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The Big Problem with Small Change

Thomas J. Sargent and François R. Velde

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Thomas J. Sargent

University of Chicago and Hoover Institution

François R. Velde

Federal Reserve Bank of Chicago

ABSTRACT

Western Europe was plagued with currency shortages from the 14th to the 19th century, at which time a 'standard formula' had been devised to cure the problem. We use a cash-in-advance model of commodity money to define a currency shortage, show that they could develop and persist under a commodity money regime, and analyze the role played by each ingredient in the standard formula. A companion paper documents the evolution of monetary theory, monetary experiments and minting technology over the course of six hundred years.

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* The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

1. Introduction

The title of this paper paraphrases one by Carlo Cipolla.[1] Like Cipolla, our subject is the process through which western monetary authorities learned how to supply small change. Along with many other writers, Cipolla described how Western Europeans long struggled to sustain a proper mix of large and small denomination coins, and to break away from the idea that a commodity coinage requires coins of *all* denominations to be full-bodied.[2]

The Carolingian monetary system, born about A.D. 800, had only one coin, the penny. At the end of the twelfth century, various states began also to create larger denomination coins. From the thirteenth to the nineteenth century, there were recurrent 'shortages' of the smaller coins. Cipolla (1956, 31) states that: 'Mediterranean Europe failed to discover a good and automatic device to control the quantity of petty coins to be left in circulation,' a failure which extended across all Europe.[3]

By the middle of the nineteenth century, the mechanics of a sound system were well understood, thoroughly accepted,[4] and widely implemented. According to Cipolla (1956, 27):

> 'Every elementary textbook of economics gives the standard formula for maintaining a sound system of fractional money: to issue on government account small coins having a commodity value lower than their monetary value; to limit the quantity of these small coins in circulation; to provide convertibility with unit money. . . . Simple as this formula may seem, it took centuries to work it out. In England it was not applied until 1816, and in the United States it was not accepted before 1853.'

Before the triumph of the 'standard formula', fractional coins were more or less full bodied, and contained valuable metal roughly in proportion to their nominal values,

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[1] 'The Big Problem of the Petty Coins', in Cipolla (1956).

[2] The mix matters because small coins can make the same transactions as large coins, but not *vice versa*. This is true so long as there is no 'limited legal tender' clause of the kind mentioned by Cipolla as a possible fourth element of his 'standard formula'.

[3] These shortages are documented in Sargent and Velde (1997b).

[4] For example, see John Stuart Mill (1857, chapter X, Section 2).

contradicting one element of the 'standard formula.' Supplies were determined by private citizens who decided if and when to use metal to purchase new coins from the mint at prices set by the government, contradicting another element of the standard formula. That system produced chronic shortages of small coins, but also occasional gluts. Gradually over the centuries, theorists proposed components of the 'standard formula'; occasionally policy makers even implemented some of them. Full implementation waited until 1816, in Britain; and, over the following 60 years, in France, Germany, the United States and other countries, culminating in the establishment of the 'Classical Gold Standard' with silver and bronze or copper coinage as subsidiary money.

Our goal in this paper is to understand what made the medieval monetary system defective, and why it took so long to implement the standard formula. We present a model of supply and demand for large and small metal coins designed to simulate the medieval and early modern monetary system, and to show how its supply mechanism lay vulnerable to alternating shortages and surpluses of small coins. We extend Sargent and Smith's (1997) model to incorporate demands and supplies of two coins differing in denomination and possibly in metal content. We specify cash-in-advance constraints to let small coins make purchases that large coins cannot. Like the Sargent-Smith model, for each type of coin, the supply side of the model determines a range of price levels whose lower and upper boundaries trigger coin minting and melting, respectively. These ranges let coins circulate above their intrinsic values. The ranges must coincide if *both* coins are to continue to circulate. The demand side of the model delivers a sharp characterization of 'shortages' of small coins. 'Shortages' make binding our additional cash-in-advance constraint—the 'penny-in-advance' constraint that requires that small purchases be made with small coins. This means that small coins must *depreciate* in value relative to large coins during shortages of small coins. Thus in our model, 'shortages' of small coins have two symptoms: (1) the quantity theory of money splits in two, one for large coins, another for small; and (2) small coins must depreciate relative to large ones in order to render binding the 'small change in advance constraint' and thereby provide a motive for money holders to economize on small change. In conjunction with the supply mechanism, the second response actually aggravates the shortage.[5]

[5] The 'standard formula' mentioned by Cipolla solves the exchange rate indeterminacy problem

We use these features of the model to account for various historical outcomes, and, among other things, to account for why debasements of small coins were a common policy response to shortages of small change. Then we modify the original arrangement by including one or more of the elements in the 'formula' recounted by Cipolla, and show the consequences. We compare these consequences with the predictions and prescriptions of contemporary monetary theorists, and with episodes in monetary history, in particular the spectacular failure of a fiduciary coinage in 17th century Spain, and the lessons drawn by contemporary observers. By way of explaining why it took so long to come to the 'standard formula', we enumerate the theoretical, technological and institutional prerequisites for its implementation.[6] In a companion paper (Sargent and Velde 1997), we document how 'theory' was ahead of technology and institutions until 1816.

In her fascinating account of Britain's adoption of the gold standard, Angela Redish (1990) confronts many of the issues studied in this paper, and traces England's inability to adopt the gold standard before the 19th century to the problem of small change. She finds that "technological difficulties (the threat of counterfeiting) and institutional immaturity (no guarantor of convertibility)" were the main obstacles. According to Redish, Matthew Boulton's steam-driven minting press of 1786 overcame the first obstacle by finally giving the government a sufficient cost advantage over counterfeiters. Redish identifies no such watershed with respect to convertibility. Though she traces the Bank of England's protracted negotiations with the Treasury over responsibility for the silver coinage, it is not clear what the obstacle was, and how it was overcome. In her conclusion, Redish suggests the earlier history of small change as a possible testing field for her theory. In some ways, we are pursuing this project here.

This paper has a companion (Sargent and Velde 1997) that traces in detail the histories of thoughts, technologies, and policies bearing on issues of coinage. That parallel study motivated the questions and modeling decisions in the present paper.

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inherent in any system with an inconvertible and less than full bodied fractional currency. Remember that Russell Boyer's (1971) original paper on exchange rate paper was titled 'Nickels and Dimes'. See Kareken and Wallace (1981) and Helpman (1981) for versions of exchange rate indeterminacy in models of multiple fiat currencies. A 'one-sided' exchange rate indeterminacy emerges from the cash-in-advance restrictions in our model, and determines salient predictions of the model.

[6] We will also discuss a sense in which the formula has redundant elements.

The remainder of the paper is organized as follows. Section 2 describes the model environment, the money supply arrangement, and the equilibrium concept. Section 3 uses 'back-solving' to indicate possible co-movements of the price level, money supplies, and national income; to illustrate perverse aspects of the medieval supply arrangements; and to interpret various historical outcomes. Section 4 uses the model to study how aspects of Cipolla's 'standard formula' remedy the perverse supply responses by making small change into tokens. Section 5 concludes with remarks about how, by learning to implement a token small change, the West rehearsed a complete fiat monetary system.

2. The Model

In a 'small country' there lives an immortal representative household that gets utility from two nonstorable consumption goods. The household faces cash-in-advance constraints. 'Cash' consists of a large and a small denomination coin, each produced by a government-regulated mint that stands ready to coin any silver brought to it by household-owned firms. The government specifies the amounts of silver in large and small coins, and also collects a flat-rate seigniorage tax on the volume of newly minted coins; it rebates the revenues in a lump sum. Coins are the only storable good available to the household. The firm can transform either of two consumption goods into the other one-for-one and can trade either consumption good for silver at a fixed international price. After describing these components of the economy in greater detail, we shall define an equilibrium. For any variable we let x denote the infinite sequence $\{x_t\}_{t=0}^{\infty}$. Table 1 compiles the main parameters of the model and their units.

Variable	Meaning	Units
ϕ	world price of silver	oz silver/cons good
b_1	intrinsic content of dollar	oz silver/dollar
b_2	intrinsic content of penny	oz silver/penny
γ_1	melting point of dollar	dollars/cons good
γ_2	melting point of penny	pennies/cons good
σ_i	seigniorage rate	(none)
${b_1}^{-1}$	mint equivalent of dollar	dollars/oz silver
${b_2}^{-1}$	mint equivalent of penny	pennies/oz silver
m_1	stock of pennies	pennies
m_2	stock of dollars	dollars
e	exchange rate	dollars/penny
p	price of cons goods	dollars/cons good

Table 1: Dramatis Personae.

The Household

The representative household maximizes

$$\sum_{t=0}^{\infty} \beta^t u\left(c_{1,t}, c_{2,t}\right) \tag{1}$$

where we assume the one-period utility function $u(c_{1,t}, c_{2,t}) = v(c_{1,t}) + v(c_{2,t} + \alpha)$ where $\alpha > 0$ and $v(\cdot)$ is strictly increasing, twice continuously differentiable, strictly concave, and satisfies the Inada conditions $\lim_{\to 0} v'(x) = +\infty$. We use consumption of good 1, c_1, to represent 'large' purchases, and c_2 to stand for small purchases. There are two kinds of cash: dollars, whose stock is m_1, and pennies, whose stock is m_2. Each stock is measured in number of coins, dollars or pennies. Both pennies and dollars can be used for large purchases, but only pennies can be used for small purchases.[7] A penny exchanges for e_t dollars. Thus, the cash-in-advance constraints are:

$$p_t\left(c_{1,t}+c_{2,t}\right) \leq m_{1,t-1}+e_t m_{2,t-1} \tag{2}$$

$$p_t c_{2,t} \leq e_t m_{2,t-1}, \tag{3}$$

where p_t is the dollar price of good i. The household's budget constraint is

$$p_t\left(c_{1,t}+c_{2,t}\right)+m_{1,t}+e_t m_{2,t} \leq \Pi_t+m_{1,t-1}+e_t m_{2,t-1}+T_t, \tag{4}$$

where Π_t denotes the firm's profits measured in dollars, and T_t denotes lump sum transfers from the government. The household faces given sequences (p, e, T), begins life with initial conditions $m_{1,-1}, m_{2,-1}$, and chooses sequences c_1, c_2, m_1, m_2 to maximize (1) subject to (2), (3), and (4).

Feasibility

A household-owned firm itself owns an exogenous sequence of an endowment $\{\xi_t\}_{t=0}^{\infty}$. In the international market, one unit of either consumption good can be traded for $\phi > 0$ units of silver,[8] leading to the following restrictions on feasible allocations:

$$c_{1,t}+c_{2,t} \leq \xi_t+\phi^{-1} S_t, \qquad t \geq 0 \tag{5}$$

where S_t stands for the net exports of silver from the country.

Coin Production Technology and the Mint

[7] The assumption that pennies can be used for the same purchases as dollars is motivated by several episodes in history during which small denominations overtook the monetary functions of large denominations with great ease.

[8] Alternatively, there is a reversible linear technology for converting consumption goods into silver.

Stocks of coins evolve according to

$$m_{i,t} = m_{i,t-1} + n_{i,t} - \mu_{i,t} \tag{6}$$

where $n_{i,t} \geq 0, \mu_{i,t} \geq 0$ are rates of minting of dollars, $i = 1$, and pennies, $i = 2$. The government sets b_1, the number of ounces of silver in a dollar, and b_2, the number of ounces of silver in a penny. The government levies a seigniorage tax on minting: for every new coin of type i minted, the government charges a flat tax at rate $\sigma_i > 0$.

The country melts or mints coins to finance net exports of silver in the amount

$$S_t = b_1 \left(\mu_{1,t} - n_{1,t}\right) + b_2 \left(\mu_{2,t} - n_{2,t}\right). \tag{7}$$

Net exports of silver S_t correspond to net imports of $\phi^{-1} S_t$ of consumption goods.

The quantities $1/b_1$ and e_t/b_2 (measured in number of dollars per minted ounce of silver) are called by Redish (1990) 'mint equivalents'. The quantities $(1-\sigma_1)/b_1, e_t(1-\sigma_2)/b_2$ are called 'mint prices', and equal the number of dollars paid out by the mint per ounce of silver.

Government

The government sets the parameters $b_1, b_2, \sigma_1, \sigma_2$, and, depending on citizens' minting decisions, collects revenues T_t in the amount

$$T_t = \sigma_1 n_{1,t} + \sigma_2 e_t n_{2,t}. \tag{8}$$

Below we shall describe other interpretations of σ_i partly in terms of the mint's costs of production. The only modification that these alternative interpretations require would be to (8).

The Firm

The firm receives the endowment, sells it, mints and melts, pays seigniorage, and pays all earnings to the household at the end of each period. The firm's profit measured in dollars is

$$\begin{aligned}\Pi_t =& p_t \xi_t + (n_{1,t} + e_t n_{2,t}) - (\sigma_1 n_{1,t} + \sigma_2 e_t n_{2,t}) \\ & - p_t \left(\frac{b_1}{\phi} n_{1,t} + \frac{b_2}{\phi} n_{2,t} \right) + p_t \left(\frac{b_1}{\phi} \mu_{1,t} + \frac{b_2}{\phi} \mu_{2,t} \right) - (\mu_{1,t} + e_t \mu_{2,t}) \,. \end{aligned} \tag{9}$$

The first term measures revenues from the sale of the endowment; the next term is revenues of minting, followed by seigniorage payments, minting costs, melting revenues and melting costs.

The firm takes the price system (p, e) as given and chooses minting and melting sequences (n_1, n_2, μ_1, μ_2) to maximize (9) subject to (6) period-by-period.

Equilibrium

A *feasible allocation* is a triple of sequences (c_1, c_2, S) satisfying (5). A *price system* is a pair of sequences (p, e). A *money supply* is a pair of sequences (m_1, m_2) satisfying the initial conditions $(m_{1,-1}, m_{2,-1})$. An *equilibrium* is a price system, a feasible allocation, and a money supply such that given the price system, the allocation and the money supply solve the household's problem and the firm's problem.

Analytical Strategy

We proceed sequentially to extract restrictions that our model places on co-movements of the price level and the money supply. The firm's problem puts restrictions on these co-movements, and the household's problem adds more.

Arbitrage Pricing Conditions

The firm's problem puts the price level inside two intervals.[9] Define $\gamma_i = \phi/b_i$ and rearrange (9):

$$\Pi_t = p_t\xi_t + \left(1 - \sigma_1 - p_t\gamma_1^{-1}\right) n_{1,t} + e_t \left(1 - \sigma_2 - p_t \left(e_t\gamma_2\right)^{-1}\right) n_{2,t} + \left(p_t\gamma_1^{-1} - 1\right) \mu_{1,t} + e_t \left(p_t \left(e_t\gamma_2\right)^{-1} - 1\right) \mu_{2,t}. \quad (10)$$

Each period, the firm chooses $n_{i,t}$, $\mu_{i,t}$ to maximize Π_t subject to non-negativity constraints $n_{i,t} \geq 0$, $\mu_{i,t} \geq 0$ and to the upper bound on melting: $m_{i,t-1} \geq \mu_{i,t}$, for $i = 1, 2$. The form of (10) immediately implies the following no-arbitrage conditions:[10]

$$n_{1,t} \geq 0; \quad = \quad \text{if} \quad p_t > \gamma_1 \left(1 - \sigma_1\right) \quad (11a)$$

$$n_{2,t} \geq 0; \quad = \quad \text{if} \quad p_t > e_t\gamma_2 \left(1 - \sigma_2\right) \quad (11b)$$

$$\mu_{1,t} \geq 0; \quad = \quad \text{if} \quad p_t < \gamma_1 \quad (11c)$$

$$\mu_{2,t} \geq 0; \quad = \quad \text{if} \quad p_t < e_t\gamma_2 \quad (11d)$$

$$\mu_{1,t} \leq m_{1,t-1}; \quad = \quad \text{if} \quad p_t > \gamma_1 \quad (11e)$$

$$\mu_{2,t} \leq m_{2,t-1}; \quad = \quad \text{if} \quad p_t > e_t\gamma_2. \quad (11f)$$

Implications of the Arbitrage Conditions for Monetary Policy

These no-arbitrage conditions constrain the mint's policy if both coins are to exist.[11] The constraints put p_t within both of two intervals, $[\gamma_1(1 - \sigma_1), \gamma_1]$ (corresponding to dollars) and $[e_t\gamma_2(1 - \sigma_2), e_t\gamma_2]$ (corresponding to pennies). See Figure 1. Only when the price level p_t is at the lower end of either interval might the associated coin be minted. Only when the price is at the upper end might that coin be melted. Therefore, if the lower ends of the intervals do not coincide, (i.e., if $\gamma_1(1-\sigma_1) \neq e_t\gamma_2(1-\sigma_2)$), then only one type

[9] See Usher (1943) for a discussion of these constraints on the price level.

[10] These restrictions must hold if the right side of (10) is to be bounded (which it must be in any equilibrium); their violation would imply that the firm could earn unbounded profits.

[11] See the related discussion in Usher (1943, 1997–201).

of coin can ever be minted. Equating the lower ends of the intervals (by the government's choice of (b_i, σ_i)) makes the mint stand ready to buy silver for the same price, whether it pays in pennies or dollars. Equating the upper ends of the intervals makes the ratio of metal contents in the two coins equal the exchange rate. If the upper ends of the intervals don't coincide, then one type of coin will be melted before the other. Pennies are said to be full-bodied if $1/b_1 = e/b_2$, that is, if the melting points (upper limits) of the two intervals coincide. Household preferences and equations (2) and (3) imply that $e_t\gamma_2 \geq \gamma_1$. If this inequality is strict, the intrinsic content of pennies is less than proportionate to their value in dollars, in which case pennies are 'light'.[12]

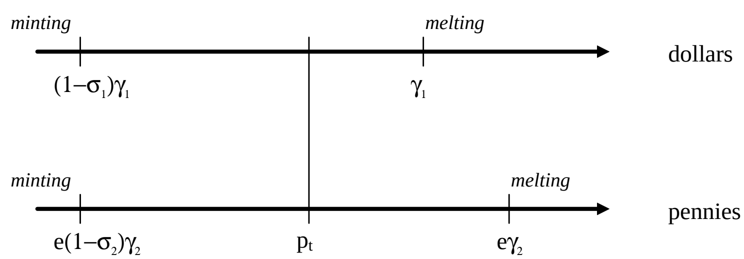

Figure 1: Constraints on the price level imposed by the arbitrage conditions.

Thus, if pennies are not full-bodied, a sufficient rise in the price level will make large coins disappear. If the mint prices differ, a sufficient fall in prices will prompt minting of

[12] By substituting $\Pi_t = p_t\xi_t$, (8), and (6) into (4) we obtain

$$p_t\,(c_{1,t} + c_{2,t}) \leq p_t\xi_t + \mu_{1,t} + e_t\mu_{2,t} - (1-\sigma_1)\,n_{1,t} - e_t\,(1-\sigma_2)\,n_{2,t}.$$

Using the no-arbitrage conditions (11) in this expression and rearranging leads to

$$c_{1,t} + c_{2,t} \leq \xi_t + \left[\gamma_1^{-1}\,(\mu_{1,t} - n_{1,t}) + \gamma_2^{-1}\,(\mu_{2,t} - n_{2,t})\right].$$

The term in square braces equals net imports of consumption goods. Thus, as usual, manipulation of budget sets at equilibrium prices recovers a feasibility condition.

only one of the two coins. The perpetual coexistence of both coins in the face of price fluctuations requires that pennies be full-bodied and that equal mint prices prevail for both coins; that is, the intervals must coincide, and therefore the seigniorage rates must be equal. This is not possible if we reinterpret the σ_i's in terms of the production costs for the two types of coins.

Interpretations of σ_i

In using (8), we have interpreted σ_i as a flat tax rate on minting of coins of type i. But so far as concerns the firm's problem and the arbitrage pricing restrictions, σ_i can be regarded as measuring *all* costs of production borne by the mint, including the 'seigniorage' it must pay to the government. On this interpretation, in setting σ_i, the government is naming the sum of the seigniorage tax rate and the mint's costs of production. If a government was unwilling to *subsidize* production of coins, then the costs of production served as a lower bound on σ_i.

The government could decide to set gross seigniorage σ_i to 0, by subsidizing the mint. In this circumstance, our two coins could coexist only if pennies were full-weight (their intrinsic content being proportional to their face values), and if the price level never deviated from γ_1.

We discuss in the companion paper the attitudes of medieval writers as well as the actual policies followed by governments. A tradition of thought advocated setting $\sigma_i = 0$ but it was not followed in practice until the 17th century; other jurists thought that σ_i should remain close to production costs, except in cases of clearly established fiscal emergency.

On the other side, restraints were placed on the government's freedom to set σ_i by potential 'competitors' to the mint, such as counterfeiters and foreign mints, and by the government's ability to enforce laws against counterfeiting and the circulation of foreign coins. Let $\tilde{\sigma}_i$ be the production costs for counterfeiters, or for arbitrageurs taking metal to foreign mints and bringing back coins (inclusive of transport costs). A wide gap between σ_i and $\tilde{\sigma}_i$ was difficult to maintain unless government's enforcement powers were strong. If they were not, $\tilde{\sigma}_i$ placed an upper bound on σ_i.[13]

A government could maintain positive seigniorage if the costs of production of licensed mints were smaller than those of competitors. For example, Montanari ([1683] 1804, 114) argues that the death penalty for counterfeiting, while impossible to enforce strictly, adds a risk premium to counterfeiters' wage bill, thereby increasing $\tilde{\sigma}_i$ when the same technology is used by all. Furthermore, if a government was able to restrict access to the mint's technology, or if it could secure the exclusive use of a better technology, it could set seigniorage above the mint's production costs, up to the level of competitors' costs.[14]

Per coin production costs differed between small and large denomination coins. The Medieval technology made it significantly more expensive to produce smaller denomination coins.[15] In situations that tied the σ_i's to the costs of production, different production costs implied different widths of our no-arbitrage intervals. This meant either that pennies had to be less than full-bodied or that the mint prices differed. In either case, price level fluctuations could arrest production or, by stimulating melting, cause the disappearance of one coin.

We have extracted the preceding restrictions from the requirement that equilibrium

[13] See Usher (1943, 201): "Seigniorage presented no special problem unless the amount exceeded the average rate of profit attractive to gold and silversmiths, or to mints in neighboring jurisdictions. Beyond this limit, the effective monopoly of coinage might be impaired by illegal coinage of essentially sound coins, or by the more extensive use of foreign coin."

[14] Philip II, king of Spain, referred explicitly to this advantage when he put a new technology to use in making small copper coinage in 1596. Mentioning the new mechanized mint of Segovia, he declared that "if we could mint the billon coinage in it, we would have the assurance that it could not be counterfeited, because only a small quantity could be imitated and not without great cost if not by the use of a similar engine, of which there are none other in this kingdom or the neighboring ones."

[15] This was true of competitors' costs as well: Montanari notes that the risk premium induced by the death penalty is the same across denominations. For arbitrageurs taking metal to foreign mints, transportation costs made the operation worthwhile only for the larger coins; the near-uniformity of Medieval seigniorage rates on gold coins, contrasted with much greater variation on smaller coinage, bears this out.

prices should not leave the firm arbitrage opportunities. We now turn to additional restrictions that the household's optimum problem imposes on equilibrium prices and quantities.

The Household's Problem

The household chooses sequences c_1, c_2, m_1, m_2 to maximize (1) subject to (4), (2) and (3), as well as the constraints $m_{i,t} \geq 0$ for $i = 1, 2$. Attach Lagrange multipliers λ_t, η_t, θ_t and $\nu_{i,t}$, respectively, to these constraints. The first-order conditions are:

$$\frac{u_{1,t}}{p_t} = \lambda_t + \eta_t \tag{12a}$$

$$\frac{u_{2,t}}{p_t} = \lambda_t + \eta_t + \theta_t \tag{12b}$$

$$-\nu_{1,t} = -\lambda_t + \beta\left(\lambda_{t+1} + \eta_{t+1}\right) \tag{12c}$$

$$-\nu_{2,t} = -e_t\lambda_t + \beta e_{t+1}\left(\lambda_{t+1} + \eta_{t+1} + \theta_{t+1}\right) \tag{12d}$$

with corresponding relaxation conditions. Conditions (12a), (12b) and (12c) lead to the following:

$$u_{1,t} \leq u_{2,t}; \quad = \text{ if } e_t m_{2,t-1} > p_t c_{2,t} \tag{13a}$$

$$\beta\frac{u_{1,t+1}}{p_{t+1}} \leq \frac{u_{1,t}}{p_t}; \quad = \text{ if } m_{1,t} > 0 \text{ and } m_{1,t-1} + e_t m_{2,t-1} > p_t\left(c_{1,t} + c_{2,t}\right) \tag{13b}$$

Subtracting (12c) from (12d), rearranging, and imposing that $\nu_{2,t} = 0$ (because of the Inada conditions on u), we find:

$$\lambda_t\left(\frac{e_t}{e_{t+1}} - 1\right) = \beta\theta_{t+1} - \nu_{1,t}. \tag{14}$$

Suppose $\theta_{t+1} > 0$ and $\nu_{1,t} = 0$: this requires $e_t > e_{t+1}$. In words, if the 'penny-in-advance constraint' is binding and positive holdings of dollars are carried over from t to $t + 1$, pennies must depreciate in terms of dollars from t to $t + 1$. This makes sense, because if the household holds money from t to $t+1$ and also wishes at $t+1$ that it had held a higher proportion of pennies (i.e., if (3) is binding at $t + 1$), it is because it chose not to because $e_t > e_{t+1}$, indicating that dollars dominate pennies in rate of return. Thus 'shortages' of pennies occur only after dollars dominate pennies in rate of return. To describe the rate of return on pennies further, write (12c) and (12d) at t, and (12a) and (12b) at $t+1$, and make the necessary substitutions to obtain:

$$\frac{e_t}{e_{t+1}} = \frac{u_{2,t+1}}{u_{1,t+1}} \geq 1. \tag{15}$$

In models with only one cash-in-advance constraint but two currencies, this equation holds with equality. Inequality (15) embodies a form of 'one-sided' exchange rate indeterminacy, and leaves open the possibility of a class of equilibrium exchange rate paths that for which the small change is not appreciating relative to large coins.

3. Equilibrium Computation via 'Back-solving'

We utilize the same 'back-solving' strategy employed by Sargent and Smith (1997) to describe possible equilibrium outcomes. Back-solving takes a symmetrical view of 'endogenous' and 'exogenous' variables.[16] It views the first-order and other market equilibrium conditions as a set of difference inequalities putting restrictions *across* the endowment, allocation, price, and money supply sequences, to which there exist many solutions. We use back-solving to display aspects of various equilibria. For example, we shall posit an equilibrium in which neither melting nor minting occurs, then solve for an associated money supply, price level, endowment, and allocation. We shall construct two examples of such equilibria, one where the penny-in-advance constraint (3) never binds, another where it occasionally does. We choose our sample equilibria to display particular adverse operating characteristics of our money supply mechanism.

Equilibria with neither melting nor minting

When neither melting nor minting occurs, $c_{1,t} + c_{2,t} = \xi_t$, and $m_{it} = m_{i,-1} \equiv m_i$, $i = 1, 2$. In this case, the equilibrium conditions of the model consist of one or the

[16] See Sims (1989, 1990) and Diaz-Giménez et al. (1992).

other of the two following sets of equations, depending on whether the penny-in-advance restriction binds:

$$\left.\begin{array}{r}\text{when (3) does not bind:}\\ m_1 + e_t m_2 = p_t \xi_t \\ e_t m_2 \geq p_t c_{2,t} \\ c_{1,t} + c_{2,t} = \xi_t \\ \dfrac{u_{2,t}}{u_{1,t}} = \dfrac{e_{t-1}}{e_t} = 1 \\ \dfrac{u_{1,t}}{p_t} \geq \beta \dfrac{u_{1,t+1}}{p_{t+1}}\end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{r}\text{when (3) binds at } t\text{:}\\ m_1 = p_t c_{1,t} \\ e_t m_2 = p_t c_{2,t} \\ c_{1,t} + c_{2,t} = \xi_t \\ \dfrac{u_{2,t}}{u_{1,t}} = \dfrac{e_{t-1}}{e_t} \geq 1 \\ \dfrac{u_{1,t}}{p_t} \geq \beta \dfrac{u_{1,t+1}}{p_{t+1}}\end{array}\right.$$

Notice how when (2) does not bind, there is one quantity theory equation in terms of the total stock of coins, but that when (3) does bind, there are two quantity theory equations, one for large purchases cast in terms of dollars, the other for small purchases in terms of the stock of pennies.

In what follows, we begin by examining equilibria where the first set of conditions applies at all times. We will show how to construct stationary equilibria. Then we will study a variation in the endowment that brings into play the second set of equations, when the penny-in-advance constraint binds, all the while maintaining the requirement that neither melting nor minting occurs. Throughout, we take the mint policy (σ_i and γ_i) as fixed.

Stationary Equilibria with no minting or melting

We describe a stationary equilibrium with constant monies, endowment, consumption rates, price level, and exchange rate.

Proposition 1. (A stationary, 'no-shortage' equilibrium)

Assume a stationary endowment ξ, and initial money stocks (m_1, m_2). Let (c_1, c_2) solve

$$u_1 = u_2 \tag{16}$$

$$c_1 + c_2 = \xi, \tag{17}$$

Then there exists a stationary equilibrium without minting or melting if the exchange rate e satisfies the following:

$$A = (\gamma_1 (1-\sigma_1), \gamma_1) \cap (e\gamma_2 (1-\sigma_2), e\gamma_2) \neq \emptyset \tag{18}$$

$$\frac{m_1 + em_2}{\xi} \in A \tag{19}$$

$$\frac{\xi em_2}{em_2 + m_1} \geq c_2. \tag{20}$$

Proof: Let $p = (m_1 + em_2)/\xi$. By (19), p will satisfy (11) in such a way that coins are neither melted nor minted. Condition (2) is then satisfied with equality and (3) is satisfied with inequality by (20). Since (3) does not bind, $\theta_t = 0$ and conditions (12) are satisfied with a constant e. ∎

Condition (18), a consequence of the no-arbitrage conditions, requires that the exchange rate be set so that there exists a price level compatible with neither minting nor melting of either coin, of the type described in proposition 1. Condition (19) means that the *total* nominal quantity of money (which depends on e) is sufficient for the cash-in-advance constraint. Condition (20) puts a lower bound on e: the share of pennies in the nominal stock must be more than enough for the penny-in-advance constraint not to bind. Conditions (19) and (20) together imply that $pc_1 \geq m_1$, that is, dollars are insufficient for large purchases.

Figure 2 depicts the determination of c_1, c_2, p, and satisfaction of the penny in advance constraint (3) with room to spare. There is one quantity theory equation cast in terms of the total money supply. That there is room to spare in satisfying (3) reveals

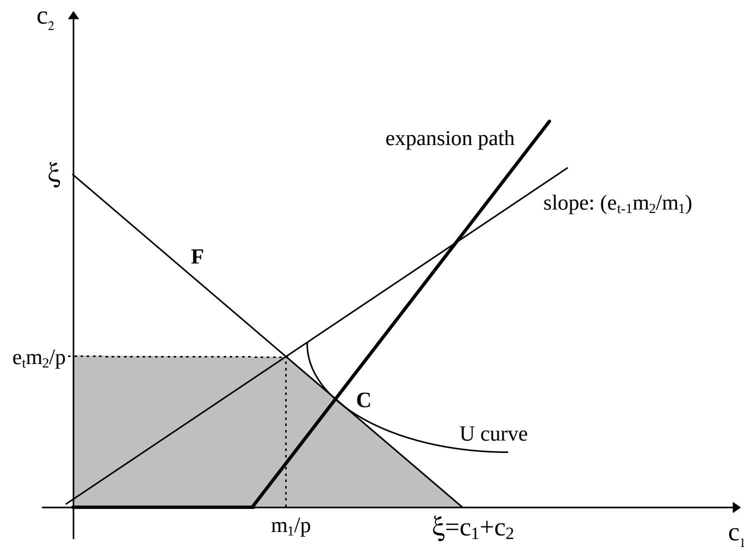

Figure 2: Stationary equilibrium, non-binding constraint.

an exchange rate indeterminacy in this setting: a range of e's can be chosen to leave the qualitative structure of this figure intact. Notice the ray drawn with slope $\frac{e_{t-1}m_2}{m_1}$. So long as the endowment remains in the region where the expansion path associated with a unit relative price lies to the southeast of this ray, the penny-in-advance constraint (3) remains satisfied with inequality. But when movements in the endowment or in preferences put the system out of that region, it triggers a penny shortage whose character we now study.

Small coin shortages

Consistent with the back-solving philosophy, we display a few possible patterns of endowment shifts that generate small coin shortages.

First, note that another way to interpret (19) and (20) is to take m_1, m_2, e as fixed and to formulate these conditions as bounds on ξ:

$$\frac{m_1 + em_2}{\gamma_1} < \xi < \frac{m_1 + em_2}{\gamma_1 (1 - \sigma_1)} \tag{21}$$

$$\frac{m_1 + em_2}{e\gamma_2} < \xi < \frac{m_1 + em_2}{e\gamma_2 (1 - \sigma_2)} \tag{22}$$

$$\xi \leq \alpha \frac{m_1 + em_2}{m_1 - em_2} \tag{23}$$

(where (23) is written for the logarithmic utility case). These equations reveal a variety of ways of generating small coin shortages with a change in ξ, starting from a given stationary equilibrium. The simplest, which we explore first, is a one-time change in ξ within the intervals defined in (21) and (22) but that violate (23): no minting or melting occurs, but a shortage ensues. Another way is as follows: a shortage of small change arises when the relative stock of pennies is insufficient relative to dollars (violation of (23)). Suppose the lower bound in (22) is lower than that in (21). Then a shift in the endowment can lower the price level to the minting point for dollars without triggering any minting of pennies, resulting in a (relative) decrease in the penny supply; for some values of the parameters, this can generate a shortage in the period following the minting of dollars.

Small coin shortage, no minting or melting

We begin by studying the situation that arises when, following an epoch where constant money supplies and endowment were compatible with a stationary equilibrium, there occurs at time t a shift in the endowment. In Figure 2, for the utility function $v(c_{1,t}) + v(c_{2,t} + \alpha)$ we drew the expansion path traced out by points where indifference curves are tangent to feasibility lines associated with different endowment levels. The expansion path is $c_2 = \max(0, c_1 - \alpha)$, and so has slope one or zero. If $\frac{em_2}{m_2} > 1$, i.e., if pennies compose a large enough fraction of the money stock, then the ray c_2/c_1 equaling this ratio never threatens to wander into the southeastern region threatening to render (3) binding, described above. However, when $\frac{em_2}{m_1} < 1$, growth in the endowment ξ can push the economy into the southeastern region, which makes (3) bind and triggers a depreciation of pennies.

Thus, suppose that ξ_t is high enough that the intersection of the expansion path with feasibility ($c_1 + c_2 = \xi_t$) is above the ray $e_{t-1}m_2/m_1$. This means that at t, our second subset of equations determines prices and the allocation. If we assume that ξ_t is such that neither minting nor melting occurs (an assumption that must in the end be verified), then equilibrium values of c_1, c_2, e, p can be computed recursively. Given e_{t-1}, the following three equations can be solved for c_1, c_2, e at t:

$$c_{1,t} + c_{2,t} = \xi_t \tag{24}$$

$$\frac{u_{2,t}}{u_{1,t}} = \frac{e_{t-1}}{e_t} \tag{25}$$

$$\frac{e_t m_2}{m_1} = \frac{c_{2,t}}{c_{1,t}}. \tag{26}$$

These can be combined into a single equation in c_2:

$$\frac{u_{1,t}\left(\xi_t - c_{2,t}, c_{2,t}\right)}{u_{2,t}\left(\xi_t - c_{2,t}, c_{2,t}\right)} = \frac{m_1}{e_{t-1}m_2}\frac{c_{2,t}}{\xi_t - c_{2,t}} \tag{27}$$

Define $f(c_2) = u_{2,t}\frac{c_2}{\xi - c_2}\frac{m_1}{e_{t-1}m_2} - u_{1,t}$. The fact that the expansion path intersects the feasibility line below the ray $e_{t-1}m_2/m_1$ implies that $f(c_2) > 0$ for $(\xi - c_2, c_2)$ on the expansion path. As $c_2 \to 0$, $\lim f(c_2) < 0$ (note that $u_2(\xi, 0) = u'(\alpha) < +\infty$). Therefore, there exists a solution c_2 to (27). Then c_1 is determined from (24), and e_t (which satisfies $e_{t-1} > e_t$ by construction) is given by (25). It remains to check that $p_t = m_1/c_1 \in A$ and that the Euler inequality $\frac{\beta u_{1,t}}{p_t} \leq \frac{u_{1,t-1}}{p_{t-1}}$ holds. There is room to satisfy the inequality if $\frac{\beta p_{t-1}}{p_t}$ is small enough.

Figure 3 depicts the situation when constraint (3) is binding but no minting or melting occurs. We use the intersections of various lines to represent the conditions (24), (25) and (26), and to determine the position of the consumption allocation at point C. The feasibility line F represents (24). Condition (26) means that the ray passing through point C has slope $e_t m_2/m_1$: this defines e_t. Finally, condition (25) can be transformed (using (26)) into the following:

$$c_1 + \frac{u_{2,t}}{u_{1,t}}c_2 = c_1 + \frac{e_{t-1}m_2}{m_1}c_1. \tag{28}$$

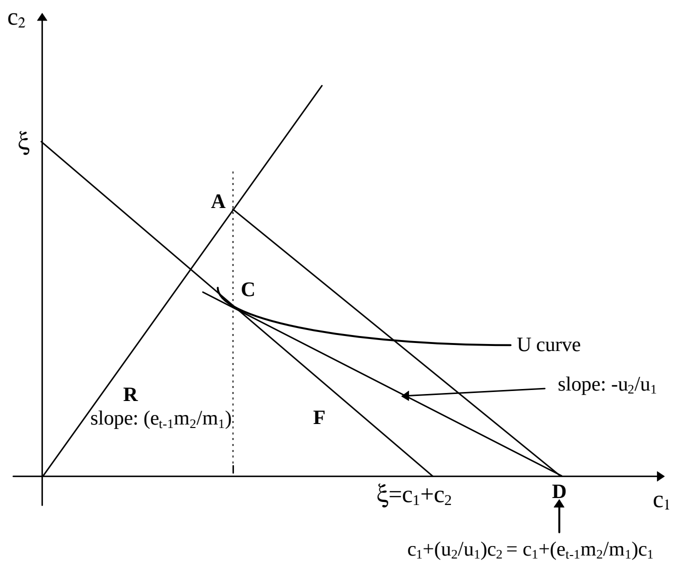

Figure 3: Effect of a shift in endowments.

Define point A to be the vertical projection of C onto the ray $e_{t-1}m_2/m_1$. The left-hand side of (28) is the point at which a line, parallel to the feasibility line and drawn through point A, intersects the x-axis. The right-hand side is the point where the tangent to the indifference curve at C intersects the x-axis. Condition (25) requires that these intersections coincide. Note that, when the constraint does not bind, points A and C coincide, and the tangent to the indifference curve coincides with the feasibility line.

Logarithmic example

Suppose in $v(\cdot) = \ln(\cdot)$ in equation (1). For this specification we can compute $p_t, e_t, c_{1,t}, c_{2,t}$ by hand for the no-minting, no-melting case in which $p_t \in A$, defined in (18). When $\frac{e_{t-1}m_2}{m_1} > \frac{\xi_t - \alpha}{\xi_t + \alpha}$, the constraint (3) does not bind; when the inequality is reversed, it does bind, requiring a decrease in e_t relative to e_{t-1}. When (3) does not bind, the time t equilibrium objects are $c_{1,t} = \frac{\xi_t + \alpha}{2}$, $c_{2,t} = \frac{\xi_t - \alpha}{2}$, $p_t = \frac{m_1 + e_{t-1}}{\xi_t}$. When (3) binds, the solutions are $c_{2,t} = \frac{e_{t-1}m_2}{m_1 - e_{t-1}m_2}\alpha, c_{1,t} = \xi_t - c_{2,t}, p_t = m_1/c_{1,t}, e_t = \frac{p_t c_{2,t}}{m_2}$, where the condition that $\frac{e_{t-1}m_2}{m_1} < \frac{\xi_t - \alpha}{\xi_t + \alpha}$ implies that $e_{t-1} > e_t$. It must be checked that $p_t \in A$ in each case.

Permanent and transitory increases in ξ

Having determined the new exchange rate e_t after a shift in the endowment ξ_t, we can determine what happens if the shift is permanent or transitory. If it is *permanent*, then the constraint (3) will continue to bind. The reason is that, since $e_t < e_{t-1}$, the ray $e_t m_2/m_1$ is in fact even lower than $e_{t-1}m_2/m_1$, which means that the expansion path remains above the ray, and the penny constraint continues to bind. This situation cannot continue without minting or melting indefinitely, however. Thus, a permanent upward shift in the endowment from the situation depicted in Figure 2 to that in Figure 3 would impel a sequence of reductions in the exchange rate until eventually the price level is pushed outside the interval A.

As for a *temporary* (one-time) increase in ξ_t, it might prompt further depreciations in the exchange rate even if the endowment immediately subsides to its original level. The reason is that the reduction in e_{t-1} induces a permanent downward shift in the $\frac{em_2}{m_1}$ ray that enlarges the (3)-is-binding southeastern region.

By shifting the interval for $(e(1-\sigma_2)\gamma_2, e\gamma_2)$ to the left, the reduction of e_t hastens the day when pennies will be melted and postpones the day when they might be minted, without a government adjustment of γ_2. Not until the advent of the 'standard formula' described in the introduction was this perverse mechanism to be set aside.

Aggravated small coin shortage through minting of dollars

A shortage of small coins can also occur as a consequence of minting, independently of the "income effect" we have described. Assume that all coins are full-bodied, so that the bounds of the intervals coincide to the right ($\gamma_1 = e\gamma_2$), but that production costs require that a higher seigniorage be levied on small coins ($\sigma_2 > \sigma_1$), so that the left boundaries do not coincide ($\gamma_1(1-\sigma_1) > e\gamma_2(1-\sigma_2)$).

In the previous section, we considered "small" increases in the endowment ξ; that is, increases that led to movements in the price level p_t *within* the intervals dictated by the arbitrage conditions. We now consider "large" increases that will induce such a fall in the price level that it reaches the minting point for large coins ($p_t = \gamma_1(1-\sigma_1)$). The structure of coin specifications and minting charges means that small coins will not be minted. As a result, m_1 increases while m_2 remains unchanged, and the ratio em_2/m_1 falls. The intuition garnered from Figure 3 suggests that, for large enough increases in m_1, trouble may occur; a shortage of small change results, because the share of pennies in the total money stock falls too far.

We begin again from a stationary equilibrium with no minting or melting. At time t, the endowment increases from ξ_0 to ξ_t, with $\xi_t > (m_1 + em_2)/\gamma_1(1-\sigma_1)$. Then minting occurs, that is, $p_t = \gamma_1(1-\sigma_1)$. From the binding constraint (2),

$$c_{1,t} + c_{2,t} = \frac{m_1 + e_t m_2}{\gamma_1 (1-\sigma_1)}.$$

As before, two situations can arise, depending on whether (3) is binding or not. We will look for equilibria where it is not binding at t, so that $u_2 = u_1$; combined with the binding cash-in-advance constraint, one can solve for $c_{1,t}$ and $c_{2,t}$. By assumption, $\xi_t - c_{1,t} - c_{2,t} > 0$: that amount is minted, and $n_{1,t} = \gamma_1(\xi_t - c_{1,t} - c_{2,t})$.

At $t+1$, it can be shown that

$$\gamma_1 (1-\sigma_1) < \frac{m_{1,t} + e_{t-1} m_2}{\xi_t} < \gamma_1$$

or, in other words, that no more minting occurs if (3) does not bind. But (3) will bind, if the following holds:

$$\frac{e_{t-1} m_2}{m_{1,t}} < \frac{\xi_t - \alpha}{\xi_t + \alpha}.$$

This turns out to be a second-degree polynomial in ξ_t, which will be positive for large enough values of ξ_t.

Thus, if enough dollars are minted, pennies become relatively short of supply.

Small coin shortages, secularly declining e

We have thus shown two ways that endowment growth can induce small coin shortages, with or without minting. We now discuss how such episodes affect the monetary system over time, in particular the relation between small and large coins.

A shortage of small coins manifests itself in a binding penny-in-advance constraint (3), and is associated with two kinds of price adjustments, one 'static', the other 'dynamic'. First, the quantity theory breaks in two to become two separate quantity theory equations, one for small, another for large coins taking the forms $p_t = m_{1,t}/c_{1,t}$ and $e_t = \frac{m_{2,t}}{p_t c_{2,t}}$. The mitosis of the quantity theory is the time-t consequence of a shortage of small coins. A second response is dynamic, and requires that $e_t < e_{t-1}$, so that pennies depreciate in terms of dollars. This response equilibrates the 'demand side' but has perverse implications because of its eventual effects on supply. For fixed b_2, a reduction in e shifts the interval $[e(1-\sigma_2)\gamma_2, \gamma_2]$ to the *left*. This hastens the occasion when pennies will be melted, and reduces the chances that pennies will again be minted.

This penny-impoverishing implication of a shortage-induced fall in e isolates a force for the government to adopt an offsetting reduction in b_2 eventually to resupply the system with a new, lighter penny. Sargent and Smith's (1997) analysis of a non-inflationary debasement can be adapted to simulate such a debasement.[17]

An alternative way to realign the two intervals in response to a shortage-induced

[17] This experiment stresses the 'circulation by tale' axiom embedded in our framework: for the purposes of the cash-in-advance constraint (3), a penny is a penny. To analyze a debasement as described in the text, one has to keep track of two kinds of pennies, old and new. See Sargent and Smith (1997) for an analysis in a one-coin system.

reduction in e would be to *raise* b_1, thereby shifting the interval for large coins also to the left. Such a small-coin-shortage-induced ‘reinforcement’ of large coins would evidently diminish the price level.

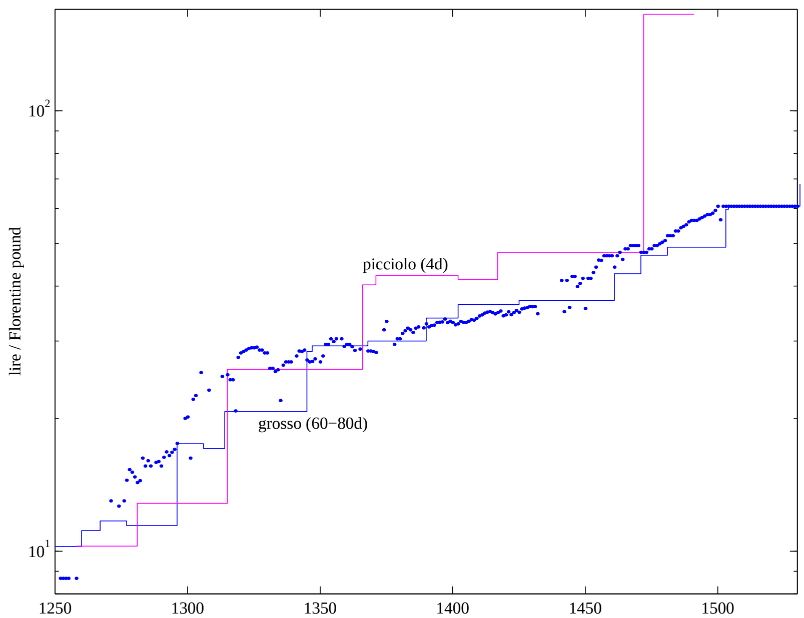

Figure 4: Plot of $1/b$ for two Florentine silver coins (1250–1530). The dots plot an index of the price (in pennies) of the gold florin, or $1/e_t$ in our model.

While the model thus identifies either a debasement of small coins or a reinforcement of large coins as a workable policy for keeping both large and small coins in existence, history records more debasements of small coins. Cipolla reports secular declines in e and b_2, in the face of long periods of stable b_1. For example, in Florence the gold florin retained a constant metal content for centuries, while petty coins were recurrently debased. Figure 4 plots the evolution of $1/b_2$ for two Florentine silver coins, the *picciolo* and the *grosso*, during the Middle Ages (for the gold Florin, b_1 remained constant). A pattern of recurrent debasements, as describe by Cipolla, is apparent. The graph also displays another piece of information, namely the price of the gold florin in terms of silver pennies:

this corresponds to $1/e_t$ in our model.[18]

The Affair of the Quattrini

Figure 5 displays fourteenth century time series for the mint equivalent $e\gamma_i$ and the mint price $e(1-\sigma)\gamma_i$ for two petty coins of Florence, the grosso and the quattrino.[19] The figure thus traces the two intervals shown in Figure 1. For most of the century the intervals overlapped, with the interval for the smaller denomination quattrino containing the interval for the grosso. A glaring exception occurred from 1371 to 1403, when the interval for the quattrino lay above that for the grosso. Our model predicts that this spells trouble, and Cipolla's account confirms it.

During this exceptional period there occurred the 'Affair of the Quattrini,' another of Cipolla's beautiful chapter titles (Cipolla 1982), sparked by a devaluation of the quattrino in 1371 designed to stimulate its coining in order to displace 'foreign' small coins flowing into Florence from Pisa. Our theory would imply during this period that (a) the price level would move upward into the interval dictated by the interval for the quattrino, that (b) quattrini would be coined, and that (c) grossi would disappear. Cipolla's account confirms implications (a) and (b), and is silent but not inconsistent with implication (c). To protect it, the florin was permitted to appreciate in terms of petty coins. The authorities faced public pressure to reverse this 'inflation'. Cipolla describes a 1381 'anti-inflation' government policy to acquire and melt quattrini and so reduce the stock of money. Within our model, this policy makes sense, though not too much could have been expected of it: a 'quantity theory' policy could move the price level downward within the band determined by b_i, σ_i, but could not drive it outside that band.

[18] Strictly speaking, our model assumes that the technological rate of transformation between the metals used in large and small coins is constant, for example, silver is used in both. In the case of the florin, the gold/silver ratio fluctuated over time; in particular, in the first half of the 14th century gold prices increased; adjusting for that variation would lower the dots in Figure 4 in that period.

[19] Between 1345 and 1365, a Florentine lira exchanged at par for 4 grossi and 60 quattrini.

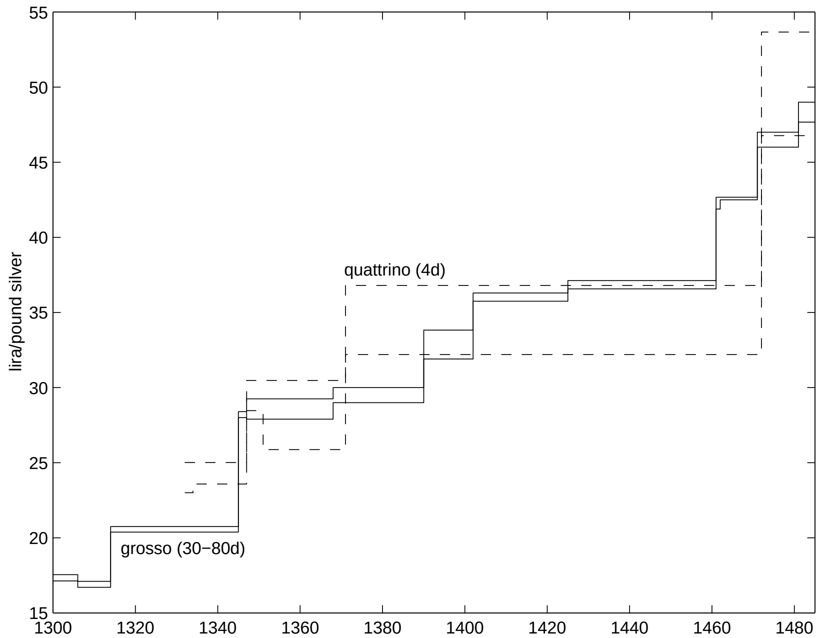

Figure 5: Evolution of the upper and lower bounds on quattrini and grossi, Florence.

4. Arrangements to eliminate coin shortages

This section describes two money supply mechanisms that, within the context of our model, eliminate shortages of small coins. After briefly describing these mechanisms, we will scrutinize them in terms of how they incorporate some or all of the ingredients in Cipolla's recipe, and will determine to what extent some ingredients of the recipe are redundant.

A 'standard formula' regime

We change the supply mechanism to implement a version of Cipolla's 'standard formula', retaining the demand side of the model. It is as if the government tells the mint to set up a 'pennies department' that operates like a 'currency board' for pennies. The rules for supplying dollars are not changed from those described above. But now the mint is required to convert pennies into dollars and dollars into pennies, upon demand and at a fixed exchange rate e, named by the government. Assume that pennies are produced costlessly by the mint, and so are truly tokens ($b_2 = 0$). The government requires the mint to carry a non-negative inventory of dollars R_t from each date t to $t+1$. The mint increases its inventory of dollars only when it buys dollars, and decreases it only when it buys pennies.

This regime imposes the following laws of motion for stocks of dollars and pennies:

$$\begin{aligned} m_{1,t} &= m_{1,t-1} + n_{1,t} - \mu_{1,t} - (R_t - R_{t-1}) \\ m_{2,t} &= m_{2,t-1} + e^{-1}(R_t - R_{t-1}) \\ R_t &\geq 0, \end{aligned}$$

where R_t is the stock of dollars held by the mint from t to $t+1$. The law of motion for dollars can be rewritten as

$$(m_{1,t} + R_t) = (m_{1,t-1} + R_{t-1}) + n_{1,t} - \mu_{1,t}.$$

For the firm, it does not matter whether it melts dollars or whether pennies are exchanged for dollars that are then melted; only the total stock of dollars $m_{1,t} + R_t$ counts.

Under this money supply mechanism, the firm's profits become:

$$\Pi_t = p_t \xi_t + n_{1,t} - \sigma_1 n_{1,t} - p_t \frac{b_1}{\phi} n_{1,t} + p_t \frac{b_1}{\phi} \mu_{1,t} - \mu_{1,t}, \tag{29}$$

subject to the constraints $n_{1,t} \geq 0$ and $m_{1,t-1} + R_{t-1} \geq \mu_{1,t} \geq 0$.

The absence of R_t from its profits signifies the firm's indifference to the choice of $m_{2,t}$ (as long as $R_t \geq 0$), because the firm always breaks even when it buys or sells pennies.[20] The firm's indifference lets the demand side of the model determine the stock

[20] The condition $R_t \geq 0$ can be ensured by choosing initial conditions such that $em_{2,0} \leq R_0$.

of $m_{2,t}$ appropriately. More precisely, one can find paths for the money stocks that are consistent with the mint's profit maximization and that satisfy the household's first-order conditions with (3) not binding.

A subset of the no-arbitrage conditions now obtains:

$$n_{1,t} \geq 0; \quad = \quad \text{if} \;\; p_t > \gamma_1 (1 - \sigma_1) \tag{30a}$$

$$\mu_{1,t} \geq 0; \quad = \quad \text{if} \;\; p_t < \gamma_1 \tag{30b}$$

$$m_{1,t-1} + R_{t-1} \geq \mu_{1,t}; \quad = \quad \text{if} \;\; p_t > \gamma_1 \tag{30c}$$

This regime forces p_t into the interval $[(1-\sigma)\gamma_1, \gamma_1]$. In particular, p_t can never rise above γ_1 because that would mean melting down all dollars, including those backing the pennies, and the economy would have no money stock. This regime also solves the exchange rate indeterminacy problem by administering a peg.

The positive stocks of R_t carried by the mint are socially wasteful, as indeed are the stocks of silver being used in large coins.

Variants of the standard formula

The preceding version of a standard formula regime omits Cipolla's stipulation to 'limit the quantity of the small coins' in circulation. Though our model renders that stipulation redundant, various writers included them, and insisted on limiting the legal tender of small coins and strictly limiting the quantity issued, often as supplements – though occasionally apparently as alternatives to pegging the exchange rate e_t by converting either coin into the other at par.

The effect of limited legal tender is evidently to modify the cash-in-advance constraint (2), because pennies are no longer accepted in payment of large purchases. Instead, two separate cash-in-advance constraints are imposed.

The 'standard formula' without convertibility: the Castilian experience

One striking feature of the 'standard formula' and its reserve requirement is the apparent wastefulness of R_t (in fact, of $m_{1,t}+R_t$). We now consider a regime in which the convertibility requirement is simply removed. As before, the rules for supplying dollars are identical to the earlier ones. Pennies are token, that is, costless to produce. The government does not require an inventory of dollars to back the pennies, but sets the seigniorage rate on pennies $\sigma_2 = 1$. The firm's profit is then (29), to be maximized subject to $n_{1,t} \geq 0$ and $m_{1,t-1} \geq \mu_{1,t} \geq 0$. Again, the firm is indifferent to the values taken by $\mu_{2,t}$ and $n_{2,t}$. Those values are set exogenously by the government, and $m_{2,t}$ follows the law of motion (6). The government's budget constraint (8) is

$$T_t = \sigma_1 n_{1,t} + e_t n_{2,t}.$$

From the firm's no-arbitrage conditions (30), we can bound p_t below, by the minting point $\gamma_1(1-\sigma_1)$; but there is no upper bound on p_t: if $p_t > \gamma_1$, $\mu_{1,t} = m_{1,t}$ and dollars disappear, but pennies remain in circulation, and the cash-in-advance constraint (or quantity theory equation) determines the price level:

$$p_t\left(c_{1,t} + c_{2,t}\right) = e_t m_{2,t-1}.$$

We start from given initial stocks $m_{1,0} + em_{2,0}$ and a constant endowment ξ, such that $\gamma_1(1-\sigma_1) < (m_{1,0} + em_{2,0})/\xi < \gamma_1$, and consider alternative policies for the path of the penny stock. Suppose that a shift in endowment occurs as in section 3, so that no minting or melting of dollars takes place. In the absence of any change in $m_{2,0}$ a penny shortage would develop. The shortage can be remedied, or prevented, by an appropriate increase in m_2, one that raises the ray of slope em_2/m_1 in Figure 2 so as to put the intersection of the expansion path and the resource constraint in the southeastern region.

Issues of pennies can proceed in the absence of any further shifts in endowments. Suppose ξ remains constant, but the government issues quantities of pennies every period.

When constraint (3) isn't binding, the price level is determined by the binding constraint (2), and the issue of pennies will lead to a (non-proportionately) rising price level, until $p_t = \gamma_1$. At that point, dollars begin to be melted, and successive increases in the penny stock displace dollars at the constant rate e, maintaining $m_{1,t} + em_{2,t} = \gamma_1 \xi$. Once all dollars have been melted, the economy becomes a standard cash-in-advance fiat currency economy, whose price level is governed by the quantity theory equation (2), namely $e_t m_{2,t-1} = p_t \xi$. Further increases in the penny stock result only in increases in the price level (and in seigniorage revenues for the government).[21]

Should a government wish to bring about the return of dollars, it can only do so by lowering the price level to $\gamma_1(1-\sigma_1)$. It could do so by acting on $m_{2,t}$: either by retiring a portion of $m_{2,t}$, a costly option envisaged by the Castilian government between 1626 and 1628 and ultimately rejected; or by changing $m_{2,t-1}$ through a redenomination, the policy followed in 1628.

The change in $m_{2,t-1}$ and the consequent change in e_t have to occur "overnight," in an unanticipated fashion, or (14) would be violated. We look for an equilibrium in which minting of dollars occurs, which requires $p_t = \gamma_1(1-\sigma_1)$. The household enters the period holding only pennies, so that (2) is $\gamma_1(1-\sigma_1)(c_{1,t}+c_{2,t}) = e_t m_{2,t-1}$. (Note that, since the household holds only pennies, (3) cannot be binding). The resource constraint becomes $c_{1,t} + c_{2,t} = \xi - \gamma_1^{-1} n_{1,t}$. The two conditions combine into

$$\gamma_1 \xi = n_{1,t} + e_t m_{2,t-1} / \left(1 - \sigma_1\right), \tag{31}$$

a joint condition on $n_{1,t}$ and e_t. The requirement that $\gamma_1 \xi > n_{1,t} > 0$ induces an interval in which e_t must lie, namely

$$e_t \in \left(0, \frac{\gamma_1 \left(1 - \sigma_1\right)}{p_{t-1}}\right)$$

where p_{t-1} was the price level prior to the monetary operation.

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[21] The ratio p_t / γ_1 can also be thought of as the premium on dollars in terms of pennies.

In the following period t_1, neither minting nor melting occurs, because (31) implies

$$\gamma_1 (1-\sigma_1) \leq \frac{n_{1,t} + e_t m_{2,t-1}}{\xi} < \gamma_1$$

However, to ensure that (3) does not bind at $t+1$, the stock of new dollars $n_{1,t} = m_{1,t+1}$ must not be too large: $m_{1,t+1} \leq p_{t+1} c_{t+1}$, or $(c_{2,t+1}/\xi) n_{1,t} \leq e_t m_{2,t-1}$. This places a further restriction on e_t, namely:

$$e_t \in \left(\frac{c_2}{c_2 + (1-\sigma_1)\xi} \frac{\gamma_1 (1-\sigma_1)}{p_{t-1}}, \frac{\gamma_1 (1-\sigma_1)}{p_{t-1}} \right) \tag{32}$$

where c_2 lies on the expansion path and on the feasibility line in Figure 2.

Condition (32) requires the government to engineer a devaluation of pennies of the "right" extent. If it is too small, no dollars are minted; if it is too large, too many dollars are minted and another increase in m_2 will be required to relieve the binding penny-in-advance constraint.

We have assumed that the government carries out this operation in an unanticipated manner. After 1628, the Castilian public surely viewed subsequent manipulations of m_2 with some suspicion. Expectations of further reforms altered the demand for pennies, but our simple model is not equipped to pursue the analysis in that direction.

Fiat Currency

To attain a version of Lucas's (1982) model of fiat money, we would suspend the original technology for producing both coins, and let them be produced costlessly. Alternatively, we can think of 'widening the bands' – i.e., driving the γ_i's to infinity. Like Lucas, we would simply award the government a monopoly for issuing coins. We get a pure quantity theory. The government fixes paths for $m_{it}, i = 1, 2$, being careful to supply enough pennies (i.e., to keep (3) from binding). Condition (2) at equality determines the price level as a function of the total money supply. Condition (3) imposes a lower bound on the quantity of pennies needed to sustain a fixed e equilibrium; equation (2) imposes an upper bound on the amount of pennies that can be issued via 'open-market operations' for dollars.

5. Concluding Remarks

We designed our model to help us understand problems with the arrangements for minting more or less full-bodied coins that prevailed for centuries throughout Western Europe. Our model ascribes rules for operating the mint that copy historical ones, and focuses on the difficulty those rules create for simultaneously maintaining two commodity currencies. The model extends insights from single-currency commodity money models,[22] where minting and melting points impose bounds within which the price level must stay to arrest arbitrage opportunities. In those one-commodity money models, when the price level falls enough (i.e., when currency becomes scarce), new coins will be minted; and when the price level rises enough, coins will be melted.[23]

With two currencies, there are distinct melting and minting points for each currency, and this causes trouble. We posit a particular model of demand for coins that, in conjunction with the two sets of melting and minting points, makes shortages of the smaller denomination coins arise when national income fluctuates. We analyze the perverse price adjustments fostered by the historical supply arrangement, how they served to aggravate shortages over time, and how they left debasement as the preferred relief.

The vulnerability to recurrent shortages of small coins that characterized the historical supply arrangement eventually prompted its repair in the form of a huge 'once-and-for-all debasement' of small coins, in the form of a permanent system of token small change. In that system, the government acts as a monopolist for small coins. It can peg the exchange rate of small for large coins, either by always choosing a proper quantity of

[22] See Sargent and Smith (1997).

[23] Sargent and Smith (1997) analyze two-commodity monies in the form of gold and silver coins, but do not link these metals to *denomination* as we do here. In particular, they have no counterpart to our constraint (3).

small ones, or by maintaining convertibility.

Establishing a system of token small change was a fateful step on the road to creating a fiat money system for *all* currency. But refining the idea of fiat money and actually implementing it were destined to take centuries. Historical episodes, such as one in Castile in the 17th century, highlight the difficulty of establishing even a token system for small coins in the face of technological imperfections, and of the pressures that governments frequently had to raise revenues.

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