Pricing Deflation Risk with U.S. Treasury Yields

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Abstract

The deflation protection option embedded in Treasury inflation-protected securities (TIPS) has received limited attention in the literature. However, since the financial crisis of 2008-2009, elevated spreads of seemingly identical TIPS bonds distinguished only by their issuance date suggest that the value of this deflation protection has been sizable. We use an arbitrage-free, Nelson-Siegel term structure model with spanned stochastic volatility to determine the value of this deflation option. The model accurately prices the deflation protection option prior to the financial crisis when its value was near zero and during the peak of the crisis in late 2008 when deflationary concerns spiked sharply. During 2009, the average value of this option was 41 basis points.

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1 Introduction

The U.S. Treasury first issued inflation-indexed bonds, which are now commonly known as Treasury inflation-protected securities (TIPS), in 1997. TIPS bonds provide inflation protection since their coupons and principal payments are indexed to the headline Consumer Price Index (CPI) produced by the Bureau of Labor Statistics.[1] In addition, TIPS bonds provide some protection against price deflation since their principal payments are not permitted to decrease below their original par value.

This deflation protection option has received limited attention in the literature, most likely since it has not been of much value in the U.S. inflationary environment since 1997. However, the sharp drops in price indexes during the financial crisis that started in the fall of 2008 increased deflationary concerns markedly, thus providing further motivation for examining the value of this protection. For example, Christensen et al. (2011a) provide evidence that one-year-ahead deflation probabilities extracted from Treasury yields spiked shortly after the Lehman bankruptcy in September 2008 and remained above 5% until April 2010. Two recent papers have used different arbitrage-free term structure models to examine the values of these deflation protection options. Grishchenko et al. (2010) use a Gaussian affine model whose two factors are nominal Treasury rates and the inflation rate observed at the monthly frequency. They found that the option value is close to zero for most months, except for the deflationary periods observed in 2003-2004 and in 2008-2009. They calculate the maximum observed option value in December 2008 to be roughly 45 cents for every $100 in TIPS bond par value (or 45 basis points).

Christensen et al. (2011a) use a "yields-only" approach based on the arbitrage-free Nelson-Siegel (AFNS) model developed by Christensen et al. (CLR, 2010) to value these deflation protection options. That model uses four factors to capture the joint dynamics of the nominal and real Treasury yield curves. The first three factors can be characterized as the level, slope, and curvature of the nominal yield curve, while the fourth factor can be characterized as the level of the real yield curve. As shown in Figure 1, the authors find that the option value, measured as the spread between two TIPS bonds of similar remaining maturity but of differing vintages, reached a maximum of almost 80 basis points in December 2008 for TIPS bonds maturing in 2013. While the model-implied option value is highly correlated with the observed TIPS spread chosen as a proxy for the deflation option value, the implied values are mainly lower than the observed values. The authors suggest that this shortcoming

[1] The actual indexation has a lag structure since the Bureau of Labor Statistics publishes price index values with a one-month lag; i.e., the index for a given month is released in the middle of the subsequent month. The reference CPI is thus set to be a weighted average of the CPI for the second and third months prior to the month of maturity. See Gürkaynak et al. (2010) for a detailed discussion as well as Campbell et al. (2009) for an overview of inflation-indexed bonds.

could be addressed by incorporating stochastic volatility into the model in the hope of better characterizing the lower tail of their model-implied distribution of inflation outcomes.

An example of such a model is developed by Christensen et al. (2011b), who introduce a class of AFNS models of nominal Treasury yields that incorporate spanned stochastic volatility; that is, the factors driving the model's volatility dynamics are the same factors as those that drive the conditional mean dynamics. This class of models is shown to retain the properties of good in-sample fit and out-of-sample forecasting performance exhibited by the AFNS model with constant volatility, but in addition, the extended model's fitted stochastic volatility series are found to capture well the data's volatility dynamics.

In this paper, we extend the CLR (2010) model of the nominal and real Treasury yield curves to incorporate spanned stochastic volatility. In particular, the volatility dynamics are specified to be driven by the nominal and real level factors in the model. Using the same Treasury yield data, the AFNS-stochastic volatility (sv) model exhibits similar in-sample fit and out-of-sample forecast performance relative to the AFNS-constant volatility (cv) model. In contrast, the two models' transformations of their conditional mean specifications into such objects of interest as five-year inflation expectations and inflation risk premiums exhibit similar dynamics, but different levels. More importantly for valuing the TIPS deflation protection option, the models exhibit important differences related to the transformations of their conditional volatility dynamics into conditional distributions of headline CPI changes.

In particular, the one-year deflation probability forecasts generated by the AFNS-sv model are generally higher than those generated by the AFNS-cv model. As might be expected, the differing deflation probabilities lead to important differences in the model-implied values of the TIPS deflation protection option. As shown in Figure 1, the AFNS-sv model generates a yield spread that more directly captures the observed spread in the last few months of 2008 and into 2009. In fact, while both sets of model-implied spreads have correlations of nearly 0.94 with the observed spread, the AFNS-sv model has a root-mean squared error over 2009 of 9.5 basis points as compared to 28.7 basis points for the AFNS-cv volatility model. These results suggest that the AFNS-sv model is well equipped to measure and price deflation risk within the Treasury market, and thus should be well equipped to price the inflation derivatives increasingly being traded in the United States.[2]

The paper is structured as follows. Section 2 presents the AFNS models with constant volatility as developed by CLR (2010) as well as the specification of the AFNS-sv model used for this study. Section 3 presents the model estimation results and diagnostics. Section 4 presents our proposed methodology for deriving the model-implied value of the deflation protection option as well as the empirical results for the two models. This section also reports

[2]See Christensen and Gillan (2011) for further discussion of U.S. inflation swaps and related liquidity issues.

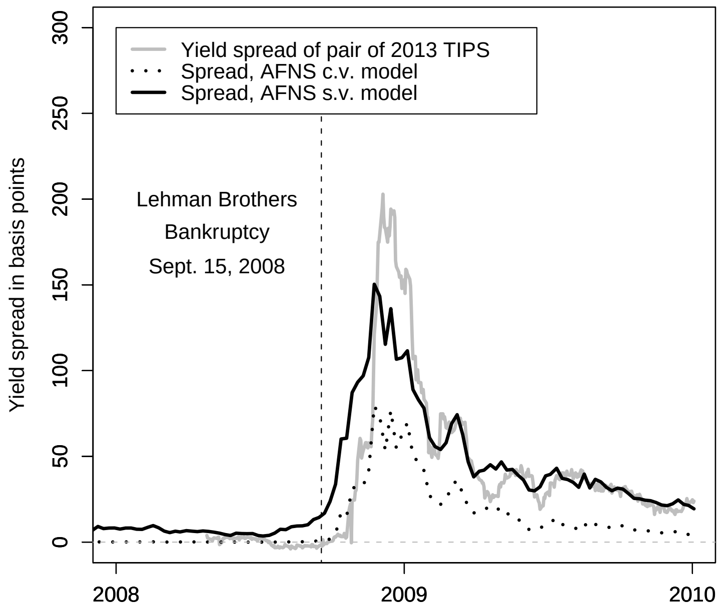

Figure 1: **Yield Spread of pair of 2013 TIPS.**
Illustration of the spread in the yield-to-maturity as reported by Bloomberg between the seasoned ten-year TIPS that matures on July 15, 2013 and the newly issued five-year TIPS that matures on April 15, 2013. Included are the model-implied five-year par-coupon yield spreads from the AFNS-cv model and the AFNS-sv model introduced in this paper.

a set of alternative model estimations presented as robustness checks. Section 5 concludes and presents directions for future research. The appendices contain additional technical details and calculations of interest.

2 AFNS Models of Nominal and Real Treasury Yield Curves

2.1 Deriving Market-Implied Inflation Expectations and Risk Premiums

We start with a review of how an arbitrage-free term structure model can decompose the difference between nominal and real Treasury yields, also known as the breakeven inflation (BEI) rate, into the sum of inflation expectations and an inflation risk premium. Define the nominal and real stochastic discount factors, denoted M_t^N and M_t^R, respectively. The no-arbitrage condition enforces a consistency of pricing for any security over time. Specifically, the price of a nominal bond that pays one dollar at time τ and the price of a real bond that pays one unit of the defined consumption basket at time τ must satisfy the conditions that

$$P_t^N(\tau) = E_t^P\left[\frac{M_{t+\tau}^N}{M_t^N}\right] \quad \text{and} \quad P_t^R(\tau) = E_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right],$$

where $P_t^N(\tau)$ and $P_t^R(\tau)$ are the observed prices of the zero-coupon, nominal and real bonds for maturity τ on day t. and $E_t^P[.]$ is the conditional expectations operator under the real-world (or $P-$) probability measure. The no-arbitrage condition also requires a consistency between the prices of real and nominal bonds such that the price of the consumption basket, denoted as the overall price level Π_t, is the ratio of the nominal and real stochastic discount factors:

$$\Pi_t = \frac{M_t^R}{M_t^N}.$$

We assume that the nominal and real stochastic discount factors have the standard dynamics given by

$$\begin{aligned} dM_t^N/M_t^N &= -r_t^N dt - \Gamma_t' dW_t^P, \\ dM_t^R/M_t^R &= -r_t^R dt - \Gamma_t' dW_t^P. \end{aligned}$$

where r_t^N and r_t^R are the instantaneous, risk-free nominal and real yields, respectively. Γ_t is a standard expression for the error covariance. By Ito's lemma, the dynamic evolution of Π_t is given by

$$d\Pi_t = (r_t^N - r_t^R)\Pi_t dt.$$

Thus, with the absence of arbitrage, the instantaneous growth rate of the price level is equal to the difference between the instantaneous nominal and real risk-free rates. (Note that there is no risk premium for the instantaneous rates, and the Fisher equation applies.) Correspondingly, we can express the price level at time t+τ can be expressed as

$$\Pi_{t+\tau} = \Pi_t e^{\int_t^{t+\tau}(r_s^N - r_s^R)ds}.$$

The relationship between the yields and inflation expectations can be expressed as follows.

The price of the nominal bond can be decomposed as

$$\begin{aligned}
P_t^N(\tau) &= E_t^P\left[\frac{M_{t+\tau}^N}{M_t^N}\right] \\
&= E_t^P\left[\frac{M_{t+\tau}^R/\Pi_{t+\tau}}{M_t^R/\Pi_t}\right] = E_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}\frac{\Pi_t}{\Pi_{t+\tau}}\right] \\
&= E_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E_t^P\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right] + cov_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}}\right] \\
&= P_t^R(\tau) \times E_t^P\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right] \times \left(1 + \frac{cov_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}}\right]}{E_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E_t^P\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right]}\right).
\end{aligned}$$

Converting this price into a yield-to-maturity, we obtain

$$y_t^N(\tau) = y_t^R(\tau) + \pi_t^e(\tau) + \phi_t(\tau), \tag{1}$$

Converting this price into a yield-to-maturity as

$$y^N(\tau) = -\frac{1}{\tau}\ln E_t^P\left[\frac{{}^N_{t+\tau}}{{}^N_t}\right], \tag{2}$$

we obtain

$$y_t^N(\tau) = y_t^R(\tau) + \pi_t^e(\tau) + \phi_t(\tau), \tag{3}$$

where the market-implied rate of inflation expected at time t for the period from t to $t + \tau$ is

$$\pi_t^e(\tau) = -\frac{1}{\tau}\ln E_t^P\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right] = -\frac{1}{\tau}\ln E_t^P\left[e^{-\int_t^{t+\tau}(r_s^N - r_s^R)ds}\right]. \tag{4}$$

The $\phi_t(\tau)$ term is the expression for the inflation risk premium. In this notation, the BEI rate is

$$BEI_t(\tau) \equiv y_t^N(\tau) - y_t^R(\tau) = \pi_t^e(\tau) + \phi_t(\tau). \tag{5}$$

2.2 Affine Term Structure Models

Given this theoretical framework, we briefly summarize the original AFNS model of nominal and real yields with constant volatility developed by CLR (2010) and introduce the extended version with stochastic yield volatility presented in full detail in CLR (2011b).

2.2.1 The AFNS Model with Constant Volatility

The joint four-factor model of nominal and real yields is a direct extension of the three-factor model developed by Christenesen, Diebold and Rudebusch (CDR, 2010) for nominal yields. In the CLR model, the state vector is denoted by $X_t = (L_t^N, S_t, C_t, L_t^R)$, where L_t^N is the level factor for nominal yields, S_t is the common slope factor, C_t is the common curvature factor, and L_t^R is the level factor for real yields. The instantaneous nominal and real risk-free rates are defined as:

$$\begin{aligned} r_t^N &= L_t^N + S_t, \\ r_t^R &= L_t^R + \alpha^R S_t. \end{aligned}$$

Note that the differential scaling of the real rates to the common slope factor is captured by the parameter α^R.

To preserve the Nelson-Siegel factor loading structure in the yield functions, the risk-neutral (or Q-) dynamics of the state variables are given by the stochastic differential equations:

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L^N,Q} \\ dW_t^{S,Q} \\ dW_t^{C,Q} \\ dW_t^{L^R,Q} \end{pmatrix},$$

where Σ is the constant covariance (or volatility) matrix.[3] Based on this specification of the Q-dynamics, nominal Treasury zero-coupon bond yields preserve the Nelson-Siegel factor loading structure as

$$y_t^N(\tau) = L_t^N + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^N(\tau)}{\tau},$$

where $A^N(\tau)$ is a maturity-dependent yield adjustment term. Similarly, real TIPS zero-coupon bond yields have a Nelson-Siegel factor loading structure expressed as

$$y_t^R(\tau) = L_t^R + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^R(\tau)}{\tau}.$$

Note that $A^R(\tau)$ is another maturity-dependent yield adjustment term. These two equations when combined in state-space form constitute the measurement equation needed for Kalman filter estimation.

To complete the model, we define the price of risk, which links the risk-neutral and real-

[3] As per CDR (2010), Σ is a diagonal matrix, and θ^Q is set to zero without loss of generality.

world yield dynamics, using the essentially affine risk premium specification introduced by Duffee (2002). The real-world dynamics of the state variables are then expressed as

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma dW_t^P, \tag{6}$$

which in its most general form can be written as

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & \kappa_{12}^P & \kappa_{13}^P & \kappa_{14}^P \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P \\ \kappa_{41}^P & \kappa_{42}^P & \kappa_{43}^P & \kappa_{44}^P \end{pmatrix} \left(\begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{P,L^N} \\ dW_t^{P,S} \\ dW_t^{P,C} \\ dW_t^{P,L^R} \end{pmatrix}.$$

This is the transition equation used in the Kalman filter estimation.

2.3 The AFNS Model with Stochastic Volatility

Financial time series, such as interest rates and bond yields, have been shown to have time-varying volatility, which is a feature not often incorporated into arbitrage-free term structure models; see Andersen and Benzoni (2006) for further discussion. To address this concern, CLR (2011b) develop a general class of AFNS models that incorporate spanned stochastic volatility according to two criteria:

(i). The time-varying volatility matrix $\Sigma(X_t)$ incorporates at least one stochastic volatility factor.

(ii). The AFNS-stochastic volatility (sv) model preserves as much of the Nelson-Siegel factor structure as possible to preserve the conditional mean properties of the AFNS-constant volatility (cv) model.

To distinguish between the various types of models that fit within the general AFNS-sv class, we use notation outlined in Dai and Singleton (2000), such that the AFNS-cv model is within the $A_0(4)$ class of models which has no volatility dynamics. As detailed in CLR (2011b), there are several possible volatility specifications within their three-factor framework, and clearly, the introduction of the fourth factor within the CLR (2010) generates an even larger set of specifications.

For this paper, we chose an $A_2(4)$ volatility specification that incorporates stochastic volatility based on the nominal and real level factors. This specification choice was motivated by a desire to focus on the longer maturity TIPS yields in our sample. The yields in the five- to ten-year maturity range should provide the greatest opportunity to examine the value of the deflation protection option. In addition, the use of the individual factors for driving their own volatility dynamics greatly simplifies the model estimations and analysis.

For the AFNS-sv model, the state vector and instantaneous risk-free rates are the same as before. To preserve the Nelson-Siegel factor loading structure and impose our volatility specification, the Q-dynamics of the state variables are given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa^Q_{L^N} & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \kappa^Q_{L^R} \end{pmatrix} \left[\begin{pmatrix} \theta^Q_{L^N} \\ 0 \\ 0 \\ \theta^Q_{L^R} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,Q} \\ dW_t^{S,Q} \\ dW_t^{C,Q} \\ dW_t^{L^R,Q} \end{pmatrix},$$

where $\kappa^Q_{L^N}$ and $\kappa^Q_{L^R}$ are constants needed to bound the volatility dynamics above zero.

The representation of the nominal zero-coupon bond yield function becomes

$$y_t^N(\tau) = g^N\left(\kappa^Q_{L^N}\right) L_t^N + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^N\left(\tau, \kappa^Q_{L^N}\right)}{\tau},$$

where $g^N\left(\kappa^Q_{L^N}\right)$ is a modified loading on the nominal level factor. Note that the slope and the curvature factor preserve their Nelson-Siegel factor loadings exactly, although the structure of the yield adjustment term $A^N\left(\tau, \kappa^Q_{L^N}\right)$ is different than before. Correspondingly, the real zero-coupon bond yield function is now

$$y_t^R(\tau) = g^R\left(\kappa^Q_{L^R}\right) L_t^R + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^R(\tau, \kappa^Q_{L^R})}{\tau},$$

where $g^R\left(\kappa^Q_{L^R}\right)$ is a modified loading on the real level factor and $A^R\left(\tau, \kappa^Q_{L^R}\right)$ is a modified yield adjustment term.[4]

To link the risk-neutral and real-world dynamics of the state variables, we use the extended affine risk premium specification introduced by Cheridito et al. (2007), as suggested by CLR

[4]Note that in our estimation, we fix $\kappa^Q_{L^N} = \kappa^Q_{L^R} = 10^{-7}$ without loss of generality.

(2011b). The maximally flexible affine specification of the P-dynamics is thus

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,P} \\ dW_t^{S,P} \\ dW_t^{C,P} \\ dW_t^{L^R,P} \end{pmatrix}.$$

To keep the model arbitrage-free, the two level factors must be prevented from hitting the lower zero-boundary. This positivity requirement is ensured by imposing the Feller conditions under both probability measures, i.e.,

$$\kappa_{11}^P \theta_1^P + \kappa_{14}^P \theta_4^P > \frac{1}{2}\sigma_{11}^2 \quad \text{and} \quad 10^{-7} \cdot \theta_{L^N}^Q > \frac{1}{2}\sigma_{11}^2,$$

and

$$\kappa_{41}^P \theta_1^P + \kappa_{44}^P \theta_4^P > \frac{1}{2}\sigma_{44}^2 \quad \text{and} \quad 10^{-7} \cdot \theta_{L^R}^Q > \frac{1}{2}\sigma_{44}^2.$$

Furthermore, to have well-defined processes for L_t^N and L_t^R, the sign of the effect that these two factors have on each other must be positive, which requires the restrictions that

$$\kappa_{14}^P \leq 0 \quad \text{and} \quad \kappa_{41}^P \leq 0.$$

These conditions ensure that the two square-root processes will be non-negatively correlated. As discussed in the estimation section below, certain of the Feller conditions turn out to be binding. In the robustness section, we present the results for model estimation with a looser set of restrictions, but as will be discussed, this approach generates values for the deflation protection option that do not match the observable data well.

The model estimation technique used in prior AFNS studies was the Kalman filter; that is, nominal and real zero-coupon yields are affine functions of the state variables such that

$$y_t(\tau) = -\frac{1}{\tau}B(\tau)'X_t - \frac{1}{\tau}A(\tau) + \varepsilon_t(\tau),$$

where $\varepsilon_t(\tau)$ are assumed to be i.i.d. Gaussian errors. The conditional mean for multi-dimensional affine diffusion processes is given by

$$E^P[X_T|X_t] = (I - \exp(-K^P(T-t)))\theta^P + \exp(-K^P(T-t))X_t, \quad (7)$$

where $\exp(-K^P(T-t))$ is a matrix exponential. In general, the conditional covariance matrix for affine diffusion processes is given by

$$V^P[X_T|X_t] = \int_t^T \exp(-K^P(T-s))\Sigma D(E^P[X_s|X_t])D(E^P[X_s|X_t])'\Sigma' \exp(-(K^P)'(T-s))ds. \quad (8)$$

Stationarity of the system under the P-measure is ensured if the real components of all the eigenvalues of K^P are positive, and this condition is imposed in all estimations. For this reason, we can start the Kalman filter at the unconditional mean and covariance matrix.[5]

However, the introduction of stochastic volatility implies that the factors are no longer Gaussian as in the AFNS-cv model since their variances are now dependent on the path of the state variables. For tractability, we chose to approximate the true probability distribution of the state variables using the first and second moments described above and use the Kalman filter algorithm *as if* the state variables were Gaussian.[6] The state equation is given by

$$X_t = (I - \exp(-K^P\Delta t))\theta^P + \exp(-K^P\Delta t)X_{t-1} + \eta_t, \quad \eta_t \sim N(0, V_{t-1}),$$

where Δt is the time between observations and V_{t-1} is the conditional covariance matrix given in Equation (8).[7] In the Kalman filter estimations, the error structure is given by

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{t-1} & 0 \\ 0 & H \end{pmatrix}\right],$$

where H is assumed to be a diagonal matrix of the measurement error standard deviations, $\sigma_\varepsilon(\tau_i)$, that are specific to each yield maturity in the data set. Furthermore, the discretization can cause the square-root processes to become negative despite the fact that the parameter sets are forced to satisfy Feller conditions and other non-negativity restrictions. Whenever

[5]In the estimation, we calculate the conditional and unconditional covariance matrices using the analytical solutions provided in Fisher and Gilles (1996), unlike in the earlier studies by CLR (2010, 2011a).

[6]A few notable examples of papers that follow this approach include Duffee (1999), Driessen (2005), and Feldhütter and Lando (2008). An unreported simulation study shows that the added bias from using the Kalman filter in estimating AFNS models with stochastic volatility generated solely through the level factor, which is a comparable case for the approach in this paper, is modest for the parameters in the K^P matrix.

[7]For the first eight years without TIPS yields in the sample, $e^{-K^P\Delta t}$, $(1 - e^{-K^P\Delta t})\theta^P$, and the conditional covariance matrix $\int_t^{t+\tau} e^{-K^P(t+\tau-s)}\Sigma D(E^P[X_s|X_t])D(E^P[X_s|X_t])'\Sigma' e^{-(K^P)'(t+\tau-s)}ds$ are calculated using the upper 3×3 part of K^P and the upper 3×1 part of θ^P.

this happens, we follow the literature and simply truncate those processes at zero; see Duffee (1999) for an example.

2.4 Deflation Probabilities Within the AFNS-sv Model

CLR (2011a) use the AFNS-cv model to generate deflation probabilities at various horizons appropriate for macroeconomic and monetary policy purposes. Similarly, the AFNS-sv model can be used to calculate deflation probabilities, although additional steps are necessary; see CLR (2011b) for a more complete description.

The change in the price index implied by the model's "yields-only" approach for the period from t to $t+\tau$ is given by

$$\frac{\Pi_{t+\tau}}{\Pi_t} = e^{\int_t^{t+\tau}(r_s^N - r_s^R)ds}.$$

To determine whether the change in the price index may be below a critical level q, we are interested in the states of the probability of

$$\frac{\Pi_{t+\tau}}{\Pi_t} \leq 1+q,$$

or, equivalently,

$$Y_{t,\tau} = \int_t^{t+\tau}(r_s^N - r_s^R)ds \leq \ln(1+q).$$

Given that $r_t^N = L_t^N + S_t$ and $r_t^R = L_t^R + \alpha^R S_t$, we are interested in the distributional properties of the process

$$Y_{0,t} = \int_0^t (r_s^N - r_s^R)ds = \int_0^t (L_s^N + S_s - L_s^R - \alpha^R S_s)ds \quad \Rightarrow \quad dY_{0,t} = (L_t^N + (1-\alpha^R)S_t - L_t^R)dt.$$

This process is then introduced into the system of equations containing the P-dynamics of the state variables X_t.

Due to the introduction of stochastic volatility into the two level factors, this is a system of equations no longer has non-Gaussian state variables. As a consequence, we must use the Fourier transform analysis described in full generality for affine models in Duffie, Pan, and Singleton (2000), as opposed to the approach detailed in CLR (2011a) for the AFNS-cv model. The intuition of this approach is to express expectations of contingent payments in a tractable, mathematical form. By simplifying these expectations to indicator variables such as $1_{(Y_{t,\tau} \leq ln(1+q))}$, event probabilities are readily generated.

3 Estimation of the AFNS Models

3.1 Data

In this paper, the nominal Treasury bond yields used are the zero-coupon yields constructed as in Gürkaynak et al. (2007).[8] These yields are constructed by fitting a zero-coupon yield curve of the Svensson (1995)-type to a large pool of underlying off-the-run Treasury bonds. As demonstrated by Gürkaynak et al. (2007), the model fits the underlying pool of bonds extremely well. By implication, the zero-coupon yields derived from this approach constitute a very good approximation to the true underlying Treasury zero-coupon yield curve. From this dataset, we use eight Treasury zero-coupon bond yields with maturities of 3-months, 6-months, 1-year, 2-years, 3-years, 5-years, 7-years, and 10-years. We use weekly Friday data and limit our sample to the period from January 6, 1995 to December 31, 2009, which provides us with 783 weekly observations. Similarly for the real Treasury yields, we use the zero-coupon bond yields constructed with the same method as detailed by Gürkaynak et al. (2010).[9] The data is available from January 1999, but due to weak liquidity in the first years of trading, we follow CLR (2010) and limit our sample to the period since 2003. We have weekly real Treasury yields from January 2, 2003 to December 31, 2009, a total of 366 observations. Since our focus is on the long-term real yields, we use the six yearly maturities from five- to ten-years.

3.2 Estimation Results

To select the best fitting specifications of the AFNS models' real-world dynamics, we use a general-to-specific modeling strategy that restricts the least significant parameter in the estimation to zero and then re-estimates the model. This strategy of eliminating the least significant coefficients is carried out up to the most parsimonious specification, which has a diagonal K^P matrix. The final specification choice is based on the values of the Akaike and Bayes information criteria as per CLR (2010).[10].

For the AFNS-cv model, the summary statistics of the model selection process are reported in Table 1. Both information criteria are minimized by specification (9), which has a K^P

[8] The Board of Governors of the Federal Reserve frequently updates the factors and parameters of this method. The output is publicly available on its website at $www.federalreserve.gov/econresdata/researchdata/feds200628_1.html$.

[9] This dataset is also maintained on the website of the Board of Governors of the Federal Reserve System. The webpage is $www.federalreserve.gov/econresdata/researchdata/feds200805_1.html$.

[10] See Harvey (1989) for further details

Alternative specifications	Goodness-of-fit statistics				
	Max log L	k	p-value	AIC	BIC
(1) Unrestricted K^P	48,263.09	40	n.a.	-96,446.18	-96,259.65
(2) $\kappa^P_{24} = 0$	48,263.09	39	1.0000	-96,448.18	-96,266.32
(3) $\kappa^P_{24} = \kappa^P_{43} = 0$	48,263.08	38	0.8875	-96,450.16	-96,272.96
(4) $\kappa^P_{24} = \kappa^P_{43} = \kappa^P_{32} = 0$	48,262.87	37	0.5169	-96,451.74	-96,279.20
(5) $\kappa^P_{24} = \ldots = \kappa^P_{12} = 0$	48,262.47	36	0.3711	-96,452.94	-96,285.07
(6) $\kappa^P_{24} = \ldots = \kappa^P_{13} = 0$	48,262.44	35	0.8065	-96,454.88	-96,291.67
(7) $\kappa^P_{24} = \ldots = \kappa^P_{31} = 0$	48,262.12	34	0.4237	-96,456.24	-96,297.69
(8) $\kappa^P_{24} = \ldots = \kappa^P_{34} = 0$	48,261.68	33	0.3482	-96,457.36	-96,303.48
(9) $\kappa^P_{24} = \ldots = \kappa^P_{14} = 0$	48,261.32	32	0.3961	**-96,458.64**	**-96,309.42**
(10) $\kappa^P_{24} = \ldots = \kappa^P_{21} = 0$	48,254.02	31	0.0001	-96,446.04	-96,301.48
(11) $\kappa^P_{24} = \ldots = \kappa^P_{42} = 0$	48,248.22	30	0.0007	-96,436.44	-96,296.55
(12) $\kappa^P_{24} = \ldots = \kappa^P_{41} = 0$	48,240.17	29	0.0001	-96,422.34	-96,287.11
(13) $\kappa^P_{24} = \ldots = \kappa^P_{23} = 0$	48,220.82	28	< 0.0001	-96,385.64	-96,255.07

Table 1: **Evaluation of Alternative Specifications of the AFNS-cv Model**. Thirteen alternative estimated specifications of the AFNS-cv model of nominal and real Treasury bond yields are evaluated. Each specification is listed with its maximum log likelihood (Max log L), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

matrix specified as

$$K^P = \begin{pmatrix} \kappa^P_{11} & 0 & 0 & 0 \\ \kappa^P_{21} & \kappa^P_{22} & \kappa^P_{23} & 0 \\ 0 & 0 & \kappa^P_{33} & 0 \\ \kappa^P_{41} & \kappa^P_{42} & 0 & \kappa^P_{44} \end{pmatrix}.$$

Table 2 contains the estimated parameters for this specification. All the off-diagonal elements are highly significant and consistent with the empirical results reported in CLR (2010).[11] In terms of dynamic properties, the nominal level factor is a persistent, slowly varying process not affected by any of the other factors. The common curvature factor is also unaffected by the other factors, but is much less persistent and more volatile. The common slope factor is in between these two extremes as it is less persistent than the nominal level factor and less volatile than the curvature factor. Finally, the real level factor is less persistent likely due to the shorter sample of real yields.

Turning to the chosen specification of the AFNS-sv model, Table 3 contains the summary statistics of the model selection. For reasons of parsimony and in light of the relatively weak

[11] The primary difference with the specification favored by CLR (2010) is that the κ^P_{14} parameter is set to zero. Note that this difference also holds relative to the specification favored in CLR (2011a), which was estimated on daily yield curve data.

K^P	$K^P_{\cdot,1}$	$K^P_{\cdot,2}$	$K^P_{\cdot,3}$	$K^P_{\cdot,4}$	θ^P		Σ
$K^P_{1,\cdot}$	0.3271	0	0	0	0.0636	$\Sigma_{1,1}$	0.0058
	(0.2546)				(0.0048)		(0.0002)
$K^P_{2,\cdot}$	1.3056	0.7437	-0.8656	0	-0.0297	$\Sigma_{2,2}$	0.0086
	(0.4772)	(0.1711)	(0.1172)		(0.0170)		(0.0002)
$K^P_{3,\cdot}$	0	0	0.7057	0	-0.0171	$\Sigma_{3,3}$	0.0299
			(0.3167)		(0.0107)		(0.0006)
$K^P_{4,\cdot}$	-4.0695	-0.6271	0	2.9894	0.0373	$\Sigma_{4,4}$	0.0069
	(0.5750)	(0.2050)		(0.3784)	(0.0051)		(0.0001)

Table 2: **Parameter Estimates for the Preferred AFNS-cv Model.**
The estimated parameters of the K^P matrix, θ^P vector, and diagonal Σ matrix are shown for the specification of the AFNS-cv model preferred according to both AIC and BIC information criteria. The estimated value of λ is 0.5146 (0.0040), while α^R is estimated to be 0.5446 (0.0060). The numbers in parentheses are the estimated parameter standard deviations. The maximum log likelihood value is 48,261.32.

Alternative specifications	Goodness-of-fit statistics				
	Max log L	k	p-value	AIC	BIC
(1) Unrestricted K^P	50,238.90	38	n.a.	-100,401.8	-100,224.6
(2) $\kappa^P_{31} = 0$	50,238.82	37	0.6892	-100,403.6	-100,231.1
(3) $\kappa^P_{31} = \kappa^P_{32} = 0$	50,238.64	36	0.5485	-100,405.3	-100,237.4
(4) $\kappa^P_{31} = \kappa^P_{32} = \kappa^P_{24} = 0$	50,238.60	35	0.7773	-100,407.2	-100,244.0
(5) $\kappa^P_{31} = \ldots = \kappa^P_{34} = 0$	50,238.20	34	0.3711	-100,408.4	-100,249.9
(6) $\kappa^P_{31} = \ldots = \kappa^P_{41} = 0$	50,237.75	33	0.3428	**-100,409.5**	-100,255.6
(7) $\kappa^P_{31} = \ldots = \kappa^P_{21} = 0$	50,236.24	32	0.0812	-100,408.5	**-100,259.2**
(8) $\kappa^P_{31} = \ldots = \kappa^P_{14} = 0$	50,224.58	31	< 0.0001	-100,387.2	-100,242.6
(9) $\kappa^P_{31} = \ldots = \kappa^P_{23} = 0$	50,216.57	30	0.0001	-100,373.1	-100,233.2

Table 3: **Evaluation of Alternative Specifications of the AFNS-sv Model** With Feller Conditions Imposed.
Seven alternative estimated specifications of the *AFNS s.v. model* with Feller conditions imposed are evaluated. Each specification is listed with its log likelihood (Max $\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

significance of the marginal parameter κ^P_{21}, we choose to focus on the specification preferred according to BIC with a mean-reversion matrix given by

$$K^P_{BIC} = \begin{pmatrix} \kappa^P_{11} & 0 & 0 & \kappa^P_{14} \\ 0 & \kappa^P_{22} & \kappa^P_{23} & 0 \\ 0 & 0 & \kappa^P_{33} & 0 \\ 0 & 0 & 0 & \kappa^P_{44} \end{pmatrix}.$$

K^P	$K^P_{\cdot,1}$	$K^P_{\cdot,2}$	$K^P_{\cdot,3}$	$K^P_{\cdot,4}$	θ^P		Σ
$K^P_{1,\cdot}$	2.9705	0	0	-2.1861	0.0451	$\Sigma_{1,1}$	0.0653
	(0.5961)			(0.8249)	(0.0027)		(0.0005)
$K^P_{2,\cdot}$	0	0.6574	-0.7244	0	-0.0088	$\Sigma_{2,2}$	0.0131
		(0.2543)	(0.1959)		(0.0163)		(0.0004)
$K^P_{3,\cdot}$	0	0	0.8698	0	-0.0026	$\Sigma_{3,3}$	0.0304
			(0.3340)		(0.0101)		(0.0007)
$K^P_{4,\cdot}$	0	0	0	0.2428	0.0277	$\Sigma_{4,4}$	0.0597
				(0.1816)	(0.0071)		(0.0009)

Table 4: **Parameter Estimates for the Specification of the AFNS-sv Model with Feller Conditions Imposed Preferred According to BIC.**
The estimated parameters of the K^P-matrix, the θ^P-vector, and the Σ-matrix for the preferred specification of the *AFNS s.v. model* of nominal and real yields with Feller conditions imposed according to BIC. The Q-related parameters are estimated at: $\lambda = 0.6179$ (0.0027), $\alpha^R = 0.3803$ (0.0080), $\theta^Q_{L^N} =$ 33,103 (14.43), and $\theta^Q_{L^R} =$ 17,844 (43.90). The numbers in parentheses are the estimated standard deviations of the parameter estimates. The maximum log likelihood value is 50,236.24.

The estimated parameters for this preferred specification are reported in Table 4.[12] Relative to the results from the AFNS-cv model, κ^P_{21} and κ^P_{41} are not significant, but κ^P_{14} is now highly statistically significant. Its negative sign shows that the nominal and real level factors are positively correlated, which is consistent with the statistical properties of the estimated level factor paths. However, interestingly, it is the real yield level factor that affects the nominal level factor, and not the other way around.

Table 5 contains summary statistics for the fitted errors from both models. For the nominal yields, the AFNS-cv model fits the very short end of the nominal yield curve relatively better than the longer maturities in the one- to ten-year maturity range. In contrast, the AFNS-sv model provides a better in-sample fit in the one- to ten-year maturity range. For the real yields, the AFNS-sv model provides a significant overall improvement in model fit relative to the AFNS-cv model.

3.3 Diagnostics: Inflation Expectations and Risk Premiums

A key property of these joint AFNS models of nominal and real yields is decompose BEI rates into inflation expectations and inflation risk premiums for further analysis. To conduct this analysis, we generate out-of-sample forecasts based on a rolling model estimation procedure. We construct 4-, 26-, and 52-week-ahead forecasts from each model for the eight nominal Treasury yield maturities and six TIPS yield maturities. We use a recursive procedure that

[12] Analysis reveals that the Feller condition pertaining to the real yield level factor, L^R_t, under the Q-measure is systematically binding. Unreported results show that the three other Feller conditions are never binding independent of the specification of the mean-reversion matrix K^P. Thus, it is mainly the Q-dynamics of L^R_t that is impacted by the imposition of the Feller conditions, most notably σ_{44} as is discussed in a later section.

Maturity in months	*AFNS c.v. model*			*AFNS s.v. model* Feller conditions			*AFNS s.v. model* No Feller conditions		
Nom. yields	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$
3	-0.57	9.78	9.76	1.38	19.68	19.69	0.95	19.66	19.66
6	0.00	0.00	0.00	0.12	8.39	8.39	-0.11	8.39	8.39
12	1.89	5.93	5.91	0.00	0.00	0.00	0.00	0.00	0.00
24	2.34	4.06	4.04	0.40	1.56	1.56	0.48	1.57	1.58
36	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00
60	-2.78	3.77	3.79	-0.29	1.28	1.34	-0.36	1.29	1.38
84	0.23	3.33	3.58	0.13	0.33	0.80	0.21	0.49	1.01
120	10.47	12.39	12.66	-1.13	4.17	4.30	-0.87	4.08	4.27
TIPS yields	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$	Mean	RMSE	$\widehat{\sigma}_{\varepsilon}(\tau_i)$
60	-5.02	21.52	21.49	-1.67	14.25	14.27	0.91	10.35	10.34
72	-3.28	12.95	12.92	-0.28	6.12	6.12	0.11	4.25	4.22
84	-1.64	5.96	5.94	0.00	0.00	0.00	-0.09	0.62	0.00
96	0.00	0.00	0.00	-0.69	4.92	4.92	0.08	3.18	3.10
108	1.64	5.19	5.17	-2.21	9.14	9.14	0.43	5.58	5.52
120	3.28	9.78	9.73	-4.44	12.99	13.00	0.81	7.56	7.50
Max log L	48,220.82			50,216.57			50,942.03		

Table 5: **Summary Statistics of the Fitted Errors.**
The mean fitted errors and root mean squared fitted errors (RMSE) for the independent-factors specification of the AFNS-cv model and AFNS-sv models are shown. All numbers are measured in basis points. The nominal yields cover the period from January 6, 1995, to December 31, 2009, while the real TIPS yields cover the period from January 3, 2003, to December 31, 2009.

re-estimates the models every week by adding one observation. That is, for the first set of forecasts, each model is estimated using the sample covering the twelve-year period from January 6, 1995 to January 5, 2007 and used to generate the forecasts of interest. For each subsequent week, a weekly observation is added to the sample and all the models are re-estimated, and another set of forecasts is constructed. The largest estimation sample for the 4-week-ahead forecasts ends on December 4, 2009 (153 forecasts in total). For the 26- and 52-week horizons, the largest samples end on July 2, 2009 and January 2, 2009 (131 and 105 forecasts), respectively.

Figure 2 illustrates the estimated market-implied expected inflation at the five-year horizon as well as the median of the five-year CPI inflation forecast from the Survey of Professional Forecasters (SPF). Both the AFNS-cv and AFNS-sv modelsproduce sharp declines in the expected inflation shortly after the failure of the investment bank Lehman Brothers in September 2008, which seems to be consistent with realized inflation; that is, headline CPI did register negative year-over-year changes during 2009 for the first time since 1955. Since the beginning of 2009, the two models suggest a gradual decline in the five-year expected

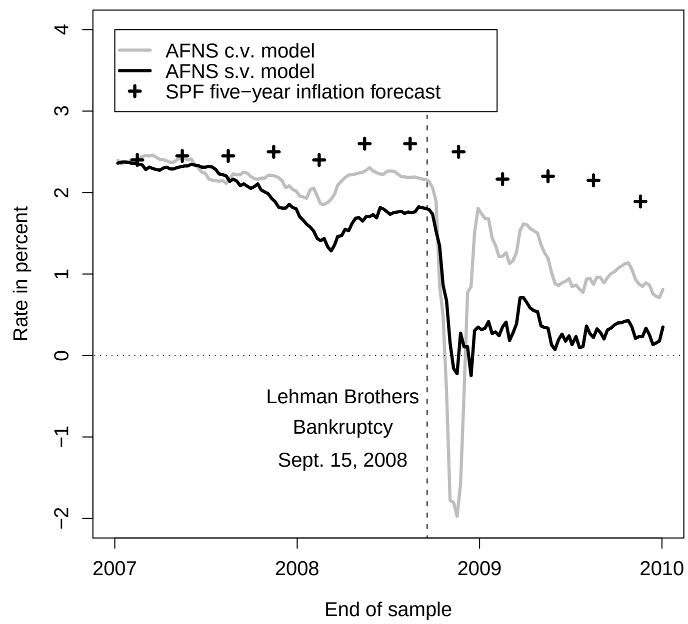

Figure 2: **Estimated Five-Year Inflation Expectations.**
Illustration of the estimated inflation expectations at the five-year horizon according to the AFNS-cv model and the AFNS-sv model. Included is the median five-year forecast of CPI inflation from the Survey of Professional Forecasters.

inflation throughout 2009 consistent with the downward trend in the survey measure. Note, though, that since 2009 there has been a systematic wedge between the estimated inflation expectations from the models that did not exist previously.

Figure 3 illustrates the estimated inflation risk premiums at the five-year horizon from the models. The inflation risk premium turned markedly negative towards the end of 2008 and remained so until mid-2009. This pattern seems to be in line with the models' inflation expectations. When there is a deflation scare as in the fall of 2008, there is little need for a risk premium on inflation; instead, the inflation risk premium became negative and thus a disadvantage since TIPS bonds' coupon payments, unlike their principal, would be adjusted downward with the decline in the general price level. However, by mid-2009, deflationary concerns had decreased in light of both extraordinary monetary policy measures and efforts to stabilize the financial system.

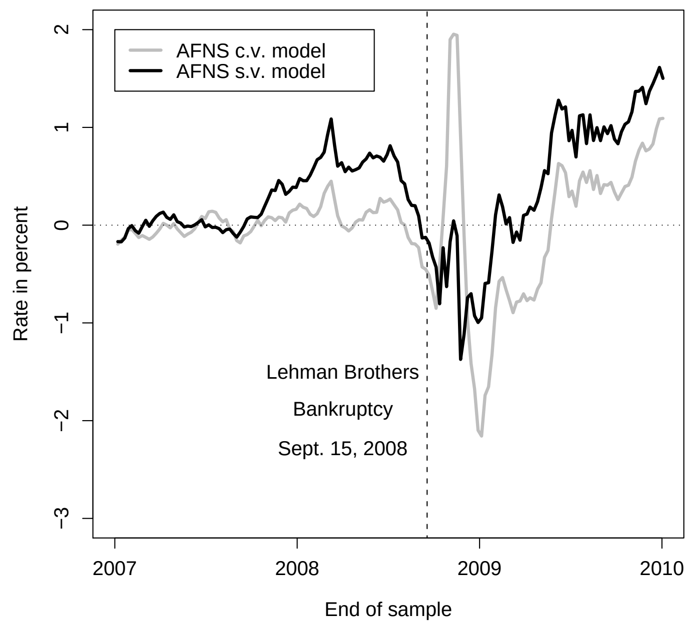

Figure 3: **Estimated Five-Year Inflation Risk Premiums.**
Illustration of the estimated inflation risk premiums at the five-year horizon according to the AFNS-cv model and the AFNS-sv model.

3.4 Diagnostics: Model-implied deflation probabilities

Another relevant comparison measure for these models is their implied probability forecasts of net deflation one year ahead, as presented in CLR (2010) and in Figure 4. The risk of deflation in 2007 and leading up to the failure of Lehman Brothers in September 2008 was basically zero under both models. In late 2009, the models assigned a high probability to net deflation over the following twelve-month period, which is consistent with the observed negative year-over-year change in headline CPI observed during these months. The probabilities from the AFNS-sv model are markedly higher than from the AFNS-cv model starting in the second quarter of 2009 through year-end 2010. These higher and more persistent probabilities are due to both the AFNS-sv model's lower inflation expectations and higher conditional volatility estimates.

4 The Value of the Deflation Protection Embedded in TIPS

The primary focus of this paper is the value of the deflation protection embedded in TIPS bonds and how, during the financial crisis of 2008 and 2009, it affected the relative prices

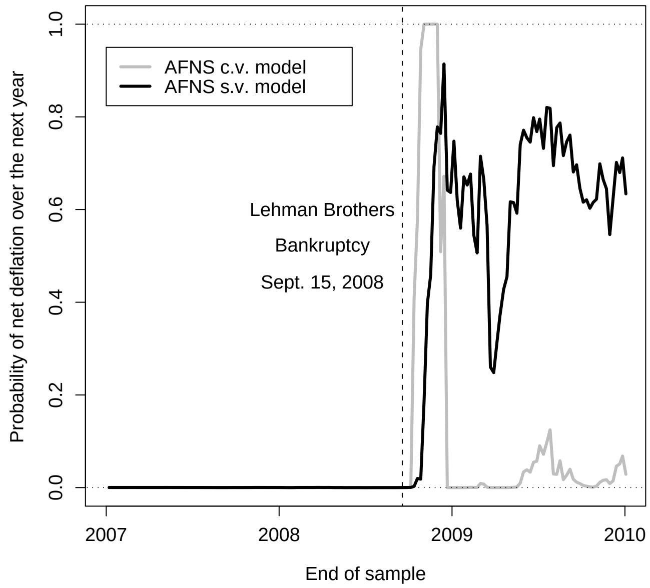

Figure 4: **Estimated One-Year Deflation Probabilities.**
Illustration of the estimated probability of negative net inflation over the following one-year period according to the AFNS-cv model and the AFNS-sv model.

of pairs of TIPS bonds differentiated only by their accrued inflation compensation. Under standard inflationary conditions, the value of the deflation protection should not play an important role in TIPS bond pricing since the probability of having negative net accrued inflation compensation at maturity is negligible; that is, the option was well out-of-the-money.[13] However, at the peak of the financial crisis in the fall of 2008, neither the perceived nor the priced probability of deflation were negligible. In that case, a wedge can develop between the prices of seasoned TIPS bonds with a significant amount of accrued inflation compensation and recently issued on-the-run TIPS bonds, which have no cumulated inflation compensation. As suggested by Wright (2009), this wedge is a proxy for the value of the TIPS deflation protection option.

To examine the ability of the proposed AFNS models to price these deflation options, we use the models' implied yield curves and deflation probabilities. We calculate the deflation options value by comparing under the risk-neutral pricing measure the prices of a newly issued TIPS bond without any accrued inflation compensation and a seasoned TIPS bond

[13]This also explains why little attention has been paid to this aspect of TIPS in the existing literature. To the best of our knowledge, Grishchenko, Vanden, and Zhang (2010) is the only other paper to explicitly address the valuation of the TIPS deflation protection within a dynamic term structure model.

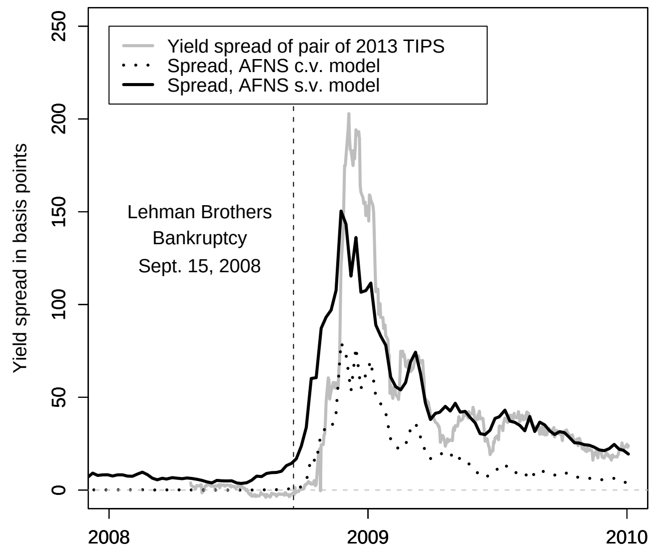

Figure 5: **Five-Year Par-Coupon Yield Spread Between Seasoned and Newly Issued TIPS.**
Illustration of the estimated five-year par-coupon yield spread between a seasoned and a newly issued TIPS according to the AFNS-cv model and the AFNS-sv model. Included is also the corresponding result from the original CLR model as well as the difference in yield-to-maturity as reported by Bloomberg between the ten-year TIPS that matures on July 15, 2013 and the five-year TIPS that matures on April 15, 2013.

with sufficient accrued inflation compensation. First, consider a hypothetical seasoned TIPS bond with T years remaining to maturity that pays an annual coupon C semi-annually. Assume this bond has accrued sufficient inflation compensation so it is impossible to reach the deflation floor before maturity. Under the risk-neutral pricing measure, the par-coupon bond satisfying these criteria has a coupon rate determined by the equation

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q \left[e^{-\int_t^{t_i} r_s^R ds} \right] + E_t^Q \left[e^{-\int_t^T r_s^R ds} \right] = 1. \tag{9}$$

The first term is the sum of the present value of the $2T$ coupon payments using the model's fitted real yield curve at day t. The second term is the discounted value of the principal payment. The coupon payment for this seasonal bond that solves this equation is denoted as C_S.

Next, consider a new TIPS bond with no accrued inflation compensation with T years to maturity. Since the coupon payments are not protected against deflation, the difference is in accounting for the deflation protection on the principal payment:

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q \left[e^{-\int_t^{t_i} r_s^R ds} \right] + E_t^Q \left[\frac{\Pi_T}{\Pi_t} \cdot e^{-\int_t^T r_s^N ds} \mathbf{1}_{\{\frac{\Pi_T}{\Pi_t} > 1\}} \right] + E_t^Q \left[1 \cdot e^{-\int_t^T r_s^N ds} \mathbf{1}_{\{\frac{\Pi_T}{\Pi_t} \leq 1\}} \right] = 1.$$

The first term is the same as before. The second term represents the present value of the principal payment conditional on a positive net change in the price index over the bond's maturity; i.e., $\frac{\Pi_T}{\Pi_t} > 1$. Under this condition, full inflation indexation applies, and the price change $\frac{\Pi_T}{\Pi_t}$ is placed within the expectations operator and weighted by the probability of accumulated inflation at time T. The third term represents the present value of the *floored* TIPS principal conditional on accumulated net deflation; i.e., when the price level change is below one, $\frac{\Pi_T}{\Pi_t}$ is replaced by a value of one to provide the promised deflation protection. Since

$$\frac{\Pi_T}{\Pi_t} = e^{\int_t^T (r_s^N - r_s^R) ds},$$

the equation can be rewritten as

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q [e^{-\int_t^{t_i} r_s^R ds}] + E_t^Q \left[e^{-\int_t^T r_s^R ds} \right] + \left[E_t^Q \left[e^{-\int_t^T r_s^N ds} \mathbf{1}_{\{\frac{\Pi_T}{\Pi_t} \leq 1\}} \right] - E_t^Q \left[e^{-\int_t^T r_s^R ds} \mathbf{1}_{\{\frac{\Pi_T}{\Pi_t} \leq 1\}} \right] \right] = 1,$$

where the last term on the left-hand side represents the net present value of the deflation protection of the principal in the TIPS contract. The par-coupon yield of a new hypothetical TIPS bond that solves this equation is denoted as C_0. The difference between C_S and C_0 is a measure of the advantage of being at the inflation adjustment floor for a newly issued TIPS bond and thus of the value of the embedded deflation protection option.

4.1 Deflation Probabilities and Model-Implied Deflation Option Values

Figure 6 illustrates the model-implied value of the TIPS deflation protection for a newly issued TIPS relative to a comparable seasoned TIPS when converted into five-year par-coupon spreads. In the figure, the model-implied values are compared to the yield difference between the seasoned ten-year 2013 TIPS bonds and the recently issued five-year 2013 TIPS bonds, as shown by the grey line. As observed by CLR (2011a), the AFNS-cv model consistently undervalues the deflation protection even though it captures its time-variation well. The AFNS-sv model is much more successful at matching the observed value of the deflation protection prior to the crisis, at the peak of the crisis, as well as in the post-crisis period. We take this as further evidence that this model more accurately captures bond investors' perceived outlook for future changes in the price level. As such and provided investors are

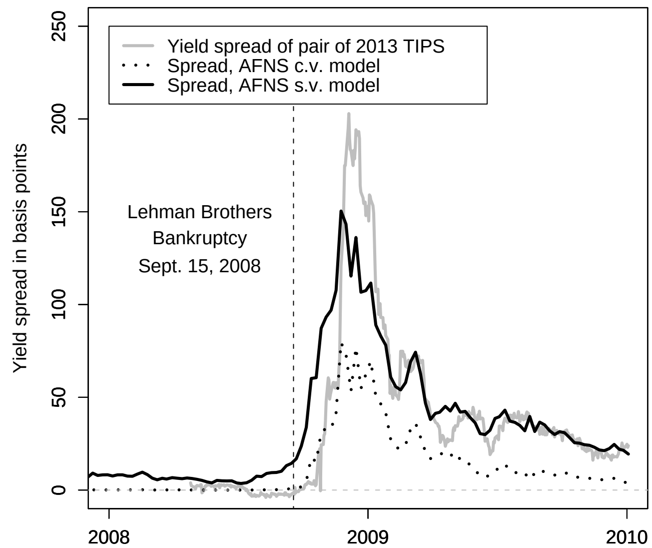

Figure 6: **Five-Year Par-Coupon Yield Spread Between Seasoned and Newly Issued TIPS.**
Illustration of the estimated five-year par-coupon yield spread between a seasoned and a newly issued TIPS according to the AFNS-cv model and the AFNS-sv model. Included is also the corresponding result from the original CLR model as well as the difference in yield-to-maturity as reported by Bloomberg between the ten-year TIPS that matures on July 15, 2013 and the five-year TIPS that matures on April 15, 2013.

rational and forward looking (they have every thinkable monetary incentive to be so) and therefore get things right, at least on average, the model could be capturing fairly accurately the dynamics for the actual inflation process. Based on this reasoning we think this model should carry some weight in judging the risk of deflation going forward. Also, this is the theoretical argument for why a "yields-only" approach like ours would work in the first place.

Figure 7 attempts to disentangle whether it was large probabilities of deflation, large deflation outcomes, or a combination of the two that drove the significant rise in the value of the TIPS deflation protection during the financial crisis. It should be noted that these are probabilities under the pricing Q-measure. The results show that the deflation fear was severe at the peak of the crisis as the priced probability of a 10 percent net decline in the general price level over the following five years was about one-in-three. At the end of our sample, markets still put more than a 10 percent chance on outcomes with 5 percent net deflation—three times its pre-crisis level. Thus, both the probability of deflation as well as its expected magnitude was sizeable throughout 2009.

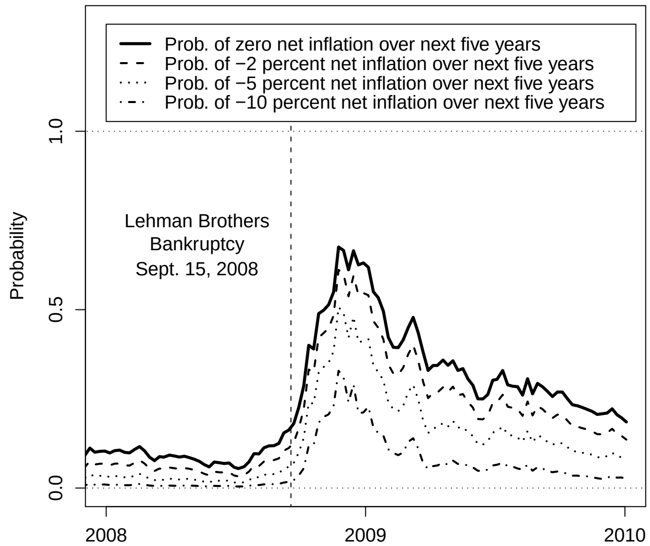

Figure 7: **Priced Probabilities of Negative Inflation Outcomes over the Next Five Years.**
Illustration of the probability of negative net inflation outcomes over the next five years under the Q-measure according to the preferred *AFNS s.v. model* with Feller conditions imposed.

In the following subsection, we present some robustness tests to support our preferred AFNS-sv model.

4.1.1 First Robustness Check: Without Feller conditions

***To be completed.

It turns out that the Feller conditions above are binding restrictions. As a result, we will also analyze the model that uses only the essentially affine risk premiums introduced in Duffee (2002). In that case, the most flexible specification of the P-dynamics we consider is

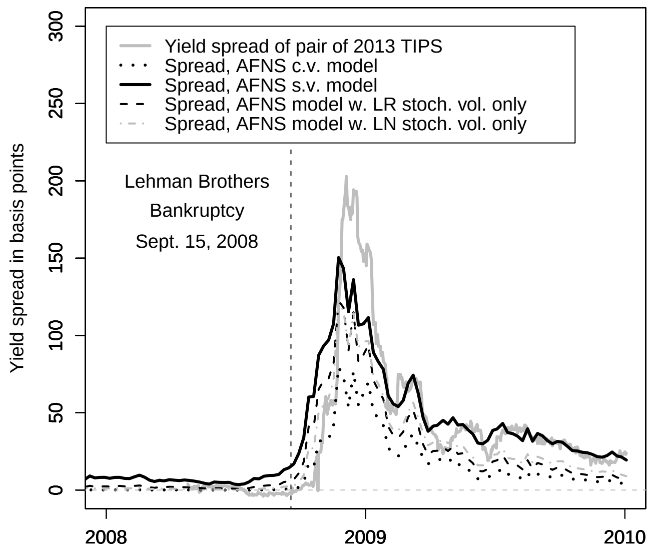

Figure 8: **Five-Year Par-Coupon Yield Spread Between Seasoned and Newly Issued TIPS.**
Illustration of the estimated five-year par-coupon yield spread between a seasoned and a newly issued TIPS according to various modifications of the original CLR model of nominal and real yields. Included is also the yield-to-maturity as reported by Bloomberg between the ten-year TIPS that matures on July 15, 2013 and the five-year TIPS that matures on April 15, 2013.

given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P \\ 0 & 0 & 0 & \kappa_{44}^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,P} \\ dW_t^{S,P} \\ dW_t^{C,P} \\ dW_t^{L^R,P} \end{pmatrix}.$$

The advantage of this risk premium specification is that we only need to require L_t^N and L_t^R not to become negative (as opposed to ensuring strict positivity as required with the extended

Alternative specifications	Goodness-of-fit statistics				
	Max $\log L$	k	p-value	AIC	BIC
(1) Unrestricted K^P	50,959.32	34	n.a.	-101,850.6	-101,692.1
(2) $\kappa^P_{32} = 0$	50,958.99	33	0.4166	-101,852.0	-101,698.1
(3) $\kappa^P_{32} = \kappa^P_{31} = 0$	50,958.59	32	0.3711	**-101,853.2**	-101,704.0
(4) $\kappa^P_{32} = \kappa^P_{31} = \kappa^P_{21} = 0$	50,956.73	31	0.0538	-101,851.5	-101,706.9
(5) $\kappa^P_{32} = \ldots = \kappa^P_{24} = 0$	50,954.42	30	0.0316	-101,848.8	**-101,708.9**
(6) $\kappa^P_{32} = \ldots = \kappa^P_{34} = 0$	50,950.25	29	0.0039	-101,842.5	-101,707.3
(7) $\kappa^P_{32} = \ldots = \kappa^P_{23} = 0$	50,942.03	28	0.0001	-101,828.1	-101,697.5

Table 6: **Evaluation of Alternative Specifications of the *AFNS s.v. Model* Without Feller Conditions Imposed.**
Seven alternative estimated specifications of the *AFNS s.v. model* without Feller conditions imposed are evaluated. Each specification is listed with its maximum log likelihood (Max $\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

affine risk premium specification). This is ensured with the following set of weaker restrictions

$$\kappa^P_{11} > 0, \quad \kappa^P_{11}\theta^P_1 > 0, \quad \kappa^P_{44} > 0, \quad \text{and} \quad \kappa^P_{44}\theta^P_4 > 0.$$

The tradeoff is twofold. First, κ^P_{14} and κ^P_{41} must be fixed at 0 as they must be equal to their counterparts under the Q-measure. Second, for L^N_t and L^R_t, there is a restriction on their mean values given by the equations

$$10^{-7} \cdot \theta^Q_{L^N} = \kappa^P_{11}\theta^P_1 \quad \text{and} \quad 10^{-7} \cdot \theta^Q_{L^R} = \kappa^P_{44}\theta^P_4.$$

By implication, θ^P_1 and $\theta^Q_{L^N}$ (and θ^P_4 and $\theta^Q_{L^R}$) cannot both vary freely when we switch from the Q-measure to the P-measure under the essentially affine risk premium structure. The way we implement these two restrictions is to let $\theta^Q_{L^N}$ and $\theta^Q_{L^R}$ vary, while θ^P_1 and θ^P_4 are determined as residuals

$$\theta^P_1 = \frac{10^{-7} \cdot \theta^Q_{L^N}}{\kappa^P_{11}} \quad \text{and} \quad \theta^P_4 = \frac{10^{-7} \cdot \theta^Q_{L^R}}{\kappa^P_{44}}.$$

In light of the binding Feller condition we also study the performance of the *AFNS s.v. model* without Feller conditions imposed as detailed in the model description. Table 6 reports the results of the model selection within this class of models. Again, for reasons of parsimony, we choose to focus on the specification preferred according to BIC. Its mean-reversion matrix

K^P	$K^P_{\cdot,1}$	$K^P_{\cdot,2}$	$K^P_{\cdot,3}$	$K^P_{\cdot,4}$	θ^P		Σ
$K^P_{1,\cdot}$	0.0714	0	0	0	0.0419	$\Sigma_{1,1}$	0.0579
	(0.1227)						(0.0005)
$K^P_{2,\cdot}$	0	0.6318	-0.6999	0	-0.0040	$\Sigma_{2,2}$	0.0129
		(0.2282)	(0.1700)		(0.0123)		(0.0003)
$K^P_{3,\cdot}$	0	0	2.0478	4.4055	0.0025	$\Sigma_{3,3}$	0.0305
			(0.3703)	(0.7086)	(0.0049)		(0.0007)
$K^P_{4,\cdot}$	0	0	0	0.7539	0.0055	$\Sigma_{4,4}$	0.2203
				(0.2587)			(0.0006)

Table 7: **Parameter Estimates for the Specification of the *AFNS s.v. Model* without Feller Conditions Preferred by BIC.**
The estimated parameters of the K^P-matrix, the θ^P-vector, and the Σ-matrix for the specification of the *AFNS s.v. model* of nominal and real yields without Feller conditions imposed preferred by BIC. The Q-related parameters are estimated at: $\lambda = 0.6156$ (0.0026), $\alpha^R = 0.6702$ (0.0101), $\theta^Q_{L^N} = 29{,}897$ (13.99), and $\theta^Q_{L^R} = 41{,}444$ (14.73). Since $\kappa^Q_{L^N} = \kappa^Q_{L^R} = 10^{-7}$, it follows that $\theta^P_1 = \frac{\kappa^Q_{L^N}\theta^Q_{L^N}}{\kappa^P_{11}} = 0.0419$ and $\theta^P_4 = \frac{\kappa^Q_{L^R}\theta^Q_{L^R}}{\kappa^P_{44}} = 0.0055$. The numbers in parentheses are the estimated standard deviations of the parameter estimates. The maximum log likelihood value is 50,954.42.

is given by

$$K^P_{BIC} = \begin{pmatrix} \kappa^P_{11} & 0 & 0 & 0 \\ 0 & \kappa^P_{22} & \kappa^P_{23} & 0 \\ 0 & 0 & \kappa^P_{33} & \kappa^P_{34} \\ 0 & 0 & 0 & \kappa^P_{44} \end{pmatrix}.$$

Across model classes, the only parameter to systematically survive elimination is κ^P_{23}. As this is the key off-diagonal element in K^Q that generates the hump-shaped factor loading for the curvature, this result might not be all that surprising. It implies that significant restrictions on the risk premium structure are required to be consistent with the data.

Table 7 reports the estimated parameters of the preferred specification (5). Note that the off-diagonal elements in the estimated K^P matrix are statistically significant. Furthermore, the two level factors and the slope factor remain persistent, while the curvature factor exhibits a much more rapid rate of mean reversion and remains more volatile than the slope factor. Finally, the estimated mean parameters accord quite well with the estimated paths of the four factors.

Figure 9 shows the estimated values of σ_{44} from the *AFNS s.v. model* with and without Feller conditions imposed. We note that its estimated value barely change when the Feller conditions are imposed because the restriction for L^R_t under the Q-measure is binding which forces it to take on a low value determined by other model parameters that only vary little, hence the very stable pattern. Once that restriction is relaxed, σ_{44} is allowed to vary. In

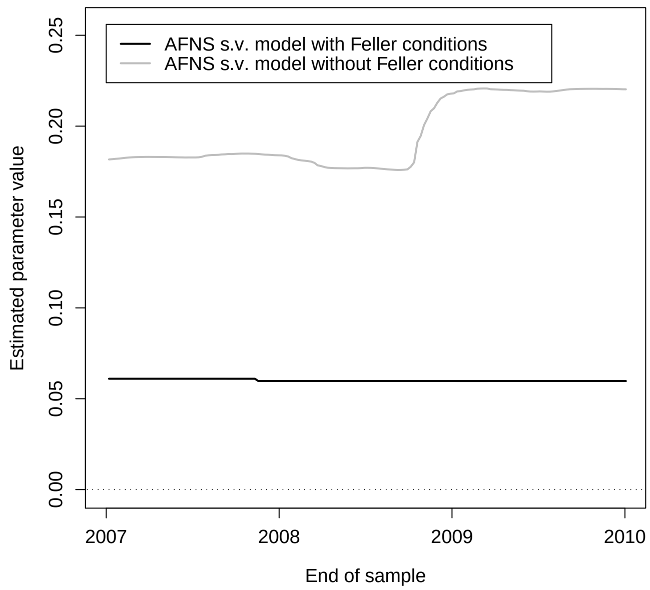

Figure 9: **Estimated Values of σ_{44}.**
Illustration of the estimated values of σ_{44} from the rolling re-estimation of the *AFNS s.v. models* with and without Feller conditions imposed.

particular, it spikes up in the fall of 2008 when general market volatility rose significantly. However, its estimated value remains high until the end of our sample. Also, we should note that there are only smaller differences in the other volatility parameters across the two models. Thus, it is mainly the Q-dynamics of L_t^R that is impacted by the imposition of the Feller conditions, most notably σ_{44}. Furthermore, this marked difference in the estimated volatility of the real level factor achieved via the imposition of the Feller conditions turns out to affect significantly the model-implied values of the TIPS deflation protection as we will see below.

4.1.2 Second Robustness Check

In the first robustness check we ask whether we need stochastic volatility in both level factors, L_t^N and L_t^R, to obtain these results or whether it would suffice with stochastic volatility in just one of them. To address that question, we analyze the two model classes where either only the nominal level factor, L_t^N, or the real level factor, L_t^R, are allowed to generate stochastic volatility.[14]

[14]The model derivations and the estimation results from these alternative model classes are available upon request.

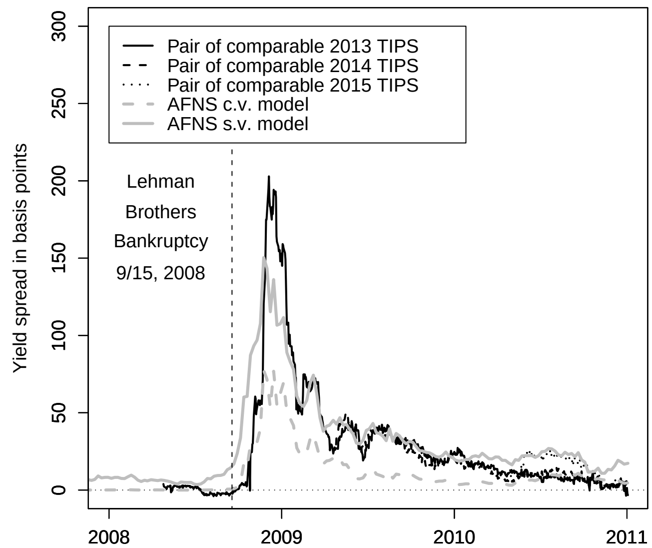

Figure 10: **Five-Year Par-Coupon Yield Spread Between Seasoned and Newly Issued TIPS.**
Illustration of the estimated five-year par-coupon yield spread between a seasoned and a newly issued TIPS according to various modifications of the original CLR model of nominal and real yields. Included is also the yield-to-maturity as reported by Bloomberg between the ten-year TIPS that matures on July 15, 2013 and the five-year TIPS that matures on April 15, 2013.

Figure 8 shows the result from these alternative model specifications and compares them to the corresponding results from the *AFNS c.v. model* and the preferred *AFNS s.v. model*. We note that both of the alternative model classes improve upon the performance of the *AFNS c.v. model*, but they still undervalue the deflation protection. Based on this evidence we conclude that it is not sufficient just to allow either the nominal or the real level factor to generate stochastic volatility in order to price the value of the TIPS deflation protection appropriately. We need both level factors to generate sufficient stochastic volatility to match the yield spreads between seasoned and newly issued TIPS observed during 2008 and 2009.

4.1.3 Third Robustness Check

***To be completed.

In our third robustness check, we update the data through the end of 2010. The model-implied values of the TIPS deflation protection measured as par-coupon yield spreads of seasoned TIPS over comparable newly issued TIPS are shown in Figure 10. Also shown are

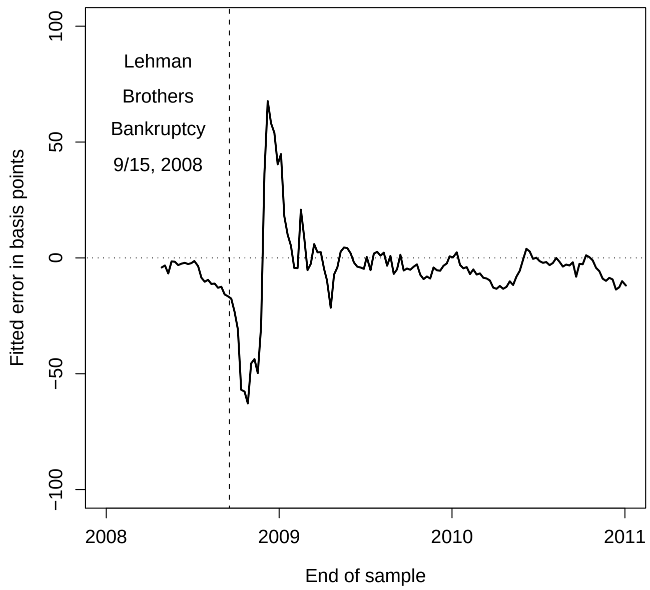

Figure 11: **Fitted Error of the Model-implied Five-year Yield Spread of a Seasoned TIPS over a Newly Issued TIPS.**
Illustration of the difference between the yield spread of the pair of TIPS that contains the on-the-run five-year TIPS and the model-implied five-year par-coupon yield spread of a seasoned TIPS over a comparable newly issued TIPS from the *AFNS s.v. model*.

the yield differences between seasoned and recently issued TIPS with maturities in 2013, 2014, and 2015, respectively.

First, the *AFNS c.v. model* is represented by the dashed grey line. Its spread continues to be below the observed spreads except at the very end of 2010. On the other hand, the preferred *AFNS s.v. model*, whose estimated par-coupon yield spread is shown with a solid grey line in Figure 10, continues to produce yield spreads very to close that of the most recently issued five-year TIPS.

To provide some statistics on the model fit, we calculate the fitted error relative to the TIPS pair containing the most recently issued five-year TIPS, we refer to this as the *on-the-run* pair and it is the closest observable to our model-implied constant-maturity yield spread.[15] Figure 11 shows the difference between the on-the-run TIPS pair and the model-implied five-year par-coupon yield spread equivalent. With the exception of a few weeks at the very peak

[15]Specifically, from April 23, 2008 to April 22, 2009 we use the 5-year TIPS with maturity in April 2013 and the 10-year TIPS with maturity in July 2013. From April 23, 2009 to April 23, 2010 we use the 5-year TIPS with maturity in April 2014 and the 10-year TIPS with maturity in July 2014. Since April 26, 2010 we use the 5-year TIPS with maturity in April 2015 and the 10-year TIPS with maturity in July 2015.

of the crisis, where liquidity effects arguably *did* play a role, the model has matched the value of the TIPS deflation protection 1) prior to the crisis, 2) during the crisis, and 3) after the crisis. The mean error is a mere -4.4 basis points, while the root mean squared errors are just 17.3 basis points.

This exercise serves two purposes. First, it highlights that the value of the TIPS deflation protection manifests itself in consecutive pairs of seasoned versus newly issued TIPS. As such, it is really in the data and does not reflect some idiosyncratic abnormal trading pattern in individual TIPS. Second, it underscores that the favored *AFNS s.v. model* is able to systematically capture bond investors' perceived risk of deflation, in particular as these TIPS yield spreads are *not* in the data used in the model estimation. The model's accuracy is entirely a byproduct of the estimated model dynamics.

We interpret these findings as evidence that the refined model is capturing the outlook for the expected inflation and the tail risk of deflation priced into the indexed and non-indexed Treasury yield curves accurately, and better than the original model. As a result, the output from the refined model should carry some weight in judging the risk of deflation going forward, a task that we now turn to.

5 Conclusion

In this paper, we introduce an extension of the joint model of nominal and real bond yields introduced in CLR (2010). In particular, we replace the model's constant volatility assumption with stochastic volatility driven by the model's nominal and real level factors. In addition, we use the dramatic changes in bond yields during the financial crisis to distinguish between models. In our view, a good model should perform well i) prior to the crisis, ii) during the crisis, and iii) during the post-crisis normalization. Our preferred AFNS-sv model works well on all three accounts, in particular it delivers very reasonable decompositions of breakeven inflation into expected inflation and inflation risk premiums. Furthermore, it is able to price surprisingly accurately the value of TIPS deflation protection as observed in the TIPS market, especially relative to the AFNS-cv model as per CLR (2011a). Based on this evidence we argue that the refined model should be useful for judging the tail risk of deflation going forward as well as for the study of bond investors' inflation expectations.

In addition, the refined model provides an example where the Feller conditions are binding, but lead to improved model performance along important dimensions as compared to models where these restrictions are relaxed. This suggests that Feller conditions can be important and useful even though they are rejected by standard in-sample statistical tests. This observation also highlights how treacherous model selection can be in to multi-dimensional dynamic term structure models as the competing model without Feller conditions performed very well along several measures normally relied upon for model selection.

Finally, and importantly, the Gaussian AFNS-cv model class is not "wrong". It is good at capturing first moments. Thus, it works well for estimating the market-implied expected inflation and the inflation risk premiums. However, it is, by its nature, unable to capture changes in the dynamics of second or higher order moments. This prevents it from measuring accurately tail risks such as the risk of deflation or the price of the deflation protection embedded in TIPS, which are important limitations in low-inflation environments like the one currently experienced in the U.S.

Appendix

A). Bold price formulas

In the *AFNS s.v. model* nominal zero-coupon bond prices are given by

$$P^N(t,T) = E_t^Q\big[\exp\big(-\int_t^T r_u^N du\big)\big] = \exp\big(B_1^N(t,T)L_t^N + B_2^N(t,T)S_t + B_3^N(t,T)C_t + B_4^N(t,T)L_t^R + A^N(t,T)\big),$$

where $B_1^N(t,T)$, $B_2^N(t,T)$, $B_3^N(t,T)$, and $B_4^N(t,T)$ are the unique solutions to the following system of ODEs

$$\begin{pmatrix} \frac{dB_1^N(t,T)}{dt} \\ \frac{dB_2^N(t,T)}{dt} \\ \frac{dB_3^N(t,T)}{dt} \\ \frac{dB_4^N(t,T)}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q \end{pmatrix} \begin{pmatrix} B_1^N(t,T) \\ B_2^N(t,T) \\ B_3^N(t,T) \\ B_4^N(t,T) \end{pmatrix}$$

$$-\frac{1}{2}\sum_{j=1}^{4}\left[\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} (B_1^N)^2 & B_1^N B_2^N & B_1^N B_3^N & B_1^N B_4^N \\ B_1^N B_2^N & (B_2^N)^2 & B_2^N B_3^N & B_2^N B_4^N \\ B_1^N B_3^N & B_2^N B_3^N & (B_3^N)^2 & B_3^N B_4^N \\ B_1^N B_4^N & B_2^N B_4^N & B_3^N B_4^N & (B_4^N)^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix}\right]_{j,j} (\delta^j)'.$$

where γ and δ (see Equations (**??**) and (**??**)) are given by

$$\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This structure implies that the factor loadings in the nominal zero-coupon bond price function are given by the unique solution to the following set of ODEs

$$\begin{aligned} \frac{dB_1^N(t,T)}{dt} &= 1 + \kappa_{L^N}^Q B_1^N(t,T) - \frac{1}{2}\sigma_{11}^2 B_1^N(t,T)^2, \quad B_1^N(T,T) = \overline{B}_1^N, \\ \frac{dB_2^N(t,T)}{dt} &= 1 + \lambda B_2^N(t,T), \quad B_2^N(T,T) = \overline{B}_2^N, \\ \frac{dB_3^N(t,T)}{dt} &= -\lambda B_2^N(t,T) + \lambda B_3^N(t,T), \quad B_3^N(T,T) = \overline{B}_3^N, \\ \frac{dB_4^N(t,T)}{dt} &= \kappa_{L^R}^Q B_4^N(t,T) - \frac{1}{2}\sigma_{44}^2 B_4^N(t,T)^2, \quad B_4^N(T,T) = \overline{B}_4^N. \end{aligned}$$

These four ODEs have the following unique solution[16]

$$
\begin{aligned}
B_1^N(t,T) &= \frac{-2[e^{\phi^N(T-t)}-1]+\overline{B}_1^N e^{\phi^N(T-t)}(\phi^N-\kappa_{L^N}^Q)+\overline{B}_1^N(\phi^N+\kappa_{L^N}^Q)}{2\phi^N+(\phi^N+\kappa_{L^N}^Q-\overline{B}_1^N\sigma_{11}^2)[e^{\phi^N(T-t)}-1]},\\
B_2^N(t,T) &= e^{-\lambda(T-t)}\overline{B}_2^N-\frac{1-e^{-\lambda(T-t)}}{\lambda},\\
B_3^N(t,T) &= \lambda(T-t)e^{-\lambda(T-t)}\overline{B}_2^N+\overline{B}_3^N e^{-\lambda(T-t)}+\Big[(T-t)e^{-\lambda(T-t)}-\frac{1-e^{-\lambda(T-t)}}{\lambda}\Big],\\
B_4^N(t,T) &= \frac{2\kappa_{L^R}^Q\overline{B}_4^N}{(2\kappa_{L^R}^Q-\overline{B}_4^N\sigma_{44}^2)e^{\kappa_{L^R}^Q(T-t)}+\overline{B}_4^N\sigma_{44}^2},
\end{aligned}
$$

where

$$\phi^N=\sqrt{(\kappa_{L^N}^Q)^2+2\sigma_{11}^2}.$$

Now, the $A^N(t,T)$-function in the yield-adjustment term in the nominal zero-coupon bond yield function is given by the solution to the following ODE

$$\frac{dA^N(t,T)}{dt} = -B^N(t,T)'K^Q\theta^Q-\frac{1}{2}\sigma_2^2B_2^N(t,T)^2-\frac{1}{2}\sigma_3^2B_3^N(t,T)^2,\quad A^N(T,T)=\overline{A}^N.$$

This ODE has the following unique solution

$$
\begin{aligned}
A^N(t,T) = \;& \overline{A}^N+\frac{2\kappa_{L^N}^Q\theta_{L^N}^Q}{\sigma_{11}^2}\ln\Bigg[\frac{2\phi^N e^{\frac{1}{2}(\phi^N+\kappa_{L^N}^Q)(T-t)}}{2\phi^N+(\phi^N+\kappa_{L^N}^Q-\overline{B}_1^N\sigma_{11}^2)(e^{\phi^N(T-t)}-1)}\Bigg]\\
&+\sigma_{22}^2\Big[\frac{1}{2\lambda^2}(T-t)-\frac{(1+\lambda\overline{B}_2^N)}{\lambda^3}[1-e^{-\lambda(T-t)}]+\frac{(1+\lambda\overline{B}_2^N)^2}{4\lambda^3}[1-e^{-2\lambda(T-t)}]\Big]\\
&+\sigma_{33}^2\Big[\frac{1}{2\lambda^2}(T-t)+\frac{1+\lambda\overline{B}_2^N}{\lambda^2}(T-t)e^{-\lambda(T-t)}-\frac{(1+\lambda\overline{B}_2^N)^2}{4\lambda}(T-t)^2e^{-2\lambda(T-t)}\\
&\quad-\frac{(1+\lambda\overline{B}_2^N)(3+\lambda\overline{B}_2^N+2\lambda\overline{B}_3^N)}{4\lambda^2}(T-t)e^{-2\lambda(T-t)}\\
&\quad+\frac{(2+\lambda\overline{B}_2^N+\lambda\overline{B}_3^N)^2+(1+\lambda\overline{B}_3^N)^2}{8\lambda^3}[1-e^{-2\lambda(T-t)}]-\frac{2+\lambda\overline{B}_2^N+\lambda\overline{B}_3^N}{\lambda^3}[1-e^{-\lambda(T-t)}]\Big]\\
&+\frac{2\kappa_{L^R}^Q\theta_{L^R}^Q}{\sigma_{44}^2}\ln\Bigg[\frac{2\kappa_{L^R}^Qe^{\kappa_{L^R}^Q(T-t)}}{(2\kappa_{L^R}^Q-\overline{B}_4^N\sigma_{44}^2)e^{\kappa_{L^R}^Q(T-t)}+\overline{B}_4^N\sigma_{44}^2}\Bigg].
\end{aligned}
$$

In the *AFNS s.v. model* the real zero-coupon bond prices are given by

$$P^R(t,T)=E_t^Q\big[\exp\big(-\int_t^T r_u^R du\big)\big]=\exp\big(B_1^R(t,T)L_t^N+B_2^R(t,T)S_t+B_3^R(t,T)C_t+B_4^R(t,T)L_t^R+A^R(t,T)\big),$$

[16]The calculations leading to this result are available upon request.

where $B_1^R(t,T)$, $B_2^R(t,T)$, $B_3^R(t,T)$, and $B_4^R(t,T)$ are the unique solutions to the following system of ODEs

$$\begin{pmatrix} \frac{dB_1^R(t,T)}{dt} \\ \frac{dB_2^R(t,T)}{dt} \\ \frac{dB_3^R(t,T)}{dt} \\ \frac{dB_4^R(t,T)}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^R \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q \end{pmatrix} \begin{pmatrix} B_1^R(t,T) \\ B_2^R(t,T) \\ B_3^R(t,T) \\ B_4^R(t,T) \end{pmatrix}$$

$$-\frac{1}{2}\sum_{j=1}^4 \left[\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} (B_1^R)^2 & B_1^R B_2^R & B_1^R B_3^R & B_1^R B_4^R \\ B_1^R B_2^R & (B_2^R)^2 & B_2^R B_3^R & B_2^R B_4^R \\ B_1^R B_3^R & B_2^R B_3^R & (B_3^R)^2 & B_3^R B_4^R \\ B_1^R B_4^R & B_2^R B_4^R & B_3^R B_4^R & (B_4^R)^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \right]_{j,j} (\delta^j)'.$$

This implies that the factor loadings in the real zero-coupon bond price function are given by the unique solution to the following set of ODEs

$$\begin{aligned}
\frac{dB_1^R(t,T)}{dt} &= \kappa_{L^N}^Q B_1^R(t,T) - \frac{1}{2}\sigma_{11}^2 B_1^R(t,T)^2, \quad B_1^R(T,T) = \overline{B}_1^R, \\
\frac{dB_2^R(t,T)}{dt} &= \alpha^R + \lambda B_2^R(t,T), \quad B_2^R(T,T) = \overline{B}_2^R, \\
\frac{dB_3^R(t,T)}{dt} &= -\lambda B_2^R(t,T) + \lambda B_3^R(t,T), \quad B_3^R(T,T) = \overline{B}_3^R, \\
\frac{dB_4^R(t,T)}{dt} &= 1 + \kappa_{L^R}^Q B_4^R(t,T) - \frac{1}{2}\sigma_{44}^2 B_4^R(t,T)^2, \quad B_4^R(T,T) = \overline{B}_4^R.
\end{aligned}$$

These four ODEs have the following unique solution[17]

$$\begin{aligned}
B_1^R(t,T) &= \frac{2\kappa_{L^N}^Q \overline{B}_1^R}{(2\kappa_{L^N}^Q - \overline{B}_1^R \sigma_{11}^2) e^{\kappa_{L^N}^Q (T-t)} + \overline{B}_1^R \sigma_{11}^2}, \\
B_2^R(t,T) &= e^{-\lambda(T-t)} \overline{B}_2^R - \alpha^R \frac{1 - e^{-\lambda(T-t)}}{\lambda}, \\
B_3^R(t,T) &= \lambda(T-t) e^{-\lambda(T-t)} \overline{B}_2^R + \overline{B}_3^R e^{-\lambda(T-t)} + \alpha^R \Big[(T-t) e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda} \Big], \\
B_4^R(t,T) &= \frac{-2[e^{\phi^R (T-t)} - 1] + \overline{B}_4^R e^{\phi^R (T-t)} (\phi^R - \kappa_{L^R}^Q) + \overline{B}_4^R (\phi^R + \kappa_{L^R}^Q)}{2\phi^R + (\phi^R + \kappa_{L^R}^Q - \overline{B}_4^R \sigma_{44}^2)[e^{\phi^R (T-t)} - 1]},
\end{aligned}$$

where

$$\phi^R = \sqrt{(\kappa_{L^R}^Q)^2 + 2\sigma_{44}^2}.$$

The $A^R(t,T)$-function in the yield-adjustment term in the real zero-coupon bond yield func-

[17]The calculations leading to this result are available upon request.

tion is given by the solution to the following ODE

$$\frac{dA^R(t,T)}{dt} = -B^R(t,T)'K^Q\theta^Q - \frac{1}{2}\sigma_2^2 B_2^R(t,T)^2 - \frac{1}{2}\sigma_3^2 B_3^R(t,T)^2, \quad A^R(T,T) = \overline{A}^R,$$

which is

$$\begin{aligned}
A^R(t,T) &= \overline{A}^R + \frac{2\kappa_{L^N}^Q\theta_{L^N}^Q}{\sigma_{11}^2}\ln\left[\frac{2\kappa_{L^N}^Q e^{\kappa_{L^N}^Q(T-t)}}{(2\kappa_{L^N}^Q - \overline{B}_1^R\sigma_{11}^2)e^{\kappa_{L^N}^Q(T-t)} + \overline{B}_1^R\sigma_{11}^2}\right] \\
&\quad +\sigma_{22}^2\Big[\frac{(\alpha^R)^2}{2\lambda^2}(T-t) - \alpha^R\frac{(\alpha^R+\lambda\overline{B}_2^R)}{\lambda^3}[1-e^{-\lambda(T-t)}] + \frac{(\alpha^R+\lambda\overline{B}_2^R)^2}{4\lambda^3}[1-e^{-2\lambda(T-t)}]\Big] \\
&\quad +\sigma_{33}^2\Bigg[\frac{(\alpha^R)^2}{2\lambda^2}(T-t) + \alpha^R\frac{\alpha^R+\lambda\overline{B}_2^R}{\lambda^2}(T-t)e^{-\lambda(T-t)} - \frac{(\alpha^R+\lambda\overline{B}_2^R)^2}{4\lambda}(T-t)^2e^{-2\lambda(T-t)} \\
&\qquad -\frac{(\alpha^R+\lambda\overline{B}_2^R)(3\alpha^R+\lambda\overline{B}_2^R+2\lambda\overline{B}_3^R)}{4\lambda^2}(T-t)e^{-2\lambda(T-t)} \\
&\qquad +\frac{(2\alpha^R+\lambda\overline{B}_2^R+\lambda\overline{B}_3^R)^2+(\alpha^R+\lambda\overline{B}_3^R)^2}{8\lambda^3}[1-e^{-2\lambda(T-t)}] \\
&\qquad -\alpha^R\frac{2\alpha^R+\lambda\overline{B}_2^R+\lambda\overline{B}_3^R}{\lambda^3}[1-e^{-\lambda(T-t)}]\Bigg] \\
&\quad +\frac{2\kappa_{L^R}^Q\theta_{L^R}^Q}{\sigma_{44}^2}\ln\left[\frac{2\phi^R e^{\frac{1}{2}(\phi^R+\kappa_{L^R}^Q)(T-t)}}{2\phi^R + (\phi^R+\kappa_{L^R}^Q - \overline{B}_4^R\sigma_{44}^2)(e^{\phi^R(T-t)}-1)}\right].
\end{aligned}$$

B). Calculation of the NPV of the TIPS principal deflation protection

In general, we are interested in finding the NPV of terminal payoffs from TIPS contingent on the cumulated inflation being below some critical value q, specifically the following difference is of interest

$$E_t^Q\Big[e^{-\int_t^T r_s^N ds}\mathbf{1}_{\{\frac{\Pi_T}{\Pi_t}\le 1+q\}}\Big] - E_t^Q\Big[e^{-\int_t^T r_s^R ds}\mathbf{1}_{\{\frac{\Pi_T}{\Pi_t}\le 1+q\}}\Big].$$

Thus, the states of the world of interest are characterized by

$$\frac{\Pi_T}{\Pi_t} \le 1+q \iff Y_{t,T} = \int_t^T (r_s^N - r_s^R)ds \le \ln(1+q).$$

Since we are pricing, we need the dynamics of the state variables under the Q-measure

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ Y_{0,t} \end{pmatrix} = \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q & 0 \\ -1 & -(1-\alpha^R) & 0 & 1 & 0 \end{pmatrix} \left[\begin{pmatrix} \theta_{L^N}^Q \\ 0 \\ 0 \\ \theta_{L^R}^Q \\ 0 \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ Y_{0,t} \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,Q} \\ dW_t^{S,Q} \\ dW_t^{C,Q} \\ dW_t^{L^R,Q} \\ dW_t^{Y,Q} \end{pmatrix},$$

where $Z_{0,t} = (L_t^N, S_t, C_t, L_t^R, Y_{0,t})$ represents the augmented state vector.

Now, define the following two intermediate functions

$$\psi^1(\overline{B}, t, T) = E_t^Q\left[e^{-\int_t^T r_s^R ds} e^{\overline{B}' Z_{t,T}}\right],$$

and

$$\psi^2(\overline{B}, t, T) = E_t^Q\left[e^{-\int_t^T r_s^N ds} e^{\overline{B}' Z_{t,T}}\right].$$

In order to calculate $\psi^1(\overline{B}, t, T)$ and $\psi^2(\overline{B}, t, T)$, we summarize the Q-dynamics by the following matrices and vectors

$$K^Q = \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q & 0 \\ -1 & -(1-\alpha^R) & 0 & 1 & 0 \end{pmatrix}, \quad \theta^Q = \begin{pmatrix} \theta_{L^N}^Q \\ 0 \\ 0 \\ \theta_{L^R}^Q \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho^N = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \rho^R = \begin{pmatrix} 0 \\ \alpha^R \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This structure implies that γ and δ in the system of ODEs provided in Equations (**??**) and

(**??**) are given by

$$\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Duffie, Pan, and Singleton (2000) it follows that

$$\psi^1(\overline{B}, t, T) = \exp(B_{\psi^1}(t,T)' Z_{t,t} + A_{\psi^1}(t,T)),$$

where $B_{\psi^1}(t,T)$ and $A_{\psi^1}(t,T)$ are the solutions to the following system of ODEs

$$\begin{aligned} \frac{dB_{\psi^1}(t,T)}{dt} &= \rho^R + (K^Q)' B_{\psi^1}(t,T) - \frac{1}{2}\sum_{j=1}^{5} (\Sigma' B_{\psi^1}(t,T) B_{\psi^1}(t,T)' \Sigma)_{j,j} (\delta^j)', \quad B_{\psi^1}(T,T) = \overline{B}, \\ \frac{dA_{\psi^1}(t,T)}{dt} &= -B_{\psi^1}(t,T)' K^Q \theta^Q - \frac{1}{2}\sum_{j=1}^{5} (\Sigma' B_{\psi^1}(t,T) B_{\psi^1}(t,T)' \Sigma)_{j,j} \gamma^j, \quad A_{\psi^1}(T,T) = 0. \end{aligned}$$

This system of ODEs can be solved analytically and the solution is provided in the following proposition.

Proposition 1:

Let the state variables be given by $Z_{t,T} = (L_t^N, S_t, C_t, L_t^R, Y_{t,T})$, and let the real instantaneous risk-free rate be given by

$$r_t^R = (\rho^R)' X_t,$$

then

$$\psi^1(\overline{B}, t, T) = \exp(B^1_{\psi^1}(t,T) L_t^N + B^2_{\psi^1}(t,T) S_t + B^3_{\psi^1}(t,T) C_t + B^4_{\psi^1}(t,T) L_t^R + B^5_{\psi^1}(t,T) Y_{t,t} + A_{\psi^1}(t,T))$$

where[18]

[18]The calculations leading to this result are available upon request.

$$
\begin{aligned}
B^1_{\psi^1}(t,T) &= \frac{-2\rho_1[e^{\phi^N_{\psi^1}(T-t)}-1]+\overline{B}^1(\phi^N_{\psi^1}-\kappa^Q_{L^N})e^{\phi^N_{\psi^1}(T-t)}+\overline{B}^1(\phi^N_{\psi^1}+\kappa^Q_{L^N})}{2\phi^N_{\psi^1}+(\phi^N_{\psi^1}+\kappa^Q_{L^N}-\overline{B}^1\sigma^2_{11})[e^{\phi^N_{\psi^1}(T-t)}-1]},\\
B^2_{\psi^1}(t,T) &= e^{-\lambda(T-t)}\overline{B}^2-[\alpha^R-(1-\alpha^R)\overline{B}^5]\frac{1-e^{-\lambda(T-t)}}{\lambda},\\
B^3_{\psi^1}(t,T) &= e^{-\lambda(T-t)}\overline{B}^3+\lambda(T-t)e^{-\lambda(T-t)}\overline{B}^2+[\alpha^R-(1-\alpha^R)\overline{B}^5]\Big\{(T-t)e^{-\lambda(T-t)}-\frac{1-e^{-\lambda(T-t)}}{\lambda}\Big\},\\
B^4_{\psi^1}(t,T) &= \frac{-2\rho_4[e^{\phi^R_{\psi^1}(T-t)}-1]+\overline{B}^4(\phi^R_{\psi^1}-\kappa^Q_{L^R})e^{\phi^R_{\psi^1}(T-t)}+\overline{B}^4(\phi^R_{\psi^1}+\kappa^Q_{L^R})}{2\phi^R_{\psi^1}+(\phi^R_{\psi^1}+\kappa^Q_{L^R}-\overline{B}^4\sigma^2_{44})[e^{\phi^R_{\psi^1}(T-t)}-1]},\\
B^5_{\psi^1}(t,T) &= \overline{B}^5,
\end{aligned}
$$

and

$$
\begin{aligned}
A_{\psi^1}(t,T) = {} & \frac{2\kappa^Q_{L^N}\theta^Q_{L^N}}{\sigma^2_{11}}\ln\Big[\frac{2\phi^N_{\psi^1}e^{\frac{1}{2}(\phi^N_{\psi^1}+\kappa^Q_{L^N})(T-t)}}{2\phi^N_{\psi^1}+(\phi^N_{\psi^1}+\kappa^Q_{L^N}-\overline{B}^1\sigma^2_{11})[e^{\phi^N_{\psi^1}(T-t)}-1]}\Big]\\
&+\sigma^2_{22}[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^2]^2\frac{1-e^{-2\lambda(T-t)}}{4\lambda^3}+\frac{\sigma^2_2}{2}\frac{[\alpha^R-(1-\alpha^R)\overline{B}^5]^2}{\lambda^2}(T-t)\\
&-\sigma^2_{22}[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^2][\alpha^R-(1-\alpha^R)\overline{B}^5]\frac{1-e^{-\lambda(T-t)}}{\lambda^3}\\
&+\sigma^2_{33}[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^3]^2\frac{1-e^{-2\lambda(T-t)}}{4\lambda^3}+\frac{\sigma^2_{33}}{2}\frac{[\alpha^R-(1-\alpha^R)\overline{B}^5]^2}{\lambda^2}(T-t)\\
&+\frac{\sigma^2_{33}}{2}[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^2]^2\Big[-\frac{1}{2\lambda}(T-t)^2e^{-2\lambda(T-t)}-\frac{1}{2\lambda^2}(T-t)e^{-2\lambda(T-t)}+\frac{1-e^{-2\lambda(t-t)}}{4\lambda^3}\Big]\\
&-\sigma^2_{33}[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^3][\alpha^R-(1-\alpha^R)\overline{B}^5]\frac{1-e^{-\lambda(T-t)}}{\lambda^3}\\
&+\sigma^2_{33}\frac{[\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^3][\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^2]}{\lambda}\Big[-\frac{1}{2\lambda}(T-t)e^{-2\lambda(T-t)}+\frac{1-e^{-2\lambda(T-t)}}{4\lambda^2}\Big]\\
&-\sigma^2_{33}\frac{[\alpha^R-(1-\alpha^R)\overline{B}^5][\alpha^R-(1-\alpha^R)\overline{B}^5+\lambda\overline{B}^2]}{\lambda}\Big[-\frac{1}{\lambda}(T-t)e^{-\lambda(T-t)}+\frac{1-e^{-\lambda(T-t)}}{\lambda^2}\Big]\\
&+\frac{2\kappa^Q_{L^R}\theta^Q_{L^R}}{\sigma^2_{44}}\ln\Big[\frac{2\phi^R_{\psi^1}e^{\frac{1}{2}(\phi^R_{\psi^1}+\kappa^Q_{L^R})(T-t)}}{2\phi^R_{\psi^1}+(\phi^R_{\psi^1}+\kappa^Q_{L^R}-\overline{B}^4\sigma^2_{44})[e^{\phi^R_{\psi^1}(T-t)}-1]}\Big],
\end{aligned}
$$

with

$$
\phi^N_{\psi^1}=\sqrt{(\kappa^Q_{L^N})^2+2\rho_1\sigma^2_{11}},\quad \rho_1=-\overline{B}^5,\quad \phi^R_{\psi^1}=\sqrt{(\kappa^Q_{L^R})^2+2\rho_4\sigma^2_{44}},\quad \text{and}\quad \rho_4=1+\overline{B}^5.
$$

Using a similar approach, it holds that

$$
\psi^2(\overline{B},t,T)=\exp(B_{\psi^2}(t,T)'Z_{t,t}+A_{\psi^2}(t,T)),
$$

where $B_{\psi^2}(t,T)$ and $A_{\psi^2}(t,T)$ are the solutions to the following system of ODEs

$$\begin{aligned}
\frac{dB_{\psi^2}(t,T)}{dt} &= \rho^N + (K^Q)' B_{\psi^2}(t,T) - \frac{1}{2}\sum_{j=1}^{5}(\Sigma' B_{\psi^2}(t,T) B_{\psi^2}(t,T)'\Sigma)_{j,j}(\delta^j)', \quad B_{\psi^2}(T,T) = \overline{B}, \\
\frac{dA_{\psi^2}(t,T)}{dt} &= -B_{\psi^2}(t,T)' K^Q \theta^Q - \frac{1}{2}\sum_{j=1}^{5}(\Sigma' B_{\psi^2}(t,T) B_{\psi^2}(t,T)'\Sigma)_{j,j}\gamma^j, \quad A_{\psi^2}(T,T) = 0.
\end{aligned}$$

This system can also be solved analytically and the solution is provided in the following proposition.

Proposition 2:

Let the state variables be given by $Z_{t,T} = (L_t^N, S_t, C_t, L_t^R, Y_{t,T})$, and let the nominal instantaneous risk-free rate be given by

$$r_t^N = (\rho^N)' X_t,$$

then

$$\psi^2(\overline{B}, t, T) = \exp(B^1_{\psi^2}(t,T) L_t^N + B^2_{\psi^2}(t,T) S_t + B^3_{\psi^2}(t,T) C_t + B^4_{\psi^2}(t,T) L_t^R + B^5_{\psi^2}(t,T) Y_{t,t} + A_{\psi^2}(t,T)),$$

where[19]

$$\begin{aligned}
B^1_{\psi^2}(t,T) &= \frac{-2\rho_1[e^{\phi^N_{\psi^2}(T-t)} - 1] + \overline{B}^1(\phi^N_{\psi^2} - \kappa^Q_{L^N})e^{\phi^N_{\psi^2}(T-t)} + \overline{B}^1(\phi^N_{\psi^1} + \kappa^Q_{L^N})}{2\phi^N_{\psi^2} + (\phi^N_{\psi^2} + \kappa^Q_{L^N} - \overline{B}^1\sigma^2_{11})[e^{\phi^N_{\psi^2}(T-t)} - 1]}, \\
B^2_{\psi^2}(t,T) &= e^{-\lambda(T-t)}\overline{B}^2 - [1 - (1-\alpha^R)\overline{B}^5]\frac{1 - e^{-\lambda(T-t)}}{\lambda}, \\
B^3_{\psi^2}(t,T) &= e^{-\lambda(T-t)}\overline{B}^3 + \lambda(T-t)e^{-\lambda(T-t)}\overline{B}^2 + [1 - (1-\alpha^R)\overline{B}^5]\Big\{(T-t)e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\Big\}, \\
B^4_{\psi^2}(t,T) &= \frac{-2\rho_4[e^{\phi^R_{\psi^2}(T-t)} - 1] + \overline{B}^4(\phi^R_{\psi^2} - \kappa^Q_{L^R})e^{\phi^R_{\psi^2}(T-t)} + \overline{B}^4(\phi^R_{\psi^2} + \kappa^Q_{L^R})}{2\phi^R_{\psi^2} + (\phi^R_{\psi^2} + \kappa^Q_{L^R} - \overline{B}^4\sigma^2_{44})[e^{\phi^R_{\psi^2}(T-t)} - 1]}, \\
B^5_{\psi^2}(t,T) &= \overline{B}^5,
\end{aligned}$$

and

[19] The calculations leading to this result are available upon request.

$$
\begin{aligned}
A_{\psi^2}(t,T) \quad = \quad & \frac{2\kappa^Q_{L^N}\theta^Q_{L^N}}{\sigma^2_{11}} \ln\Big[\frac{2\phi^N_{\psi^2} e^{\frac{1}{2}(\phi^N_{\psi^2}+\kappa^Q_{L^N})(T-t)}}{2\phi^N_{\psi^2} + (\phi^N_{\psi^2} + \kappa^Q_{L^N} - \overline{B}^1\sigma^2_{11})[e^{\phi^N_{\psi^2}(T-t)} - 1]}\Big] \\
& + \sigma^2_{22}[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^2]^2 \frac{1-e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma^2_{22}}{2}\frac{[1-(1-\alpha^R)\overline{B}^5]^2}{\lambda^2}(T-t) \\
& - \sigma^2_{22}[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^2][1-(1-\alpha^R)\overline{B}^5]\frac{1-e^{-\lambda(T-t)}}{\lambda^3} \\
& + \sigma^2_{33}[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^3]^2 \frac{1-e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma^2_{33}}{2}\frac{[1-(1-\alpha^R)\overline{B}^5]^2}{\lambda^2}(T-t) \\
& + \frac{\sigma^2_{33}}{2}[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^2]^2\Big[-\frac{1}{2\lambda}(T-t)^2 e^{-2\lambda(T-t)} - \frac{1}{2\lambda^2}(T-t)e^{-2\lambda(T-t)} + \frac{1-e^{-2\lambda(t-t)}}{4\lambda^3}\Big] \\
& - \sigma^2_{33}[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^3][1-(1-\alpha^R)\overline{B}^5]\frac{1-e^{-\lambda(T-t)}}{\lambda^3} \\
& + \sigma^2_{33}\frac{[1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^3][1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^2]}{\lambda}\Big[-\frac{1}{2\lambda}(T-t)e^{-2\lambda(T-t)} + \frac{1-e^{-2\lambda(T-t)}}{4\lambda^2}\Big] \\
& - \sigma^2_{33}\frac{[1-(1-\alpha^R)\overline{B}^5][1-(1-\alpha^R)\overline{B}^5 + \lambda\overline{B}^2]}{\lambda}\Big[-\frac{1}{\lambda}(T-t)e^{-\lambda(T-t)} + \frac{1-e^{-\lambda(T-t)}}{\lambda^2}\Big] \\
& + \frac{2\kappa^Q_{L^R}\theta^Q_{L^R}}{\sigma^2_{44}} \ln\Big[\frac{2\phi^R_{\psi^2} e^{\frac{1}{2}(\phi^R_{\psi^2}+\kappa^Q_{L^R})(T-t)}}{2\phi^R_{\psi^2} + (\phi^R_{\psi^2} + \kappa^Q_{L^R} - \overline{B}^4\sigma^2_{44})[e^{\phi^R_{\psi^2}(T-t)} - 1]}\Big],
\end{aligned}
$$

with

$$
\phi^N_{\psi^2} = \sqrt{(\kappa^Q_{L^N})^2 + 2\rho_1\sigma^2_{11}}, \quad \rho_1 = 1 - \overline{B}^5, \quad \phi^R_{\psi^2} = \sqrt{(\kappa^Q_{L^R})^2 + 2\rho_4\sigma^2_{44}}, \quad \text{and} \quad \rho_4 = \overline{B}^5.
$$

With these results at our disposal, we can turn our attention to the pricing of the deflation protection in the TIPS contract. From Duffie, Pan, and Singleton (2000) it follows that

$$
\begin{aligned}
E^Q_t\Big[e^{-\int_t^T r^R_s ds} e^{\overline{B}' Z_{t,T}} \mathbf{1}_{\{b' Z_{t,T} \leq z\}}\Big] \quad = \quad & \frac{\psi^1(\overline{B}, t, T)}{2} \\
- \quad & \frac{1}{\pi}\int_0^\infty \frac{Im\{e^{-ivz}\psi^1(\overline{B} + ivb, t, T)\}}{v} dv
\end{aligned}
$$

and

$$
\begin{aligned}
E^Q_t\Big[e^{-\int_t^T r^N_s ds} e^{\overline{B}' Z_{t,T}} \mathbf{1}_{\{b' Z_{t,T} \leq z\}}\Big] \quad = \quad & \frac{\psi^2(\overline{B}, t, T)}{2} \\
- \quad & \frac{1}{\pi}\int_0^\infty \frac{Im\{e^{-ivz}\psi^2(\overline{B} + ivb, t, T)\}}{v} dv.
\end{aligned}
$$

Since we interested in the condition

$$
Y_{t,T} = \int_t^T (r^N_s - r^R_s) ds \leq \ln(1+q),
$$

the expectations above should be evaluated at

$$b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad z = \ln(1+q).$$

Furthermore, we have zero boundary values at maturity so

$$\overline{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

A similar approach can be used to calculate the NPV of the TIPS deflation protection within the *AFNS c.v. model* (see CLR 2011b for details).

The functions $\frac{Im\{e^{-ivz}\psi^1(\overline{B}+ivb,t,T)\}}{v}$ and $\frac{Im\{e^{-ivz}\psi^2(\overline{B}+ivb,t,T)\}}{v}$ that need to be integrated in order to calculate the NPV of the TIPS deflation protection have already converged to zero for values of v above 500, so we approximate the infinite integral in the pricing formulas by capping v at 1000 to err on the side of conservatism and use a step size of $\Delta v = 0.01$ in the numerical approximation, which is sufficient since the functions are clearly smooth.

5.1 Deflation Probabilities Within the AFNS-sv Model

CLR (2011a) use the AFNS-cv model to generate deflation probabilities at various horizons appropriate for macroeconomic and monetary policy purposes. Similarly, the AFNS-sv model can be used to calculate deflation probabilities, although additional steps are necessary as described more fully in CLR (2011b). The change in the market-implied price index for the period from t until $t+\tau$ is given by

$$\frac{\Pi_{t+\tau}}{\Pi_t} = e^{\int_t^{t+\tau}(r_s^N - r_s^R)ds}.$$

We want to calculate the probability of the event that the change in the price index is below a certain critical level q. By implication, we are interested in the states of the world where

$$\frac{\Pi_{t+\tau}}{\Pi_t} \leq 1+q,$$

or, equivalently,

$$\int_t^{t+\tau}(r_s^N - r_s^R)ds \leq \ln(1+q).$$

Since the nominal and real instantaneous short rates are given by

$$\begin{aligned} r_t^N &= L_t^N + S_t, \\ r_t^R &= L_t^R + \alpha^R S_t, \end{aligned}$$

we are interested in the distributional properties of the following process

$$Y_{0,t} = \int_0^t (r_s^N - r_s^R)ds = \int_0^t (L_s^N + S_s - L_s^R - \alpha^R S_s)ds \quad \Rightarrow \quad dY_{0,t} = (L_t^N + (1-\alpha^R)S_t - L_t^R)dt.$$

In general, the P-dynamics of the state variables X_t are given by

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma D(X_t)dW_t^P.$$

Adding the Y_t-process to this system, leaves us with a five-factor SDE of the following form[20]

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dY_{0,t} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P & 0 \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P & 0 \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \\ 0 \end{pmatrix} dt - \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P & 0 \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P & 0 \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P & 0 \\ -1 & -(1-\alpha^R) & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ Y_{0,t} \end{pmatrix} dt$$

$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{L^N,P} \\ dW_t^{S,P} \\ dW_t^{C,P} \\ dW_t^{L^R,P} \\ dW_t^{Y,P} \end{pmatrix},$$

where $Z_{0,t} = (L_t^N, S_t, C_t, L_t^R, Y_{0,t})$ represents the augmented state vector.

This is a system of non-Gaussian state variables. As a consequence, we cannot use the approach detailed in CLR (2011b). Instead, we use the Fourier transform analysis described in full generality for affine models in Duffie, Pan, and Singleton (DPS, 2000). DPS provide a formula for calculating contingent expectations of the form

$$G_{\overline{B},\bar{b}}(y; Z_{t,t}, t, T) = E^P\left[e^{-\int_t^T \rho_\psi' Z_{s,T} ds} e^{\overline{B}' Z_{t,T}} \mathbf{1}_{\{\bar{b} Z_{t,T} \leq y\}} \Big| \mathcal{F}_t\right].$$

[20]The shown case is for the model with the Feller conditions imposed.

If we define

$$\psi(\overline{B}; Z_{t,t}, t, T) = E^P\left[e^{-\int_t^T \rho_\psi' Z_{s,T} ds} e^{\overline{B}' Z_{t,T}}\right] = e^{B_\psi(t,T)' Z_{t,t} + A_\psi(t,T)},$$

where $B_\psi(t,T)$ and $A_\psi(t,T)$ are solutions to the corresponding system of ODEs outlined in Equations (**??**) and (**??**), then DPS show that

$$G_{\overline{B},\overline{b}}(y; Z_{t,t}, t, T) = \frac{\psi(\overline{B}; Z_{t,t}, t, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[e^{-ivy}\psi(\overline{B} + iv\overline{b}; Z_{t,t}, t, T)]}{v} dv.$$

Here, we are interested in the cumulative probability function of $Y_{t,T}$ conditional on $Z_{t,t}$, that is, we are interested in the function $E^P[\mathbf{1}_{\{Y_{t,T} \leq y\}} | \mathcal{F}_t]$. From the result above it follows that we get the desired probability function if we fix

$$\overline{B} = 0, \quad \overline{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_\psi = 0, \quad \text{and} \quad y = \ln(1+q).$$

5.1.1 The Priced Probability of Deflation

The actual probability of deflation calculated above is determined by the estimated mean-reversion matrix K^P under the P-measure. Thus, it reflects the actual time series dynamics of the state variables. The priced probability of deflation, on the other hand, reflects the implicit probability of deflation needed to match the observed bond prices. Due to risk premia that reflect bond investor risk aversion, this measure can be different from the actual deflation probability.

To calculate the priced probability of deflation, we replace the P-dynamics above with the Q-dynamics. Thus, the system of SDEs becomes

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dY_{0,t} \end{pmatrix} = \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_{L^N}^Q \\ 0 \\ 0 \\ \theta_{L^R}^Q \\ 0 \end{pmatrix} dt - \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q & 0 \\ -1 & -(1-\alpha^R) & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ Y_{0,t} \end{pmatrix} dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{L^N,Q} \\ dW_t^{S,Q} \\ dW_t^{C,Q} \\ dW_t^{L^R,Q} \\ dW_t^{Y,Q} \end{pmatrix},$$

while everything else remains the same.

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