

Information and Liquidity*

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Abstract

We develop a model with money and alternative assets, both of which can be used as media of exchange, although potentially with a different liquidity (probability of being accepted). We first take liquidity to be exogenous and show that even agents/transactions that never use cash are affected by inflation. We then endogenize liquidity through recognizability – money is perfectly recognizable, but not everyone is informed enough to distinguish real from counterfeit claims to other assets. Given heterogeneous costs of becoming informed, we determine who accepts what in equilibrium, and study the interplay between endogenous liquidity and monetary policy.

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Aringosa walked back to his black briefcase, opened it, and removed one of the bearer bonds. He handed it to the pilot. “What’s this?” the pilot demanded. “A ten-thousand-euro bearer bond drawn on the Vatican Bank.” The pilot looked dubious. “It’s the same as cash.” “Only cash is cash,” the pilot said, handing the bond back. Dan Brown, *The Da Vinci Code*.

1 Introduction

What does monetary policy do and what are the effects of on prices and allocations? To a first approximation, central banks control the rate of return on currency.¹ In theories where currency is essential for at least some transactions, when its rate of return declines agents try to economize on their holdings of it, so money demand falls, and given the supply, this lowers the value of money and this makes these transactions more difficult. The reason agents hold money in the first place, despite a spread between its return and that on alternative assets, is presumably that it some *liquidity* advantage: it helps to facilitate said transactions. This is old news. What is less clear is the following: Which transactions are affected by monetary policy – only transactions where money is essential, or others as well? And who is affected – only agents holding cash, or others as well?

In reality, some (many, more and more) transactions take place without the use of currency. Are these immune from central bank policy? Our answer is, *No*. In equilibrium, as long as someone is holding the cash – and of course, somebody must be, even if in the limit it is only commercial banks holding it for settlement purposes – then there should be general equilibrium effects on the rates of return on some (most, all) other assets when the central bank adjusts the rate of return on central bank money. A plausible scenario is that when the rate of return on currency falls, agents attempt to adjust their portfolios by substituting out of currency, which affects equilibrium portfolios, rates of return, and potentially all transactions, not only those

¹In standard theory, the no-arbitrage condition known as the Fisher equation tells us that it is equivalent for the central bank to target the growth rate of money supply, the inflation rate, or the nominal interest rate: a higher value for any of these policy instruments lowers the return on currency. While one might be able to write down models where the Fisher equation does not hold, or where it makes a difference whether we target money growth, inflation or interest rates for some other reason, this is beside the point for the issues at hand.

that use currency directly.

Consider what happens to money and banking when inflation goes up. Given the interest rates on demand deposits, individuals at the margin prefer to hold less M0 and more M1. As the supply of demand deposits increase, their interest rate falls and agents at the margin prefer to hold more M2. At the end of the day, equilibrium portfolios are less liquid: agents have less real wealth in cash, more in their saving accounts, and potentially more or less in their checking accounts. This affects the transactions process in several ways. Suppose an opportunity to consume or invest comes along, but for whatever reason it requires M0 – they guy won't take a check – or maybe M1 but not M2 – he'll take cash or a check but needs it now. Your portfolio, which depends on inflation, determines whether the deal goes through (more generally, its size). Of course, whether the guys accepts M0, M1 or M2 may also depend on equilibrium portfolios. In general, monetary policy has many lots of implications for the equilibrium structure of portfolios, transactions, prices and allocations.

This position is not entirely novel. It is related e.g. to anyof the old-fangled “portfolio theories” of the demand for money, and Wallace (1980, p. 64) speaks to it directly in the context of overlapping generations models:

Of course, in general, fiat money issue is not a tax on all saving. It is a tax on saving in the form of money. But it is important to emphasize that the equilibrium rate-of-return distribution on the equilibrium portfolio does depend on the magnitude of the fiat money-financed deficit. ... In all these models, the real rate-of-return distribution faced by individuals in equilibrium is less favorable the greater the fiat money-financed deficit. Many economists seem to ignore this aspect of inflation because of their unfounded attachment to Irving Fisher's theory of nominal interest rates. (According to this theory, (most?) real rates of return do not depend on the magnitude of anticipated inflation.) The attachment to Fischer's theory of nominal

interest rates accounts for why economists seem to have a hard time describing the distortions created by anticipated inflation. The models under consideration here imply that the higher the fiat money-financed deficit, the less favorable the terms of trade – in general, a distribution – at which present income can be converted into future income. This seems to be what most citizens perceive to be the cost of anticipated inflation.

Although these ideas ring true, the problem is that they are not so easy to formalize. There are many questions unanswered in the above discussion. How can the Fisher equation not hold? What is it that allows different assets, especially money and other assets, to bear different rates of return? In the models Wallace mentions, it is *not* differences in liquidity. It is difficult to generate a portfolio of assets with differential liquidity in any standard frictionless model, even overlapping generations models, because there is no role for liquidity in those models, and no-arbitrage conditions must equate the rates of returns across assets after adjusting for risk.² To model liquidity, one needs to “look frictions in the face” (Hicks 1935) by making explicit assumptions about spatial, temporal, and informational problems, which is what more modern monetary theory is all about.³

We introduce multiple assets in a benchmark modern monetary model, the one in Lagos and Wright (2005), hereafter LW, and derive the liquidity properties of the assets endogenously. Lagos and Rocheteau (2005) also consider multiple assets in LW – in their case, money and capital, but they are equally liquid. If the first-best capital stock is sufficiently large, they show money is not essential, but if this stock is too low the economy overaccumulates capital absent

²Notice that Wallace talks about “saving” and defines returns in terms of the rate “at which present income can be converted into future income” – there is no mention of the transactions role or medium of exchange role of assets.

³An alternative approach to modelling liquidity (e.g. Glosten and Milgrom 1985 and Kyle 1985) considers bilateral transactions between either a buyer or seller and a middleman with asymmetric information. These models generally assume that the buyer or seller has more information about expected future payoffs of the asset than the middleman. These models fall silent on the questions we care about here – e.g. explaining liquidity differences between cash and other assets.

money and hence money enhances efficiency. But money and capital must bear the same return. Geromichalos, Licari and Suarez-Lledo (2006) study LW with money and real assets in fixed supply, like the claims to “trees” in a standard Lucas (1978) asset-pricing model. They show money is essential if and only if the supply of the real asset is too low, and discuss the effect of policy on asset prices, rates of return, and allocations. But again, all assets are equally liquid and hence must bear the same return.

Differential liquidity was considered in a related model by Lagos (2005), with two assets meant to resemble stocks and bonds. There, when assets are valued both for their rates of return and for their use as media of exchange, even if they are equally liquid, one can go some distance toward resolving much-discussed puzzles in the asset pricing literature; and if they are differentially liquid, one can go a lot further. However, his liquidity differentials are exogenous, and he does not discuss monetary policy at all, mainly because there is no money in that model. Differential liquidity between money and capital was considered by Aruoba, Waller and Wright (2007), but although there were some words to the effect that recognizability was relevant, these differentials are basically exogenous in that model. The goal here is to take the recognizability idea more seriously to make differential liquidity endogenous.

The first thing we do is to generalize the environment Geromichalos et al. to allow for exogenous liquidity differentials, by simply assuming that some agents accept only money. This allows more interesting outcomes (e.g. in Geromichalos et al. monetary equilibrium only exists for negative inflation rates), and allows us to formalize some of the above points concerning monetary policy. In particular, we show that even agents/transactions that never use cash are affected by inflation. We then endogenize liquidity by incorporating information explicitly. Suppose money is perfectly recognizable, but not everyone is informed enough to distinguish real from counterfeit claims to other assets. Assuming agents have differential costs of becoming informed, we determine who becomes informed and who accepts which assets in equilibrium,

and analyze how this depends on policy.

2 A Benchmark Model

Time is discrete and continues forever. There is a $[0, 1]$ continuum of infinitely-lived agents. As in Lagos and Wright (2005), hereafter LW, each period these agents participate in in two distinct markets: a frictionless centralized market CM, and a decentralized market DM where agents meet anonymously and there is a standard double coincidence problem. These frictions make a medium of exchange essential (see Kocherlakota 1998, Wallace 2001, and Aliprantis et al. 2007 for in-depth discussions of anonymity and essentiality). At each date in the CM there is a consumption good x that agents can produce using labor h according to $x = h$, and utility is $U(x) - h$.⁴ In the DM there is a different good q that gives utility $u(q)$ and is produced at disutility cost $c(q)$. We assume $U'(0) = u'(0) = \infty$, and we define the efficient quantities x^* and q^* by $U'(x^*) = 1$ and $u'(q^*) = c'(q^*)$.

There are for now two assets. First there are claims to a real asset in fixed supply A , just like the ‘trees’ in the Lucas (1978) asset-pricing model, in the sense that a unit of the asset gives off a constant dividend δ each period. Second, there is fiat money, the supply of which grows according to $\hat{M} = \gamma M$ (for any variable, including M , we let \hat{M} denote its value next period). Money $\hat{M} - M = (\gamma - 1)M$ is injected or withdrawn using lump sum transfers or taxes (although it is equivalent here to assume the government uses new money it to buy x in the CM since we have quasi-linear utility). In what follows, we assume $\gamma > \beta$ where β is the rate of time preference; we do consider the limit where $\gamma \rightarrow \beta$, which is the Friedman rule. Let ϕ be the CM price of money and ψ the CM price of the real asset, both in terms of x .

As shown in Lagos and Rocheteau (2005) and Geromichalos et al. (2006), an agent with a units of the real asset may not necessarily bring all of it to the DM when the terms of trade

⁴It is easy to generalize these assumptions by allowing many goods and more general technologies in the CM, but for our purposes it suffices to concentrate on the simplest formulation. What is important for tractability, if not for the overall logic of the theory, is that CM utility is quasi-linear.

in the DM are determined through bargaining (with, for simplicity, perfect information in the sense that agents can observe each other's asset holdings). Bargaining entails a hold-up problem: when an agent brings additional assets into a match, the terms of trade can become increasingly unfavorable. Thus, an agent may choose to acquire a large quantity of the real asset for its productive value, but only bring a portion to the DM as a medium of exchange. There is no similar effect on m , because it yields no dividend. Since money only possesses value as a medium of exchange, there is no reason to acquire it and not bring it to the DM. So, in general, agents in the CM choose a portfolio comprised of m units of money, a_1 units of the asset they do not take to the DM, and a_2 units that they do take to the DM.

Let $V(m, a_1, a_2)$ denote the value function of an agent entering the DM and $W(y)$ the value function of an agent entering the CM with a portfolio worth $y = \phi m + (\delta + \psi)(a_1 + a_2)$ (clearly, in the frictionless CM, it is only the total value of the portfolio y that is relevant, not its composition). Then

$$\begin{aligned} W(y) &= \max_{x, h, \hat{m}, \hat{a}_1, \hat{a}_2} \{U(x) - h + \beta V(\hat{m}, \hat{a}_1, \hat{a}_2)\} \\ \text{s.t. } x &= h + y - \phi \hat{m} - \psi(\hat{a}_1 + \hat{a}_2) + T, \end{aligned}$$

where $T = (\gamma - 1)M$ is the lump sum transfer. If we substitute for h , it is immediate that the value function is linear, $W'(y) = 1$, and the first order conditions are:⁵

$$x : U'(x) = 1 \tag{1}$$

$$\hat{m} : \phi \geq \beta V_1(\hat{m}, \hat{a}_1, \hat{a}_2), = \text{ if } \hat{m} > 0 \tag{2}$$

$$\hat{a}_1 : \psi \geq \beta V_2(\hat{m}, \hat{a}_1, \hat{a}_2), = \text{ if } \hat{a}_1 > 0 \tag{3}$$

$$\hat{a}_2 : \psi \geq \beta V_3(\hat{m}, \hat{a}_1, \hat{a}_2), = \text{ if } \hat{a}_2 > 0 \tag{4}$$

Notice that $x = x^*$ and $(\hat{m}, \hat{a}_1, \hat{a}_2)$ do not depend on y . For simplicity, assume there is a unique

⁵We are assuming an interior solution for h ; see LW for conditions to guarantee this is valid in these kinds of models.

solution $(\hat{m}, \hat{a}_1, \hat{a}_2)$ to (2)-(4) satisfying the second order conditions (this is necessarily true under assumptions like those in LW).

In the DM, there is a fixed probability λ of a bilateral meeting in which an agent is a consumer, and an equal probability of a bilateral meeting in which he is a producer. We distinguish two types of meetings where an agent is a buyer: with probability ρ we have what we call a type 2 meeting, where the seller accepts either m or a_2 ; and with probability $1 - \rho$ we have a type 1 meeting, where the seller accepts only m . For now $\rho \in (0, 1)$, and is exogenous; later ρ is endogenous. In either type of meeting, the seller cares only about the total value of the buyer's assets that he is willing to accept. Therefore, in a type 2 meeting, the seller considers the total value of cash and assets that the buyer has available $\phi m + (\psi + \delta)a_2$, while in a type 1 meeting he only looks at ϕm . Of course, the amount that a buyer ultimately pays p in any meeting is constrained by what he has available: $p_j \leq y_j$, where $y_1 = \phi m$ and $y_2 = \phi m + (\psi + \delta)a_2$.

Given these assumptions, we now describe our bargaining solution for the DM.⁶ Consider a type j meeting between a buyer with (m, a_1, a_2) and a seller with $(\tilde{m}, \tilde{a}_1, \tilde{a}_2)$. The former pays p_j to the latter for q_j units of the good, determined by the generalized Nash solution

$$\max [u(q_j) + W(y - p_j) - W(y)]^\theta [-c(q_j) + W(\tilde{y} + p_j) - W(\tilde{y})]^{1-\theta} \quad (5)$$

subject to $p_j \leq y_j$, where y and \tilde{y} describe total wealth of the buyer and seller, and y_j describes the wealth available to the buyer in this particular meeting. This bargaining problem is easy, mainly because $W(y)$ is linear, and leads to the following:

Lemma 1. *The solution to (5) is*

$$q_j = \min \{z^{-1}(y_j), q^*\} \quad \text{and} \quad p_j = \min \{y_j, y^*\},$$

⁶The terms of trade in these type of models can be determined in a number of ways without changing the basic results. Aruoba, Rocheteau and Waller (2007) analyze several alternative bargaining solutions, Rocheteau and Wright (2005) analyze price taking and price posting, and Galenianos and Kircher (2007) or Dutta, Julien and King analyze auctions (in version that has some multilateral meetings).

where y_j is the wealth available to the buyer in a type j meeting, the function z is given by

$$z(q) \equiv \frac{\theta u'(q)c(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)},$$

q^* is defined by $u'(q^*) = c'(q^*)$, and $y^* = z(q^*)$. Also, $z'(q) > 0$ for all $q < q^*$.

Given this, the DM value function satisfies

$$V(m, a_1, a_2) = \lambda_0 W(y) + \lambda_1 [u(q_1) + W(y - p_1)] + \lambda_2 [u(q_2) + W(y - p_2)] + k,$$

where $\lambda_0 = 1 - \lambda$, $\lambda_1 = \lambda(1 - \rho)$, $\lambda_2 = \lambda\rho$ and k is a constant. Notice that when the agent is not a buyer, he may or may not be a seller, and this effects his continuation value W , but since Lemma 1 implies the terms of trade do not depend on a seller's state and W is linear, we can represent this by $W(y)$ plus a term k that does not depend on y . In particular, we do not need to discuss when an individual seller accepts a and when he does not, or which individual sellers accept a and which do not; what is relevant for asset demand and hence for equilibrium here is only what happens when an individual is a buyer, not what happens when he is a seller.

Differentiating V , after inserting the derivatives of q_j with respect to (m, a_1, a_2) which we get from Lemma 1, we have

$$V_1(m, a_1, a_2) = \phi [1 + \lambda_1 \ell(q_1) \mathbf{1}\{y_1 < y^*\} + \lambda_2 \ell(q_2) \mathbf{1}\{y_2 < y^*\}] \quad (6)$$

$$V_2(m, a_1, a_2) = \psi + \delta \quad (7)$$

$$V_3(m, a_1, a_2) = (\psi + \delta) [1 + \lambda_2 \ell(q_2) \mathbf{1}\{y_2 < y^*\}], \quad (8)$$

where $\mathbf{1}\{\Phi\}$ is the indicator function that equals 1 iff Φ is true, and $\ell(q) \equiv \frac{u'(q)}{z'(q)} - 1$. Notice that $\ell(q)$ represents a *liquidity premium*: $\ell(q_j)$ is the value of an additional unit of asset in a type j meeting, over and above its return if it is simply carried forward to the next CM. We assume $\ell'(q) < 0$.⁷ Updating (6)-(8) and inserting $V_i(\hat{m}, \hat{a}_1, \hat{a}_2)$ into (2)-(4), we arrive at conditions

⁷This assumption generally involves third derivatives of the utility function, but LW shows it always holds if θ is close to 1, and it holds for any θ if c is linear and u' is log concave. These same conditions also guarantee that there is a unique solution to the first order conditions.

determining demand for the three assets:

$$m : \phi \geq \beta \hat{\phi} [\lambda_1 \ell(\hat{q}_1) \mathbf{1}\{\hat{y}_1 < y^*\} + \lambda_2 \ell(\hat{q}_2) \mathbf{1}\{\hat{y}_2 < y^*\} + 1], = \text{ if } \hat{m} > 0 \quad (9)$$

$$a_1 : \psi \geq \beta(\hat{\psi} + \delta), = \text{ if } \hat{a}_1 > 0 \quad (10)$$

$$a_2 : \psi \geq \beta(\hat{\psi} + \delta) [\lambda_2 \ell(\hat{q}_2) \mathbf{1}\{\hat{y}_2 < y^*\} + 1], = \text{ if } \hat{a}_2 > 0 \quad (11)$$

We are now ready to discuss equilibria. Generally, an equilibrium can be defined in terms of time paths for asset holdings (m, a_1, a_2) , asset prices (ϕ, ψ) , the DM terms of trade (p_j, q_j) for $j = 1, 2$, and the CM allocation (x, h) for every individual, satisfying the utility maximization conditions derived above, the bargaining solution, and the obvious market clearing conditions. Given the other variables, the CM allocation is obvious: we know $x = x^*$ from (1), and we can get h from the budget equation. Hence, (x, h) will remain implicit in the following discussion. A special case is a steady state equilibrium, where the real variables (q_1, q_2) are constant over time, which from the bargaining solution implies ϕm and ψa_2 are constant as well, and in particular $\phi/\hat{\phi} = \hat{M}/M = \gamma$. We restrict attention to steady states in much of what follows, and to monetary equilibria, where $\phi > 0$, $\hat{m} > 0$, $q_1 > 0$ and (9) holds with equality.

From Lemma 1, q_j is an increasing function of y_j and $q_1 \leq q_2 \leq q^*$. In fact, it is easy to show that $q_j \leq \bar{q}$, where \bar{q} is the q that maximizes the buyer's surplus $u(q) - p = u(q) - z(q)$, and $\bar{q} \leq q^*$ with strict inequality unless $\theta = 1$.⁸ Notice that $\ell(\bar{q}) = 0$. The next result is that $a_2 > 0$ in any equilibrium (not only steady state) as long as $\lambda_2 > 0$; intuitively, since it is costly to carry cash when $\gamma > \beta$, agents do not carry enough to get \bar{q} , and so it is always optimal to bring at least a little of the real asset to the DM in order to get closer to \bar{q} .

Lemma 2. *If $\rho > 0$, then $a_2 > 0$ in any equilibrium.*

Proof: Suppose $a_2 = 0$. Then $q_1 = q_2 \equiv q_0$. Given $\gamma > \beta$, we know $q_0 < \bar{q} \leq q^*$ by standard

⁸See Geromichalos et al. (2007) for details. Intuitively, let $\bar{y} = z(\bar{q}) \leq y^*$. Then the buyer's surplus is decreasing in y for $y > \bar{y}$, because the value of what he pays y increases by more than the value of what he gets q . And $\bar{y} < y^*$ unless $\theta = 1$ due to a standard holdup problem: unless the buyer has all the bargaining power, the q that maximizes his surplus is not the q^* that maximizes the total surplus.

results. Since $q_0 < q^*$, from (9) at equality

$$(\lambda_1 + \lambda_2)\ell(q_0) + 1 = \phi/\beta\hat{\phi} = \gamma/\beta > 1,$$

which implies $\ell(q_0) > 0$. Since we are supposing $a_2 = 0$, feasibility implies $a_1 = A > 0$, and (10) holds at equality. Thus, $\psi = \beta(\hat{\psi} + \delta)$. Then (11) implies $\lambda_2\ell(q_0) \leq 0$, a contradiction. ■

We now have $m > 0$ and $a_2 > 0$, and it remains to determine whether $a_1 = 0$ or $a_1 > 0$. To answer this, let $\tilde{q} < \bar{q}$ be defined by $\ell(\tilde{q}) \equiv (\gamma - \beta)/\beta\lambda_1$, and let

$$\bar{A} \equiv [z(\bar{q}) - z(\tilde{q})](1 - \beta)/\delta > 0.$$

The next result, the proof of which is in the Appendix, tells us that $A \leq \bar{A}$ (the real asset is relatively scarce) implies $a_1 = 0$, and $A > \bar{A}$ (the real asset is plentiful) implies $a_1 > 0$. Given A , the latter case becomes more likely when γ or ρ decreases, and when β , δ or λ increases.

Proposition 1. (i) If $A \leq \bar{A}$ there exists a unique steady state monetary equilibrium, and in this equilibrium, (q_1, q_2) solves

$$A\delta = [z(q_2) - z(q_1)]\{1 - \beta[\lambda_2\ell(q_2) + 1]\} \quad (12)$$

$$\gamma = \beta[\lambda_1\ell(q_1) + \lambda_2\ell(q_2) + 1], \quad (13)$$

prices are $\phi = z(q_1)/M$ and $\psi = [z(q_2) - z(q_1)]/A - \delta$, and the portfolio is $(m, a_1, a_2) = (M, 0, A)$. (ii) If $A > \bar{A}$ there exists a unique steady state equilibrium, and in this equilibrium, $(q_1, q_2) = (\tilde{q}, \bar{q})$, prices are $\phi = z(\tilde{q})/M$ and $\psi = \beta\delta/(1 - \beta)$, and the portfolio is $(m, a_1, a_2) = (M, A - \bar{A}, \bar{A})$.

Before using Proposition 1 to discuss the economics of the model, we need a little more notation. Imagine an asset that costs 1 unit of x in the CM and pays $1 + r$ units of x in the next CM, but cannot be brought to (or traded in) the DM. Its return, the risk free real interest rate, is pinned down by $1 + r = 1/\beta$. Now imagine an asset that costs 1 dollar in the CM and pays

$1 + i$ dollars in the next CM, and similarly cannot be traded in the DM. Its return, the risk free nominal rate, is $1 + i = \phi/\hat{\phi}\beta$. Hence, $1 + i = (1 + r)\phi/\hat{\phi}$. This is the Fisher equation, which can be interpreted as a simple no-arbitrage condition between real and nominal assets traded in the CM but not the DM. Given this, we can equivalently discuss monetary policy in terms of either the nominal rate i or the inflation rate $\phi/\hat{\phi}$, which in steady state equals the rate of money growth γ .

We now rewrite (12)-(13), the equilibrium conditions for the case where $A \leq \bar{A}$, as

$$(1 + r)A\delta = [z(q_2) - z(q_1)][r - \lambda_2\ell(q_2)] \quad (14)$$

$$i = \lambda_1\ell(q_1) + \lambda_2\ell(q_2). \quad (15)$$

Let $q_2 = \alpha(q_1)$ and $q_1 = \mu(q_2)$ denote the implicit functions characterized by (14) and (15), respectively. We establish in the Appendix that μ is decreasing while α is increasing, and that they intersect for some $q_1 \in [0, \bar{q}]$, as seen in Figure 1. For $A \leq \bar{A}$, as in Figure 1-A, the intersection of α and μ determines the equilibrium $(q_1, q_2) \in [0, \bar{q}]^2$, from which we can recover prices and the portfolio from Proposition 1. For $A > \bar{A}$, as in Figure 1-B, the intersection of α and μ occurs at $q_2 > \bar{q}$, and the equilibrium is $(q_1, q_2) = (\bar{q}, \bar{q})$. Equilibrium is thus conveniently characterized by the intersection of $q_1 = \mu(q_2)$ and $q_2 = \bar{\alpha}(q_1) = \min\{\alpha(q_1), \bar{q}\}$.

When $A \leq \bar{A}$, $q_2 < \bar{q}$ and a bears a *liquidity premium*, $\ell(q_2) > 0$. In this case (11) implies $\psi > \beta\delta/(1 - \beta) = \delta/r$ - i.e. the price of the asset exceeds the present value of its dividend stream, because it reflects not only the asset's fundamental value, but also its value as a medium of exchange. To be precise, in steady state (11) at equality yields

$$\psi = \frac{\beta\delta[1 + \lambda_2\ell(q_2)]}{1 - \beta[1 + \lambda_2\ell(q_2)]} = \frac{\delta}{r} \left[1 + \frac{(1 + r)\lambda_2\ell(q_2)}{r - \lambda_2\ell(q_2)} \right],$$

which exceeds the fundamental price δ/r whenever $\lambda_2 > 0$ or $q_2 < \bar{q}$.⁹ On the other hand, when $A > \bar{A}$, we have $q_2 = \bar{q}$ and the price of the asset is equal to its fundamental value, $\psi = \delta/r$.

⁹An alternative but equivalent way to price a in equilibrium with $A < \bar{A}$ comes from the bargaining solution, which says $z(q_1) = M\phi$ and $z(q_2) = A(\psi + \delta) + M\phi$, and hence implies $A\psi = z(q_2) - z(q_1) - \delta A$.

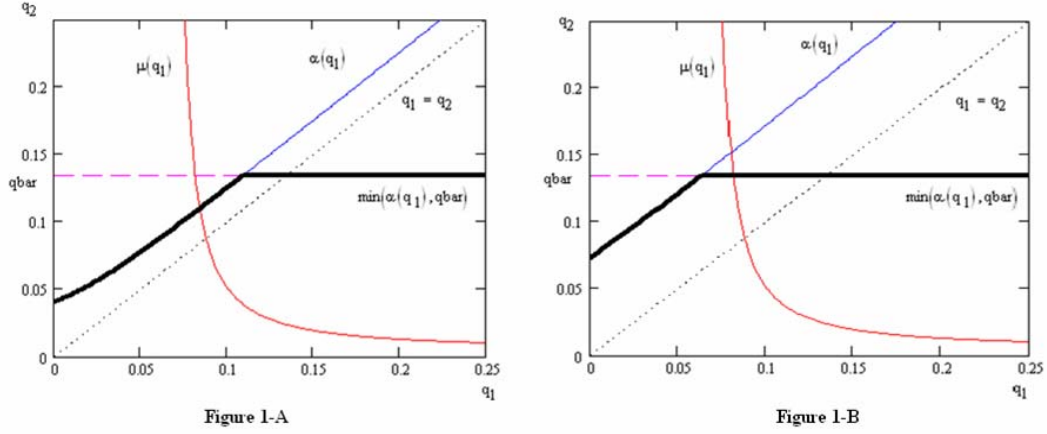


Figure 1:

The reason that the asset carries no liquidity premium in this case is that $A > \bar{A}$ implies $a_1 > 0$, so that agents at the margin are indifferent between holding a for its dividend stream alone and as a medium of exchange. Of course, m bears a liquidity premium in any equilibrium where $q_1 > 0$, since its fundamental value is 0.

When $A < \bar{A}$, we obtain the following relationships between the equilibrium and the exogenous variables.

$x =$	i	A	δ	λ	ρ
$\frac{\partial q_1}{\partial x}$	-	-	-	+	-
$\frac{\partial q_2}{\partial x}$	-	+	+	+	?
$\frac{\partial \phi}{\partial x}$	-	-	-	+	-
$\frac{\partial \psi}{\partial x}$	+	-	?	?	+

Table 1: Comparative Statics when $A < \bar{A}$

Formal derivations are in the Appendix, but the results are easy to see in Figure 1-A. For example, an increase in i shifts the μ curve southwest but leaves α unchanged, leading to a reduction in both q_1 and q_2 . Intuitively, as the nominal rate increases, agents try to economize on money, reducing its CM price ϕ and hence its DM value $q_1 = z^{-1}(\phi M)$. Given this, agents

substitute into the real asset a , which raises its price ψ , but the net effect is to lower $q_2 = z^{-1}(\phi M + \psi A)$.¹⁰

An interesting implication of this result is that the observed return on a between meetings of the CM, $1 + \delta/\psi$, decreases with i ; Geromichalos et al. (2007) interpret this kind of result as saying inflation is bad for stock market returns. So the Fisher equation apparently does *not* hold for this asset: its observed return is not independent of nominal interest or inflation rates. The reason of course is that i affects the demand for m , which affects the demand for a , and hence its price and return. Note that this would not happen if a were never used as a medium of exchange, since $\rho = 0$ entails the usual result $\psi = \delta/r$, and $\partial\psi/\partial i = 0$. It is when assets have *some* liquidity premium that the idea behind the Fisher equation (real returns do not depend on inflation) goes wrong.

Other results of interest include the following. An increase in δ shifts the α curve northwest but leaves μ unchanged, leading to a fall in q_1 and a rise in q_2 . Intuitively, as dividends increase, agents substitute out of m and into a , which affects both their DM and CM values. An increase in A is similar. Changing λ or ρ shifts both curves – e.g. an increase in λ (more frequent meetings) shifts μ to the right and α to the left, although in the Appendix we show the net effect on both q_1 and q_2 is positive. As ρ increases (more sellers accept a) q_1 decreases, but the effect on q_2 is ambiguous. We can show $\partial\phi/\partial\rho < 0$ and $\partial\psi/\partial\rho > 0$: as more sellers accept a in the DM, agents want to substitute out of m and into a , which affects their CM prices. Other effects can be similarly discussed.

The sensitivity analysis above holds under the condition $A < \bar{A}$. When $A \geq \bar{A}$ we have $q_2 = \bar{q}$ and $q_1 = \tilde{q}$ where \tilde{q} is such that $\tilde{q} = \mu(\bar{q})$, or $\ell(\tilde{q}) = i/\lambda(1 - \rho)$. It is easy to see that $\partial\tilde{q}/\partial i < 0$, $\partial\tilde{q}/\partial\lambda > 0$, and $\partial\tilde{q}/\partial\rho < 0$, while neither A nor δ affect q_1 . The value of q_2 is unaffected by a perturbation to any of the five exogenous parameters. Therefore, the CM price

¹⁰Notice that in the limit as $i \rightarrow \infty$, the μ curve becomes vertical at $q_1 = 0$, and we get $q_2 = \alpha(0)$ as a nonmonetary equilibrium.

of m is decreasing in i and ρ and increasing in λ , while as we already have remarked, the CM price of a is tied down by its fundamental value $\psi = \beta\delta/(1 - \beta)$. Also, notice that $\tilde{q} < \bar{q}$ if $i > 0$, but $\tilde{q} \rightarrow \bar{q}$ as $i \rightarrow 0$. In fact, as $i \rightarrow 0$, we have $\bar{A} \rightarrow 0$, which means we must be in an equilibrium where $a_1 > 0$. This simply says that at the Freidman rule $i = 0$, we get $q_1 = q_2 = \bar{q}$ and all assets pay the same return, $1 + r = 1/\beta$. Hence, $i = 0$ is the optimal policy, although it does not give the first best outcome $q = q^*$ unless $\theta = 1$, due to the holdup problem.

To close this section, notice that there is a well-defined sense in which money is essential in this economy as long as $\lambda_1 = \lambda(1 - \rho) > 0$ (there are some meetings where the seller does not accept a) for any values of the other parameters. To be precise, expected utility is higher in the monetary equilibrium than in the nonmonetary equilibrium. In the nonmonetary equilibrium, $q_1 = 0$ and $q_2 = \bar{\alpha}(0)$. In any monetary equilibrium, q_1 is higher, and so is q_2 at least as long as it is below \bar{q} . Interestingly, we not only have more trade on the extensive margin in the monetary equilibrium, because now you can buy even from sellers who do not accept claims to a , we also have more trade on the intensive margin when $q_2 < \bar{q}$, since you get more even when sellers do accept claims to a . We summarize these and some other key results of this section as follows.

Proposition 2. *Money is essential if $\lambda_1 > 0$. As long as $A < \bar{A}$, the real asset bears a liquidity premium $\ell(q_2) > 0$, and its price exceeds the fundamental value δ/r ; in this case, an increase in γ or i reduces the demand for m and hence ϕ , which increases the demand for a and hence ψ , and lowers the observed return between meetings of the CM, $1 + \delta/\psi$. An increase in γ or i decreases q even in meetings where the seller accepts a . The optimal policy is $i = 0$.*

3 Extension: A Cashless Market

One of the questions we set out to address is whether a change in inflation affects only those who use cash, or others as well. In the benchmark model, all agents hold cash, so we cannot address this question. Hence, we present here a simple extension to show change in monetary policy can

affect agents that hold no currency. To this end, suppose there are two distinct decentralized markets, B and C , where a fraction b and $1 - b$ of the agents go between meetings of the CM.¹¹ In market B , sellers accept both a and m , while in market C some transactions will be cash-only: as in the baseline model, a fraction ρ of sellers accept a while the remaining $1 - \rho$ do not. As before, let $\lambda_1 = (1 - \rho)\lambda$ and $\lambda_2 = \rho\lambda$, where λ is the exogenous arrival rate.

Since market C is identical to the DM in the benchmark model, the first order conditions are identical to (9) - (11). In market B , the matching technology is characterized by $\tilde{\lambda}$, which may or may not equal λ , and there is only one type of meeting, since all sellers accept both m and a . Let \tilde{q}_2 and \tilde{p}_2 denote quantities and prices traded in this market.¹² The DM value function can be characterized by

$$V^B(m, a_1, a_2) = (1 - \tilde{\lambda})W(y) + \tilde{\lambda} [u(\tilde{q}_2) + W(y - \tilde{p}_2)],$$

with first order conditions

$$V_1^B(m, a_1, a_2) = \phi [\tilde{\lambda}\ell(\tilde{q}_2) + 1] \tag{16}$$

$$V_2^B(m, a_1, a_2) = \psi + \delta \tag{17}$$

$$V_3^B(m, a_1, a_2) = (\psi + \delta) [\tilde{\lambda}\ell(\tilde{q}_2) + 1] \tag{18}$$

By substituting into the usual first order conditions from the CM, we have

$$m : \phi \geq \beta\phi' [\tilde{\lambda}\ell(\tilde{q}_2) + 1], = \text{if } m > 0 \tag{19}$$

$$a_1 : \psi \geq \beta(\psi' + \delta), = \text{if } a_1 > 0 \tag{20}$$

$$a_2 : \psi \geq \beta(\psi' + \delta) [\tilde{\lambda}\ell(\tilde{q}_2) + 1], = \text{if } a_2 > 0 \tag{21}$$

We concentrate on the case where $q_2, \tilde{q}_2 < \bar{q}$ (otherwise, ψ is determined solely by the

¹¹Assume for concreteness that they are permanently assigned to one of these markets; things would be basically the same if they were randomly assigned each period.

¹²We adopt the convention that the variables associated with agents in market B will have $\tilde{}$ and those in market C will not.

fundamental value). Recall that $\rho < 1$ implies that agents in market C will bring a strictly positive quantity of cash, so (9) holds with equality.

Lemma 3. *In any equilibrium with $\psi > \beta\delta/(1 - \beta)$, both $a_2 > 0$ and $\tilde{a}_2 > 0$.*

Proof: When $q_2, \tilde{q}_2 < \bar{q}$, $a_2 + \tilde{a}_2 = A$, and so clearly either $a_2 > 0$ or $\tilde{a}_2 > 0$. Suppose first that $a_2 > 0$ and $\tilde{a}_2 = 0$. From the Inada conditions, $\tilde{a} = 0 \Rightarrow \tilde{m} > 0$. We also have shown that $\rho < 1 \Rightarrow m > 0$. Therefore, from the first order conditions on m and \tilde{m} , we have

$$\lambda_1 \ell(q_1) + \lambda_2 \ell(q_2) + 1 = \tilde{\lambda} \ell(\tilde{q}_2) + 1,$$

and since $a_2 > 0$ we know

$$\begin{aligned} \psi &= \beta(\psi' + \delta)[\lambda_2 \ell(q_2) + 1] \\ &= \beta(\psi' + \delta)[\tilde{\lambda} \ell(\tilde{q}_2) + 1 - \lambda_1 \ell(q_1)] \\ &< \beta(\psi' + \delta)[\tilde{\lambda} \ell(\tilde{q}_2) + 1]. \end{aligned}$$

This contradicts (21). Now suppose the opposite, that $a_2 = 0$ and $\tilde{a}_2 > 0$. Since $a_2 = 0$, we know $q_1 = q_2 \equiv q_0$ for some $q_0 < \bar{q}$. Therefore, from (9),

$$\phi = \beta\phi'[\lambda \ell(q_0) + 1].$$

Moreover, $a_2 = 0$ implies the price of the asset is greater than its marginal value, so that generically

$$\psi > \beta(\psi' + \delta)[\lambda \ell(q_0) + 1].$$

Since $\tilde{a}_2 > 0$,

$$\begin{aligned} \psi &= \beta(\psi' + \delta)[\tilde{\lambda} \ell(\tilde{q}_2) + 1] \\ &\Rightarrow \tilde{\lambda} \ell(\tilde{q}_2) > \lambda \ell(q_0) \\ &\Rightarrow \phi < \beta\phi'[\tilde{\lambda} \ell(\tilde{q}_2) + 1], \end{aligned}$$

which contradicts (19). ■

Lemma 4. *Agents in market B carry no cash, $\tilde{m} = 0$.*

Proof: Since $a_2 > 0$ and $\tilde{a}_2 > 0$, we know that $\lambda_2 \ell(q_2) = \tilde{\lambda} \ell(\tilde{q}_2)$ from (11)-(21), and therefore

$$\begin{aligned}\phi &= \beta \phi' \left[\lambda_1 \ell(q_1) + \lambda_2 \ell(q_2) + 1 \right] \\ &= \beta \phi' \left[\lambda_1 \ell(q_1) + \tilde{\lambda} \ell(q_2) + 1 \right] \\ &> \beta \phi' \left[\tilde{\lambda} \ell(\tilde{q}_2) + 1 \right].\end{aligned}$$

Hence the first order condition on \tilde{m} is not binding, and $\tilde{m} = 0$. ■

Note that

$$\begin{aligned}z(q_1) &= \phi m \\ z(q_2) &= \phi m + (\psi + \delta) a_2 \\ z(\tilde{q}_2) &= \phi m + (\psi + \delta) \tilde{a}_2\end{aligned}$$

and when $a_1 = \tilde{a}_1 = 0$, we know $A = (1 - b)a_2 + b\tilde{a}_2$. Given this, we have the following two conditions on asset prices that will complete our characterization of equilibrium:

$$z(q_1) = \phi M \tag{22}$$

$$(1 - b)z(q_2) + bz(\tilde{q}_2) = (1 - b)\phi M + (\psi + \delta)A \tag{23}$$

As before, we can use these relationships to derive the following expression relating the price of assets to the quantities exchanged:

$$\psi + \delta = \frac{(1 - b)[z(q_2) - z(q_1)] + bz(\tilde{q}_2)}{A}. \tag{24}$$

Therefore, equilibrium values (q_1, q_2, \tilde{q}_2) are characterized by three equations:

$$\gamma = \beta \left[\lambda_1 \ell(q_1) + \lambda_2 \ell(q_2) + 1 \right] \tag{25}$$

$$A\delta = \left\{ (1 - b)[z(q_2) - z(q_1)] + bz(\tilde{q}_2) \right\} \left\{ 1 - \beta[\lambda_2 \ell(q_2) + 1] \right\} \tag{26}$$

$$A\delta = \left\{ (1 - b)[z(q_2) - z(q_1)] + bz(\tilde{q}_2) \right\} \left\{ 1 - \beta[\tilde{\lambda} \ell(\tilde{q}_2) + 1] \right\} \tag{27}$$

The important thing to note is that changes in monetary policy effect q_1 and q_2 through (??), and this will effect both ψ and \tilde{q}_2 . Again, appealing to the Fisher equation, we can rewrite these conditions as:

$$i = (1 - \rho)\lambda\ell(q_1) + \rho\lambda\ell(q_2) \quad (28)$$

$$(1 + r)A\delta = \left\{ (1 - b)[z(q_2) - z(q_1)] + bz(\tilde{q}_2) \right\} [r - \rho\lambda\ell(q_2)] \quad (29)$$

$$(1 + r)A\delta = \left\{ (1 - b)[z(q_2) - z(q_1)] + bz(\tilde{q}_2) \right\} [r - \tilde{\lambda}\ell(\tilde{q}_2)]. \quad (30)$$

Finally, since there is a one-to-one mapping between q_2 and \tilde{q}_2 , let us denote $q_2 = h(\tilde{q}_2) \equiv \ell^{-1}\left[\frac{\tilde{\lambda}}{\rho\lambda}\ell(\tilde{q}_2)\right]$, so that the equilibrium conditions reduce to two equations in (q_1, \tilde{q}_2) :

$$i = (1 - \rho)\lambda\ell(q_1) + \tilde{\lambda}\ell(\tilde{q}_2) \quad (31)$$

$$(1 + r)A\delta = \left\{ (1 - b)[z(h(\tilde{q}_2)) - z(q_1)] + bz(\tilde{q}_2) \right\} [r - \tilde{\lambda}\ell(\tilde{q}_2)]. \quad (32)$$

Using similar techniques to the previous analysis, we can show $\partial\tilde{q}_2/\partial i < 0$. That is, an increase in the inflation rate causes agents in market C to shift their portfolios away from cash and towards the real asset, thus driving the price of assets up. In equilibrium, this results in agents in market B purchasing less of the asset, and receiving less in exchange in the DM.

4 Endogenous Liquidity

We now endogenize the liquidity or acceptability of the assets. Suppose that each agent possesses the technology required to create counterfeit money and assets at no cost. Moreover, suppose that all agents recognize cash and are endowed with the ability to spot a counterfeit note without incurring any additional costs. However, all agents are *not* ex-ante familiar with claims to a real asset. Instead, an agent must invest in a costly technology in order to verify the authenticity of such a claim. In particular, assume that agent $i \in [0, 1]$ must incur cost $\kappa(i)$ in order to verify the authenticity of real assets, where we arrange agents in order of increasing costs so that $\kappa'(i) \geq 0$. We assume $\kappa(i)$ is continuous. In equilibrium, an agent who has not invested

in the verification technology will not accept real assets, since the buyer will surely offer her a counterfeit. Therefore, the fraction of agents that incur cost $\kappa(i)$ is equivalent to the fraction of agents that accepts a , ρ .¹³

Given this framework, in addition to the portfolio selection problem he makes in the CM, an agent must also choose whether or not to invest in this verification technology. The return from this investment depends on the number of agents who bring assets into the DM; but the incentive for agents to bring assets depends in turn on the number of sellers who accept assets. Therefore, coordination will play a key role in any equilibrium. Of course, there will always be an equilibrium in which no sellers invest in the technology and no buyers bring assets into the DM. However, it's not obvious that an equilibrium exists in which both money and the asset circulate as a medium of exchange. Suppose, for example, that the distribution of verification costs is such that the maximum cost for any agent to acquire the technology, $\kappa(1)$, is close to zero. If buyers bring even a small amount of the asset to the market, then, there will be a strictly positive value to a seller of having the technology.

If the maximum cost is less than this benefit, all sellers will acquire the technology. But in this case, the asset is universally accepted and money has no advantage over taking the asset into the DM, and since there is a positive return on the asset, no agent will hold money. Thus, if the cost of acquiring the technology is too low, there will only be equilibrium in which only money is accepted in the DM (that is, no sellers acquire the technology), and another in which only the asset is traded in the DM (all sellers acquire the technology). Alternatively, if the lower bound of $\kappa(0)$ is sufficiently high, then there might be no level of DM asset holdings such that any sellers would invest in the technology. Thus for there to be interesting equilibria in which

¹³This setup has several analogs in practice. Perhaps the most obvious is the choice consumers (producers) face between using (accepting) debit cards. A consumer chooses how much cash to carry and how much to leave in interest-bearing bank accounts that can be drawn upon using a debit card. A producer always accepts cash, but has to choose whether or not to invest in a machine that allows him to verify a debit card. A seller would never simply write down the numbers on a debit card without using the verification technology. Moreover, it is clear that producers face heterogeneous costs for the use of this technology (large stores often pay smaller fees e.g.).

both money and assets are used in the DM there must be constraints on the distribution of costs to sellers.

For any $\rho \in [0, 1]$, we can define the premium from investing in the technology as

$$\Pi(\rho) \equiv \beta\lambda \left\{ \left[z[q_2(\rho)] - c[q_1(\rho)] \right] - \left[z[q_1(\rho)] - c[q_1(\rho)] \right] \right\}. \quad (33)$$

That is, the expected benefit from investing in the technology is equal to the extra utility a seller receives from being in a type 2 meeting – as opposed to a type 1 meeting – discounted and weighted by the probability that he is a seller. Note here that $q_1(\rho)$ and $q_2(\rho)$ are well-defined objects, as described in Proposition 1. The decision rule, then, for a seller with verification cost κ is to invest if $\Pi(\rho) \geq \kappa$ and not otherwise. An equilibrium with $\rho^* > 0$, then, satisfies the condition that $\Pi(\rho^*) \geq \kappa(\rho^*)$, with equality if $\rho^* < 1$.

Constructing an argument for the existence of an equilibrium is not difficult. Define the cumulative distribution function $F = \kappa^{-1}(i)$ and the equilibrium mapping $\mathcal{E} : [0, 1] \rightarrow [0, 1]$ as $\mathcal{E}(\rho) = F[\Pi(\rho)]$. An equilibrium is thus a fixed point of this mapping. One can show that \mathcal{E} is a continuous mapping from a compact, convex set into itself, so that Brouwer's fixed point theorem assures an equilibrium. However, as discussed above, we would like to know when an equilibrium with $\rho \in (0, 1)$ exists. To this end, we consider conditions on κ that would ensure us an interior fixed point.

Since the value of investing in the technology is bounded above, a sufficiently high upper bound $\bar{\kappa}$ on the support of κ would guarantee that $\rho^* < 1$. More specifically, let

$$\bar{\Pi} = \beta\lambda \left\{ [z(\bar{q}) - c(\bar{q})] - [z(0) - c(0)] \right\} = \beta\lambda [z(\bar{q}) - c(\bar{q})].$$

That is, the maximal possible value of Π would be attained when agents carry enough assets to purchase the optimal quantity \bar{q} but no cash. For any ρ , we have $\Pi(\rho) \leq \bar{\Pi}$. Therefore, if $\bar{\kappa} > \bar{\Pi}$, then $\kappa(1) > \Pi(1)$. In words, if there are some agents with sufficiently high costs of verification we can be assured that there is not an equilibrium in which $\rho = 1$.

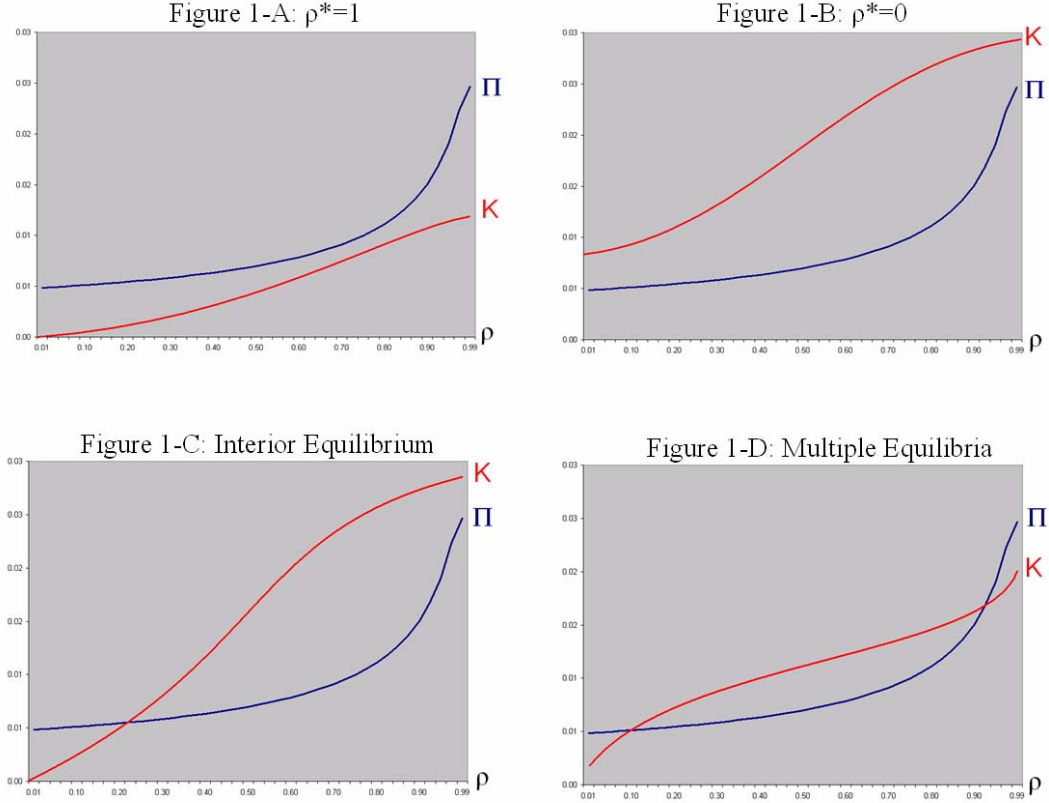


Figure 2:

Similarly we know that the value of investing in the technology is bounded below. In particular, it can be shown that

$$\lim_{\rho \rightarrow 0^+} \Pi(\rho) \equiv \underline{\Pi} > 0.$$

If $\kappa(\epsilon) < \underline{\Pi}$ for some ϵ sufficiently small (or zero), we are assured that $\rho^* > 0$. Therefore, if we impose sufficient structure on the distribution costs, we are assured of an equilibrium in which both cash and assets circulate as media of exchange, though assets are less liquid than cash. Such an equilibrium need not be unique. For illustrative purposes, let us parameterize the

model by $u(q) = q^{1/2}$, $c(q) = q$, and pick some parameter values.¹⁴ In Figure 2-A through 2-D we illustrate four distinct possibilities: a unique equilibrium with $\rho^* = 1$, a unique equilibrium with $\rho^* = 0$, a unique equilibrium with $\rho^* \in (0, 1)$, and multiple equilibria with $\rho^* \in (0, 1)$.

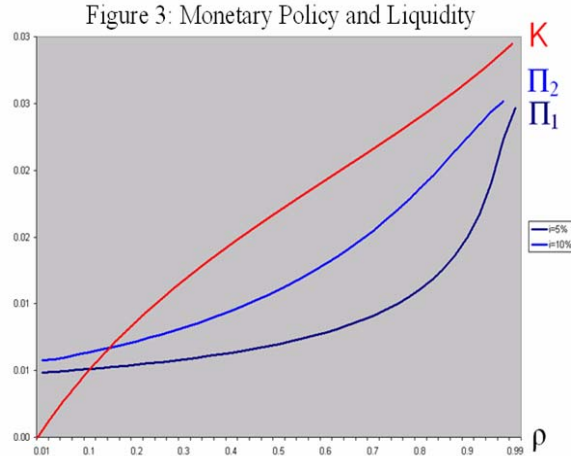


Figure 3:

Lastly, consider the effect of monetary policy. In Figure 3 we compare equilibria under two regimes, $i = 0.05$ and $i = 0.1$. As illustrated, the effect of higher i is that buyers allocate more of their wealth to a , which makes more sellers choose to (pay to be able to) accept a as a means of payment. This prediction seems consistent with observed behavior during periods of high inflation. During Argentina's struggle with hyper-inflation, inflation-indexed bonds circulated as a medium of exchange. Similarly, many countries have resorted to the dollar during high inflation. Note that while inflation served only a negative role in the model with exogenous liquidity, there is a potential welfare-improving role for inflation in this model with endogenous liquidity: inflation moves agents away from using only money, which is after all a dominated medium of exchange.

¹⁴In Figures 2 and 3, we set $\theta = 0.5$, $\gamma = 1.0$, $\beta = 0.95$, $\lambda = 0.35$, $A = 0.1$, and $\delta = 0.02$.

5 Conclusion

Monetary policy controls the rate of return on a single asset: currency. However, in adjusting the rate of return on currency, the policymaker is able to affect both the rate of return *and* the liquidity properties of other assets as well. In particular, as the rate of return on cash falls, agents reallocate their wealth to alternative assets, hence driving up the price and down the rate of return of these assets. As a result, even those agents not holding cash are affected by monetary policy. Moreover, as agents shift their portfolios to assets other than currency, there is greater incentive for sellers to pay to be able to accept these assets as a medium of exchange. In turn, as the rate of return on currency falls, other assets could potentially become more liquid, further increasing their price.

Several extensions come to mind. First, while the bargaining framework employed in the DM is common in models of this sort, it carries some undesirable features. For one, tractability requires we assume both parties are perfectly informed about the other's portfolio. As a result, a hold-up problem emerges, and we must sort out how much agents leave in the CM and how much they carry into the DM. It may be worth considering alternative mechanisms for the terms of trade, as has been done in models where money is the only liquid asset. However, regardless of this mechanism, the model seems useful for addressing several interesting issues. The most obvious issue is the relationship between inflation and the use of alternative means of payment, such as debit cards, or maybe foreign currency.

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7 Appendix

Proof of Proposition 1 We only present the case $A \leq \bar{A}$; the other case is similar. First, by the implicit function theorem, we have $\mu'(q_1) = -\beta\lambda_1'\ell(q_1)/\lambda_2'\ell(q_2) < 0$ and

$$\alpha'(q_1) = \frac{-z'(q_1) \{1 - \beta[\lambda_2\ell(q_2) + 1]\}}{\beta\lambda_2e'(q_2) [z(q_2) - z(q_1)] - z'(q_1) \{1 - \beta[\lambda_2\ell(q_2) + 1]\}} > 0.$$

Now let q' satisfy $\ell(q') = \frac{\gamma-\beta}{\beta\lambda_1} + \frac{\lambda_2}{\lambda_1}$, with $q' < \tilde{q} \leq \bar{q}$. Since $\ell'(q) < 0$ and $\lim_{q \rightarrow \infty} \ell(q) = -1$, it is easy to see that $\lim_{q_1 \rightarrow q'} \mu(q_1) = \infty$. One can also show $\lim_{q_1 \rightarrow q'} \alpha(q_1) < \infty$. Therefore, $\mu(q') > \alpha(q')$.

Now consider (13) with $q_1 = \bar{q}$, so that $\gamma/\beta = \lambda_2\ell(q_2) + 1$. This implies $\ell(q_2) \leq 0$, so $\mu(\bar{q}) \leq \bar{q}$. Now consider (12) with $q_2 = \bar{q}$, so that $A\delta = [z(\bar{q}) - z(q_1)](1 - \beta)$. This implies $\alpha^{-1}(\bar{q}) < \bar{q}$, and $\alpha(\bar{q}) > \bar{q} \geq \mu(\bar{q})$. Since $\mu' < 0$, $\alpha' > 0$, $\mu(q') > \alpha(q')$ for some $q' < \bar{q}$, and $\alpha(\bar{q}) \geq \mu(\bar{q})$, we conclude that there exists a unique equilibrium pair (q_1, q_2) with $q_1 > 0$ and $q_2 \leq \bar{q}$ that satisfy (12) and (13).

It is left to show that (12) and (13) are equivalent to the conditions for equilibrium with $m > 0$, $a_1 = 0$, and $a_2 > 0$. Since $m > 0$, (9) must be met with equality in equilibrium. Since $\gamma = \phi/\phi'$, clearly (13) and (9) are equivalent. Since $a_2 > 0$, (11) must also hold with equality. We know that $a_1 = 0 \Rightarrow a_2 = A$. Also, $z(q_1) = \phi M$ and $z(q_2) = \phi M + (\psi + \delta)A$ implies $\psi = [z(q_2) - z(q_1)]/A - \delta$. Substituting this into (12) yields (11). ■

Proof of Proposition 2. (\Rightarrow) Suppose (q_1, q_2) characterize a case 2 equilibrium, so that $q_2 = \bar{q}$, $q_1 = \hat{q}$, $a_1 = A_1 > 0$ and $A_2 < A$. We know that

$$\psi + \delta = \frac{\delta}{1 - \beta} = \frac{z(q_2) - z(q_1)}{A_2}.$$

The desired result follows immediately. Now suppose

$$\frac{z(\bar{q}) - z(\hat{q})}{A} < \frac{\delta}{1 - \beta}.$$

Let $q_2 = \bar{q}$. Clearly $\exists q_1 = \hat{q}$ such that (15) holds. Moreover, let $\psi = \frac{\beta\delta}{1-\beta}$, and define $A_2 = \frac{(1-\beta)(z(\bar{q})-z(\hat{q}))}{\delta}$ so that $A_1 = A - A_2$. It is trivial to verify that this is a case 2 equilibrium. ■

Sensitivity analysis with $A \leq \bar{A}$: Let Δ denote the determinant of

$$\begin{bmatrix} \lambda_1 l'(q_1) & \lambda_2 l'(q_2) \\ [\lambda_2 \ell(q_2) - r] z'(q_1) & [r - \lambda_2 \ell(q_2)] z'(q_2) - [z(q_2) - z(q_1)] \lambda_2 l'(q_2) \end{bmatrix}$$

From (14), equilibrium with $q_2 \geq q_1$ requires $r - \lambda_2 \ell(q_2) \geq 0$, so $\Delta < 0$. Then we have:

$$\begin{aligned} \frac{\partial q_1}{\partial i} &= \frac{[r - \rho \lambda \ell(q_2)] z'(q_2) - [z(q_2) - z(q_1)] \rho \lambda \ell'(q_2)}{\Delta} < 0 \\ \frac{\partial q_2}{\partial i} &= \frac{[r - \rho \lambda \ell(q_2)] z'(q_1)}{\Delta} < 0 \\ \frac{\partial q_1}{\partial \delta} &= \frac{-(1+r)A\rho\lambda\ell'(q_2)}{\Delta} < 0 \\ \frac{\partial q_2}{\partial \delta} &= \frac{(1+r)A(1-\rho)\lambda\ell'(q_1)}{\Delta} > 0 \\ \frac{\partial q_1}{\partial A} &= \frac{-(1+r)\delta\rho\lambda\ell'(q_2)}{\Delta} < 0 \\ \frac{\partial q_2}{\partial A} &= \frac{(1+r)\delta(1-\rho)\lambda\ell'(q_1)}{\Delta} > 0 \\ \frac{\partial q_1}{\partial \rho} &= \frac{\lambda[\ell(q_1) - \ell(q_2)][r - \rho\lambda\ell(q_2)]z'(q_2) - [z(q_2) - z(q_1)]\rho\lambda\ell'(q_2)\lambda\ell(q_1)}{\Delta} < 0 \\ \frac{\partial q_2}{\partial \rho} &= \frac{\lambda[\ell(q_1) - \ell(q_2)][r - \rho\lambda\ell(q_2)]z'(q_1) + [z(q_2) - z(q_1)](1-\rho)\lambda\ell'(q_1)\lambda\ell(q_2)}{\Delta} \\ \frac{\partial q_1}{\partial \lambda} &= \frac{(1-\rho)\ell(q_1)[z(q_2) - z(q_1)]\rho\lambda\ell'(q_2) - [(1-\rho)\ell(q_1) + \rho\ell(q_2)][r - \rho\lambda\ell(q_2)]z'(q_2)}{\Delta} > 0 \\ \frac{\partial q_2}{\partial \lambda} &= \frac{(1-\rho)\ell(q_2)[z(q_2) - z(q_1)]\rho\lambda\ell'(q_1) - [(1-\rho)\ell(q_1) + \rho\ell(q_2)][r - \rho\lambda\ell(q_2)]z'(q_1)}{\Delta} > 0 \end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial i} &= \frac{[z(q_2) - z(q_1)]\rho\lambda\ell'(q_2)z'(q_1)}{A\Delta} > 0 \\
\frac{\partial \phi}{\partial i} &= \frac{z'(q_1)}{M} \frac{\partial q_1}{\partial i} < 0 \\
\frac{\partial \psi}{\partial \delta} &= \frac{[z(q_2) - z(q_1)](1 - \rho)\lambda\ell'(q_1)\rho\lambda\ell'(q_2) + [1 + \rho\lambda\ell(q_2)][(1 - \rho)\lambda\ell'(q_1)z'(q_2) + \rho\lambda\ell'(q_2)z'(q_1)]}{\Delta} \\
\frac{\partial \phi}{\partial \delta} &= \frac{z'(q_1)}{M} \frac{\partial q_1}{\partial \delta} < 0 \\
\frac{\partial \psi}{\partial A} &= \frac{[z(q_2) - z(q_1)](1 - \rho)\lambda\ell'(q_1)\rho\lambda\ell'(q_2)}{A\Delta} < 0 \\
\frac{\partial \phi}{\partial A} &= \frac{z'(q_1)}{M} \frac{\partial q_1}{\partial A} < 0 \\
\frac{\partial \psi}{\partial \rho} &= \frac{[z(q_2) - z(q_1)][(1 - \rho)\lambda\ell'(q_1)\lambda\ell(q_2)z'(q_2) + \rho\lambda\ell'(q_2)\lambda\ell(q_1)z'(q_1)]}{\Delta} > 0 \\
\frac{\partial \phi}{\partial \rho} &= \frac{z'(q_1)}{M} \frac{\partial q_1}{\partial \rho} < 0 \\
\frac{\partial \psi}{\partial \lambda} &= \frac{(1 - \rho)[z(q_2) - z(q_1)][\ell(q_2)\ell'(q_1)z'(q_2) - \ell(q_1)\ell'(q_2)z'(q_1)]}{\Delta} \\
\frac{\partial \phi}{\partial \lambda} &= \frac{z'(q_1)}{M} \frac{\partial q_1}{\partial \lambda} > 0
\end{aligned}$$

Sensitivity analysis with a cashless market:

$$\begin{aligned}
\frac{\partial \tilde{q}_1}{\partial i} &= \frac{(1 - b)\left\{z'[h(\tilde{q}_2)]h'(\tilde{q}_2)[r - \tilde{\lambda}\ell(\tilde{q}_2)]\right\} - \left\{(1 - b)[z(h(\tilde{q}_2)) - z(q_1)] + bz(\tilde{q}_2)\right\}\tilde{\lambda}\ell'(\tilde{q}_2)}{\Delta} < 0 \\
\frac{\partial \tilde{q}_2}{\partial i} &= \frac{(1 - b)z'(q_1)[r - \tilde{\lambda}\ell(\tilde{q}_2)]}{\Delta} < 0
\end{aligned}$$

where $\Delta < 0$ is given by

$$\begin{aligned}
\Delta &= (1 - \rho)\lambda_1\ell'(q_1)\left\{(1 - b)\left[z'[h(\tilde{q}_2)]h'(\tilde{q}_2)[r - \tilde{\lambda}\ell(\tilde{q}_2)]\right] + (1 - b)z'(q_1)[r - \tilde{\lambda}\ell(\tilde{q}_2)]\tilde{\lambda}\ell'(\tilde{q}_2)\right. \\
&\quad \left. - \left[(1 - b)[z(h(\tilde{q}_2)) - z(q_1)] + bz(\tilde{q}_2)\right]\tilde{\lambda}\ell'(\tilde{q}_2)\right\}.
\end{aligned}$$