E \subseteq \Pi_F$. Now unlike the case where F is ambiguity neutral, it is possible to rank the financing contracts as the following proposition shows.

Proposition 4. *If $I > E_{p-\varepsilon}(\theta)$, then E cannot finance the project.*

If $\theta_d \leq I \leq E_{p-\varepsilon}$ *then* E *will accept to finance the project with any* d *in the set* $\mathcal{H}$.[3] *Moreover, all the contracts is the set* $\mathcal{H}$ *are dominated (from* E*'s perspective) by the contract*

$$D^* = \left(\theta_d + \frac{1}{p-\varepsilon}(I-\theta_d), \theta_d\right).$$

If $0 \leq I \leq \theta_d$*, then* E *will accept to finance the project with any* d *in the set* $\mathcal{H}$*. Moreover, all the contracts is the set* $\mathcal{H}$ *are dominated (from* E*'s perspective) by the risk free contracts of the form*

$$D = (\gamma, \gamma) \quad \textit{with } I \leq \gamma \leq \theta_d.$$

Notice that in all subcases, the only dominating contracts are the one which are optimal when E uses SEU to make decisions and and F uses IBEU to make decisions. The above proposition says that if we only select the dominating contracts, the ambiguity aversion of E is irrelevant in the context of our problem.[4]

Notice that the assumption $\Pi_E \subseteq \Pi_F$ seems crucial for the above proposition. Under this assumption the structure of the set of E's prior is irrelevant and everything is as if E has a single prior and SEU preferences. It can be shown that if we assume instead that $\Pi_F \subseteq \Pi_E$ we will have multiple contract which are not comparable as in the case where F has a single prior (section 5.2).

5.3.2 MEU Choices

Here again, the assumptions on the sets Π_E and Π_F are going to be important. If the set Π_F is the largest then E will issue risk free securities when they satisfy the financing

[3]Notice that when $\theta_d \leq I \leq E_{p-\varepsilon}$ the set $\mathcal{H}$ can also be defined as

$$\mathcal{H} = \{D \mid D \leq \theta \text{ and } E_{p-\varepsilon}(D) \geq I\}$$

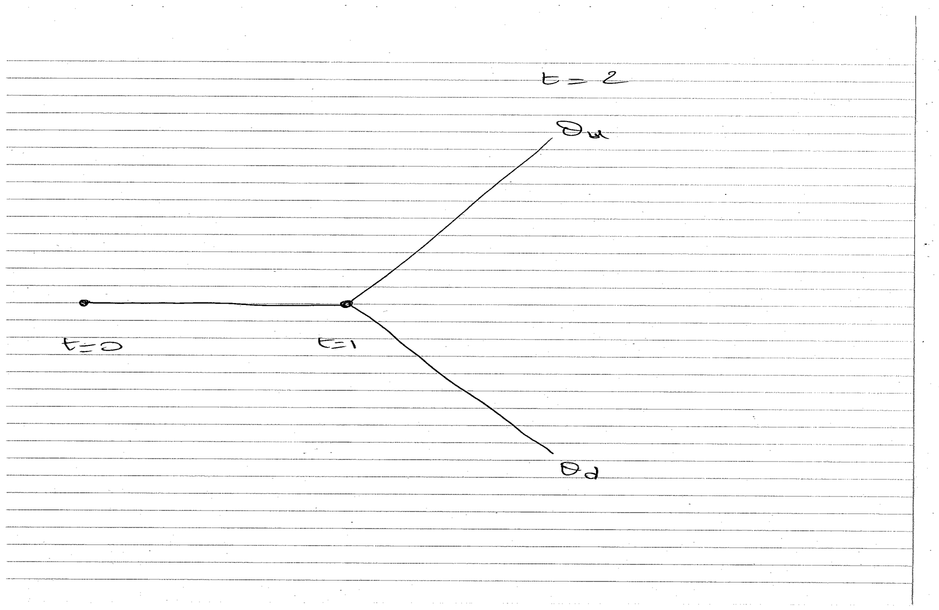
[4]However it seems that this true only when we assume that $\Pi_E \subseteq \Pi_F$. If we make the opposite assumption then we will have probably something similar to the case when E is ambiguity averse and F is ambiguity neutral (multiple optima).

constraints. If this is not possible they will issue a state contingent security that gives up all the firm in the down state. If the set Π_E is the largest then, E will sell the whole firm.

[To be completed]

6 Dynamic contracting under ambiguity

An entrepreneur (E) has access to an investment opportunity generating a cash flow θ_u in the "up" state and θ_d in the "down" state at time $t = 2$. The cash flow of the project is then $\theta = (\theta_u, \theta_d)$. A cost of I needs to be paid to begin the project and E does not have enough cash to pay for it. To simplify the model,we assume that the E has no personal wealth and if he wants to start the project, he must issue some securities to the savers (S) in order to start the project. The securities are issued at $t = 0$ and they have an option feature that can be exercised at the intermediate time $t = 1$.



We suppose that F has multiple priors in the set $\Pi = [p - \varepsilon, p + \varepsilon]$ and we will consider both MEU and IBEU preferences. At this stage, we do not commit to any preferences for E (we just discuss valuation and not contracting).

We will consider two type of primitive contracts. Risky debt has the form

$$B^\beta = (B_u^\beta = \theta_d + \beta(\theta_u - \theta_d), B_d^\beta = \theta_d) \text{ with } 0 \le \beta \le 1.$$

When $\beta = 0$, the debt is safe and when $\beta > 0$ the debt is risky and has a face value B_u^β.

Equity has the form

$$Q^\alpha = (\alpha\theta_u, \alpha\theta_d) \text{ with } 0 \le \alpha \le 1.$$

Because the payoff in the up state is always larger for the class of securities that we consider, we know that if F makes decisions based on IBEU or MEU, E will be able to raise the amount

$$E_{p-\varepsilon}(D), \text{ for } D = B^\beta or D = Q^\alpha.$$

Notice that the amount raised is equal under IBEU or MEU.

Now we turn to the cases where E issues a security with an option feature. Consider the security $BQ^{\alpha,\beta}$ which gives to the savers the possibility to convert the bond B^α to the equity Q^α. The conversion decision must be taken at time $t = 1$. Similarly, we consider the security $QB^{\alpha,\beta}$ which gives the savers to convert the equity to a bond at time $t = 1$. With both securities, F must decide if they exercise the option at time $t = 1$ by comparing the value under a particular prior π of the bond

$$\pi(\theta_d + \beta(\theta_u - \theta_d)) + (1 - \pi)(\theta_d) = \theta_d + \pi\beta(\theta_u - \theta_d)$$

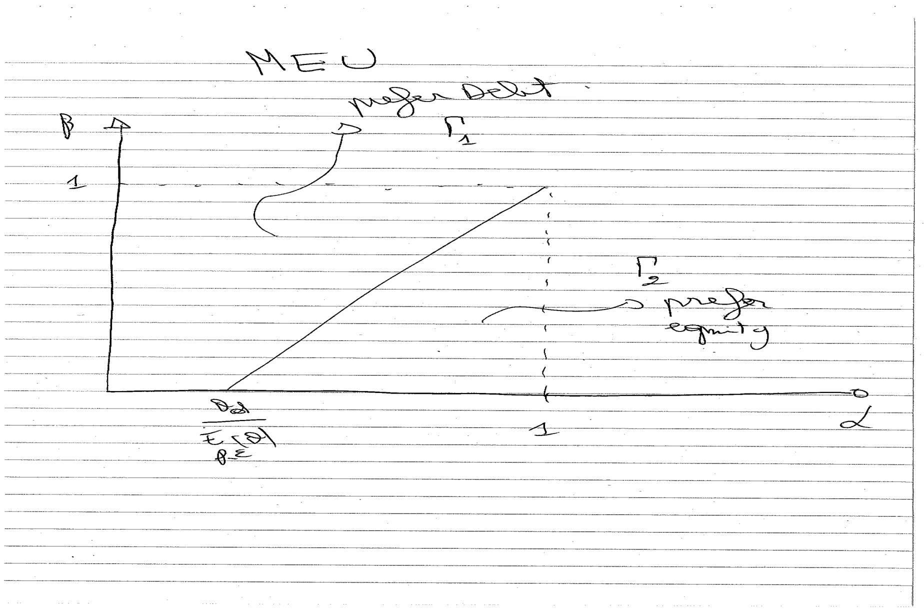
with the value of the equity

$$\alpha\pi\theta_u + \alpha(1 - \pi)\theta_d.$$

With MEU, the decision is based on the worst prior $p - \varepsilon$ and the indifference frontier in the plan (α, β) is given by

$$\alpha = \frac{\theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

The frontier splits the plan into a region Γ_1 where debt is preferred and a region Γ_2 where equity is preferred.



Therefore, both securities $BQ^{\alpha,\beta}$ and $QB^{\alpha,\beta}$ will have the same price which is given by

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \theta_d + (p-\varepsilon)\beta(\theta_u - \theta_d) \text{ if } (\alpha,\beta) \in \Gamma_1$$

and

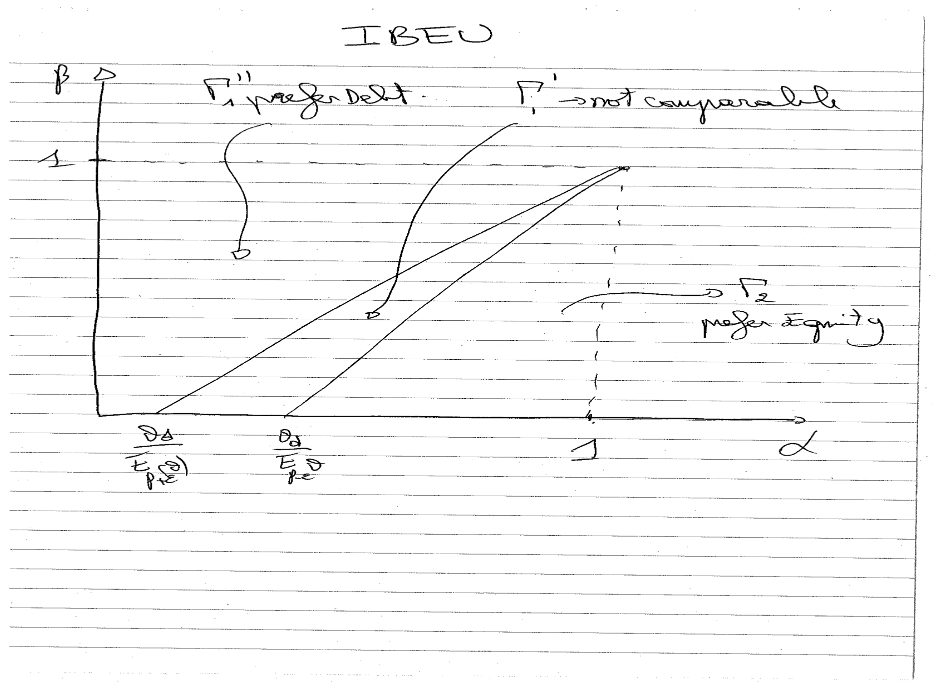
$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \text{ if } (\alpha,\beta) \in \Gamma_2$$

With IBEU, the plan is split into three regions Γ_1', Γ_1'' and Γ_2.

In the region Γ_1'' debt is preferred and in region Γ_2 equity is preferred. Region Γ_1' is a region where equity and debt are not comparable (under some prior debt is preferred whereas under some other priors equity is preferred). The region Γ_1' is described by the equation

$$\frac{\theta_d + (p+\varepsilon)\beta(\theta_u - \theta_d)}{E_{p+\varepsilon}(\theta)} \leq \alpha \leq \frac{\theta_d + (p-\varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

In the regions Γ_1'' and Γ_2 the proceeds are given as before



$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \begin{cases} \theta_d + (p-\varepsilon)\beta(\theta_u - \theta_d) \text{ if } (\alpha,\beta) \in \Gamma_1'' \\ \alpha E_{p-\varepsilon}(\theta) \text{ if } (\alpha,\beta) \in \Gamma_2 \end{cases}$$

But on the region Γ_1', the amount raised by E may change with the security that is issued. When pricing Security $BQ^{\alpha,\beta}$ with $(\alpha,\beta) \in \Gamma_1'$ at time $t = 0$, F knows that he will not exercise the conversion option because he will have a bond as a status quo. As a result, he will price this security as a straight bond. More formally, for any $(\alpha,\beta) \in \Gamma_1'$

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \theta_d + (p-\varepsilon)\beta(\theta_u - \theta_d).$$

Similarly, F will price the security $QB^{\alpha,\beta}$ as an equity and for any $(\alpha,\beta) \in \Gamma_1'$,

$$\text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta).$$

This result suggests that when issuing convertible equities (an equity with the option to convert it to a bond), it is possible E will be able to raise more money when F make decisions based on MEU than when they make decision based on IBEU.

Proposition 5. *For any $(\alpha, \beta) \in \Gamma_1'$, the proceeds from selling $QB^{\alpha,\beta}$ are larger when savers are MEU than when they are IBEU. More formally,*

$$Proceeds^{IBEU}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \leq Proceeds^{MEU}(QB^{\alpha,\beta}) = \theta_d + (p-\varepsilon)\beta(\theta_u - \theta_d)$$

The result in this proposition is a manifestation of the asymmetry that we already observed in the expansion/contraction options example of section 3. The proposition says that some securities with option features will be overvalued by MEU savers relative to IBEU savers.

Proposition5 illustrates an important difference between IBEU and MEU. Consider a hybrid security A offering the ownership of security X with the option to convert it one period to security Y. Assume securities X and Y pay the cash flows after the first period. Consider the alternative hybrid security B offering the ownership of security Y with the option to convert it one period to security X. Assume that there is no information revelation about cash flows between the issuance date and the option exercise date.

When F uses SEU with the prior $\pi = p$, the *ex ante* valuation of the two securities is clearly identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{ Max } \{E_p(X), E_p(Y)\}$$

Notice that the security valuation at $t = 0$ is also identical if it was possible to commit to a particular policy exercise. From the perspective of time $t = 0$, the SEU saver will also pick the security delivering the highest expected payoff under the probability p even when commitment is possible.

When F is MEU, the *ex ante* valuation of the two securities is also identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{ Max } \left\{\inf_{\pi} E_\pi(X), \inf_{\pi} E_\pi(Y)\right\}$$

The security valuation at time $t = 0$ does not change if it was possible for saver to commit to a particular policy decision. In this case, the MEU saver will still pick the security that offers the highest expected payoff according to the worst probability measure.

With IBEU the valuation of security A can be different from the valuation of Security B. Specifically, when the primitive security X is not comparable with security Y, the inertia assumption shows that

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(X)$$

whereas

$$\text{Proceeds}(B) = \inf_{\pi} E_{\pi}(Y)$$

This is a first important difference between MEU and IBEU: it seems that IBEU is sensitive to the sequencing of the options of hybrid securities whereas sequencing is irrelevant for SEU and MEU. The sequence of option is relevant for IBEU because it is induces a particular path of status quo which in turn break down the indifference in the exercise choice.

The second difference is that when commitment on the exercise policy is possible, the valuation of the security A can be different form the valuation of the security A in the absence of commitment. An interesting situation occurs when X is not comparable to 0 whereas X dominates 0.[5] In this case, the commitment exercise policy for the hybrid security A is to convert it to Y because the commitment solution uses 0 as a status quo. As a result, the valuation of security A under commitment is given by

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(Y)$$

and it is different from its valuation when commitment is not possible.

[5]This can occur when $X = (-1, 3)$ and $Y = (1, 2)$. With a large enough set of priors, we see that X is not comparable with 0. The security Y always dominates 0. Again with a large enough set of priors X is not comparable with Y.

7 Conclusion

We have examined the way in which ambiguity aversion, modelled as multiple priors, affects financing decisions. There are several approaches to modelling decision making in the presence of multiple priors and we consider two specific approaches: Maxmin Expected Utility (MEU) and Inertia Based Expected Utility (IBEU). Both approaches deliver novel impacts for financing choices that in turn provide testable implications. Future empirical work may help determine if an when the different approaches are used in practice.

A Appendix: Proofs

Proof of Proposition 1

We start with the observation that an optimal security must bind the constraint

$$\inf_{\pi \in \Pi_F} E_\pi(D) = I$$

If $\inf_{\pi \in \Pi_F} E_\pi(D) > I$, then we can decrease the payoff of d by a small amount in one of the two states and we will then improve E's utility.[6]

Now, if $I \leq \theta_d$, then $d = (I, I) \in \mathcal{D}$ and E's utility is maximized because Inequality (18) binds. On the other hand, if $I > E_{p-\varepsilon}(\theta)$, then E is not going to be able to finance the project even if he sells the entire firm.

If $\theta_d \leq I \leq E_{p-\varepsilon}(\theta)$ then it possible to finance the project. Assume that E finance with the project with a security that has $d_d < \theta_d$. It is then necessary to have $d_u > \theta_d$ to be able to raise I (because $I \geq \theta_d$). Because $d_u > D_d$, we have $I = \inf_{\pi \in \Pi_F} E_\pi(D) = E_{p-\varepsilon}(D)$. We will now show that a small modification of d allows E to increase the project valuation. Consider the security d' defined by

$$D'_d = D_d + \eta, \quad D'_u = D_u - \eta \frac{p - \varepsilon}{1 - p + \varepsilon}$$

where η is a very small number. By construction, issuing d' allows to raise exactly I because $\inf_{\pi \in \Pi_F} E_\pi(D') = E_{p-\varepsilon}(D) = I$. It is also easy to verify that,

$$U_E(D') = U_E(D) + \eta \frac{\varepsilon}{1 - p + \varepsilon}$$

[6] To clarify this point, suppose that $d_u > D_d$, then the inf of $E_\pi(D)$ is attained at $\pi = p - \varepsilon$. Consider the security $D' = (D_u - \eta, D_d)$ with η being a small number so that $E_{p-\varepsilon}(D') \geq I$. It can be checked that E's utility derived from issuing d' is given by

$$U_E(D') = U_E(D) + \varepsilon\eta > U_E(D).$$

and therefore the entrepreneur prefers to issue d'. A similar reasoning can be used when $D_u > D_d$ by considering the security $D'' = (D_u, D_d - \eta)$.

and thus $u_E(D') > U_E(D)$. We conclude then that the optimal security satisfies $d_d = \theta_d$ and the financing constraint requires that

$$d_u = \theta_d + \frac{1}{p - \varepsilon}(I - \theta_d).$$

∎

Proof of Proposition 2

We have already observed that any security d in the set $\mathcal{G}$ is going to generate a positive u_E^{π} under any π in the set of E's priors Π_E and thus the entrepreneur is happy to start the project by issuing d.

To show that these contracts are not comparable, let us consider a security d in $\mathcal{G}$. We have $u_E^{\pi}(D) = -I + E_p(D) + E_{\pi}(\theta - D)$ for any $\pi \in \Pi_E$. Define the security $d' = (D_u + \eta, D_d)$ with η small enough so that $d' \in \mathcal{G}$. Direct calculations show $u_E^{\pi}(D') = -I + E_p(D) + E_{\pi}(\theta - D) + \eta(p - \pi)$ and thus

$$U_E^{\pi}(D') - U_E^{\pi}(D) = \eta(p - \pi)$$

which can be positive for some π and negative for some other π provided that p is the interior of Π_E. We thus conclude that d and d' are not comparable for E. ∎

Proof of Proposition 3

If E finance the firm by issuing $d \in \mathcal{G}$, then he derives the utility

$$U_E(D) = -I + E_p(D) + \inf_{\pi} E_{\pi}(\theta - D).$$

If he issues $d = \theta$, then the derived utility is

$$U_E(\theta) = -I + E_p(\theta).$$

We can see that for any $d \in \mathcal{D}$ we have

$$U_E(\theta) - U_E(D) = E_p(\theta - D) - \inf_{\pi} E_\pi(\theta - D) \geq 0$$

and therefore $u_E(\theta) = \max_{D \in \mathcal{G}} U_E(D)$ and it optimal to sell the whole firm. ∎

Proof of Proposition 4

Let us start with the case $\theta_d \leq I \leq E_{p-\varepsilon}(\theta)$. Notice first that in this case, the set $\mathcal{H}$ is above the 45 degree line and thus any $d \in \mathcal{H}$ satisfies $d_d \leq D_u$. Thus each time E's finances the project with $d \in \mathcal{H}$, he will get the proceeds $E_{p-\varepsilon}(D)$. Now, it is convenient to stratify $\mathcal{H}$ as

$$\mathcal{H} = \bigcup_{I \leq \gamma \leq E_{p-\varepsilon}(\theta)} \mathcal{H}_\gamma, \text{ where } \mathcal{H}_\gamma = \{D \mid 0 \leq D \leq \theta \text{ and } E_{p-\varepsilon}(D) = \gamma\}.$$

Let us first prove that any $d \in \mathcal{H}_\gamma$ is dominated by $\overline{D}$ which is the unique element of $\mathcal{H}_\gamma$ satisfying $\overline{D}_d = \theta_d$. Notice that $\overline{D} = (D_u - \alpha, \theta_d)$ where $\alpha > 0$ solves the equation

$$\alpha(p - \varepsilon) - (\theta_d - D_d)(1 - p + \varepsilon) = 0.$$

For any prior $\pi \in \Pi_E$,

$$U_E^\pi(\overline{D}) - U_E^\pi(D) = E_\pi(D) - E_\pi(\overline{D}) = \pi\alpha - (1 - \pi)(\theta_d - D_d)$$

Because $\Pi_E \subseteq \Pi_F$, we have $\pi \geq p - \varepsilon$ for any $\pi \in \Pi_E$ by comparing the last two equalities we see that $u_E^\pi(\overline{D}) - U_E^\pi(D) \geq 0$. We conclude that $\overline{D}$ is preferred to d. The second step is to show that any $d \in \mathcal{H}_\gamma \cap \{D \mid D_d = \theta_d\}$ for some $\gamma \in [I, E_{p-\varepsilon}(\theta)]$ is dominated by D^*. First observe that financing the firm with a security $d \in \mathcal{H}_\gamma \cap \{D \mid D_d = \theta_d\}$ yield the utility

$$U_E^\pi(D) = -I + E_{p-\varepsilon}(D) + E_\pi(\theta - D) = -I + \gamma + E_\pi(\theta - D)$$

for any $\pi \in \Pi_E$. On the other hand if E finances the project with D^*, he gets the utility

$$U_E^\pi(D^*) = -I + E_{p-\varepsilon}(D^*) + E_\pi(\theta - D^*) = E_\pi(\theta - D^*)$$

for any any $\pi \in \Pi_E$. Therefore

$$U_E^\pi(D^*) - U_E^\pi(D) = I - \gamma + E_\pi(D^* - D) = I - \gamma + \pi(D_u - D_u^*)$$

Using the fact that $(D^*, D) \in \mathcal{H}_I \times \mathcal{H}_\gamma$ and the fact that $D_d^* = D_d = \theta_d$ we get

$$U_E^\pi(D^*) - U_E^\pi(D) = (\gamma - I)\left[\frac{\pi}{p - \varepsilon} - 1\right]$$

which is positive for any $\pi \in \Pi_E$, again because $\pi \geq p - \varepsilon$ since we assumed that $\Pi_E \subseteq \Pi_F$. To summarize, we have shown that D^* is preferred to any other implementable contract in $(\mathcal{H})$ and thus we can consider that the only "stable" contract is D^*.

Let us now turn to the case $I \leq \theta_d$. This case is different because the 45 degrees crosses the set $\mathcal{H}$ and splits it into two subsets

$$\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$$

where $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) contains all the elements $d \in \mathcal{H}$ satisfying $d_u \geq D_d$ (resp. $d_u < D_d$). We will show here that while E is happy to finance the project with any security in the set $\mathcal{H}$ he still prefers to finance the firm with a risk free security.

First, let us mention that if E finances the project with a security of the form $d = (\gamma, \gamma)$ with $\gamma \in [I, \theta_d]$ then he will get the utility

$$U_E^\pi(D) = -I + E_\pi(\theta)$$

under the prior π. As a result, E is indifferent (prior by prior) between any two risk free contracts in $\mathcal{H}$. We will now focus on showing the dominance of the contract $d^f = (\theta_d, \theta_d)$ over all other contracts.

If E finances the project with contract $d \in \mathcal{H}^+$, the financiers will pay $\inf_{\pi \in \Pi_F} E_\pi(D) = E_{p-\varepsilon}(D)$ and E gets the utility

$$U_E^\pi(D) = -I + E_{p-\varepsilon}(D) + E_\pi(\theta - D)$$

for any prior $\pi \in \Pi_E$.

On the other hand, if E instead finances the project with the risk free security d^f he will get the utility

$$U_E^\pi(D^f) = -I + E_\pi(\theta)$$

for any prior $\pi \in \Pi_E$ and thus

$$U_E^\pi(D^f) - U_E^\pi(D) = E_\pi(D) - E_{p-\varepsilon}(D).$$

Noticing that $\pi \geq p - \varepsilon$ and $d_u \geq D_d$ yields

$$U_E^\pi(D^f) \geq U_E^\pi(D) \text{ for all } \pi \in \Pi_E$$

meaning that E prefers d^f to any contract in $\mathcal{H}^+$.

Now, if E finances the project with $d \in \mathcal{H}^-$, the financier F pays $\inf_{\pi \in \Pi_F} E_\pi(D) = E_{p+\varepsilon}(D)$ he gets the utility

$$U_E^\pi(D) = -I + E_{p+\varepsilon}(D) + E_\pi(\theta - D)$$

for any prior $\pi \in \Pi_E$. Thus

$$U_E^\pi(D^f) - U_E^\pi(D) = E_\pi(D) - E_{p+\varepsilon}(D)$$

and recalling that $\pi \leq p + \varepsilon$ and $d_u \leq D_d$ gives

$$U_E^\pi(D^f) \geq U_E^\pi(D) \text{ for all } \pi \in \Pi_E.$$

∎

Proof of Proposition 5

To be added.

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