

Adverse Selection and Liquidity Distortion in Decentralized Markets*

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Abstract

This paper studies the competitive equilibrium outcome in decentralized asset markets when both search frictions and adverse selection play a role. In a dynamic environment with heterogeneous sellers and buyers, I show how adverse selection leads to the downward distortion of equilibrium market liquidity. As our setup captures two important dimensions in the trading market, price and liquidity, it shows how price and liquidity are jointly determined as an equilibrium outcome and further sheds lights on market segmentation. The model predicts a strong link between the market liquidity and the underlying uncertainty stemming from adverse selection and provides an explanation for the existence of massive illiquidity. It further allows for a richer analysis of how sorting patterns are determined in such an environment and how different market distortion may arise when sellers' motives for sale are unknown to the market.

Key words: Liquidity; Search frictions; Adverse selection; Over-the-Counter; Market segmentation

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1 Introduction

It is commonly believed that massive illiquidity in asset markets has been a catalyst for the current financial crisis. Illiquid markets make it difficult for companies to access capital and likewise for investors to find a place to put their money to work. The real question is why markets remain illiquid even when there is a positive gain from trade. Furthermore, market liquidity usually presumes that there are buyers on the other side. However, as the recent crisis has demonstrated, the possibility of a "buyers strike" arises. This concept seems contradictory to the standard notion that prices should adjust downward to the level at which buyers will be willing to enter the market. These phenomena show that for a model to speak on this issues, it is important to understand both the trading price and the market liquidity. To this end, this paper studies a dynamic environment in which the decentralized trading markets are subject to both search frictions and adverse selection. It demonstrates how illiquidity arises endogenously as an competitive equilibrium outcome and analyzes the impact of market distortions on both the price and the market liquidity.

As lot of assets are traded in decentralized markets, traders must search for the counterparty. Traders therefore care about both the selling price as well as liquidity, which, in line with the Over-the-Counter and monetary search literature, is defined as the expected search time. This definition here emphasizes the idea that market liquidity is provided by buyers on the other side of the market. How fast a seller can cash his assets will depend crucially on how many buyers are out there, ie, how tight the market is. The other key element in our framework is adverse selection, which can not be overemphasized. For example, the difficulty in assessing the fundamental value of asset-backed securities, which therefore leads to the adverse selection problem, has been one of the prevailing explanations for the recent crisis. Introducing search frictions with adverse selection is not only realistic but further allows the analysis on the impact of information frictions on the market liquidity (the extensive margin of trade) besides the trading price (the terms of trade). This makes this paper distinct from the long literature on adverse selection, which only focuses on the price discount. In particular, as our setup employes the competitive search equilibrium where uninformed principals (buyers) post prices to attract informed agents (sellers) and sellers direct their search toward their preferred market, traders sort themselves into different market environments taking into account both price and liquidity. Hence, the model gives predictions on price, market liquidity (trading volume) as well as market segmentation.

The first result shown in our basic model illustrates that with the existence of adverse

selection, the equilibrium market liquidity will be downward distorted when compared to an environment with complete information. In fact, as an equilibrium outcome, prices will not adjust downward and fewer buyers will enter. In particular, the market with a higher quality asset will suffer a more distorted market tightness when compared to the benchmark with complete information. The key intuition is that holding different quality assets results in different liquidity preference. This is essentially the mechanism behind this paper, which demonstrates an agent's type is revealed by his choice of market. The same mechanism also plays a role in a contemporaneous work by Guerrieri and Shimer (2011), who studies the competitive equilibrium outcome in a dynamic asset market without search frictions. In both Guerrieri and Shimer (2011) and our basic model, liquidity distortion works as a screening mechanism at the equilibrium. This outcome is unique and no pooling equilibrium, which involves the price distortion in the standard lemon model, can be sustained. Furthermore, in such an environment, I establish a strong link between the market liquidity (tightness) and the underlying uncertainty stemming from adverse selection. It is shown that the underlying dispersion, or more precisely, the possible range of underlying asset qualities, plays an important role in determining the equilibrium market liquidity: the higher the dispersion, the more illiquid the market. Contrary to the standard lemon model, what matters is the dispersion (range) of the asset instead of the expected value.

It is important to note, however, that such an equilibrium outcome relies on the fact that buyers' willingness to pay aligns with sellers' preference over liquidity, which in general is not necessarily the case. For example, in a setting when sellers' motives for sale are unknown to the market, the types who are willing to wait longer are not necessarily the ones with more valuable assets. Therefore, the corresponding mechanism must adjust. In a more general setting, we first identify the key condition for the existence of our baseline result and then show how a combination of both price and liquidity distortion arises when this condition does not hold. Utilizing this result, I investigate a setting when sellers' liquidity position are their private information and show how this setup can be nested in our general model. Interestingly, a semi-pooling equilibrium, behaving like a fire sale, arises endogenously in such an environment, which is then different from Guerrieri and Shimer (2011) and Guerrieri et al. (2010). In such an equilibrium, certain types of sellers will choose to enter a submarket that is liquid but has a heavy price discount. On the other hand, different types of sellers will enter a submarket with liquidity distortion but with a high selling price. It is important to note that although the impact of market illiquidity and the price discount is well understood, most analyses are conducted

assuming their existence; little is known about why each occurs in the first place. For example, the standard model for lemons which focuses in price discounts is only one special type of market distortion. Our framework allows us to answer the question as to why some asset markets remain illiquid while some markets suffer price discount. The contribution is therefore twofold: it is first to establish how price and market liquidity are jointly determined by adverse selection and by the market's perceived motives for selling. Second, it demonstrates that depending on sellers' liquidity preference as well as buyers' willingness to pay, different types of market segmentations and hence different forms of market distortions, arise as an equilibrium outcome.

The key ingredient of our model is the endogenous market liquidity stemming from search frictions and adverse selection in the competitive decentralized trading markets. Theoretically, our work is closest to Guerrieri et al. (2010), who apply the notion of competitive search equilibrium to an environment with adverse selection in a static environment. As discussed in Guerrieri et al. (2010), this equilibrium concept is similar to the refined equilibrium concept developed in Gale (1992) and Gale (1996). In the above works, uninformed principals post (exchange) contracts, and it is important that the contract satisfies the sorting condition so that agents are screened. This assumption however does not hold here, as the contract space is limited to the payment in our trading environment. Therefore, the only instrument to screen agents here is the market liquidity. This difference allows us to use a different approach to characterize the equilibrium by establishing the decentralized competitive equilibrium as a mechanism design problem. This method simplifies the equilibrium characterization to solving a differential equation and it further facilitates the analysis for a more general environment. A market designer in such environment needs to match agents from two sides of the market, designing the price and market liquidity for each submarkets subject to both sellers' and buyers' optimality constraints. We first identify the conditions under which the least-cost separating equilibrium is obtained, which ties our result to Guerrieri et al. (2010). However, it is important to understand that the screening mechanism in such environment is a combination of a downward distorted market liquidity and an upward price schedule. Hence, it is necessary that buyers are willing to pay more for a more patient seller, which is governed by the monotonicity in the matching value. Economically, the monotonicity condition says that the more patient types generate higher matching value to buyers, which however does not hold in general. With this new approach, we further expand our analysis to an environment when the monotonicity in the matching value is relaxed and show how the equilibrium behaves differently. In particular, a semi-pooling equilibrium arises in

such an environment and, in general, it is not unique.

Furthermore, our setup is dynamic and is designed to handle two-sided heterogeneity with a general payoff function, so that it can be easily applied to a more general trading environment. Allowing for heterogeneous buyers takes into account the competition among buyers as well as their diversity, which is the hallmark of economic exchange. It therefore sheds light on the sorting pattern. In particular, it is well known that, supermodularity in matching value is not enough to guarantee positive assortative matching (PAM) in a search framework. For example, Eeckhout and Kircher (2010) studies the sorting of heterogeneous agents in a competitive search trading market with complete information and identifies the condition under which PAM obtains. We show that, however, with adverse selection, positive assortative matching is guaranteed by the supermodularity in matching value.

This paper is related to two lines of literature which focuses on search friction and adverse selection in asset markets separately. First of all, the literature focusing on the effect of search friction in asset markets includes the OTC literature put forth by Duffie et al. (2005), Duffie et al. (2007)¹ and the monetary search literature² (for example, Kiyotaki and Wright (1993), Trejos and Wright (1995)). Compared to the above literature, this paper focuses on how equilibrium market liquidity, ie, the market tightness, is determined in a direct search framework, which is developed in the labor search literature, put forth by Moen (1997), Burdett et al. (2001) and Mortensen and Wright (2002). Two important elements in such a framework are 1) buyers commit to the posting price and 2) sellers *direct* their search toward their preferred markets. This framework is important as it allows us to analyze how traders sort themselves into different submarket. Furthermore, we consider the environment in which sellers have private information about their asset quality. There are few works which also consider both search frictions and adverse selection and most are done in a framework of random search. For example, in a model of production and exchange under private information, Williamson and Wright (1994) studies the impact on the time it takes to buy and sell and shows how the introduction of fiat money can improve welfare. Also, a recent work by Chiu and Koepl (2011) studies trading dynamics and policy implication in OTC market in a random search setting, where a pooling equilibrium is obtained.

Our work is also related to the large literature on the classic market for lemons Akerlof

¹Building on their model, the emerging literature studies the effects of liquidity in search models of asset pricing. For example, Weill (2008), Lagos and Rocheteau (2009)

²Williamson and Wright (2008) provides a detailed survey for this line of literature.

(1970) and on dynamic adverse selection models. In most cases, all trades are assumed to take place at one price so a pooling equilibrium and therefore a price discount is obtained, for example, Eisfeldt (2004) and Daley and Green (2009). Compared to this line of literature, our setting introduces another important dimension of the market distortion: liquidity. In fact, in our basic model, a pooling equilibrium can not be sustained as long as buyers have freedom to post the price and sellers can direct their search to each market (price) and a fully separated equilibrium is the unique outcome. This therefore implies several distinct features: 1) All markets, in the equilibrium, are priced and open if and only if there is a positive gain from trade for the *worst* possible asset; 2) Nevertheless, some high quality asset markets are rather illiquid (or even close to frozen) so it is hard for sellers to get rid of their assets; 3) More importantly, different dispersion of the asset quality will have a first order effect on market illiquidity. Contrary to the standard adverse selection problem, which predicts that the equilibrium outcome highly depends on the expected value of assets because of the feature of a pooling equilibrium, what matters in our framework is the dispersion of asset quality. Furthermore, in a more general environment, we show how both price discount and liquidity distortion can arise resulting from different types of market segmentation. A submarket involving pooling exists and its behavior then shares similar feature with this line of literature. However, the novelty is show how different market segmentations may arise as an equilibrium outcome, without imposing it.

The rest of the paper is organized as follows. Section 2 introduces the basic model and characterizes the equilibrium outcome. Section 3 extends the basic model to allow for heterogenous buyers, resale, and the non-monotonicity in the matching value. Section 4, we further consider a setup when sellers' motives for sale are unobserved by the market. Section 5 and Section 6 discuss efficiency, sorting behaviors, and other related implications. For example, we show that why some types of financial securities, paying similar cash flows, are more liquid than others. Furthermore, applying the developed method to explain firms' capital reallocation, our framework provides the micro-foundation of the reallocation pattern documented in Eisfeldt and Rampini (2006) and allows for a richer analysis of how this market friction respond to varied economic shocks.

2 A Basic Model

There is a continuum of sellers who own one asset with different quality indexed by $s \in S$, which is a seller's private information. Assume that $S = [s_L, s_H] \subset R_+$ and $G^0(s)$ denote

the measure of sellers with asset quality weakly below s at $t = 0$. While holding the asset s , the seller enjoys a flow payoff s but must at the same time pay a holding cost c as long as the asset remains unsold. One can think of this as a simple way to model a seller's need to "cash" the asset. As explained in Duffie et al. (2007), we could imagine this holding cost to be a shadow price for ownership due to, for example, (a) low liquidity, that is, a need for cash; (b) high financing cost; (c) adverse correlation of asset returns with endowments; or (d) a relatively low personal use for the asset, for example, for certain durable consumption goods such as homes. For now, one should think the holding cost c as an easy way to generate the gain from trades. As shown in our general model, the main result holds for a general payoff. There is a large continuum of homogenous buyers. That is, we assume that the measure of buyers is strictly larger than sellers and free-entry condition holds on the buyers' side. A buyer who owns the asset s enjoys a flow payoff s . In order to buy the asset, the buyer needs to enter the market to search for the seller, incurring a search cost, $k > 0$ for the duration of the search. The measure of buyers who decide to enter the market is endogenously determined by free-entry condition. For our basic model, we assume traders leave the market once the trade takes place.

All agents are infinitely lived and discount at the interest rate, r . Time is continuous. The setup borrows the direct search framework. Buyers (uninformed principals) post a trading price and sellers direct their search toward their preferred market. All traders have rational expectations in the equilibrium market tightness, the buyer-seller ratio, associated with each market $\theta(p)$, which will be endogenously determined in the equilibrium. As standard, in each submarket, matching is bilateral and traders are subject to the random matching function. A seller who enters the submarket $(p, \theta(p))$ matches a buyer with the Poisson rate $m(\theta(p))$. The idea that relatively more buyers make it easier to sell is captured by assuming $m(\cdot)$ is a strictly increasing function in θ . On the other hand, a buyer at the market $(p, \theta(p))$ meets a seller with the rate $q(\theta(p))$, where $q(\cdot)$ is assumed to be a strictly decreasing function in θ . That is, when there are relatively more buyers, it becomes harder for buyers to trade. Trading in pairs further requires that $m(\theta) = \theta \cdot q(\theta)$. Particularly, throughout this paper, we assume that the matching function takes Cobb-Douglas form so that $m(\theta) = \theta^\rho$ where $1 > \rho > 0$. Our result is robust to a different form of search technology with standard assumptions³.

³That is, $m(\cdot)$ is twice continuously differentiable and strictly concave.

2.1 Benchmark: Complete information

We first establish the benchmark with complete information, which is the canonical competitive search model put forth by Moen (1997). In our particular setup, buyers simply post a trading price and sellers direct their search toward their preferred market. Moreover, following the interpretation of Mortensen and Wright (2002), one can imagine the competitive search equilibrium as if there is a market maker who can costlessly set up a collection Θ of submarkets. Each market can be characterized by a pair $(\theta(p), p)$, which is known ex ante to participants. Given the posting price and the market tightness in each market, each trader then selects the most preferred submarket in which to participate (search). With the assumption that there is perfect competition among market makers, the market maker's problem is then to maximize traders' utilities.

Sellers' and buyers' expected utilities who enter the market with the pair (θ, p) can be expressed as follows, respectively:

$$rV(s, \theta, p) = s - c + m(\theta)(p - V(\theta, p, s))$$

$$rU_b(s, \theta, p) = -k + \frac{m(\theta)}{\theta} \left(\frac{s}{r} - p - U_b(\theta, p, s) \right)$$

Free entry condition for buyers is assumed. That's, buyers' entry and exit decisions are instantaneous and they will adjust until free entry condition holds. With perfect information, one can solve the equilibrium independently for each asset s . The market maker's optimization problems for each asset s is:

$$\max_{p, \theta} U(s) = \max_{p, \theta} \frac{s - c + pm(\theta)}{r + m(\theta)}$$

$$st : U_b(s) = \frac{m(\theta) \left(\frac{s}{r} - p \right) - \theta k}{r\theta + m(\theta)} = 0$$

One can easily see that θ_{FB} solves following FOC:

$$\frac{c}{k} = \frac{1}{\rho} (r\theta_{FB}^{1-\rho} + (1 - \rho)\theta_{FB}) \quad (1)$$

Notice that θ_{FB} is an increasing function of the cost ratio, $\frac{c}{k}$. Namely, it is relatively easier for sellers to meet the buyer, and it takes longer for the buyer to find the seller when the holding cost is higher. Also, the first best solution is independent of the asset quality. The intuition is clear since the gain from trade is simply the holding cost, which is independent of the asset quality. The price of each asset is then: $p^{FB}(s) = \frac{s}{r} - \frac{k\theta_{FB}}{m(\theta_{FB})}$,

the expected value of the asset minus the expected searching cost paid by buyers. One can easily check that IR constraint holds for all types of sellers. Obviously, first-best allocations can not be implemented in the environment with adverse selection. Facing the same market tightness, sellers always want to pretend a higher type so that they can get a higher payment.

2.2 Equilibrium with Adverse Selection

We now turn to the environment with adverse selection, that is, sellers have private information about the asset quality. As in the complete information environment, buyers/sellers choose the price they would like to offer/accept, and all traders have rational beliefs about the ratio of buyers to sellers $\theta(p)$ in each market p . The key difference is that, given sellers' types are unobserved, buyers now form rational beliefs about the distribution of sellers' types in each market p , which determines the expected asset quality they receive in each submarket. It is important to note that given the expected asset quality in each submarket, free-entry condition determines the measure of active buyers in each market independently of the distributions of sellers in other submarkets. As a result, the equilibrium market tightness function $\theta(\cdot)$ does not depend on the distributions of sellers in other submarkets. This property is important as it simplifies our analysis to stationary equilibria where the set of offered prices P^* and the market tightness function $\theta(\cdot)$ are time invariant even though the aggregate distribution might change over time.

To elaborate, consider the set of offered price and the function $\theta(\cdot)$ which are time invariant. Each market is then characterized by a pair $(p, \theta(p))$. It is sufficient for sellers to choose their searching decision and, obviously, sellers' strategy are stationary facing the time invariant $(p, \theta(p))$. Sellers' trading decisions then determine the expected asset quality in each submarket, which pin down the buyers' expected value of buying an asset in each market $p : \bar{J}(p) = \int J(\tilde{s})\mu(\tilde{s}|p)d\tilde{s}$, where $\mu(s|p)$ denotes the probability of a type- s seller conditional on a match in the market p and $J(s)$ denotes buyers' value of buying the asset from type- s seller. Notice that given sellers' searching decisions are stationary and the matching is random in each submarket, the composition of sellers' types is therefore stationary as well as buyers' expected matching value $\bar{J}(p)$. Furthermore, the free entry condition guarantees that at each point of time the measure of active buyers generates the correct ratio $\theta(p)$ in each submarket such that $p = \bar{J}(p) - \frac{k\theta(p)}{m(\theta(p))}$ for all $p \in P^*$ and therefore the market tightness function $\theta(p)$ is then stationary. Finally, one still needs to show that the set of offered prices P^* is time invariant. As it will become clear in

the later discussion, the set of offered prices depends on sellers' equilibrium utilities and the *range* of underlying asset quality; both of them are time invariant in the constructed environment. Hence, the above discussion shows that traders' *strategy* are stationary in such an environment even though the aggregate distribution might change over time and affect aggregate statics. Note that in a setting of competitive search models with heterogeneous agents, it is well known that the type distribution does not play a role as the standard result in the literature is the full separation, for example, Moen (1997). Shi (2009) further establishes the block recursive property in the environment when search on the job is allowed. The outlined argument here shares the same spirit with Shi (2009) but with the following modifications: 1) The possibility of (semi) pooling is allowed. In that case, the distribution of sellers' types in other submarkets does not play a role, as implied by block recursive property, but, clearly, the distribution within each market matters, which is governed by $\mu(\cdot|p)$. 2) As it will become clear later, the determination of θ in each market depends on the *range* of underlying distribution, which is the key feature stemming from adverse selection. Nevertheless, as shown in the above discussion, to solve for the equilibrium, it is enough to characterize the set of active markets P^* , the equilibrium market tightness function $\theta(\cdot)$, and the composition $\mu(s|p)$ in each market. This property, eliminating the role of the aggregate distribution, makes our dynamic environment tractable. The following section then characterizes traders' decision in such stationary equilibria, neglecting the role of aggregate distribution, and one can easily back up the aggregate dynamics afterward.

Clearly, no trade takes place at prices below zero and above $J(s_H)$, and we define the set of feasible prices as $P = [0, J(s_H)]$. An equilibrium consists of a set of offered price P^* , a market tightness function $\theta(\cdot)$ and traders' trading decisions. First of all, buyers' and sellers' expected payoff, entering a active markets $(p, \theta(p))$, can be expressed as following:

$$\begin{aligned} rU_b(p, \mu, \theta(p)) &= -k + \frac{m(\theta(p))}{\theta(p)} \left(\int \frac{\tilde{s}}{r} \mu(\tilde{s}|p) d\tilde{s} - p - U_b \right) \\ rV(p, \theta(p), s) &= s - c + m(\theta(p))(p - V(p, \theta(p), s)) \end{aligned}$$

Given the markets which are open, which can be characterized as $(p, \theta(p))$, sellers direct their search toward their preferred market and can always choose the option of no trade, denoted by \varnothing^4 . The equilibrium expected utilities of seller s then must satisfy:

⁴To make the choice of no trade consistent with the rest of our notation, let $\varnothing_p = \bar{P} > J(s_H)$ denote a nonexistent price which is higher than the feasible price and the trading probability at \varnothing_p is zero, $\theta(\varnothing_p) = 0$. Hence, a seller achieves his outside option $\frac{s-c}{r}$ if $\varnothing = \arg \max V(p, \theta(p), s)$.

$$V^*(s) = \max_{p \in P^* \cup \emptyset} V(p, \theta(p), s)$$

We now need specify the belief out of the equilibrium path. Our equilibrium concept adopts Guerrieri et al. (2010), which is also similar to the refined Walrasian general-equilibrium approach developed in Gale (1992)⁵. The spirit follows the market utility property used in the competitive search equilibrium literature⁶. When a buyer contemplates a deviation and offers a price p which has not been posted, $p \notin P^*$, he has to form a belief about the market tightness and the types he will attract, taking sellers' utilities $V^*(s)$ as given. First of all, a buyer expects a positive market tightness only if there is a type of seller who is willing to trade with him. Moreover, he expects to attract the type s who is most likely to come until it is no longer profitable for them to do so. Formally, define:

$$\begin{aligned} \theta(p, s) &\equiv \inf\{\tilde{\theta} > 0 : V(p, \tilde{\theta}, s) \geq V^*(s)\} \\ \theta(p) &\equiv \inf_{s \in S} \theta(p, s) \end{aligned} \tag{2}$$

By convention, $\theta(p, s) = \infty$ when $V(p, \tilde{\theta}, s) \geq V^*(s)$ has no solution, mainly $\theta(p, s) = \infty$ for any $p < V^*(s)$. Intuitively, we can think of $\theta(p)$ as a *lowest* market tightness for which he can find such a seller type. Now let $T(p)$ denote the set of types which are most likely to choose p :

$$T(p) = \arg \inf_{s \in S} \{\theta(p, s)\}$$

Therefore, this suggests that, given $\theta(p)$, p is optimal for every type $s \in T(p)$ but not optimal for $s \notin T(p)$. Hence, the buyer's assessment about $\mu(s|p)$ for any posting price p needs to satisfy the following restriction:

$$\text{For any price } p \notin P^* \text{ and type } s, \mu(s|p) = 0 \text{ if } s \notin T(p) \tag{3}$$

In the case when $T(p)$ is unique, a buyer then expects this deviation will only attract seller $T(p)$ and therefore $\mu(s|p) = 1$ if $s = T(p)$ and $\mu(s|p) = 0$ for $\forall s \notin T(p)$. To simplify the notation, let μ_p denote the sellers' distribution $\mu(\cdot|p)$ conditional on the market p .

⁵See Guerrieri et al. (2010) for the detailed discussion regarding its relationship with different refinement developed in the previous literature.

⁶Burdett, Shi and Wright (2001) prove that a competitive search equilibrium is the limit of a two stage game with finite numbers of homogeneous buyers and sellers, which can be understood as a micro-foundation for the market utility property.

Definition 1 An equilibrium consists of a set of offered price P^* , a function of seller's expected utilities $V^*(s)$, a market tightness function in each market p , $\theta(\cdot) : P \rightarrow [0, \infty]$, the conditional distribution of sellers in each submarket $\mu : S \times P^* \rightarrow [0, 1]$, such that the following conditions hold:

E1 (optimality for sellers): let

$$V^*(s) = \max\left\{\frac{s-c}{r}, \max_{p' \in P^*} V(p', \theta(p'), s)\right\}$$

and for any $p \in P^*$ and $s \in S$, $\mu(s|p) > 0$ implies $p \in \arg \max_{p' \in P^* \cup \emptyset} V(p', \theta(p'), s)$

E2 (optimality for buyers and free-entry): for any $p \in P^*$

$$0 = U_b(p, \theta(p), \mu_p)$$

;and there does not exist any $p' \in P$ such that $U_b(p', \theta(p'), \mu_{p'}) > 0$, where $\theta(p')$ and $\mu(s|p')$ satisfies (2) and (3)

As explained earlier, the aggregate distribution does not play a role. The law of motion of the stock of sellers in each market is given by the transaction outflow. On the other hand, buyers' participation must generate the *correct* buyer-seller ratio $\theta(p)$ at each point time in all the submarkets according to *E2*. To characterize the equilibrium, one does not need to track the aggregate distribution and, therefore, the role of traders' distribution is eliminated in the above definition. Nevertheless, one can back up traders' distribution over time after solving the equilibrium above as shown in the later discussion.

2.3 Characterization

We now show that the equilibrium outcome can be characterized as the solution of a mechanism design problem which takes into account both sellers' and buyers' optimality condition. Intuitively, one can think of a market designer who promises the price and the market tightness in each submarket so that sellers truthfully report their type, that is, condition *E1* has to hold. Moreover, a *feasible* mechanism must satisfy the market clear condition. In other words, the market tightness must equal the ratio of the measure of buyers who are willing to pay p to the measure of types- s sellers who are willing to accept p . Meanwhile, given that buyers can post the price freely in the decentralized markets, any price schedule recommended by the market designer has to be optimal for buyers. Otherwise, buyers will deviate by posting price other than the ones recommended by the mechanism designer. This point is characterized by condition *E2*.

Overview of the solution: Our approach therefore follows two steps: First, we characterize the set of feasible mechanism $\alpha \in A$, which satisfies $E1$ and free-entry condition (Proposition 1). Second, we use $E2$ to identify the necessary condition for which the solution to the mechanism can be decentralized in equilibrium. This result enables us to pin down the unique candidate among the set of feasible mechanism A , which is a fully separating one. At the end, we show that this candidate is indeed the solution, that is, $E2$ and participating constraints are all satisfied.

To find out the set of mechanism that satisfies sellers' IC constraints, we setup the problem as a mechanism design problem (of an imaginary market designer). By the revelation principle, it will be without loss of generality to focus direct revelation mechanisms. A direct mechanism is a pair (θ, p) where $\theta : S \rightarrow R_+$ is the market tightness function and a price function $p : S \rightarrow R_+$. The mechanism is interpreted as follows. A seller who reports his type $\hat{s} \in S$ will then enter the market with the pair $(\theta(\hat{s}), p(\hat{s}))$. Hence, the value of seller s announces his type \hat{s} , denoted by $V(\hat{s}, s)$, can then be expressed as:

$$rV(\hat{s}, s) = s - c + m(\theta(\hat{s}))(p(\hat{s})) - V(\hat{s}, s)$$

The seller's optimal search problem can be rearranged as:

$$V^*(s) = \left\{ \frac{s - c}{r}, \max_{\hat{s}} \frac{s - c + p(\hat{s})m(\theta(\hat{s}))}{r + m(\theta(\hat{s}))} \right\}$$

Notice that a seller can always choose not to participate and he will get his autarky utility $\frac{s-c}{r}$ in that case. For convenience, one can think of not entering the market as if choosing the market where the matching rate is zero. As the mechanism has to satisfy sellers' IR constraint, we set $\theta(s) = 0$ whenever IR constraint is binding. Sellers' optimal search problem can be re-written as the requirement that $s \in \arg \max_{s' \in S} V(\hat{s}, s)$. First of all, we can prove that any mechanism which satisfies $E1$ can be characterized with following proposition:

Proposition 1 *The pair of function $\{\theta(\cdot), p(\cdot)\}$ satisfies sellers' optimality condition (E1) if and only if following conditions are satisfied:*

$$\frac{1}{r + m(\theta^*(s))} \text{ is non-decreasing} \quad (\text{M})$$

$$V^*(s) = \frac{u(s) + p^*(s) \cdot m(\theta^*(s))}{r + m(\theta^*(s))} = V^*(s_l) + \int_{s_l}^s V_s(\theta^*(\tilde{s}), \tilde{s}) d\tilde{s} \quad (\text{ICFOC})$$

$$V^*(s) \geq \frac{u(s)}{r} \quad (\text{IR})$$

Proof. Standard proof in mechanism design literature (Milgrom and Segal (2002)). See Appendix. In this basic model, $u(s) = s - c$. ■

Define $B(p') \equiv \{s \in S \mid p(s) = p'\}$, buyers' expected asset quality in the market p is calculated as the conditional expectation: $E[s \mid s \in B(p)]$. Furthermore, for any feasible mechanism, the free entry condition must hold for buyers, which means that the mechanism needs to satisfy the following constraint:

$$p = \frac{E[s \mid s \in B(p)]}{r} - \frac{k\theta(p)}{m(\theta(p))}$$

Proposition 1 and the free-entry condition then define the set of *feasible* mechanisms, A . Market clear condition is guaranteed by free-entry condition. Namely, buyers will entry until the "right" market tightness is satisfied. Moreover, let $V(\alpha, s)$ denote the expected payoff to a type- s seller under the mechanism $\alpha \equiv (p, \theta)$. Each mechanism $\alpha \in A$ is then composed of a price schedule $p^\alpha(\cdot)$, market tightness $\theta^\alpha(\cdot)$, and corresponding sellers' utilities $V^\alpha(\cdot)$. This set then includes all possible pooling as well as separating equilibrium. *Nevertheless, not all of them can be sustained in the decentralized equilibrium.* A decentralized equilibrium has to satisfy buyers' optimality condition. Hence, $(p, \theta, V(s; \alpha))$ is only an equilibrium if there is no profitable deviation for buyers to open a new market p' , where the off-equilibrium belief is specified in (2) and (3), as discussed earlier. When a buyer considers to open a new market $p' \notin \text{range of } P$, they expect to only attract the type who is most likely to come, $T(p')$, as defined (3). To facilitate the analysis, we first prove following lemma:

Lemma 1 *Given any the mechanism $\alpha \in A$, which includes a price function $p^\alpha : S \rightarrow R_+$, market tightness function $\theta^\alpha : S \rightarrow R_+$, and sellers utilities $V^\alpha : S \rightarrow R_+$, for any price $p' \notin \text{range of } p$, the unique type who will come to this market p' is given,*

$$\begin{aligned} T(p') &= s^+ \cup s^- \\ \text{where } s^- &= \inf\{s \in S \mid p' < p^\alpha(s)\} \\ s^+ &= \sup\{s \in S \mid p' > p^\alpha(s)\} \end{aligned}$$

Proof. Notice that $p^\alpha(\cdot)$ is non-decreasing for $\forall \alpha \in A$ given (M). Therefore, $T(p')$ is uniquely defined⁷. Namely, the type which is most likely to come is unique. For any

⁷With the expectation when some types of sellers are out of the market. In this case, there then exists a marginal type s^* such that $\theta(s) = 0$ for $\forall s > s^*$. For any $s > s^*$ and $p' > \frac{u(s)}{r}$, type- s will come to the market even when $\theta(p', s) \rightarrow 0$. Hence $T(p')$ is then a set of these types of sellers. Nevertheless, it will not change our equilibrium result and it will become clear later that a buyer will deviate even when he expects the *worst* type among those set.

$p' \notin \text{range } p^\alpha$, by definition, $\theta(p', s) \equiv \inf\{\tilde{\theta} > 0 : U(p, \tilde{\theta}, s) \geq V(s; \alpha)\}$. Therefore, for any $p' > V(s; \alpha)$, $\theta(p', s)$ solves: $U(p', \theta, s) - V(s; \alpha) \equiv G(p', \theta, s)$,

$$\begin{aligned} \frac{d\theta(p', s)}{ds} &= -\left(\frac{dG/ds}{dG/d\theta}\right) \propto \frac{1}{r + m(\theta^\alpha(s))} - \frac{1}{r + m(\theta(p', s))} \\ &= \begin{cases} < 0 \text{ if } p' > p^\alpha(s), (\because \theta(p', s) < \theta^\alpha(s)) \\ > 0 \text{ if } p' < p^\alpha(s), (\because \theta(p', s) > \theta^\alpha(s)) \end{cases} \end{aligned}$$

Recall that, posting a new price p' , a buyer should expect the lowest market tightness $\theta(p') = \inf_s\{\theta(p', s)\}$ and the type $T(p') = \arg \inf\{\theta(p', s)\}$. Above relation then implies that, for example, if a buyer deviates to posting a new price p' which is lower than all the existing price, so that $s^- = s_L$ and $s^+ = \emptyset$, he should attract only the lowest type, given that $\theta(p', s)$ is increasing in s and, therefore, $s_L = \arg \inf_s \theta(p, s) = T(p')$. Similar argument holds for any $p' \notin \text{range of } p^\alpha$. ■

With this condition, we can then prove following lemma:

Claim 1 *There is no pooling submarket in equilibrium when $h_s(s) > 0$*

Proof. See Appendix. Let $h(s)$ denote buyers' flow value over s and $h(s) = s$ in our basic model. Intuitively, a buyer can post a new price p' which is only slightly higher than the original pooling price. In that case, he only pays a little bit more but gets the best type in the original pooling for sure (as implied from lemma 1), which therefore generates a profitable deviation. ■

Claim 1 then allows us to focus on a fully separating equilibrium. In each market, (θ, p, s) , the price schedule then has to satisfy:

$$p(s) = \frac{s}{r} - \frac{k\theta(s)}{m(\theta(s))} \quad (4)$$

Substituting this payment schedule into (ICFOC) :

$$V(s) = \frac{s - c + \left(\frac{s}{r} - \frac{k\theta}{m(\theta)}\right)m(\theta^*(s))}{r + m(\theta^*(s))} = V(s_l) + \int_{s_l}^s U_s(\theta^*(\tilde{s}), \tilde{s})d\tilde{s}$$

One can then get differential equation of $\theta^*(s)$:

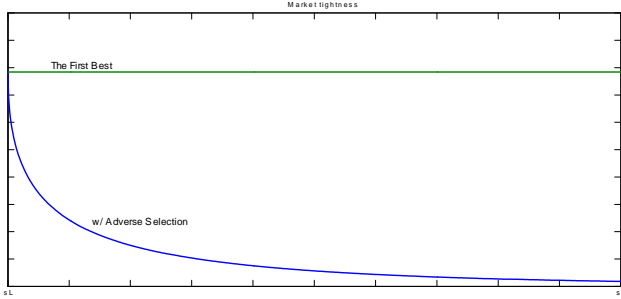
$$\left[c + \frac{k}{\rho} \left((\rho - 1)\theta - \frac{r\theta}{m(\theta)}\right)\right] \frac{d\theta}{ds} = -\frac{\theta}{\rho r} (r + m(\theta)) \quad (5)$$

Therefore, the market tightness function $\theta^*(\cdot)$ has to be the solution of (5) subject to the monotonic condition, (M) in order to satisfy the incentive compatible constraints and

free-entry condition. Left hand side of (5) is monotonically decreasing in θ and reaches zero at θ^{FB} . Therefore, for any initial condition $\theta_0 > \theta^{FB}$, the solution will be explosive and violate the monotonic solution. (5) is a separable nonlinear first-order differential equation with a family solution form:

$$s = C + \int \frac{1}{f(\theta)} d\theta$$

where $f(\theta) = \frac{-\frac{\theta}{\rho r}(r+m(\theta))}{[c+\frac{k}{\rho}((\rho-1)\theta-\frac{r\theta}{m(\theta)})]}$. One can understand the qualitative properties the solutions by constructing a simple phase diagram. The solution is illustrated as below, which can be understood by observing that for any $\theta \in (0, \theta^{FB})$, a) $f(\theta) < 0$; b) $f'(\theta) > 0$ and c) $\lim_{\theta \rightarrow 0} f(\theta) = 0$.



Equilibrium Market Liquidity $\theta^*(s)$

With the following initial condition⁸, we are able to pin down the unique solution which satisfies both sellers' and buyers' optimality constraints.

Claim 2 *In a full separated equilibrium, the lowest type has to achieve his first-best utility. That is, the initial condition is:*

$$\theta(s_L) = \theta^{FB}(s_L) \quad (6)$$

Proof. See Appendix for detail. The intuition is clear: a downward distorted market tightness is to preventing a lower-type from mimicking a higher-type. Therefore, it should be clear that there is no reason to distort θ for the lowest type. ■

The mechanism can be summarized as following. Because of the asymmetric information, sellers will then face a lower meeting rate, $\theta^*(s) < \theta^{FB}$ for all s but get a higher

⁸One can see that standard condition of the uniqueness does not hold with this initial condition. In fact, there will be two solutions. However, the other solution increases with s and therefore violates our monotonic condition.

transfer $p^*(s) = \frac{s}{r} - \frac{k\theta^*}{m(\theta^*)} > p^{FB}(s)$. There will be also less buyers participating the market, who needs to pay a higher price but with relatively high meeting rate. To note that, this result holds for any arbitrary distribution of sellers. Traders' participation and therefore, the trading volume, which is governed by the meeting rate, is endogenously determined. Also, we can easily check that IR constraint holds for all sellers⁹ and, indeed, buyers will not deviate by a opening market p' other than those which are already open. The argument is following: First, note that the price function is continuous. Denote (p_L, p_H) as the lower bound and the upper bound support of function $p(s)$ constructed above. From Lemma 1, if buyers post the price $p' > p^H$, he will only attract the highest type. One can easily show that it involves more distortion and hence not profitable. Similarly, if posting $p' < p^L$, buyer will attract the lowest type and buyers' utilities will decreases due to the distortion. Namely, it confirms that no profitable deviation exists for buyers.

3 Generalization

The goal of this section is to study a more general economic environment where traders have different valuation of the asset. Similar as before, there is a mass of heterogenous sellers who are indexed by a type $s \in S$ that is sellers' private information. The flow payoff of the asset s to the seller is now given by $u(s)$, where u is a continuously differentiable function, $u : S \rightarrow R_+$. The indices s that are ordered such that they increase the utility of sellers: $u'(s) > 0$. On the other side of the market, there is a large mass of buyers. The flow payoff of an asset bought from seller s is given by $h(s)$ and h is a strictly positive function. I now make following assumptions on traders' preferences and will discuss how these assumptions can be relaxed in the later section.

Assumption 1: $h(s)$ is (1a) a continuously differentiable function and (1b) strictly increasing in s , $h_s(s) > 0$

Assumption 2: $g(s) = h(s) - u(s) > 0$ for $\forall s \in S$

Monotonicity in the matching value (1b) is an important assumption for our basic result. In particular, it is crucial to Lemma 2 under which we show that there is no pooling submarkets and an unique full-separated equilibrium is obtained under this assumption.

⁹Define $G(s) = V(s) - \frac{s-c}{r}$ and $\lim_{s \rightarrow \infty} G(s) = 0$, given $\lim_{s \rightarrow \infty} \theta(s) = 0$. From (ICFOC) :

$$\frac{dg(s)}{ds} = \frac{1}{r + m(\theta^*(s))} - \frac{1}{r} < 0 \text{ for all } \theta(s) > 0$$

Hence $V(s) > \frac{s-c}{r}$ for $\forall s < \infty$.

As shown in our basic model, one can think of s represent the quality of an asset, which gives both sellers and buyers a higher payoff. In the later section, I consider an environment when this assumption does not hold and show how the equilibrium outcome behaves differently. The second assumption simply guarantees that there is a gain from trade.

Given these two assumption, it is straightforward to see all our previous results holds. The only difference is now that equilibrium solution of the market tightness $\theta^*(s)$ needs to solve a more general form of differential equation, which is given by:

$$\left[(h(s) - u(s) - k \left(\frac{r + m(\theta) - \theta m'(\theta)}{m'(\theta)} \right)) \right] \frac{d\theta}{ds} = -(r + m(\theta)) \cdot \frac{\theta h_s(s)}{\rho r} \quad (7)$$

and the corresponding price schedule has to satisfies:

$$p(s) = \frac{h(s)}{r} - \frac{k\theta^*(s)}{m(\theta^*(s))} \quad (8)$$

According to Claim 2, the initial condition is given by $\theta^*(s_L) = \theta^{FB}(s_L)$, where the first best market tightness $\theta^{FB}(s)$ solves:

$$\frac{h(s) - u(s)}{k} = \frac{r + m(\theta) - \theta m'(\theta)}{m'(\theta)} \quad (9)$$

Claim 3 1) *The first best solution $\{\theta^{FB}(s), p^{FB}(s)\}$ is not implementable when $h_s > 0$;*
 2) *The equilibrium market tightness $\theta^*(s)$ with the initial condition $\theta^*(s_L) = \theta^{FB}(s_L)$ is downward distorted compared to the first best, that is,*

$$\theta^*(s) < \theta^{FB}(s) \text{ for } \forall s > s_L$$

Proof. See Appendix. ■

Observing from (7), one can see $\frac{d\theta^*(s)}{ds} < 0$ given that $\theta^*(s) < \theta^{FB}(s)$ and $h_s > 0$. Therefore, $\theta^*(s)$ is decreasing and hence condition (M) is satisfied. As before, given $\theta^*(s)$, the equilibrium price is pinned down by (8). The equilibrium can then be summarized as follows: type- s sellers enter the submarket which is characterized by $(p(s), \theta(s))$ and buyers who enter the market $(p(s), \theta(s))$ pay the price $p(s)$, expecting type- s asset. That is, $\mu(s|p(s')) = 1$ if $s' = s$ and zero otherwise. The transaction outflow for each type- s asset is then determined by the matching rate $\theta^*(s)$, following the law of motion $\frac{dg^t(s)}{dt} = -m(\theta(s))g^t(s)$, where $g^t(s)$ is the size of type- s sellers at time t . On the other hand, the measure of active buyers in each market is endogenously determined by the ratio $\theta^*(s)$, that is, $\mu_B^t(s) = \theta^*(s) \cdot g^t(s)$. Hence, as discussed earlier, one can back up traders' aggregate distribution accordingly. Furthermore, as a full-separated equilibrium is obtained in such

an environment and the distribution does not have impact on equilibrium price and market tightness (conditional on the same range), allowing for the new inflow of sellers only changes the law of motion of the stock in each market but not the market characterization $(p(s), \theta(s))$.

3.1 Heterogeneous Buyers

The setup is now chosen to allow for heterogenous buyers in the market so that it can be easily applied to trading environments with two-sided heterogeneity. Many decentralized markets have this feature. Understanding the trading pattern is crucial since it determines the allocation and therefore welfare. For example, in the factor market, the resource allocation determines aggregate productivity. Different companies might have different technology to utilize the assets (machine or capital). Productivity of the assets is determined by assets allocation, which is mainly governed by both the pattern of trade and the equilibrium liquidity. With this generalization, the model further sheds light on the sorting pattern. We show that, supermodularity in the matching value is enough to guarantee positive sorting, which is a distinct feature compared to the environment without adverse selection.

Consider that there are two types of buyers, $b^i \in \{b^h, b^l\}$ and buyers' type are observable. For simplicity, we assume that the measure of each type is larger than the one of sellers and the outside option of buyer b^i is given by $\phi(b^i)$ ¹⁰. The flow payoff of an asset owned by buyer b^i and bought from seller s is given by $h(b^i, s)$, where h shares the assumption as our basic model. The indices s and b^i that are ordered such that they increase the utility of sellers: $h(b^h, s) > h(b^l, s)$. For example, $h(b^i, s)$ represents the payoff produced by the firm with technology b^i and asset quality s . The simple functional form widely used for a macroeconomic model with heterogenous firms is usually given by $h(b^j, s) = b^j s$, which can be seen as the productivity. Furthermore, we assume that there is complementarity in the matching function.

Assumption 3: $h_s(b^h, s) - h_s(b^l, s) > 0$

Obviously, if $\phi(b^h) < \phi(b^l)$, sellers then always obtain higher value if trading with the higher type buyer and one can easily show that, facing resulting (p, θ) , lower type buyer will not enter the market. In that case, the environment can be trivially solved just like as with homogenous buyers. The following analysis focus on the relevant environment in

¹⁰This assumption is made to simplify the analysis. One can interpret this as a partial equilibrium where we take the level of buyers' utilities as given.

which both type of buyers are *active* in the market when there is no adverse selection. One can establish the benchmark similar as before. It is well known that the equilibrium outcome can be thought of as a competitive market maker who promises traders the price, the market tightness, as well as the trading pattern. The equilibrium will then consist of a price function $p^{FB}(s)$, a market tightness function $\theta^{FB}(s)$, trading pattern $j^{FB}(s)$ and the corresponding sellers' utility function $V^{FB}(s)$, which solve following the optimization problem:

$$V^{FB}(s) = \max_{j,p,\theta} \left\{ \frac{u(s) + m(\theta)p}{r + m(\theta)} : U_b(p, \mu, \theta, b^j) = \phi(b^j) \right\}$$

The algebra detail is left in the appendix. In words, given buyer type b^i , one can solve the optimization problem as before. Similarly, the first best market tightness should be a function of the ratio of the gain from trade over the searching cost, which are type dependent and is denoted as $R(j, s) \equiv \frac{g(j,s)}{k+r\phi(b^j)}$. Let the function $V^{FB}(s, b^j)$ represent seller's utilities if traded with type b^j under perfect information. The first best utilities $V^{FB}(s)$ can be seen as a upper envelope of $V^{FB}(s, b^h)$ and $V^{FB}(s, b^l)$. That is, $V^{FB}(s) = \max_j \{V^{FB}(j, s)\}$. As it will become clear later, the interesting case is when there exists a marginal type $s^{FB} \in S$ who is indifferent to trading with high type and low type buyers and for a seller with assets $s < s^{FB}$, he will only trade with a lower type buyer and vice versa for sellers with assets $s > s^{FB}$ ¹¹. The following section therefore focuses on such an environment and discusses how other cases can be solved accordingly.

As sellers do not care about buyers' types, sellers' expected value in each market is the same as before. On the other hand, type-j buyers' expected payoff can be expressed as follows:

$$rU_b(p, \mu, \theta(p), b^j) = -k + \frac{m(\theta(p))}{\theta(p)} \left(\int \frac{h(b^j, \tilde{s})}{r} \mu(\tilde{s}|p) d\tilde{s} - p - U_b \right)$$

Definition 2 *An equilibrium consists of a set of offered price P^* , a function of seller's expected utilities $V^*(s)$, a market tightness function in each market p , $\theta(\cdot) : P \rightarrow [0, \infty]$, the conditional distribution of sellers in each submarket $\mu : S \times P^* \rightarrow [0, 1]$ and the trading pattern $j^* : P \rightarrow \{h, l\}$, such that the following conditions hold:*

¹¹It is clear from (??) that $R(h, s^{FB}) < R(l, s^{FB})$, given $\phi^h > \phi^l$. Therefore, $\theta(h, s^{FB}) < \theta(l, s^{FB})$ and $p(h, s^{FB}) < p(l, s^{FB})$. Namely, there will be two separating markets for the asset s^{FB} . These two markets are different from the trading price and the liquidity, between which the seller s^{FB} is indiscriminate. High type buyers will pay more for the good with shorter waiting time in one market and, vice versa for the low type buyers in the other market.

E1 (optimality for sellers): let

$$V^*(s) = \max\left\{\frac{s-c}{r}, \max_{p' \in P^*} V(p', \theta(p'), s)\right\}$$

and for any $p \in P^*$ and $s \in S$, $\mu(s|p) > 0$ implies $p \in \arg \max_{p' \in P^* \cup \emptyset} V(p', \theta(p'), s)$

E2 (optimality for buyers and free-entry): for any $p \in P^*$ and $j \in \{h, l\}$

$$U_b(p, \mu_p, \theta(p), b^j) \leq \phi^j$$

with equality if $p \in P^*$ and $j = j^*(p)$; and there does not exist any $p' \in P$ such that $U_b(p', \theta(p'), \mu_{p'}, b^j) > \phi^j$, where $\theta(p')$ and $\mu(s|p')$ satisfies (2) and (3)

Clearly, IC constraints for sellers are the same as before, that is, Proposition 1 still holds. The only difference is that we need to make sure the buyers' optimality condition will hold for both types. In particular, facing the price and market tightness recommended by the market maker, a buyer will benefit neither from going to the markets which belong to the other buyers, nor from opening a market which has not been open. The mechanism can be interpreted as follows: given $(p(s), \theta(s))$, a seller reports his type \hat{s} optimally; meanwhile, $j^*(s)$ denotes the sorting pattern recommended by the market maker, who recommends buyers $j^*(s)$ post the price $p(s)$, that is, entering the market $(p(s), \theta(s))$. The sets of types who trade with the lower type buyer, $\Omega_L = \{s : j^*(s) = l\}$, and of those who trade with the high type, $\Omega_H = \{s : j^*(s) = h\}$, are disjointed and satisfy $\Omega_L \cup \Omega_H \equiv S$. Then, define s^* as the marginal type $j^*(s^*) = \{l, h\}$. Obviously, some lessons learned from the basic model are still applied: there is no submarket involving pooling under assumption 1b) and, hence, we can focus on the full separation on the sellers' sides. From buyers' view points, each market can therefore be characterized as a pair of (p, θ, s) . Given (p, θ, s) , buyers will choose to go to the preferred markets and expect to trade with seller s .

Moreover, once we identify the set of sellers who trade with buyers j , Ω_j , the market tightness can be solved as in the case in which there is only one type of buyer j . Given Ω_j , the solution of $\theta(s; j)$ needs to the following differential equation, which is similar to (7) but taking into account that buyers' heterogeneity

$$\left[(h(b^j, s) - u(s) - r\phi(b^j)) + \frac{k + r\phi(b^j)}{\rho} \left((\rho - 1)\theta - \frac{r\theta}{m(\theta)} \right) \right] \frac{d\theta}{ds} = -(r + m(\theta)) \cdot \frac{\theta}{\rho} \frac{h_s(b^j, s)}{r} \quad (10)$$

As before, the corresponding price schedule $p(j, s)$ is then pinned down with the free entry condition:

$$p(s, j) = \frac{h(a^j, s)}{r} - \frac{(k + r\phi^j)\theta(s; j)}{m(\theta(s; j))} - \phi_j \quad (11)$$

Notice that solutions can be easily characterized once we have the initial condition for $\theta(s; j)$. Therefore, the key remaining task is essentially finding out the set Ω_j , that is, the marginal type s^* and identifying the initial condition $\{\theta_L^0, \theta_H^0\}$, which gives $\theta(s_L; j) = \theta_L^0$ and $\theta(s^*; j) = \theta_H^0$. For notation convenience, let $p^j(s), \theta^j(s)$ denote the price and the market tightness in the market with buyer type j . In equilibrium, it must be the case that the buyer j will not enter the market where $j^*(s) \neq j$. Hence, following constraints must be satisfied:

$$\begin{aligned} U_b(p^h, \theta^h, s, b^l) &< \phi^l \text{ for } j^*(s) = h \\ U_b(p^l, \theta^l, s, b^h) &< \phi^h \text{ for } j^*(s) = l \end{aligned}$$

To facilitate the analysis, define $\tilde{\theta}(s)$ to solve the following:

$$\begin{aligned} \phi^l &= U_b(p^h, \theta, s, b^l) \\ &= U_b(p^h, \theta, s, b^h) - \frac{q(\theta)}{r+q(\theta)} \left(\frac{h(b^h, s) - h(b^l, s)}{r} \right) \\ &= \phi^h - \frac{q(\theta)}{r+q(\theta)} \left(\frac{h(b^h, s) - h(b^l, s)}{r} \right) \end{aligned} \tag{12}$$

where $q(\theta) = \frac{m(\theta)}{\theta}$. Given that $h(b^h, s) - h(b^l, s)$ increases with s , $\tilde{\theta}(s)$ increases with s . This function then plays an important role in determining buyers' incentive constraint. Entering the high-type buyers' markets, the difference in utilities gain is characterized by the second term, $\frac{q(\theta)}{r+q(\theta)} \left(\frac{h(b^h, s) - h(b^l, s)}{r} \right)$, which captures low types' disadvantage. The impact of this disadvantage is higher when the expected waiting time for buyers is shorter, that is, for the higher $q(\theta)$ and hence the lower θ . As a result, for any $\theta < \tilde{\theta}(s)$, the low type will not mimic high type to enter the market. Similarly, when a high-type buyer contemplates entering a low-type market, he will only enter when $\theta < \tilde{\theta}(s)$ so that his advantage is high enough to compensate¹². Hence, we can conclude the following claim:

Claim 4 *In equilibrium, the market (p, θ, s) attracts high-type buyers but not low-type buyers if $\theta < \tilde{\theta}(s)$; similarly, the market (p, θ, s) attracts low-type buyers but not high-type buyers if $\theta > \tilde{\theta}(s)$.*

Denote the function $\theta_j^{FB}(s), V_j^{FB}(s)$ as the market tightness and sellers' utility, respectively, when trading with buyer j with *complete information*. We next prove that the equilibrium can be characterized by following proposition.

¹²One can show that the utility of a high-type buyer entering a low-type market is: $U_b(p^l, \theta, s, a^h) = \phi^l + \frac{q(\theta)}{r+q(\theta)} \frac{h(a^h, s) - h(a^l, s)}{r}$, which is bigger than ϕ^h iff $\theta < \tilde{\theta}(s)$.

Proposition 2 *The unique solution to the mechanism is a market tightness function $\theta : S \rightarrow R_+$, a price schedule $P : S \rightarrow R_+$, a marginal type s^* , a pair of initial condition $\{\theta_L^0, \theta_H^0\}$, where:*

$$\begin{aligned}\theta^*(s) &= \begin{cases} \theta(s, l; \theta_L^0), & \text{for } s \leq s^* \\ \theta(s, h; \theta_H^0), & \text{for } s \geq s^* \end{cases}; P^*(s) = \begin{cases} p(s, l), & \text{for } s \leq s^* \\ p(s, h), & \text{for } s \geq s^* \end{cases} \\ V^*(s) &= V_L^{FB}(s_L) + \int_{s_L}^s \frac{u'(\tilde{s})}{r + m(\theta^*(\tilde{s}))} d\tilde{s}\end{aligned}$$

where $\theta(s, j; \theta_j^0)$ is the solution to (10) with the initial condition: $\theta(s_L, l) = \theta_L^0, \theta(s^*, h) = \theta_H^0$, and corresponding $p(j, s)$ is defined in (11).

a) The initial condition θ_l^0 :

$$\theta_l^0 = \theta_l^{FB}(s_L)$$

b) The marginal types:

$$s^* = \begin{cases} s^A, & \text{if } \tilde{\theta}(s^A) \geq \theta_H^{FB}(s) \\ s^B, & \text{if } \tilde{\theta}(s^A) < \theta_H^{FB}(s) \end{cases}$$

where (s^A, s^B) is the unique¹³ solution to the following equation:

$$\begin{aligned}s^A &: V(l, s) = V_L^{FB}(s_L) + \int_{s_L}^s \frac{u'(\tilde{s})}{r + m(\theta^*(\tilde{s}, l; \theta_L^0))} d\tilde{s} = V_H^{FB}(s) \\ s^B &: \tilde{\theta}(s) = \theta(s, l; \theta_L^0)\end{aligned}$$

c) The initial conditions θ_H^0 :

$$\theta_H^0 = \begin{cases} \theta_H^{FB}(s^*), & \text{if } s^* = s^A \\ \tilde{\theta}(s^*), & \text{if } s^* = s^B \end{cases}$$

Proof. See Appendix. ■

As explained earlier, once we can separate the buyers from different markets, we can apply the method for homogenous buyers separately. Therefore, the equilibrium solution is expected to be a combination of two. However, it has to be combined in a particular way so that traders' optimality conditions hold. In appendix, we prove that the constructed solution above is the unique solution.

¹³Observe that (s^A, s^B) is unique (and all smaller than s^{FB}). Notice that, $V(l, s), V_H^{FB}(s), \tilde{\theta}(s), \theta_H^{FB}(s)$ are all well defined and monotonically increases in s and $\theta(s, l; \theta_L^0)$ is strictly decreasing in s . Given that $\tilde{\theta}(s_L) \leq \theta_L^{FB}(s_L)$ under the assumption $V_L^{FB}(s_L) > V_H^{FB}(s_L) \implies s^B$ always exists and is unique.

3.2 Extension: Resale

The basic model assumes that once a buyer buys the asset, he keeps it forever. If a buyer hits by liquidity shock in the future, he has motives to sell his asset to exchange for cash and will then re-enter the market as sellers. Clearly, taking this into account, buyers' expected profit will also depends on the resale value. To capture preference for asset ownership possibly switch overtime and the impact of liquidity shock on the equilibrium price and market liquidity, this section extends our model to allow for resale. To be precise, the flow value of owning the asset decreases, dropping from $h(s)$ to $u(s)$, when the owner hit by the liquidity shock which arrives at the Poisson arrival rate δ . In our basic model, this simply means that the owner now needs to pay the holding cost and hence he naturally becomes the seller in the market. In the example of the capital market, one can interpret this as a firm receives a negative shock of his technology, he will then need to disinvestment. The contingent value of the ownership can now be rewritten as:

$$rJ(s) = h(s) + \delta(V(s) - J(s))$$

where $V(s)$ is the expected value of a type- s seller. Obviously, the expression of a buyers' expected value searching in the market is the same as before. All methods developed in our main model remain valid and the key difference is that the value of holding the asset, which is a function of the resale value $V(s)$, will then be determined in the equilibrium. Nevertheless, one can see that the monotonic condition still holds given that $h(s) + \delta V(s)$ strictly increases with s ¹⁴. Hence, one should expect a full-separated equilibrium is the unique outcome as shown in our previous discussion and our previous approach remain the same. From the free entry condition, the price schedule then has to satisfy the following constraint:

$$p(s) = J(s) - \frac{k\theta}{m(\theta)} = \frac{h(s) + \delta V(s)}{r + \delta} - \frac{k\theta(s)}{m(\theta(s))} \quad (13)$$

The equilibrium outcome can then be solved as before. The only difference is that we now have a different differential equation and, of course, different first best solution, i.e, different initial condition. The differential equation can be derived by substituting the above price schedule into (*ICFOC*) and differentiate respect to s in both sides, which yields:

$$\left[(h(s) - u(s) + \frac{k}{\rho}((\rho - 1)\theta - \frac{(r + \delta)\theta}{m(\theta)})) \right] \frac{d\theta}{ds} = - \left(\frac{r + \delta + m(\theta)}{r + \delta} \right) \cdot \frac{\theta}{\rho} \left(h_s(s) + \frac{\delta u'(s)}{r + m(\theta)} \right) \quad (14)$$

¹⁴This is true as $V^*(s)$ is necessarily increasing in s according to Proposition 1.

One can easily check that the basic version, ie, equation (7), is simply the case when $\delta = 0$. The initial condition is then given by the first best solution $\theta_\delta^{FB}(s)$ in such an environment, which is defined as follows:

$$\begin{aligned} V_\delta^{FB}(s) &= \max_{\theta} \frac{r + \delta}{r} \left(\frac{u(s)}{r + \delta} + \frac{m(\theta) \left(\frac{h(s) - u(s)}{r + \delta} \right) - k\theta}{r + \delta + m(\theta)} \right) \\ \theta_\delta^{FB}(s) &= \arg \max V_\delta^{FB}(s) \end{aligned}$$

Notice that the first best solution can be solved as before, given by (9) but with the discount factor $r + \delta$ instead of r . Note that $\theta_\delta^{FB}(s)$ is decreasing in δ . The intuition is clear since no buyers would want to enter the market if they need to sell it again soon. Since the trading surplus is decreasing in δ , the higher δ , the less entry and hence the lower equilibrium market tightness. Given the solution $(p^*(s), \theta^*(s))$, one can solve for the steady state in such an environment. In particular, for any type of asset, the steady state ratio of the holders to the sellers is pinned down by $\frac{m(\theta^*(s))}{\delta}$. That is, due to the downward distortion of $\theta^*(s)$, the better the asset, the larger portion of which stays in the bad hands.

3.3 Non-monotonicity in the Matching value

The previous analysis shows that no pooling submarkets can exist in such an environment. This is a key feature of the model. It is important to note that it relies on the assumption, $h_s(\cdot) > 0$. The intuition is that if a buyer strictly prefers a higher type s , he will benefit from posting a price which is slightly higher than the original pooling price so that no pooling can exist. Obviously, if this assumption is violated, a pooling equilibrium can then be sustained. Furthermore, as shown in the Proposition 1, any IC allocation $\theta^*(s)$ has to satisfy the condition (M). Observing from (10), if $h_s < 0$, the solution $\theta^*(s)$ can then be increasing, which means that the allocation is no longer incentive compatible. Intuitively, the screening mechanism is a combination of a downward distorted liquidity and an upward price scheme. By the nature of the matching, the market maker needs to make sure buyers are willing to pay for the price, which relies on the fact that buyers also prefer a higher s . In other words, the screening mechanism works when buyers' willingness to pay aligns with sellers' willingness to wait. When these two values do not match, the screening is not implementable. In fact, as we will show in the next section, this setting has an interesting implication when sellers' motives for sale are their private information. The goal for now is to first establish how our previous analysis can be adjusted in a more general environment when the monotonicity condition is violated. Mainly, I show how a

semi-pooling equilibrium can be constructed in a more general and abstract setting. In the later section, we apply this result to analyze the environment when sellers' motives for sale are their private information.

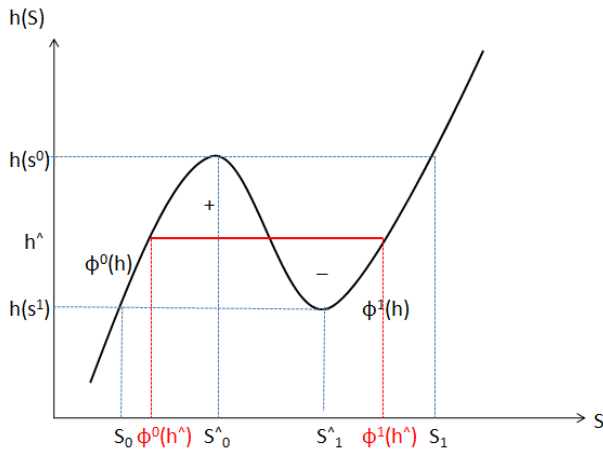
For simplicity, assume the case with homogenous buyers. The pooling outcome will obviously depend on the distribution and in general will not be unique. In fact, from our previous discussion, we know that any mechanism $\alpha = (p^\alpha, \theta^\alpha, V^\alpha)$ that promises a subset of sellers $S' = [s_1, s_2] \subset S$ the same price schedule and market tightness (ie, semi-pooling) has to satisfy Proposition 1. Note that Proposition 1 remains intact regardless of the assumption on $h(\cdot)$. I now consider two approaches to establish a semi-pooling equilibrium, which mainly can be understood in the graphs below respectively. Notice that, depending on the distribution, in general the equilibrium is not unique and the condition for its existence is also identified below for each construction.

3.3.1 Pooling Types

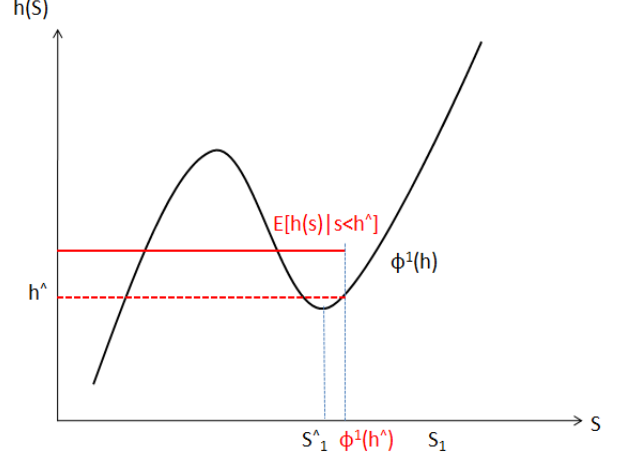
The first approach can be constructed as follows: The first step is to "flatten" buyers' valuations by pooling types. Intuitively, if we can flatten buyer's valuation over s by bunching certain types together in a way that buyers' valuation is (weakly) monotonic in s , our previous analysis still applies (conditionally). However, flattening $h(\cdot)$ directly is convenient to work with but does not necessarily guarantee a non-increasing on $\theta^*(s)$. Therefore, the sufficient condition for which $\theta^*(s)$ is non-increasing is also identified as below.

Consider a continuously differentiable function $h(s)$, assuming that $h(s_L) < h(s_H)$ and the curve has a finite number of interior peaks on $[s_L, s_H]$. Pick a single interior peak \hat{s}_0 , then there is also a single interior trough \hat{s}_1 . As shown in the figure below, the inverse image of the interval $[h(\hat{s}_1), h(\hat{s}_0),]$ is composed of two intervals, $[s_0, \hat{s}_0]$ and $[\hat{s}_1, s_1]$, over which $h(\cdot)$ is increasing, and one interval, $[\hat{s}_0, \hat{s}_1]$ over which $h(\cdot)$ is decreasing. Let $\phi_0(h)$ and $\phi_1(h)$ denote the inverse functions of h over the intervals $[s_0, \hat{s}_0]$ and $[\hat{s}_1, s_1]$. Last, let $\hat{h} \in [h(\hat{s}_1), h(\hat{s}_0)]$ solve:

$$H(h) \equiv h - \int_{\phi_0(h)}^{\phi_1(h)} h(\tilde{s}) dG(\tilde{s}) = 0$$



(A) Reconstructing $h(\tilde{s})$ by bunching types



(B) Fire Sale Equilibrium

Given \hat{h}^{15} , a non-decreasing function $\tilde{h}(\cdot)$ can then be reconstructed as follows: let $\tilde{h}(s) = \hat{h}$ for $s \in [\phi_0(\hat{h}), \phi_1(\hat{h})]$ and $\tilde{h}(s) = h(s)$ otherwise. Suppose that there are more than one interior peak and we could independently different bunching levels where $\hat{h}_1 \leq \hat{h}_2$, a non-decreasing function $\tilde{h}(\cdot)$ can be constructed in a similar way. If treating the two bunching regions separately yields $\hat{h}_1 > \hat{h}_2$, we must then merge the two into a single bunching level. No matter what, if a non-decreasing function $\tilde{h}(\cdot)$ can be obtained: let $\theta^*(s)$ solves (10) with the initial condition $\theta^*(s_L) = \theta^{FB}(s_L)$ at the interval $\tilde{h}(s) = h(s)$ for $\forall s \leq \phi_0(\hat{h})$ and, at the pooling interval, set $\theta^*(s) = \theta^*(\phi_0(\hat{h}))$ for $\forall s \in [\phi_0(\hat{h}), \phi_1(\hat{h})]$. Next, we should let $\theta^*(s)$ solves (10) and set $\theta^*(\phi_1(\hat{h})) = \theta^*(\phi_0(\hat{h}))$ for the interval $[\phi_1(\hat{h}), s_H]$. However, the initial condition for $\theta(\phi_1(\hat{h}))$ has to satisfy the following condition so that the solution for the last interval is non-increasing:

$$\theta^*(s_0) < \theta^{FB}(\phi_1(\hat{h}))$$

Therefore, a semi-pooling equilibrium exists and is characterized by the constructed $\theta^*(s)$ and the corresponding price, $p^*(s) = \frac{\tilde{h}(s)}{r} - \frac{k\theta^*(s)}{m(\theta^*(s))}$. With the above construction, one can easily see that free-entry condition is satisfied. In particular, in the pooling market, $E[h(s)|s \in [s_0, s_1]] = \hat{h}$ by construction. Although $\tilde{h}(\cdot)$ is not differentiable at the kink, ie. at the boundary point s_0 and s_1 , the left and the right limit exists so (10) still applies. In other words, our method applies to any monotonically increasing $\tilde{h}(\cdot)$, even when $\tilde{h}(\cdot)$ has discontinuities of the first kind.

¹⁵ \hat{h} might not be unique and its existence is guaranteed as long as $h(s_1) > h(s_L)$

3.3.2 Equilibrium with Fire Sales

We now focus on a particular type of semi-pooling equilibrium, which has distinct features from the previous analysis. Consider a continuously differentiable function $h(s)$ on the interval $[s_L, s_H]$ and that is strictly increasing in s after some point $\hat{s} \in [s_L, s_H]$ and let $\phi_1(h)$ denote the inverse function of h mapping to $s \geq \hat{s}$, which is illustrated in the figure above.

Proposition 3 *A semi-pooling equilibrium \hat{h} with fire sales exists if following conditions hold:*

$$\begin{aligned} \hat{\mu}(\hat{h}) &\equiv \int_{s_L}^{\phi_1(\hat{h})} h(\tilde{s}) dG(\tilde{s}) \geq \hat{h} \text{ and} \\ V(s_L, \theta_p, p_\mu) &\geq V^{FB}(s_L) \end{aligned}$$

The marginal type is $s^* = \phi_1(\hat{h})$ and the pooling market with the market tightness θ_p is given by:

$$\begin{aligned} \theta_p &= \max_{\theta} \left\{ \theta | V(s^*, p_\mu, \theta) = V^{FB}(s^*) \text{ and } p_\mu = \frac{\hat{\mu}(\hat{h})}{r} - \frac{k\theta_p}{m(\theta_p)} \right\} \\ \theta^*(s) &= \begin{cases} \theta_p & \forall s \in [s_L, \phi_1(\hat{h})] \\ \theta(s; \theta_0(\hat{s})) & \forall s \geq \phi_1(\hat{h}) = s^* \end{cases} \end{aligned}$$

where, $\theta^*(s)$ denotes the solution of (10) with the initial condition $\theta_0(\hat{s}) = \theta^{FB}(\hat{s})$; and the equilibrium price follows:

$$p(s) = \begin{cases} p_\mu = \frac{\hat{\mu}}{r} - \frac{k\theta_p}{m(\theta_p)} & \forall s \in [s_L, \phi_1(\hat{h})] \\ \frac{h(s)}{r} - \frac{k\theta^*(s)}{m(\theta^*(s))} & \forall s \geq \phi_1(\hat{h}) = s^* \end{cases}$$

In words, there exists an marginal type $s^* = \phi_1(\hat{h})$ such that he is indifferent between trading in the pooling market and the market $(p^{FB}(s^*), \theta^{FB}(s^*))$ by construction. In particular tightness θ_p is set to make sure this marginal type indifferent and furthermore, it is upward distorted, that is, $\theta_p \geq \theta^{FB}(s^*)$. For any sellers $s < s^*$, they enter the pooling market (p_μ, θ_p) ; and for any $s > s^*$, they enter a separated market denoted by $(p(s), \theta(s))$ as in our basic model. The proof is straightforward from our previous discussion, so it is omitted. Notice that in such an equilibrium, there is a jump on the equilibrium price whenever $\hat{\mu}(\hat{h}) > \hat{h}$. However, one can easily see that buyers will not deviate by posting any price $p \notin \text{range}P$ as long as the stated conditions are satisfied. These two conditions guarantee that a buyer will not benefit from raising the price ($p' = p_\mu + \varepsilon$) to attract $\phi_1(\hat{h}) = \hat{s}$ nor lowering the price ($p' = p_\mu - \varepsilon$) to attract s_L as he obviously can not do better given $V(s_L, \theta_p, p_\mu) \geq V^{FB}(s_L)$ and $V(\hat{s}) = V^{FB}(\hat{s})$.

4 Obscure Motives for Sale

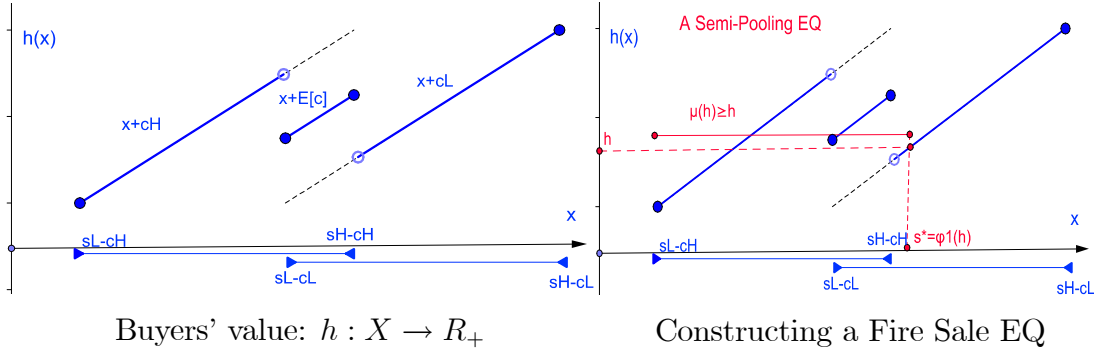
In our baseline model, market liquidity essentially acts as a screening mechanism, similar to Guerrieri et al. (2010). As holding different quality assets results in different liquidity preferences, an agent's type is revealed by his choice of market. The crucial assumption in such an environment, however, is that agents' liquidity positions (i.e. the holding cost in our basic model) are observed. In an environment in which sellers' exact liquidity positions are not known by the market. For example, as pointed out by Tirole (2010), this situation is relevant when there are difficulties involved in apprehending banks' liquidity positions. Any incentive compatible mechanism must then accommodate this effect. Otherwise, sellers would benefit from appearing fragile in order to get a better price. In order to understand how market liquidity might be affected not only by adverse selection and by the market's perceived motives for selling, this section considers an extension when sellers' holding cost is not known to the market. In other words, there are two dimensions of sellers' types: the asset quality (common value) and the liquidity position (private value). The goal of this section is to understand how market liquidity and price are determined in such an environment. We first show how this setup can be nested in our general model and then discuss conditions that are crucial in order to determine whether the least distorted separation is the unique equilibrium outcome.

The setup is similar to our basic model but with the extension that a seller type now has two components: $z^i = (s^i, c^i) \in Z \equiv S \times C$. The support of s^i is the real interval $S \equiv [s_L, s_H] \subset R_+$ as before, but the support of c^i is some arbitrary set C which can assume discrete or continuous values. A seller's payoff of holding the asset is then governed by both the cash flow s and his liquidity position c . Define type $x = s - c \in X \equiv \{s - c \mid s \in S \text{ and } c \in C\}$, representing an agent's value of holding the asset. Clearly, the mechanism discriminates only on the basis of sellers' payoffs of owning the asset. In other words, two agents with the same type \tilde{s} must enter the the same market, irrespective of any other unobservable characteristics that might differentiate the two agents in terms of their attractiveness to buyers. Therefore, one can understand this setup in our general model by following two reinterpretations. First of all, x is now the relevant sellers' type. The utility of seller x , who reports his type \hat{x} , entering the market with $(\theta(\hat{x}), p(\hat{x}))$ can then be expressed as follows:

$$rV(\hat{x}, x) = \max_{\hat{x} \in X} \frac{x + m(\theta(\hat{x})) \cdot p(\hat{x})}{r + m(\theta(\hat{x}))}$$

Second, as buyers care only about the asset quality (i.e the common value), a buyer's expected value of buying the asset from type x can then be seen as $h(x) = E[s|s - c = x]$, where $h : X \rightarrow R_+$. With the above interpretation, we can now apply our previous analysis. Obviously, what matters is $h(\cdot)$. Depending on the distribution, if $h(\cdot)$ is monotonically increasing, the equilibrium can be solved easily as before. In that case, sellers' private information on their liquidity positions essentially generates some noise, but the main result still hold. That is, a fully-separated equilibrium will be the unique equilibrium outcome. For a simple illustration, suppose $c^i \sim U(c_L, c_H)$ and $s^i \sim U(s_L, s_H)$, one can show that $h(\cdot)$ monotonically increases on the interval $[s_L - c_H, s_H - c_L]$ ¹⁶. Hence, $\theta^*(\cdot)$ can be solved as before, and the corresponding price is given by $p(x) = \frac{h(x)}{r} - \frac{k\theta^*(x)}{m(\theta(x))}$.

We now consider a more interesting case when $h(\cdot)$ is not monotonically increasing. Suppose there are two possible liquidity positions for sellers $C \equiv \{c_H, c_L\}$, where $c_H > c_L > 0$. Let λ denote the probability of a seller who owns the asset s suffers a worse liquidity position c_H . For simplicity, assume this probability is the same across asset quality s^i and $s^i \sim U(s_L, s_H)$. The value of $h(\cdot)$ can then be understood in the left figure below:



As shown in the previous discussion, multiple semi-pooling equilibria can exist. More importantly, a fire sale equilibrium can exist. Interestingly and probably counterintuitively, overall equilibrium market liquidity in fact can increase when sellers' holding cost are unobserved. Note that, however, an increase in market liquidity does not necessarily mean welfare improvement, which we will analyze in the later discussion. To illustrate this point, a fire sale equilibrium is now characterized as below and the condition for its

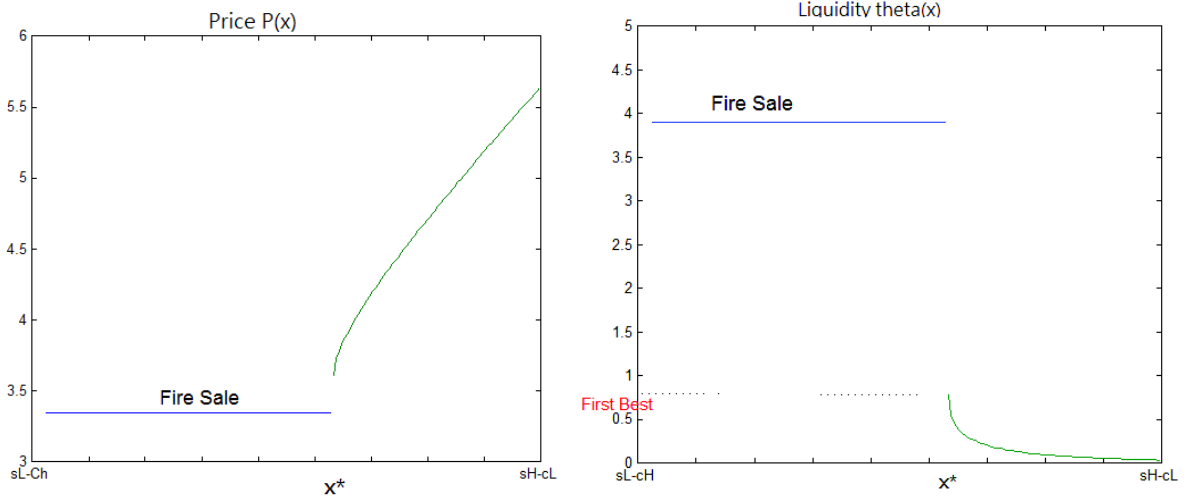
¹⁶The function $h(\tilde{s})$ has two kinks at $(\tilde{s}_1, \tilde{s}_2)$, which, depending on the parameters, is given by:
$$\left\{ \begin{array}{ll} \Delta c < \Delta s & \Delta c \geq \Delta s \\ \tilde{s}_1 = s_L + \Delta c & \tilde{s}_1 = s_L + \Delta s \\ \tilde{s}_2 = s_L + \Delta s & \tilde{s}_2 = s_L + \Delta c \end{array} \right\}$$

existence is also established.

Buyers' value function $h(\cdot)$ is strictly increasing after $x_1 = s_H - c_H$. According to the Proposition 5, a fire sale equilibrium \hat{h} can be found whenever there exists \hat{h} such that the following conditions are satisfied: $\hat{\mu}(\hat{h}) \equiv \int_{s_L}^{\phi_1(\hat{h})} h(x) d\tilde{G}(x) \geq \hat{h}$ and $V(s_L, \theta_p, p_\mu) \geq V^{FB}(s_L)$, which is illustrated in the right figure above. Notice that the set X is the domain of both function $h(\cdot)$ and $\tilde{G}(\cdot)$. We start with the case when $s_H - s_L > c_H - c_L$ so that both $h(\cdot)$ and $\tilde{G}(\cdot)$ have full support, as it will become clear that a fire sale equilibrium always exists when $s_H - s_L < c_H - c_L$. Therefore the following characterization can be directly applied.

Proposition 4 *Given λ , there exists $\bar{\Delta}(\lambda) > 0$ and $\bar{k} > 0$ such that, a fire sale equilibrium always exists when the dispersion of underlying liquidity position $\Delta c \equiv c_H - c_L > \bar{\Delta}(\lambda)$ and $k > \bar{k}$*

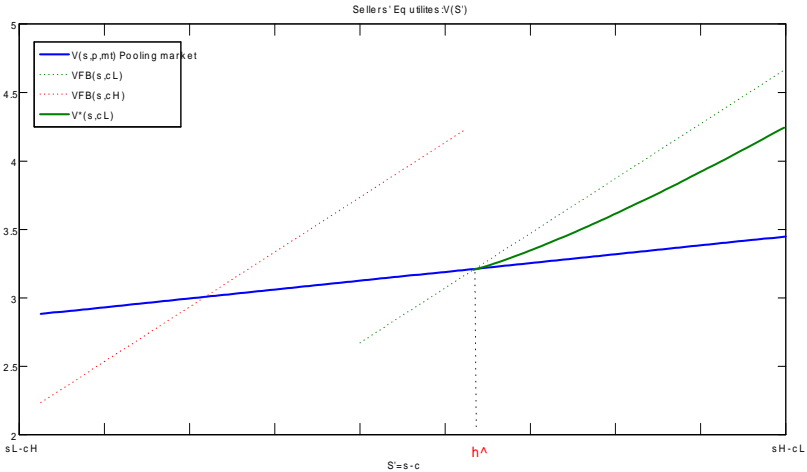
For illustration purpose, an example of a fire sale equilibrium is characterized as follows:



We now give an example when assumed parameters satisfy the conditions, ie, $\exists x^* > s_H - c_H$, such that a fire sale equilibrium can be constructed as shown in the Proposition 5. The above figures shows the equilibrium price and market tightness $p : X \rightarrow R_+$ and $\theta : X \rightarrow R_+$ in each submarket. There exists a submarket which behaves like a fire sale, in which the price is heavily discount while market is liquid. This market attract all sellers $x < \phi_1(\hat{h}) = x^*$. In this market, the price is given by:

$$p^p = \frac{\mu(\hat{h})}{r} - \frac{k\theta^p}{m(\theta^p)}$$

That is, buyers pay the average value among the pool of sellers, $\frac{\mu(\hat{h})}{r}$, minus the expected searching cost in this market. Hence, as in the standard pooling equilibrium, sellers with better assets sell their asset with a high price discount. Furthermore, there is another factor driving down the price— the upward distorted market tightness θ^p . Sellers’ utilities in such an equilibrium is shown below. Notice that in the pooling submarket, sellers with a worse asset are better off compared to benchmark as they effectively receive subsidies from sellers with a better asset. In the rest of submarkets, sellers with a better asset are worse off for the same reason in our basic model— downward distortion market liquidity.



Sellers’ Utilities

Notice that if in an environment where traders’ liquidity positions are unobserved while resale is allowed, the original screening mechanism will also break down even though sellers are all homogenous. The reason is the same as above. In this case, a full pooling equilibrium might arise, which will then share similar features with the standard lemon problem.

Remark There are two ways to understand why our result is different from Guerrieri et al. (2010), where the least-cost separating equilibrium is the unique outcome. There are two main assumptions in Guerrieri et al. (2010): monotonicity and sorting. If types are ranked by their asset quality s , then monotonicity condition holds but the sorting condition does not. As in our trading environment (with limited contract space), the only way to screen agents is through their waiting preference, and there is no hope to separate agents who have better asset quality but with high holding cost from those who have a low-quality asset but low holding cost given that the net value of the holding asset

hence the liquidity preference is the same between them. On the other hand, if agents are ranked by the value of holding the asset \tilde{s} as in our previous analysis, then agents can be screened by the combination of (p, θ) . However, monotonicity condition is then not satisfied.

5 Implications

5.1 The Dispersion of Asset Quality

As shown in the previous analysis, the equilibrium solution does not depend on the assumed distribution. In particular, given that the equilibrium market tightness is pinned down by the initial condition—which must equal the first best market tightness of the *lowest* quality asset—the dispersion, more precisely, the possible range of underlying asset quality plays an important role in determining the market liquidity. This implication can be understood from the basic model, even when the gain from trade is constant across assets. The impact of asset range on the market liquidity is summarized by the following proposition.

Proposition 5 *Given any submarket with asset quality s , its resulting equilibrium liquidity, $\theta^*(s)$, increases with the quality of the worst asset of the whole market s_L . Formally, let $\theta(s; \theta_0(s_L))$ be the solution of (5) satisfying the initial condition $\theta(s_L) = \theta_0$. For any $s'_L < s_L$, then*

$$\theta(s; \theta(s'_L)) < \theta(s; \theta(s_L)) \text{ for } \forall s$$

Proof. Denote $\theta'_0 = \theta(s_L, \theta_0(s'_L))$, representing the equilibrium liquidity of market s_L when the worst asset quality is s'_L . Given that $\theta'_0 < \theta_{FB}(s_L) = \theta_0(s_L)$, the proposition is then simply a result of the Comparison Theorem. ■

It is important to note that this effect only exists in an environment with adverse selection, as we have shown that, with complete information, market liquidity is only a function of its own gain from trade. However, with adverse selection, the market liquidity of all submarkets are connected in the sense that the liquidity in each serves as a screening device to separate assets so that buyers can make sure the asset is worth the paying price. This result then links the impact of underlying asset distribution and market liquidity.

Another interpretation of this result is the impact of transparency. To this end, we now allow assets to be subject to varying severities of the adverse selection problem.

Formally, there are two different asset markets $i \in \{a, b\}$ and the payoff of each asset can be expressed as:

$$d_i = y_i + \sigma_i s$$

where (y_i, σ_i) are publicly observed and s is the owners' private information, where $s \in [s_L, s_H]$ with some distribution $G(s)$. Since (y_i, σ_i) are publicly observed, we can imagine there are two separate markets for each asset, and buyers can choose to go to one of them. Each market can now be characterized as (p^i, θ^i) , where $i \in \{a, b\}$. We can use similar methods to solve for $p^i(s; \sigma^i)$ and $\theta^i(s; \sigma^i)$. Clearly, a higher σ_i has an similar effect to that resulting from a higher dispersion. It can be shown easily that the more transparent asset is more liquid. That is, for any $\sigma_b < \sigma_a$, $\theta^*(s; \sigma_b) > \theta^*(s; \sigma_a) \forall s$.

As the key element in our model is endogeneity of market liquidity, the result also sheds light on the patterns in cross market liquidity. One important question which has been asked in the literature is, why assets paying similar cash flow can have significant differences in their liquidity. We can answer this question from the comparative static exercise above. Loosely speaking, an uncertainty of underlying asset quality has a significant effect on the overall market liquidity, even when the expected value remains unchanged. It therefore implies that assets paying identical cash flows can differ significantly from their liquidity, transaction cost, and price. All of above are endogenously determined in the equilibrium and can be understood as follows:

$$p^i(s) = \underbrace{\frac{d_i}{r}}_{\text{present value}} - \underbrace{\frac{k\theta^i(s; \sigma^i)}{m(\theta^i(s; \sigma^i))}}_{\text{transaction cost}} + \underbrace{\frac{\delta}{r} \left(\frac{c + k\theta^i(s; \sigma^i)}{r + \delta + m(\theta^i; \sigma^i)} \right)}_{\text{resale value}}$$

For any two asset paying the same cash flows, that is, the same $\frac{d}{r}$, their transaction costs as well as resale values, which are both functions of their own liquidity $\theta^i(s; \sigma^i)$, are pinned down by each asset's own underlying dispersion. They can therefore result in a completely different liquidity pattern.

5.2 Policy Implication: Buyback

This section focuses on the buyback policy, as we know that cleaning up the toxic asset will show a significant improvement in overall market liquidity. The idea of cleaning up the toxic asset in the market is not new. In particular, Tirole (2011) shows that the intervention needs to take into account traders' participation constraints and the government always strictly overpays for the worst asset. In our framework, since traders

can choose either to trade their assets in the over-the-counter market specified earlier or to join the government's scheme, traders only join the scheme if and only if they are leaving the market. Hence, though we do know that cleaning up the toxic asset would improve the market liquidity, what is important is to understand what *price* has to be paid in order to clean the market. Suppose now the government offers the price p_g to whomever shows up in the discount windows. Anticipating that traders' utilities would increase in the future after government intervention, the original price and market tightness is then no longer incentive-compatible. Intuitively, a seller can now choose to hold on to the asset and claim a higher type in the future. Therefore, to solve for the equilibrium, a mechanism designer needs to take into account that sellers participate in the scheme only if they get at least as much as what they would have obtained in the decentralized market.

In the equilibrium, the sets of types who join in government intervention, Ω_g , and of those who stay in the decentralized market, Ω_d , are disjointed and satisfy $\Omega_g \cup \Omega_d \equiv S$. Agents' utilities can be expressed as:

$$V(s) = \max\left\{\frac{s-c}{r}, \max_{p'} U(p', \theta(p'), s), p_g\right\}$$

One can therefore pin down the condition for the marginal participant type s^* :

$$V(s^*) = p_g$$

That is, the marginal type has to be indifferent between trading in the OTC market and obtaining the transfer from the government right away. Let (p^*, θ^*) denote the price and the market tightness that the marginal participant type will be facing if he goes to the market. Obviously,

$$\begin{aligned} U_g(s) &= p_g > \max_{p'} U(p', \theta(p'); \Omega_d) \text{ for } \forall s < s^* \\ U_g(s) &= p_g < \max_{p'} U(p', \theta(p'); \Omega_d) \text{ for } \forall s > s^* \end{aligned}$$

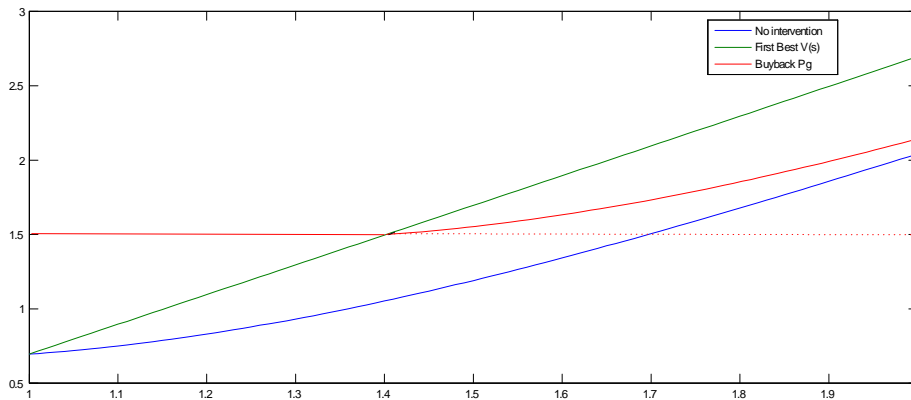
Also, from the previous discussion, we know how to solve the equilibrium outcome $p(\cdot), \theta(\cdot)$, given Ω_d . The key task is to pin down the marginal type, which is characterized as follows:

Proposition 6 *A competitive search equilibrium with government buyback price p_g , is a marginal participant type s^* which solves*

$$V(s^*) = p_g = V^{FB}(s^*)$$

and a pair of $(p(s; \Omega_d), \theta(s; \Omega_d))$ that satisfies (4), (5), (6), as a solution to the market maker's constrained incentive-efficient problem.

The figure below shows traders' utilities with the buyback policy p_g , represented by the red line. Given any price offered by the government, the marginal types s has to solve $V^{FB}(s^*) = p_g$, represented by the intersection of p_g and the green line, which in turn is represented by $V^{FB}(s)$.



Equilibrium with Buyback Policy P_g

5.3 Reallocation and Macroeconomic Performance

As the massive ongoing microeconomics restructuring and factor reallocation is crucial to aggregate performance, there is clearly a strong link between how the economy is doing and how well factors markets are functioning. To this end, this section focuses on the reallocation of firms' corporate assets, given that capital is one of the important factors determining aggregate productivity. As documented in the empirical literature, changes in ownerships of firms' corporate assets— for example, product lines, plants, machines, and other business units— affect productivity. More precisely, capital typically flows from less to more productive firms, and the productivity of acquired capital increases. Furthermore, as suggested by the empirical findings, market thinness generates frictions that are a large impediment to the efficient reallocation of capital, even within a well-defined asset class, in which capital is moderately specialized. We then provide a possible explanation as to why the market for used capitals is relatively thin. Severe trading delays would result in resource mismatch and have a negative impact on aggregate performance. This paper then

elucidates the underlying source of market frictions and its link to economic fluctuation

To illustrate our result, we now specify the function governing traders' payoff, nested in our general framework. The flow value of the capital is simply the product of the capital quality s and its use of technology a_j , that is, $h(a_j, s) = a_j s$. More productive firms will be the natural buyers. In the economy, there are two profitable technologies $j \in \{H, L\}$, where $a_H > a_L$, and the owner of the technology j has the outside option ϕ_j , where $\phi_H > \phi_L$. Firms who receive a negative shock, at the arrival rate of δ , become unproductive and only produce a flow payoff $a_o s$, where $a_o < a_L < a_H$, that is, $u(s) = a_o s$. Let $H_t^j(s)$ represent the measure of capital s owned by the firm with technology j and let $\mu_t(s)$ denote the measure of sellers who own capital s . The aggregate productivity of the capital can then be defined as follows:

$$\bar{A}_t = \int \{a_o s \mu_t(s) + a_l s H_t^l(s) + a_h s H_t^h(s)\} ds \quad (15)$$

Therefore, both the aggregate TFP and the cross-sectional distribution of active firms are endogenously determined in our model, crucially depending on how well the economy can allocate its resource to better hands. Interestingly, probably counter-intuitively, at the macro level, Eisfeldt and Rampini (2006) have documented that the capital reallocation is procyclical while the cross-sectional dispersion of the productivity is countercyclical. Concluding from this finding, they suggest that the reallocation friction is countercyclical and we need a better foundation for that. Contrary to most macro models assuming exogenous adjustment costs, one advantage of our framework is allowing for a richer analysis of how this market friction responds to varied economic shocks and a better understanding of its aggregate implication.

First of all, we consider the shock to the underlying dispersion. From the previous analysis, it is clear that an increase in dispersion would increase the market frictions. Hence, the resulting resource mismatch would generate a drop in TFP and further increase the cross-sectional dispersion of productivity. It therefore provides an explanation for the coexistence of the countercyclical dispersion and procyclical reallocation, as documented in the empirical literature. It is definitely an interesting extension to endogenously size the underlying dispersion and worth further exploration.

In line with the growing literature on uncertainty, the shock to the downward uncertainty is also analyzed. Intuitively, a higher δ , that is a higher level of instability of business condition would decrease investors' willingness to enter. Hence, the market liquidity also decreases with δ , resulting in worse aggregate performance and a higher level of dispersion across firms. The story here, however, is different from Bloom (2009).

Bloom (2009) shows that, with existence of capital adjustment costs, higher uncertainty (measured as a shock to the second moment) expands firms' inactive regions because it increases the real-option value of waiting. This concern then slows down the reallocations from low to high productivity firms. Instead of relying on an exogenous adjustment cost, firms who receive a negative shock, do want to exit but have a hard time finding an investor who is willing to buy their capital in our model. This idea also explains why few firms exit in bad time, as documented in Lee and Mukoyama (2008)

It is important to note that, however, an increase in the downward uncertainty, also results in a higher demand for the reallocation. Hence, the net effect on capital reallocation is ambiguous. As emphasized by Bachmann and Bayer (2009), a large countercyclical second moment shock would be incompatible with procyclical investment dispersion. Though, the shock on the downward uncertainty considered here is different from the second moment shock, it is important to understand that it also has two driving forces¹⁷ and that its net effect depends on assuming parameters. As a full calibration is beyond the scope of this paper, the main purpose of this exercise is to understand how uncertainty affects market frictions, as it can generate significant fluctuations in an environment with adverse selection.

6 Discussion

6.1 On Efficiency

Does this decentralized equilibrium outcome necessarily solve the centralized planner's problem? The answer is obvious from our solution method. As explained earlier, among the set of feasible mechanisms A defined from Proposition 1, the decentralized outcome is the one satisfying the buyers' optimality constraint, $E2$. Namely, given that buyers have the freedom to post new prices in the decentralized market, $E2$ is the additional constraint compared to the social planner's problem. This logic clearly implies that a social planner can always (weakly) do better than the market. In fact, in our basic model, a full pooling equilibrium always achieves the first best welfare level as long as it is sustainable. The main reason is that the first best solution of market tightness is independent of types. A

¹⁷To see this, the total amount of reallocation at the steady state can be expressed as:

$$\frac{\delta}{\delta + m(\theta^*(s; \delta))} \cdot m(\theta^*(s; \delta))$$

pooling equilibrium, simply subsidizing some at the expense of others, therefore does not incur any distortions as long as participating constraints of the highest types are satisfied. To be precise, a full pooling equilibrium maximizes social welfare as in the environment under complete information if and only if 1) the gain from trade is independent of types and 2) the IR constraint of the highest type is satisfied.

The above point then leads us to the next question: is the decentralized equilibrium outcome Pareto efficient? The answer can also be understood from our basic model. First of all, notice that the outcome of separating equilibrium does not depend on the distribution of types. On the other hand, traders' utilities in any kind of pooling equilibrium will obviously depend on the distribution. Intuitively, the highest type in a pooling equilibrium suffers a lower price because the market is only willing to pay the expected value of the asset. Nevertheless, in a separating equilibrium, he enjoys a much higher trading price but must suffer a long waiting time, decreasing his expected utilities. One extreme case would be that a full pooling equilibrium drives the highest type out of the market, which can happen when there are too many bad assets; therefore, the highest type is obviously better off in the separating equilibrium. In this particular case, the separating equilibrium is not Pareto ranked by the full pooling equilibrium¹⁸. Another extreme case would be that average quality is high enough so that even the highest type is better off in a full pooling equilibrium. The separating equilibrium then is Pareto ranked by the full pooling equilibrium. Hence, the answer to the efficiency properties of equilibrium will depend on the distribution assumed, which is actually a straightforward task once some particular distribution is given. This point then explains why the competitive search equilibrium is Pareto inefficient for some parameter values, as shown in Guerrieri et al. (2010). The important lesson is that the equilibrium outcome is not necessarily constrained Pareto efficient and the main reason is that pooling cannot be sustained even it is desirable due to the competitive nature of markets. And the distortions in market tightness resulting from separating equilibrium are rather costly.

6.2 On Sorting Behavior

Shi (2001) and Eeckhout and Kircher (2010) have shown that the complementarity in production is not enough to guarantee positive assortative matching (PAM) in an environment with complete information. The intuition is that, given that the social surplus increases with types, it could be optimal to match high-type seller with a low-type buyer

¹⁸However, depending on distribution, it could be ranked by a semi-pooling equilibrium.

by promising him a tight market, that is, a higher utilization. The above intuition still holds in our framework with complete information. However, with adverse selection, we prove that the supermodularity of the matching value necessarily induces PAM in the equilibrium.

Proposition 7 *In the competitive search equilibrium with adverse selection, the equilibrium trading pattern $j^*(s)$ satisfies PAM, that is, for $s' > s$, $j^*(s) = h \implies j^*(s') = h$ under the assumption $h_s(a^h, s) - h_s(a^l, s) > 0$,*

Proof. Suppose Not. There exists $s' > s$ such that $j^*(s) = h$ and $j^*(s') = l$. According to Claim 1, the equilibrium market tightness must satisfy: $\theta^*(s) \leq \tilde{\theta}(s)$ and $\theta^*(s') \geq \tilde{\theta}(s')$. Moreover, from the monotonic condition, (M), $\theta^*(s) \geq \theta^*(s')$. The above relation then implies $\tilde{\theta}(s) \geq \tilde{\theta}(s')$. This is a contradiction to the fact that $\tilde{\theta}(s)$ is strictly increasing with s under the assumption $h_s(a^h, s) - h_s(a^l, s) > 0$. (Recall $\tilde{\theta}(s)$ solves $\phi^l = \phi^h - \frac{q(\theta)}{r+q(\theta)}(\frac{h(a^h, s) - h(a^l, s)}{r})$) ■

To understand this result, recall that the reason as to why a higher type can be better off when trading a low-type buyer is that he can be compensated by a higher utilization. That is, given that a lower type buyer is more willing to wait, it could be optimal for a high type seller to choose to trade with a lower type buyer, enjoying a lower gain from trade but a tighter market compared to trading with a high type buyer. Hence, contingent on negative assortative matching (NAM), a high type seller must be compensated with a higher market tightness compared to low type sellers. This situation, however, can not be sustained in an environment with adverse selection, as it violates the monotonic condition. Namely, it is not incentive compatible for the sellers. Notice that in an environment with complete information, a high type seller prefer to trader faster as his gain from trade is higher. Nevertheless, with adverse selection, when all sellers are facing the same market price schedule and market tightness, a high type seller becomes the one who is more patient in the sense that he will prefer the combination of a higher price and a lower market tightness as contrary to a low type seller. This implies that it would be optimal to match a high type seller with a buyer who is more willing to offer a higher price and less willing to wait. Obviously, a high type buyer is more willing to do this. Hence, a lower type buyer no longer has his advantage to trade with a high type seller as in the case with complete information.

Our solution developed earlier starts with the environment with PAM and $V^{FB}(l, s_L) > V^{FB}(h, s_L)$. However, according to the above Proposition, one should expect that those conditions can be relaxed. First of all, suppose $V^{FB}(h, s_L) > V^{FB}(l, s_L)$, so it is clear

that $j^*(s_L) = h$ from Lemma 3 and, clearly, from the above Proposition, $j^*(s) = h$ for $\forall s \in S$. Hence, we can simply solve the model as if there are only high-type buyers in the market, regardless of positive or negative assortative matching under complete information. Suppose that we are now in the environment with NAM, that is, for $s' > s$, $V^{FB}(l, s) - V^{FB}(h, s) > 0 \implies V^{FB}(l, s') - V^{FB}(h, s') > 0$ and $V(l, s_L) > V(h, s_L)$, implying that only low type buyers are active in the case with complete information. Although we do not provide the formal solution for this case, our conjecture tells us that the solution should take similar pattern as the developed method. And, depending on the range of S , it could be the case that $j^*(s) = h$ for some $s' > s$. The above argument shows that adverse selection essentially makes a higher type buyer more likely to stay the market compared to the case with complete information. Notice that this phenomenon can be understood for our main results as well, given that the marginal type decreases in the case of adverse selection, that is, $s^* < s^{FB}$. Hence, more sellers end up trading with high-type buyers with adverse selection.

This result might seem at first counter-intuitive but, in fact, it is simply the flip side of market illiquidity. Adverse selection creates a downward distortion of market liquidity, that is, a low ratio of buyers over sellers in the market. This distortion makes it hard for a seller to find a buyer; on the other hand, it also makes it easier for a buyer to find a seller, shortening a buyer's wait time and expected search cost. Given that high-type buyers like to secure trade with high probability and they are willing to pay for this, the environment effectively makes a high type buyer more competitive, compared to low-type buyers.

6.3 Search Frictions and Rationing

One key feature of our results is that market liquidity acts as a screening mechanism in the setup of a competitive search equilibrium. Search friction has a natural interpretation in decentralized assets market and over-the-counter market, where traders' matching rate depends on how tight the market is. Interestingly, Guerrieri and Shimer (2011) independently obtained similar results in an environment without search frictions, where illiquidity results from the rationing in each submarket. In the limiting behavior of their economy as the period length shortens, they show that the sale rate per unit of time can be seen as convergence to a Poisson arrival rate. Therefore, although with different interpretations, the trade-off faced by sellers in both our models are driven by different trading/matching rates and prices across markets. This then raises the question, what is

the role of search friction and, economically, how and to what extent, is it different from rationing?

Notice that the approach of considering the limit economy as trading becomes more frequent does not shed light on the relation between search friction and rationing. In order to address this issue, one can understand the rationing as one special matching technology in the sense that the short side of the markets gets matched with certainty. This point has first made by Eeckhout and Kircher (2010), which considers a convergent sequence of search technologies in a static economy. Therefore, one way to show how search frictions vanish is to apply their approach in a discrete time version of our model and shows how the distortion varies as search friction vanishes. Here, we propose another way to see how search friction vanishes directly in our original continuous time setup. As show in Guerrieri and Shimer (2011), as an equilibrium outcome, buyers always match with probability one as they're the short side of the market. What is similar is that buyers in our model can trade relatively fast when facing a downward distorted market tightness. The key difference, however, lies on the fact that buyers in their model do not care about the market tightness conditional on they're at the short side of the market. Effectively, at the equilibrium, there is no congestion effect on the buyers' side. This effect is governed by the elasticity of the matching function with respect to measure of buyers in the standard codd-douglas matching function, that is, the parameter ρ in our model. Therefore, one can understand the result in Guerrieri and Shimer (2011) as the limit economy in our model when $\rho \rightarrow 1$. Notice that the first best market tightness increases with ρ as buyers care less about the market tightness. Hence the first best market tightness should increase. At the limit, the arrival rate for the lowest type goes to infinity when $\frac{c}{k} > 1$, which corresponds to the standard discrete setup where the trading probability equals one (no rationing) given the gain from trade is larger than the entry cost.

7 Conclusion

(To be added)

8 Appendix

8.1 Omitted Proof

(A) Proof of Proposition 1:

Proof. One can observe that $U(\theta, s)$ satisfies following properties: that 1) $U_2(\theta, s)$ exists; 2) has an integrable bound: $\sup_{s \in S} |U_2(\theta, s)| \leq \frac{M}{r}$ for all s , where $M = u'(s_L)$ for; 3) $U(\theta, \cdot)$ is absolutely continuous (as a function of s) for all θ ; 4) $\theta^*(s)$ is nonempty. Following the mechanism literature, (see Milgrom and Segal (2002)), let

$$V(s) = \max_{\hat{s}} U(\theta(\hat{s}), s) = \max \frac{u(s) + p(\theta(\hat{s}))m(\theta(\hat{s}))}{r + m(\theta(\hat{s}))}$$

then any selection $\theta(s)$ from $\theta^*(s) \in \arg \max_{\theta'} U(\theta', s)$,

$$V(s) = V(s_l) + \int_{s_l}^s U_s(\theta^*(\tilde{s}), \tilde{s}) d\tilde{s}$$

Namely, (*ICFOC*) is the necessary condition for any IC contract. To prove the sufficiency, define function: $x = q(\theta) = \frac{1}{r+m(\theta)}$ and $q^{-1}(x) = \theta$. Also, since $\theta > 0$, $0 < x \leq \frac{1}{r}$. One can then easily see $U(x, s)$ satisfies the strict single crossing difference property under the assumption $u'(s) > 0$. For any $x' > x$ and $s' > s$:

$$U(x', s') - U(x', s) + U(x, s) - U(x, s') = x'(u(s') - u(s)) - x(u(s') - u(s)) > 0$$

Therefore, $U(x', s') - U(x, s') > U(x', s) - U(x, s)$. Given that $U(x, s)$ satisfies SSCD condition, then any non-decreasing $x(s)$ combined with (*ICFOC*) are also sufficient conditions for the achievable outcome. Hence, $x(s) = \frac{1}{r+m(\theta^*(s))}$ has to solve subject to the non-decreasing constraint. Namely, the market tightness function $\theta^*(\cdot)$ has to be non-increasing. ■

B) Proof of Claim 1: No pooling

Proof. Suppose Not: There exists a subset of sellers $s \in S' = [s_1, s_2] \subset S$ are in the same market $(p_\alpha, \theta_\alpha)$. From the free entry condition,

$$p_\alpha = \frac{E[s|s \in S']}{r} - \frac{k\theta_\alpha}{m(\theta_\alpha)}$$

and denote $V^\alpha(s_2) = V(p_\alpha, \theta_\alpha, s_2)$ as the utilities of the highest type seller in the market, and define the pair (p_2, θ_2) solves:

$$\left\{ \begin{array}{l} p_2 = \frac{s_2}{r} - \frac{k(\theta_2)}{m(\theta_2)} \\ V(p_2, \theta_2, s_2) = V^\alpha(s_2) \end{array} \right\}$$

Given (p_2, θ_2) solves above relations and $\frac{E[s|s \in S']}{r} < \frac{s_2}{r}$, there exists $p' = p_2 + \varepsilon$ such that $p_\alpha < p' < p_2$ and θ' which solve $V(p', \theta', s_2) = V^\alpha(s_2)$. From lemma 1, $T(p') = s_2$. Namely, if a buyer deviates to posting p' , only the highest type in the original pooling market will come. And, as explained, the expected market tightness is defined from (2), that is, $\theta' = \theta(p')$. Obviously, $U_b(p', \theta', s_2) > U_b(p_2, \theta_2, s_2) = 0$. Contradiction. ■

C) Proof of Claim 2: the lowest type always receives his first best utility.

Proof. Suppose not, pick any initial condition $\theta'_0 \in (0, \theta^{FB}(s_L))$ and denote its corresponding market tightness as $\theta'(s; \theta'_0)$ and price schedule $p'(s)$. One can easily show that there exists $\tilde{p} = p'(s_L) - \varepsilon$ and, from Lemma 1, $T(\tilde{p}) = s_L$. That is, a buyer can open a new market with lower price and expect the lowest type to come. Due to the violation of the tangent condition at $(p'(s_L), \theta'(s_L))$ when $\theta'(s_L) \neq \theta^{FB}(s_L)$ and $V'(s_L) < V^{FB}(s_L)$, buyers' utility can be improved, $U(\tilde{p}, \theta(\tilde{p}), s_L) > \phi^L$. Contradiction. ■

D) Proof of Heterogenous buyers

Before proving the constructed solution is indeed the solution, we first prove that the following claim holds:

Claim 5 $\forall \theta' < \theta$, if $V(s', p, \theta) - V(s', p', \theta') \geq 0$ then $V(s, p, \theta) - V(s, p', \theta') > 0$ for $\forall s' > s$

Proof. $V(s, p, \theta) - V(s, p', \theta') = \{V(s', p, \theta) + \frac{u(s)-u(s')}{r+m(\theta)}\} - \{V(s', p', \theta') + \frac{u(s)-u(s')}{r+m(\theta')}\} \geq (u(s') - u(s))(\frac{1}{r+m(\theta')} - \frac{1}{r+m(\theta)}) > 0$

Proof. a) Sellers' optimality: NTS: Given $(p(s), \theta(s))$, $s = \arg \max_{\hat{s}} V(s, p(\hat{s}), \theta(\hat{s}))$. First of all, we need to show that monotonic condition holds. The solution $\theta(s)$ is essentially the combination of $\theta(s; l)$ and $\theta(s; h)$, which are both non-increasing as long as the initial condition θ_j^0 is smaller than $\theta_j^{FB}(s_0)$. Therefore, $\theta(s)$ is also non-increasing as long as $\theta_H^0 \leq \theta(s^*, l)$, which holds by construction. From proposition 1, it is clear that facing $(p(s, l), \theta(s, l))$, IC holds for sellers $s < s^*$, and, similarly, given $(p(s, h), \theta(s, h))$, IC holds for sellers $s \geq s^*$. What is left to prove is that those sellers will not benefit from entering the markets $\{p(s'), \theta(s')\}$ for $\forall s' \geq s^*$. Clearly, given that $V(s^*, p(s^*, l), \theta(s^*, l)) = V(s^*, p(s^*, h), \theta(s^*, h)) > V(s^*, p(s', h), \theta(s', h))$ and $\theta(s') < \theta(s^*)$ for all $s' \geq s^*$, from claim 2:

$$V(s, p(s), \theta(s)) > V(s, p(s'), \theta(s')) \text{ for } \forall s' > s$$

■ ■

Similarly, one can use the fact that $V(s^*, p(s^*, h), \theta(s^*, h)) \geq V(s^*, p(s), \theta(s))$ and $\theta(s) < \theta(s')$ for all $s \leq s^*$ to prove $V(s', p(s'), \theta(s')) > V(s', p(s), \theta(s))$ for $\forall s < s^*$

b) Buyers' optimality: In order to make sure there is no profitable deviation for buyers, following two conditions much hold: b-1) $U_b(p(s), \theta(s), s, a^j) < \phi^j$ if $j^*(s) \neq j$. Namely, given the markets which are already open, buyers will not enter the market to trade with seller s , if $j^*(s) \neq j$. Note that this is an additional condition we need to prove with heterogenous buyers; b-2) There does not exist $p' \notin \text{support } P$, such that $U_b(p', \theta(p'), s', a^j) < \phi^j$, where $\theta(p')$ is defined as (2) and $s' \in T_s$ is the type of a seller who is most likely to come. That is, the buyer will not benefit from posting a price p' which is not recommended by the market maker.

b-1) First of all, we show that low type of buyers will not enter the market with sellers s , where $j^*(s) = h$. Observe that by construction, $\theta(s; h, \theta_H^0) < \tilde{\theta}(s)$ for $\forall s > s^*$. This is true as long as $\theta_H^0 \leq \tilde{\theta}(s^*)$, given that $\theta(s; h, \theta_H^0)$ decreases with s and $\tilde{\theta}(s)$ increases with s . The condition is satisfied since, by construction, if $\tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$, $\theta_H^0 = \theta_H^{FB}(s^A) \leq \tilde{\theta}(s^A)$; otherwise, $\theta_H^0 = \tilde{\theta}(s^B)$. Hence, from claim 1, low type will not enter the market with the pair $\{p^*(s), \theta^*(s)\}$ for $\forall s > s^*$. Similarly, by construction, $\theta(s; l, \theta_L^0) \geq \theta(s)$ for $\forall s \leq s^*$. Hence, high-type buyers will not enter the the market with the pair $\{p^*(s), \theta^*(s)\}$ for $\forall s < s^*$

b-2) Let $\{\bar{p}, \underline{p}\}$ as the upper bound and the lower bound of the support constructed price schedule P . Apply Lemma 1, for any new posting $p' > \bar{p}$, $T_s(p') = s_H$ and $p' < \underline{p}$, $T_s(p') = s_L$. Obviously, the lower type will not benefit from opening $p' < \underline{p}$ since (p^*, θ^*) is the first-best solution. The higher type obviously will not benefit from attracting the lowest type seller, given $V^{FB}(s, h) < V^{FB}(s, l)$ and, therefore, $U_b(p', \theta', s_L; V^{FB}(s_L, l)) < \phi^h$. Similarly, as before, neither the high type will benefit from posting $p' > \bar{p}$ to attract the highest type with a even higher distortion of market liquidity, nor the lower type buyer. Notice that $p(\cdot)$ has a jump discontinuity at s^* when $s^* = s^A$. In particular, there are two markets opened for the seller s^* , and, among two of them, sellers are indifferent. Given any $p' \in (p(s^*, l), p(s^*, h))$, $T(p') = s^*$ according to Lemma 1. Given that the type who is most likely come is s^* , it is clear that low type buyer will not raise the price $p' > p(s^*, l)$ to attract the same seller. Nor the high type will benefit from posting $p' < p(s^*, h)$ since the pair of $p(s^*, h)$ and $\theta(s^*, l) = \theta_H^{FB}(s)$ has already maximized the joint surplus (first best).

Proof. Uniqueness: From Claim 2, we know that $\theta_L^0 = \theta_L^{FB}(s)$. Hence, to prove the uniqueness, we essentially need to show that marginal types s^* constructed above and its corresponding θ_H^0 is unique. First of all, by construction, $V_H^{FB}(s) < V(s, l)$ for $\forall s < s^A$, it is clear that a high-type buyer will not enter the market for $s < s^A$, given that the highest utilities he can offer to the seller is lower than the one offered by a low type buyer.

Also, from Claim 1, s^* can not be larger than s^B in the equilibrium otherwise there exists $s < s^*$ such that $\theta(s, l) < \tilde{\theta}(s)$, which implies that it is profitable for a higher type to enter this market. Hence, the only possible range is $[s^A, s^B]$. Given that there are two different cases, depending on the relation between $\tilde{\theta}(s^A)$ and $\theta_H^{FB}(s^A)$, we will prove the uniqueness separately for each case. Before that, we first prove formally that s^A is unique. Notice that s^A is the intersection of $V_H^{FB}(s)$ and $V(s, l)$. Therefore, the unique of s^A is obtained as long as $V_H^{FB}(s) - V(s, l)$ satisfies single crossing condition. As a result of the following inequalities, we can conclude that $V_H^{FB}(h, s') - V(l, s') > V_H^{FB}(h, s) - V(l, s)$ for any $s' > s$

$$\begin{aligned} & V_H^{FB}(s') - (V_L^{FB}(s) - V(l, s)) - V(l, s') \\ > & V_H^{FB}(s') - (V_L^{FB}(s') - V(l, s')) - V(l, s') \\ > & V_H^{FB}(s) - (V_L^{FB}(s) - V(l, s)) - V(l, s) \end{aligned}$$

The first inequality follows from the fact that $V_L^{FB}(s') - V(l, s') > V_L^{FB}(s) - V(l, s)$, that is, the utility of a high type seller decreases more than the one of a low type, resulting from a higher distortion¹⁹. The second inequality follows from the condition of PAM, that's, $V_H^{FB}(s') - V_L^{FB}(s') > V_H^{FB}(s) - V_L^{FB}(s)$. Moreover, from the discussion of b-2), we can conclude the following claim: (Claim 3) If there is a discontinuity in $\theta^*(\cdot)$ at s^* , which necessarily induces a discontinuity in $p(\cdot)$, it has to be the case that $\theta_H^0 = \theta_H^{FB}(s^*)$. Namely, θ_H^0 must equal its first best value $\theta_H^{FB}(s^*)$ when $\theta_H^0 \neq \theta(s^*, l)$; otherwise, there is a profitable deviation for a high type buyer, who will deviate by posting a new price $p' = p(\theta_H^0) - \varepsilon \notin \text{range of } P$ to attract s^* .

(CASE 1) $s^* = s^A$ when $\tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$: $\tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$ immediately implies that $\theta(s^A, l) \geq \tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$ ²⁰ and the equality holds iff $\tilde{\theta}(s^A) = \theta_H^{FB}(s^A)$. Pick $s_m \in (s^A, s^B]$ as the marginal type. By definition, the marginal type must be indifferent among two markets, that is, indifferent between $(\theta(s_m, l), p(s_m, l))$ and $(\theta_H^0, p(s_m, h))$. Given that $s_m > s^A \implies V_H^{FB}(s_m) > V(s_m, l)$, It has to be case that $\theta_H^0 < \theta_H^{FB}(s_m)$, that is, there must be a downward distortion²¹ in market tightness otherwise s_m is strictly better off going

¹⁹One can show that $\theta_L^{FB}(s) - \theta^*(s, l)$ increases with s , given that $\theta_L^{FB}(s)$ increases with s while $\theta^*(s, l)$ decreases with s .

²⁰Note that $\theta(s^A, l)$ is the intersection of a low-type buyer's utility at ϕ^l and the utility of a seller s^A with the level of $V_H^{FB}(s^A)$. Given that $\tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$ and the tangent condition of $V_H^{FB}(s^A)$ and the utility of a high-type buyer is satisfied, $\theta(s^A, l)$ must be larger than $\tilde{\theta}(s^A)$. This is because that, by construction, the utility curve of $U_b(s, p, \theta) = \phi^l$ lies below $U_b(s, p, \theta) = \phi^h$ for any $\theta < \tilde{\theta}(s)$ and above $U_b(s, p, \theta) = \phi^h$ for any $\theta > \tilde{\theta}(s)$.

²¹Clearly, it has to be downward distortion instead of upward one since the monotonic condition from

to the market with a high type buyer. According to Claim 3, this can not be sustained in equilibrium if there is a discontinuous in $\theta^*(\cdot)$. Namely, the only possible case is that $\theta_H^0 = \theta(s_m, l) < \theta_H^{FB}(s^m)$. However, given that $\theta(s^A, l) \geq \theta_H^{FB}(s^A)$, $\theta(s_m, l)$ decreases with s and $\theta_H^{FB}(s)$ increases with s , $\theta(s_m, l) < \theta_H^{FB}(s^m) \implies \exists s' \in (s^A, s^m)$ such that $\theta(s', l) = \theta_H^{FB}(s') > \tilde{\theta}(s')$. The above relation implies that $V(p^l(s'), \theta(s', l)) > V_H^{FB}(s')$, which contradicts with the fact that $V_H^{FB}(h, s') - V(l, s') > 0$ for $\forall s' > s^A$. Therefore, we show that $s^* = s^A$ is the unique solution when $\tilde{\theta}(s^A) \geq \theta_H^{FB}(s^A)$.

(CASE 2) $s^* = s^B$ when $\tilde{\theta}(s^A) < \theta_H^{FB}(s^A)$. First of all, $\tilde{\theta}(s^A) < \theta_H^{FB}(s^A)$ implies that $\theta_H^{FB}(s^A) > \theta(s^A, l) > \tilde{\theta}(s^A)$. Same as before, the only possible range for s_m is $[s^A, s^B]$. Given that $\tilde{\theta}(s)$ and $\theta_H^{FB}(s)$ increase with s , while $\theta(s, l)$ decreases with s and $\theta_H^{FB}(s^A) > \theta(s^A, l) > \tilde{\theta}(s^A)$, it has to be the case that $\theta_H^{FB}(s_m) > \theta(s_m, l) > \tilde{\theta}(s_m)$ for $\forall s_m \in [s^A, s^B]$. Moreover, according to Claim 1, the initial condition θ_H^0 has to be smaller than $\tilde{\theta}(s_m)$. Therefore, $\theta(s_m, l) > \tilde{\theta}(s_m) \geq \theta_H^0$, which necessarily results in a discontinuity of $\theta^*(\cdot)$ at s_m . The resulting discontinuity and $\theta_H^0 < \theta_H^{FB}(s_m)$ violates Claim 3. Contradictions. The above argument also confirms why only s^B and $\theta_H^0 = \theta(s^B, l) = \tilde{\theta}(s^B)$ is the unique solution in this case, guaranteeing the continuity of $\theta^*(s)$. ■

E) Proof of Claim 3: From the FOC of the first best solution, one can solve $\frac{d\theta^{FB}(s)}{ds} = \frac{h'(s) - u'(s)}{k \frac{(1-\rho)}{\rho} (1+\theta^{-\rho})} \equiv f_2(\theta, s)$. Observe from the differential equation, $\frac{d\theta^*(s)}{ds} = f(\theta, s) \rightarrow -\infty$ at $f(\theta^{FB}(s), s)$ given $h_s > 0$. Hence, we know that $\theta^*(s) \leq \theta^{FB}(s)$ for some $s_1 > s_L$. Suppose now $\theta^*(s) > \theta^{FB}(s)$ for some s , which implies that these two functions must cross at some point $(\hat{s}, \theta^{FB}(\hat{s}))$ and the slope of the crossing point must be the case that $f_2(\theta^{FB}(\hat{s}), \hat{s}) < f(\theta^{FB}(\hat{s}), \hat{s}) = -\infty$. Contradiction.

F) Proof of Proposition 6 (to be added)

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sellers' IC won't hold for any $\theta_H^0 > \theta_H^{FB}(s)$.

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