Intermediary Leverage Cycles and Financial Stability

Tobias Adrian and Nina Boyarchenko*

September 24, 2012

Abstract

We present a theory of financial intermediary leverage cycles within a dynamic model of the macroeconomy. Intermediaries face risk based funding constraints that give rise to procyclical leverage. The pricing of risk varies as a function of intermediary leverage, and asset return exposure to intermediary leverage shocks earns a positive risk premium. Relative to an economy with constant leverage, financial intermediaries generate higher consumption growth and lower consumption volatility in normal times, at the cost of endogenous systemic financial risk. The severity of systemic crisis depends on intermediaries' leverage and net worth. Regulations that tighten funding constraints affect the systemic risk-return trade-off by lowering the likelihood of systemic crises at the cost of higher pricing of risk.

---

*Adrian: Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045, tobias.adrian@ny.frb.org. Boyarchenko: Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045, nina.boyarchenko@ny.frb.org. The views expressed here are the authors' and are not representative of the views of the Federal Reserve Bank of New York or the Federal Reserve System. The authors thank Michael Abrahams and Daniel Green for excellent research assistance; David Backus, Olivier Blanchard, Markus Brunnermeier, Mikhail Chernov, John Cochrane, Douglas Diamond, Darrell Duffie, Xavier Gabaix, Ken Garbade, Lars Peter Hansen, Nobu Kiyotaki, Ralph Koijen, John Leahy, David Lucca, Monika Piazzesi, Martin Schneider, Jules Van Binsbergen, Charles-Henri Weymuller, and Michael Woodford for helpful comments; and seminar participants at the Chicago Institute for Theory and Empirics, the Federal Reserve Bank of New York, the Federal Reserve Board of Governors, Goethe University and Duke University (Fuqua School of Business) for feedback.

1 Introduction

The financial crisis of 2007-09 demonstrated the role of financial intermediaries in the amplification of fundamental shocks, spurring a renewed interest in policies toward financial stability. In this paper, we develop a dynamic stochastic general equilibrium model that captures the leverage cycle of financial intermediaries in an empirically relevant way. The model features endogenous solvency risk of the financial sector, allowing us to study the impact of macroprudential policies on the systemic risk-return trade-off.

We depart from the emerging literature[1] on dynamic macroeconomic models with financial intermediaries by assuming that intermediaries have to hold equity in proportion to the riskiness of their total assets. This assumption is empirically motivated, and gives rise to empirical predictions about the intermediation and pricing of credit that alternative theories do not capture. In particular, in equilibrium, the risk-based funding constraint of intermediaries generates the procyclical leverage behavior emphasized by Adrian and Shin (2010a). In addition, the share of intermediated credit is strongly procyclical, an empirical fact documented by Adrian et al. (2011a).

In our theory, financial intermediaries have two roles. While both households and intermediaries can own existing firms' capital, only intermediaries can generate new capital via investment. This assumption captures financial institutions ability to allocate capital and monitor borrowers. The second role of intermediaries is to provide risk bearing capacity. Due to their risk sensitive leverage constraint, the effective risk aversion of intermediaries is fluctuating, giving rise to the intermediary leverage cycle. Equilibrium dynamics for the whole economy are functions of two intermediary state variables: their leverage and their net worth. These two state variables capture the two sources of uncertainty in the model, which are productivity shocks and liquidity shocks of households. Movements in the leverage state variable are closely tied to the liquidity shocks that households experience, as these leverage shocks represent funding liquidity shocks to intermediaries.[2] In empirical work, Adrian et al.

[1] Brunnermeier and Sannikov (2011, 2012), He and Krishnamurthy (2012a,b), Gertler and Kiyotaki (2012), and Gertler et al. (2011) all have recently proposed equilibrium theories with a financial sector.

[2] Notice that, unlike in Rampini and Viswanathan (2012), the second variable is leverage, and not household wealth, which mirrors recent empirical work. The liquidity shock that is giving rise to the importance of leverage as a state variable is similar to the shocks emphasized in the banking literature by Allen and Gale (1994), Diamond and Dybvig (1983), and Holmström and Tirole (1998).

(2011b) and Adrian et al. (2010) show that fluctuations in intermediary leverage are tightly linked to the time series and cross section of asset risk premia. Adrian et al. (2011b) show that asset return exposure to leverage shocks earn a positive risk premium, while Adrian et al. (2010) find that asset risk premia vary systematically with intermediary leverage over time.

Costly adjustments to the real capital stock lead to the intermediary leverage cycle, which in turn translates into an endogenous amplification of shocks. While fundamental shocks are assumed to be homoskedastic, equilibrium asset prices and equilibrium consumption growth exhibit stochastic volatility. When adverse shocks to intermediary balance sheets are sufficiently large, intermediaries experience systemic solvency risk and need to restructure. We assume that such systemic risk occurs when intermediary's net worth falls below a threshold. In that case, intermediaries deleverage by writing down debt, giving rise to negative payoffs to the households who own intermediary debt. Whether systemic financial crisis are benign, or whether they generate severe drops in real consumption depends on the severity of the shocks, the leverage of intermediaries, and their relative net worth. Our model gives rise to the "volatility paradox" of Adrian and Brunnermeier (2010): Times of low volatility tend to be associated with a buildup of leverage, which in turn increases forward-looking systemic risk. We also study the systemic risk-return trade-off: Low prices of risk today tend to be associated with larger forward-looking systemic risk measures, suggesting that measures of asset price valuations are useful indicators for systemic risk assessments.

In a benchmark model with a constant intermediary leverage constraint, the resulting equilibrium growth of investment, price of capital, and the risk-free rate are constant. Fluctuations in output of the benchmark economy are entirely due to productivity shocks, and output is fully insulated from liquidity shocks. In contrast, in our model with a risk based funding constraint, liquidity shocks spill over to real activity, and productivity shocks are amplified. When funding constraints are risk based, intermediaries provide consumption smoothing services to households during normal times that result in higher growth rates, at the cost of occasional realizations of systemic risk states with large consumption drops. The tightness of risk based intermediary constraints thus regulate a systemic risk return tradeoff.

Since our theory captures important empirical regularities about the dynamic interactions between the financial sector and the macroeconomy, it provides a conceptual framework for financial stability policies. In this paper, we focus primarily on one form of prudential policy, which concerns the tightness of intermediaries' funding constraint and can be interpreted as capital regulation. Our paper is among the few that consider the role of (macro)prudential policies in dynamic equilibrium models explicitly (see Goodhart et al. (2012), Angelini et al. (2011), and Bianchi and Mendoza (2011) for alternative settings). Our main findings are intuitive. We show that households' welfare dependence on the tightness of the intermediaries capital constraint is inversely U-shaped: very loose constraints generate excessive risk taking of intermediaries relative to household preferences, while very tight funding constraints inhibit intermediaries' risk taking and investment. Furthermore, forward-looking systemic default risk also has a U-shaped relationship with the tightness of the funding constraint.This tradeoff maps closely into the debate on optimal regulation. In addition, we show that the degree of forward looking systemic risk relates to the tightness of the capital constraints in a U-shaped fashion.

The rest of the paper is organized as follows. We describe the model in Section 2. The equilibrium interactions and outcomes are outlined in Section 3. We investigate the creation of systemic risk and its welfare implications in Section 4. Section 5 concludes. Technical details are relegated to the appendix.

1.1 Related Literature

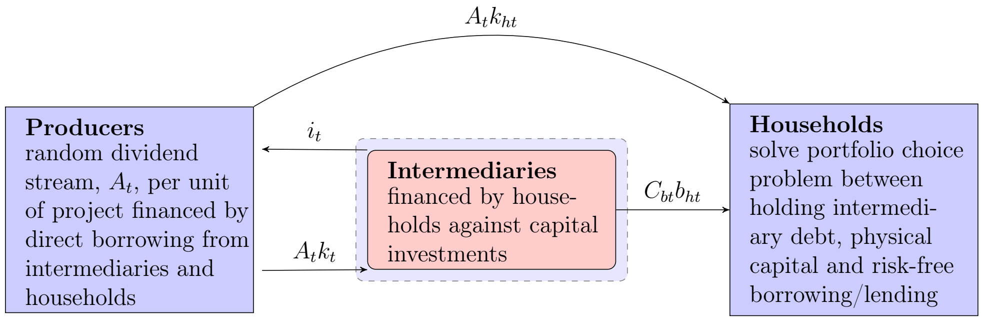
This paper is related to several strands of the literature. Geanakoplos (2003) and Fostel and Geanakoplos (2008) show that leverage cycles can cause contagion and issuance rationing in a general equilibrium model with heterogeneous agents, incomplete markets, and endogenous collateral. Brunnermeier and Pedersen (2009) further show that market liquidity and traders' access to funding are co-dependent, leading to liquidity spirals. Our model differs from that of Fostel and Geanakoplos (2008) as our asset markets are dynamically complete and debt contracts are not collateralized. The leverage cycle in our model comes from the risk-based leverage constraint of the financial intermediaries and is intimately related to the funding liquidity of Brunnermeier and Pedersen (2009). Unlike in their model, however, the funding liquidity that matters in our setup is that of the financial intermediaries, not that of speculative traders.

This paper is also related to studies of amplification in models of the macroeconomy. The seminal paper in this literature is Bernanke and Gertler (1989), which shows that the condition of borrowers' balance sheets is a source of output dynamics. Net worth increases during economic upturns, increasing investment and amplifying the upturn, while the opposite dynamics hold in a downturn. Kiyotaki and Moore (1997) show that small shocks can be amplified by credit restrictions, giving rise to large output fluctuations. Instead of focusing on financial frictions in the demand for credit as Bernanke–Gertler and Kiyotaki–Moore do, our theory focuses on frictions in the supply of credit. Another important distinction is that the intermediaries in our economy face leverage constraints that depend on current volatility, which give rise to procyclical leverage. In contrast, the leverage constraints of Kiyotaki–Moore are state independent.

Gertler and Kiyotaki (2012) and Gertler et al. (2011) extend the accelerator mechanism of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) to financial intermediaries. Gertler et al. (2011) consider a model in which financial intermediaries can issue outside equity and short-term debt, making intermediary risk exposure an endogenous choice. Gertler and Kiyotaki (2012) further extend the model to allow for household liquidity shocks as in Diamond and Dybvig (1983). While these models are similar in spirit to that presented in this paper, our model is more parsimonious in nature and allows for endogenous defaultable debt. We can thus investigate the creation of systemic default and the effectiveness of macroprudential policy in mitigating these risks.

Our theory is closely related to the work of He and Krishnamurthy (2012a,b) and Brunnermeier and Sannikov (2011, 2012), who explicitly introduce a financial sector into dynamic models of the macroeconomy. While our setup shares many conceptual and technical features of this work, our points of departure are empirically motivated. We allow households to invest via financial intermediaries as well as directly in the capital stock, a feature strongly supported by the data, which gives rise to important substitution effects between directly granted and intermediated credit. In the setup of He–Krishnamurthy, investment is always intermediated. Furthermore, our model features procyclical intermediary leverage, while theirs is countercyclical. Finally, systemic risk of the intermediary sector is at the heart of

Figure 1: Economy structure



our analysis, while He–Krishnamurthy and Brunnermeier–Sannikov focus primarily on the amplification of shocks.

The interactions between the households, the financial intermediaries, and the productive sector lead to a highly nonlinear system. We consider the nonlinearity a desirable feature, as the model is able to capture strong amplification effects. Our theory features both endogenous risk amplification (where fundamental volatility is amplified as in Danielsson et al. (2011)), as well as the creation of endogenous systemic risk.

2 A Model

We begin with a single consumption good economy, where the unique good at time $t > 0$ is used as the numeraire. There are three types of agents in the economy: producers, financial intermediaries, and households. We abstract from modeling the decisions of the producers and focus instead on the interaction between the intermediary sector and the households. The basic structure of the economy is represented in Figure 1.

2.1 Production

We consider an economy with two active types of agents: financially sophisticated intermediaries and unsophisticated households. While both types of agents can own capital, only financial intermediaries can create new capital through investment. We denote by K_t the aggregate amount of capital in the economy at time $t \geq 0$ and assume that each unit of capital produces A_t units of the consumption good. The total output in the economy at time t is given by

$$Y_t = A_t K_t,$$

where the stochastic productivity of capital $\{A_t = e^{a_t}\}_{t\geq 0}$ follows a geometric diffusion process of the form

$$da_t = \bar{a}dt + \sigma_a dZ_{at},$$

where $(Z_{at})_{0\leq t<+\infty}$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each unit of capital in the economy depreciates at a rate λ_k, so that the capital stock in the economy evolves as

$$dK_t = (I_t - \lambda_k)\, K_t dt,$$

where I_t is the reinvestment rate per unit of capital in place. Thus, output in the economy evolves according to

$$dY_t = \left(I_t - \lambda_k + \bar{a} + \frac{\sigma_a^2}{2} \right) Y_t dt + \sigma_a Y_t dZ_{at}.$$

Notice that the quantity $A_t K_t$ corresponds to the "efficiency" capital of Brunnermeier and Sannikov (2012), with a constant productivity rate of 1.
There is a fully liquid market for physical capital in the economy, in which both the financial intermediaries and the households are allowed to participate. To keep the economy scale-invariant, we denote by $p_{kt}A_t$ the price of one unit of capital at time t in terms of the consumption good.

2.2 Household sector

There is a unit mass of risk-averse, infinitely lived households in the economy. We assume that the households in the economy are identical, such that the equilibrium outcomes are determined by the decisions of the representative household. The households, however, are exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. This reduced-form approach allows us to remain agnostic as to the exact source of this second shock: With this specification, it can arise either from time-variation in the households' risk aversion or from time-variation in households' beliefs. In particular, we assume that the representative household evaluates different consumption paths $\{c_t\}_{t\geq 0}$ according to

$$\mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t+\rho_h t)} \log c_t dt\right],$$

where ρ_h is the subjective time discount of the representative household, and c_t is the consumption at time t. Here, $\exp(-\xi_t)$ is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we assume that $\{\xi_t\}_{t\geq 0}$ evolves as a Brownian motion, correlated with the productivity shock, Z_{at}:

$$d\xi_t = \sigma_\xi \rho_{\xi,a} dZ_{at} + \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} dZ_{\xi t},$$

where $\{Z_{\xi t}\}$ is a standard Brownian motion of $(\Omega, \mathcal{F}_t, \mathbb{P})$, independent of Z_{at}. In the current setting, with households constrained in their portfolio allocation, $\exp(-\xi_t)$ can be interpreted as a time-varying liquidity preference shock, as in Allen and Gale (1994), Diamond and Dybvig (1983), and Holmström and Tirole (1998) or as a time-varying shock to the preference for early resolution of uncertainty, as in Bhamra et al. (2010a,b). In particular, when the households receive a positive $d\xi_t$ shock, their effective discount rate increases, leading to a higher demand for liquidity.

The households finance their consumption through holdings of physical capital, holdings of risky intermediary debt, and short-term risk-free borrowing and lending. Unlike the intermediary sector, the households do not have access to the investment technology. Thus, the physical capital k_{ht} held by households evolves according to

$$dk_{ht} = -\lambda_k k_{ht} dt.$$

When a household buys k_{ht} units of capital at price $p_{kt}A_t$, by Itô's lemma, the value of capital evolves according to

$$\frac{d\left(k_{ht}p_{kt}A_t\right)}{k_{ht}p_{kt}A_t} = \frac{dA_t}{A_t} + \frac{dp_{kt}}{p_{kt}} + \frac{dk_{ht}}{k_{ht}} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t} \right\rangle.$$

Each unit of capital owned by the household also produces A_t units of output, so the total return to one unit of household wealth invested in capital is

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht}p_{kt}A_t}dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(k_{ht}p_{kt}A_t\right)}{k_{ht}p_{kt}A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t}dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}.$$

For future use, notice that, with this notation, we have

$$\frac{dp_{kt}}{p_{kt}} = \left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2} - \sigma_a\left(\sigma_{ka,t} - \sigma_a\right)\right)dt \\ + \left(\sigma_{ka,t} - \sigma_a\right)dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}.$$

In addition to direct capital investment, the households can invest in risky intermediary debt. To keep the balance sheet structure of the financial institutions time-invariant, we assume that the bonds mature at a constant rate λ_b, so that the time t probability of a bond maturing before time $t + dt$ is $\lambda_b dt$. Notice that this corresponds to an infinite-horizon version of the "stationary balance sheet" assumption of Leland and Toft (1996). Thus, the risky debt holdings b_{ht} of households follow

$$db_{ht} = \left(\beta_t - \lambda_b\right)b_{ht}dt,$$

where β_t is the issuance rate of new debt. The bonds pay a floating coupon $C_{bt}A_t$ until maturity, with the coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time t in terms of the consumption good given by $p_{bt}A_t$. Hence, the total return from one unit of household wealth invested in risky debt is

$$dR_{bt} = \underbrace{\frac{\left(C_{bt} + \lambda_b\right) A_t b_{ht}}{b_{ht} p_{bt} A_t}}_{\text{dividend}-\text{price ratio}} dt + \underbrace{\frac{d\left(b_{ht} p_{bt} A_t\right)}{b_{ht} p_{bt} A_t}}_{\text{capital gains}} \equiv \mu_{Rb,t} dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi t}.$$

When a household with total wealth w_{ht} buys k_{ht} units of capital and b_{ht} units of risky intermediary debt, it invests the remaining $w_{ht} - p_{kt}k_{ht} - p_{bt}b_{ht}$ at the risk-free rate r_{ft}, so that the wealth of the household evolves as

$$dw_{ht} = r_{ft} w_{ht} + p_{kt} A_t k_{ht} \left(dR_{kt} - r_{ft} dt\right) + p_{bt} A_t b_{ht} \left(dR_{bt} - r_{ft} dt\right) - c_t dt. \tag{1}$$

We assume that the household faces no-shorting constraints, such that

$$k_{ht} \geq 0$$
$$b_{ht} \geq 0.$$

Thus, the household solves

$$\max_{\{c_t, k_{ht}, b_{ht}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt\right], \tag{2}$$

subject to the household wealth evolution 1 and the no-shorting constraints.

2.3 Financial intermediary sector

There is a unit mass of risk-neutral, infinitely lived financial intermediaries in the economy. In the basic formulation, we assume that all the financial intermediaries are identical and thus the equilibrium outcomes are determined by the behavior of the representative intermediary. We abstract from modeling the dividend payment decision ("consumption") of the intermediary sector and assume that an intermediary invests maximally if the opportunity arises. In particular, financial intermediaries create new capital through capital investment. Denote by k_t the physical capital held by the representative intermediary at time t and by $i_t A_t$ the investment rate per unit of capital. Then the stock of capital held by the representative intermediary evolves according to

$$dk_t = (\Phi(i_t) - \lambda_k)\, k_t dt.$$

Here, $\Phi(\cdot)$ reflects the costs of (dis)investment. We assume that $\Phi(0) = 0$, so in the absence of new investment, capital depreciates at the economy-wide rate λ_k. Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow Brunnermeier and Sannikov (2012) in assuming that investment carries quadratic adjustment costs, so that Φ has the form

$$\Phi(i_t) = \phi_0 \left(\sqrt{1 + \phi_1 i_t} - 1 \right),$$

for positive constants ϕ_0 and ϕ_1.

Each unit of capital owned by the intermediary produces $A_t(1 - i_t)$ units of output net of investment. As a result, the total return from one unit of intermediary capital invested in physical capital is given by

$$dr_{kt} = \underbrace{\frac{(1 - i_t) A_t k_t}{k_t p_{kt} A_t} dt}_{\text{dividend−price ratio}} + \underbrace{\frac{d(k_t p_{kt} A_t)}{k_t p_{kt} A_t}}_{\text{capital gains}} = dR_{kt} + \left(\Phi(i_t) - \frac{i_t}{p_{kt}} \right) dt.$$

Compared to the households, the financial intermediary earns an extra return to holding firm capital to compensate it for the cost of investment. This extra return is partially passed on to the households as coupon payments on the intermediaries' debt.

It should be noted that financial intermediaries serve two functions in our economy. First, they generate new investment. Second, they provide capital that provides risk-bearing capacity to the households. Compare this with the notion of intermediation of He and Krishnamurthy (2012a,b,c). In their model, intermediaries allow households to access the risky investment technology: Without the intermediary sector, the households can only invest in the risk-free rate. Instead, the households enter into a profit-sharing agreement with the intermediary, with the profits distributed according to the initial wealth contributions.

The intermediaries finance their investment in new capital projects by issuing risky floating coupon bonds to the households. Denoting by β_t the issuance rate of bonds at time t, the stock of bonds b_t on a representative intermediary's balance sheet evolves as

$$db_t = (\beta_t - \lambda_b)\, b_t dt.$$

Each unit of debt issued by the intermediary pays C_{bt} units of output until maturity and one unit of output at maturity. The total net cost of one unit of intermediary debt is therefore given by

$$dr_{bt} = \underbrace{\frac{(C_{bt} + \lambda_b - \beta_t p_{bt})\, A_t b_t}{b_t p_{bt} A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\,(b_t p_{bt} A_t)}{b_t p_{bt} A_t}}_{\text{capital gains}} = dR_{bt}.$$

Thus, the cost of debt to the intermediary equals the return on holding bank debt for the households.

Consider now the budget constraint of an intermediary in this economy. An intermediary in this economy holds capital investment projects (k_t) on the assets side of its balance sheet and has bonds (b_t) on the liability side. In mathematical terms, we can express the corresponding budget constraint as

$$p_{kt} A_t k_t = p_{bt} A_t b_t + w_t, \tag{3}$$

where w_t is the implicit value of equity in the intermediary. Thus, in terms of flows, the intermediary's equity value evolves according to

$$dw_t = k_t p_{kt} A_t dr_{kt} - b_t p_{bt} A_t dr_{bt}. \tag{4}$$

The key assumption of this paper concerns the funding of the intermediary. We assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk (VaR) constraint of Danielsson et al. (2011). In particular, we assume that

$$\alpha\sqrt{\frac{1}{dt}\left\langle k_t d\left(p_{kt}A_t\right)\right\rangle^2} = w_t, \tag{5}$$

where $\langle\cdot\rangle^2$ is the quadratic variation operator. That is, we assume that the intermediaries are restricted to retain enough equity to cover a certain fraction of losses on their assets. Unlike a traditional VaR constraint, this does not keep the volatility of intermediary equity constant, leaving the intermediary sector exposed to solvency risk. The risk-based capital constraint implies a time-varying leverage constraint θ_t, defined by

$$\theta_t = \frac{p_{kt}A_t k_t}{w_t} = \frac{1}{\alpha\sqrt{\frac{1}{dt}\left\langle\frac{d(p_{kt}A_t)}{p_{kt}A_t}\right\rangle^2}}.$$

Thus, the per-dollar total VaR of assets is negatively related to intermediary leverage, a feature documented by Adrian and Shin (2010a).

The risk based capital constraint of intermediaries is directly related to the way in which financial intermediaries manage market risk. Trading operations of major banks – most of which are undertaken in the security broker-dealer subsidiaries – are managed by allocating equity in relation to the VaR of trading assets. Constraint 5 directly captures such behavior. Banking books, on the other hand, are managed either according to credit risk models, or using historical cost accounting rules with loss provisioning. Although the constraint 5 does not directly capture these features of banks' risk management, empirical evidence suggests that the risk based funding constraint is a good behavioral assumption for bank lending. In particular, the proxy for the tightness of credit supply conditions reported by the Senior Loan Officer Survey of the Federal Reserve is highly correlated with measures of aggregate volatility such as the VIX. (see Figure 2). A higher level of asset volatility is thus associated with tighter lending conditions of commercial banks, which constraint 5 captures.

We assume that the risk based capital constraint 5 always binds. This can be viewed as a technological constraint in the context of our model. The assumption of a constraint that is always binding is in sharp contrast to the models of Brunnermeier and Sannikov (2011, 2012) where banks' management of the buffer relative to their funding constraint

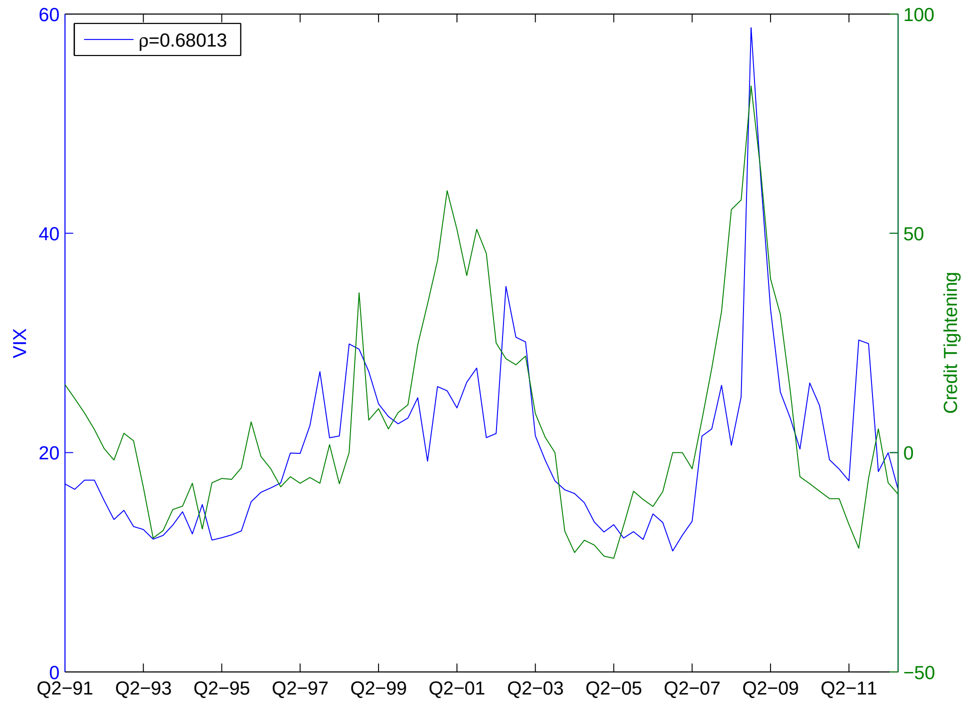


Figure 2: Market volatility and credit supply conditions. VIX refers to the Chicago Board Options Exchange (CBOE) market volatility index. The credit tightening indicator refers to the measure of lending standards for commercial and industrial loans to large and medium firms, as reported in the Board of Governors of the Federal Reserve System Senior Loan Officer Opinion Survey. ρ is the linear correlation between the two series. Source: Haver DLX.

is the central feature of financial intermediaries' behavior. In that alternative approach, there is no risk based leverage constraint. Instead, intermediaries manage their leverage so as to make sure that they have a big enough buffer to make their debt instantaneously risk free. The intertemporal risk management of the intermediary is then driving their effective risk aversion, pinning down their leverage and balance sheet growth. In contrast, in our approach, intermediaries always leverage to the maximum, and do not have to make intertemporal decisions about the tightness of their funding constraint. We choose our assumption for its power in generating empirical predictions that are closely aligned with the data. Furthermore, there is anecdotal evidence that intermediaries tend to leverage maximally.

The parameter α determines how much equity the intermediary has to hold for each dollar of asset volatility. We view this parameter α as a policy parameter that is pinned down by regulation. α can be interpreted as the tightness of risk based capital requirements, similar to the capital requirements of the Basel Committee on Banking Supervision.

The representative intermediary maximizes equity holder value to solve

$$\max_{\{k_t,\beta_t,i_t\}} \quad \mathbb{E}\left[\int_0^{\tau_D} e^{-\rho t} w_t dt\right], \tag{6}$$

subject to the dynamic intermediary budget constraint 4 and the risk-based capital constraint constraint 5. Here, ρ is the subjective discount rate of the intermediary and τ_D is the (random) time at which the representative intermediary becomes distressed and has to be restructured. We assume that distress occurs when the intermediary equity falls below an exogenously specified threshold, so that

$$\tau_D = \inf_{t\geq 0}\left\{w_t \leq \bar{\omega} p_{kt} A_t K_t\right\}.$$

Notice that, since the distress boundary grows with the scale of the economy, the intermediary can never outgrow the possibility of distress. When the intermediary is restructured, the management of the intermediary changes. The new management defaults of the debt of the previous intermediary, reducing leverage to $\underline{\theta}$, but maintains the same level of capital as before. The inside equity of the new intermediary is thus

$$w_{\tau_D^+} = \bar{\omega}\frac{\theta_{\tau_D}}{\underline{\theta}} p_{k\tau_D} A_{\tau_D} K_{\tau_D}.$$

Finally, we define the term structure of distress risk to be

$$\delta_t\left(T\right) = \mathbb{P}\left(\tau_D \leq T \middle| \left(w_t, \theta_t\right)\right).$$

Here, $\delta_t\left(T\right)$ is the time t probability of default occurring before time T. Notice that, since the fundamental shocks in the economy are Brownian, and all the agents in the economy have perfect information, the local distress risk is zero. We refer to the default of the intermediary as systemic risk, as there is a single intermediary in the economy, so its distress is systemic.

In our simulations, we use parameter values for $\bar{\omega}$ that are positive (not zero), thus viewing intermediaries default state as a restructuring event.

2.4 Equilibrium

Definition 2.1. *An equilibrium in this economy is a set of price processes* $\{p_{kt}, p_{bt}, C_{bt}\}_{t\geq 0}$*, a set of household decisions* $\{k_{ht}, b_{ht}, c_t\}_{t\geq 0}$*, and a set of intermediary decisions* $\{k_t, \beta_t, i_t, \theta_t\}_{t\geq 0}$ *such that the following apply:*

1. *Taking the price processes and the intermediary decisions as given, the household's choices solve the household's optimization problem 2, subject to the household budget constraint 1.*

2. *Taking the price processes and the household decisions as given, the intermediary's choices solve the intermediary optimization problem 6, subject to the intermediary wealth evolution 3 and the risk-based capital constraint 5.*

3. *The capital market clears:*

$$K_t = k_t + k_{ht}.$$

4. *The risky bond market clears:*

$$b_t = b_{ht}.$$

5. *The risk-free debt market clears:*

$$w_{ht} = p_{kt} A_t k_{ht} + p_{bt} A_t b_{ht}.$$

6. *The goods market clears:*

$$c_t = A_t \left(K_t - i_t k_t \right).$$

Notice that the bond markets' clearing conditions imply

$$p_{kt}A_tK_t = w_{ht} + w_t.$$

Notice also that the aggregate capital in the economy evolves as

$$dK_t = -\lambda_k K_t dt + \Phi\left(i_t\right) k_t dt.$$

3 Solution

To solve for the equilibrium, we introduce two additional state variables. In particular, we define the fraction of the total wealth in the economy held by the financial intermediaries as

$$\omega_t = \frac{w_t}{w_t + w_{ht}} = \frac{w_t}{p_{kt}A_tK_t}.$$

With this definition, the share of total wealth in the economy held by the households is $(1 - \omega_t)$. The second state variable we use is θ_t, the leverage of the intermediary sector. The vector of state variables in the economy is then

$$\left(\theta_t, \omega_t\right).$$

Notice that, by construction, the household belief shocks are expectation-neutral, and thus their level is not a state variable in the economy. Similarly, we have defined prices in the economy to scale with the level of productivity, A_t, so productivity itself is not a state variable in the scaled version of the economy. We will characterize the equilibrium outcomes in terms of these variables, with the equilibrium conditions determining the time series evolution of θ_t and ω_t in terms of the primitive shocks in the economy, $(Z_{at},\ Z_{\xi,t})$. In particular, we will make use of the following representations

$$\frac{d\omega_t}{\omega_t} = \mu_{\omega t}dt + \sigma_{\omega a,t}dZ_{at} + \sigma_{\omega\xi,t}dZ_{\xi t}$$
$$\frac{d\theta_t}{\theta_t} = \mu_{\theta t}dt + \sigma_{\theta a,t}dZ_{at} + \sigma_{\theta\xi,t}dZ_{\xi t}.$$

Notice that, by observing the evolution of A_t, as well as the two state variables in the economy, we can isolate the time series evolution of the shocks to household beliefs, $(Z_{\xi t})_{t\geq 0}$. Notice finally that the *VaR* constraint implies

$$\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2.$$

Thus, the riskiness of the return to holding capital increases as intermediary leverage decreases. This relation forms the crux of the volatility paradox discussed in detail below: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. Thus, the riskiness of the return to holding capital increases as intermediary leverage decreases. This relation forms the crux of the volatility paradox discussed in detail below: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. We plot the theoretical and the empirical trade-off between leverage growth and volatility in Figure 3. Clearly, higher levels of the VIX tend to precede declines in broker-dealer leverage (right panel). In the model, this translates into a negative relationship between contemporaneous volatility and expected leverage growth (left panel). In the left panel of Figure 3, we show that this relationship survives both as intermediary wealth is decreased (going from solid to dashed lines) and as the funding constraint is tightened (going from blue to red). The negative relationship between broker-dealer leverage and the VIX is further investigated in Adrian and Shin (2010b,a).[3] While the evidence from Figure 3 is from broker dealers, we would argue that it also has an empirical counterpart for the banking book. As discussed earlier, the lending standards of banks vary tightly with the VIX, indicating that new lending of commercial banks is highly correlated with measures of market volatility.

[3]While Adrian and Shin (2010b) show that fluctuations in primary dealer repo—which is a good proxy for fluctuations in broker-dealer leverage — tend to forecast movements in the VIX, Figure 3 shows that higher levels of the VIX precede declines in broker-dealer leverage. We use the lagged VIX as the VIX is implied volatility and hence a forward-looking measure (though the negative relationship also holds for contemporaneous VIX). Adrian and Shin (2010a) use the *VaR* data of major securities broker-dealers to show a negative association between broker-dealer leverage growth and the *VaRs* of the broker dealers. All of these additional empirical results are fully consistent with our setup.

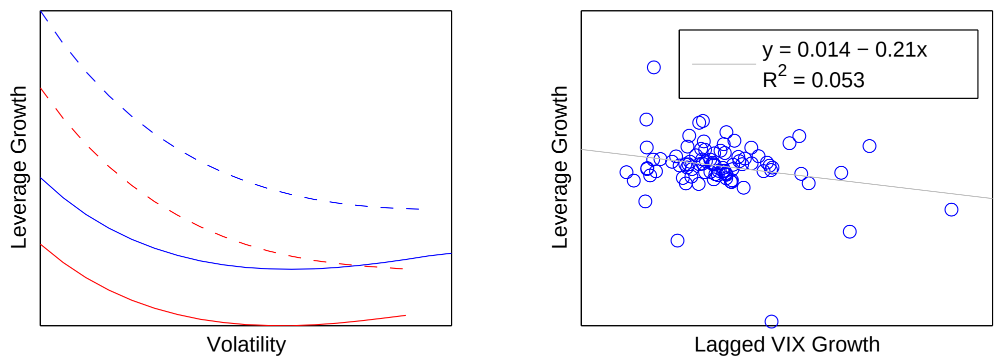


Figure 3: The trade-off between the growth rate of leverage of financial institutions and local volatility.The left panel investigates the shift in the trade-off as intermediary wealth share is decreased (going from the solid line to the dashed line) and the risk-based capital constraint is tightened, increasing α (going from the blue to the red lines). Data on broker-dealer leverage comes from Flow of Funds Table L.129.

3.1 Capital evolution

Recall from the intermediary's leverage constraint that

$$\theta_t = \frac{p_{kt} A_t k_t}{w_t}.$$

Using our definition of ω_t, we can thus express the amount of capital held by the financial institutions as

$$k_t = \frac{\theta_t w_t}{p_{kt} A_t} = \theta_t \omega_t K_t.$$

Applying Itô's lemma, we obtain

$$dk_t = \omega_t K_t d\theta_t + \theta_t K_t d\omega_t + \theta_t \omega_t dK_t + K_t \langle d\theta_t, d\omega_t \rangle .$$

Recall, on the other hand, that the intermediary's capital evolves as

$$dk_t = (\Phi(i_t) - \lambda_k) k_t dt.$$

Equating coefficients, we obtain

$$\sigma_{\theta a,t} = -\sigma_{\omega a,t}$$

$$\sigma_{\theta \xi,t} = -\sigma_{\omega \xi,t}$$

$$\mu_{\theta t} = \underbrace{\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right)}_{\text{asset growth rate}} - \mu_{\omega t} + \underbrace{\sigma^2_{\theta a,t} + \sigma^2_{\theta \xi,t}}_{\text{risk adjustment}}.$$

Thus, intermediary leverage is perfectly negatively correlated with the share of wealth held by the financial intermediaries. This reflects the fact that capital stock is not immediately adjustable, so changes in the value of intermediary assets translate one-for-one into changes in intermediary leverage. Notice further that the intermediary faces a trade-off in the growth rate of its leverage, $\mu_{\theta t}$, and the growth rate of its wealth share in the economy, $\mu_{\omega t}$.

Figure 4 plots the growth of the share of intermediated credit as a function of total credit growth, showing the strong positive relationship in the model and the data. This positive relationship has been previously documented in Adrian et al. (2011a) and shows the procyclical nature of intermediated finance. The middle panel of Figure 4 shows the procyclical nature of the leverage of financial intermediaries. Leverage tends to expand when balance sheets grow, a fact that has been documented by Adrian and Shin (2010b) for the broker-dealer sector and by Adrian et al. (2011a) for the commercial banking sector. The lower panel shows that the procyclical leverage translates into countercyclical equity growth, both in the data and in the model. We should note that the procyclical leverage of financial intermediaries is closely tied to the risk-based capital constraint. In fact, previous literature has found it a challenge to generate this feature (see Brunnermeier and Sannikov (2011, 2012), He and Krishnamurthy (2012a,b), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Gertler and Kiyotaki (2012), and Gertler et al. (2011)).

It should be noted that the empirical evidence from Figure 4 mixes evidence from various financial institutions. The share of intermediated credit is capturing all credit extended to the non-financial sector, including bank lending and market based lending via bonds, mortgages, and other securities. The lower two panels are taken from broker dealer subsidiaries, all of which are part of bank holding companies since the financial crisis of 2007-09. In fact, very

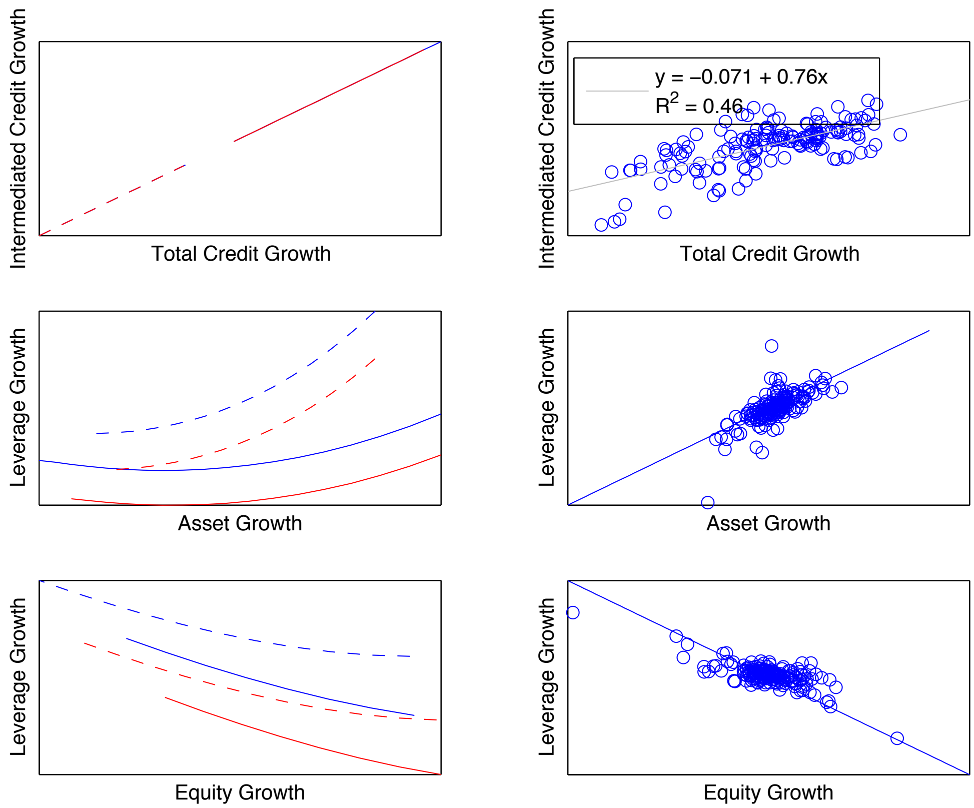


Figure 4: Intermediary balance sheet evolutionTop panels: The share of intermediated credit increases as a function of total credit extended to nonfinancials, showing the procyclicality of financial intermediation. Middle panels: the trade-off between the growth rate of intermediary leverage and intermediary asset growth predicted by the model (left panel) and observed empirically (right panel), showing the procyclical leverage of financial intermediaries. Lower panels: the trade-off between the growth rate of intermediary leverage and intermediary equity growth predicted by the model (left panel) and observed empirically (right panel). The figure investigates the shift in the trade-off as intermediary wealth share is decreased (going from the solid line to the dashed line) and the risk-based capital constraint is tightened, increasing α (going from the blue to the red lines). Data on total credit to the nonfinancial corporate sector and the share of intermediated finance come from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets come from Flow of Funds Table L.129.

similar charts can be generated by using the total holding company balance sheets of the largest banking organizations. However, the picture is somewhat cleaner for the broker dealer subsidiaries, who are marking to market their balance sheets (as is the case in the theory). In contrast, the commercial bank subsidiaries are not marking to market their balance sheet, and the procyclical leverage relationship is therefore slightly more noisy.

In all three panels of Figure 4, the movement from the solid to the dashed lines represents a decrease in intermediary wealth share. In the lower two panels, such a decline in intermediary wealth corresponds to a steepening of the corresponding relationship. Notice also that tightening of the funding constraint (so that α is increased going from the blue lines to the red lines) does not change the shape of the relation but, rather, shifts the relations to lower ranges of intermediary leverage growth.

3.2 Household's problem

Consider now solving for the optimal consumption and investment rules of the household, given asset prices. We have the following result.

Lemma 1. *The household's optimal consumption choice satisfies*

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

In the unconstrained region, the household's optimal portfolio choice is given by

$$\begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} = \left(\begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mu_{Rk,t} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}$$
$$- \sigma_\xi \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} \rho_{\xi,a} \\ \sqrt{1-\rho_{\xi,a}^2} \end{bmatrix}.$$

Proof. See Appendix A.1. □

Thus, the household with the time-varying beliefs chooses consumption as a myopic investor but with a lower rate of discount. The optimal portfolio choice of the household, on the other hand, also includes a hedging component for variations in the Radon-Nikodym derivative, $\exp(-\xi_t)$. Since intermediary debt is locally risk-less, however, households do not self-insure against intermediary default. Appendix A.1 provides also the optimal portfolio choice in the case when the household is constrained. In our simulations, the household never becomes constrained as the intermediary wealth never reaches zero.

Notice that, with the notation introduced above, we also have

$$\pi_{kt} = \frac{p_{kt} A_t k_{ht}}{w_{ht}} = \frac{p_{kt} A_t \left(K_t - k_t\right)}{\left(1 - \omega_t\right) p_{kt} A_t K_t} = \frac{1 - \theta_t \omega_t}{1 - \omega_t}$$
$$\pi_{bt} = 1 - \pi_{kt} = \frac{\omega_t \left(\theta_t - 1\right)}{1 - \omega_t}.$$

Thus, the household holds a non-zero amount of intermediary debt while intermediary leverage exceeds one, and a non-zero amount of capital while the unlevered value of intermediary capital share is less than one. Using this result, we can express the excess return to holding capital as

$$\mu_{Rk,t} - r_{ft} = \underbrace{\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right) \frac{1 - \theta_t \omega_t}{1 - \omega_t}}_{\text{compensation for own risk}} + \underbrace{\left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{\omega_t \left(\theta_t - 1\right)}{1 - \omega_t}}_{\text{compensation for risk of correlated asset}}$$
$$+ \underbrace{\sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1 - \rho_{\xi,a}^2}\right)}_{\text{compensation for beliefs risk}}.$$

Thus, the excess return on holding capital directly has three components. The first compensates households for the direct risk of holding a claim to the volatile output stream, while the second compensates households for the riskiness of holding the correlated asset (risky intermediary debt). The remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative, $\exp\left(-\xi_t\right)$.

Similarly, the excess return to holding risky intermediary debt is given by

$$\mu_{Rb,t} - r_{ft} = \underbrace{\left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) \frac{\omega_t \left(\theta_t - 1\right)}{1 - \omega_t}}_{\text{compensation for own risk}} + \underbrace{\left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{1 - \theta_t \omega_t}{1 - \omega_t}}_{\text{compensation for risk of correlated asset}}$$
$$+ \underbrace{\sigma_\xi \left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1 - \rho_{\xi,a}^2}\right)}_{\text{compensation for beliefs risk}}.$$

As with the excess return to direct capital investment, the excess return on risky intermediary debt has three components. The first compensates households for the direct risk of holding a claim to the volatile coupons, while the second compensates households for the riskiness

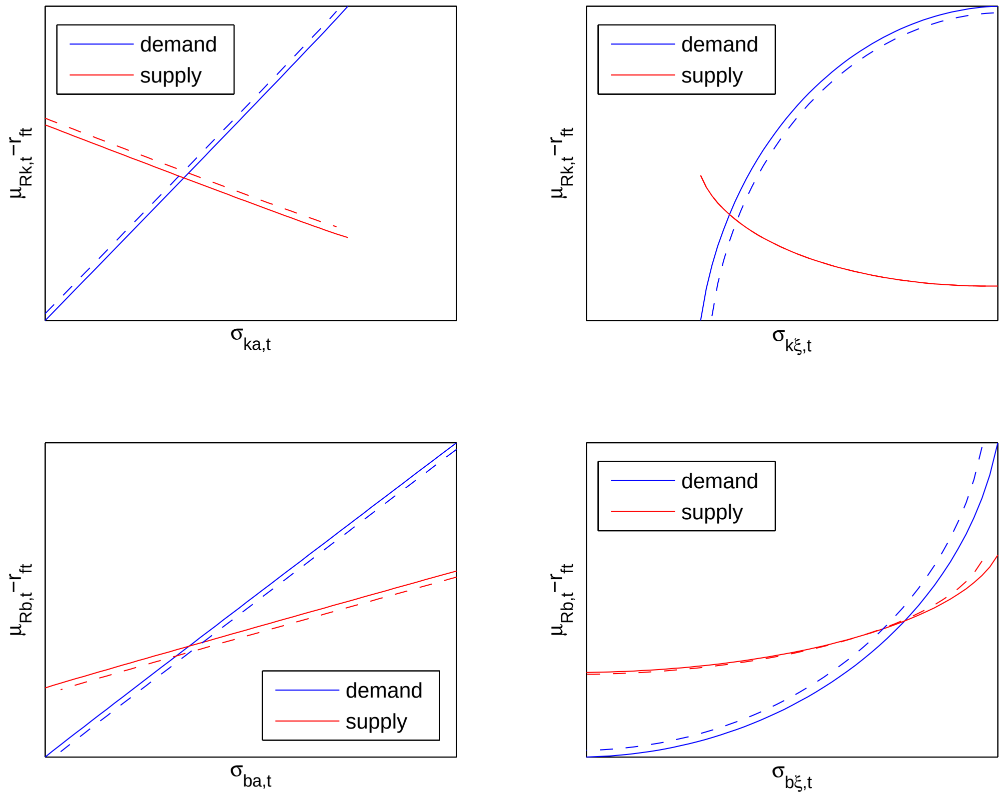


Figure 5: Asset markets clearing. "Demand" refers to the asset demand by the households; "supply" refers to the asset demand by the financial institutions. The upper panels investigate the shift in the capital market clearing equilibrium as the risk-based capital constraint is tightened, increasing α (going from the solid lines to the dashed lines). The lower panels investigate the shift in the debt market clearing equilibrium as the risk-based capital constraint is tightened, increasing α (going from the solid lines to the dashed lines).

of holding the correlated asset (direct capital investment). As with capital, the remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative, $\exp(-\xi_t)$.

In Figure 5, we plot the equilibrium risk-return trade-off for capital and intermediary debt, and investigate how the equilibrium outcome changes as the risk-based capital constraint becomes tighter. In the capital market, an increase in α leads to an increase in expected excess return and a decrease in sensitivity to the liquidity shock. In the debt market, an increase in α leads to a decrease in expected excess return, as well as a decrease in the sensitivity to productivity shocks and household liquidity shocks. Intuitively, as the risk-

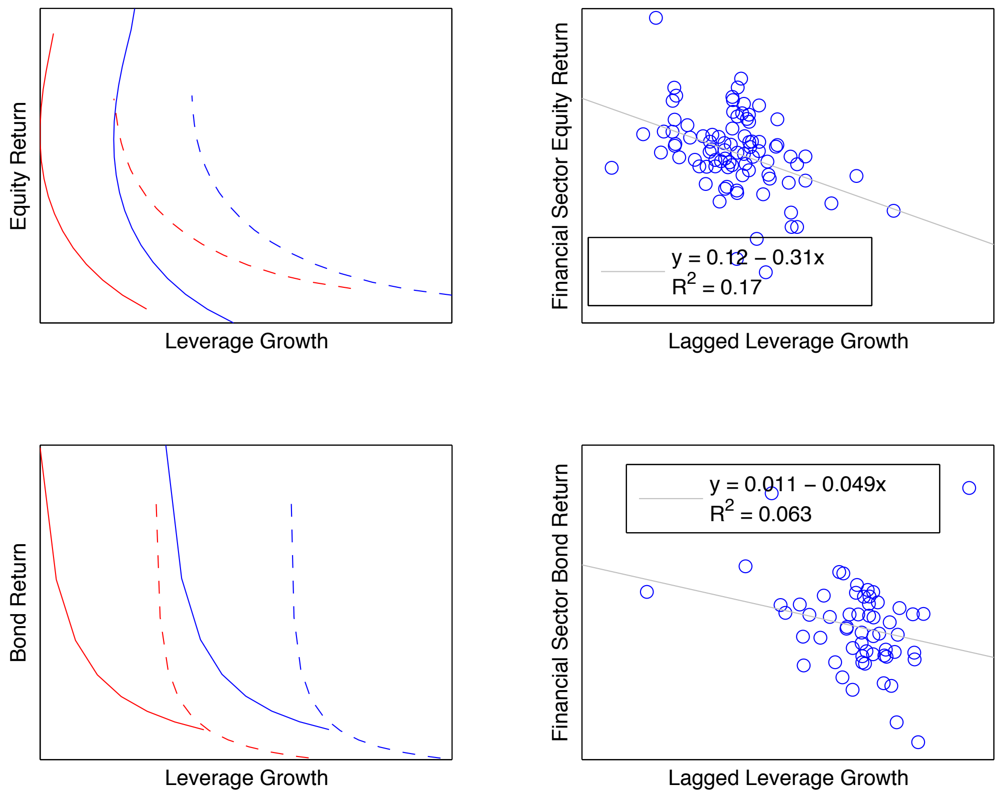


Figure 6: Expected excess returns. Upper-left panel: the trade-off between the excess return to holding capital and the growth rate of intermediary leverage. Lower-left panel: the trade-off between the excess return to intermediary debt and the growth rate of intermediary leverage. Upper-right panel: the quarterly return to holding the S&P Financial Index as a function of lagged annual broker-dealer leverage growth. Lower-right panel: the quarterly return to holding the Barclays Bond Financial Index as a function of lagged annual broker-dealer leverage growth. The figures in the left panels investigate the shift in equilibrium as intermediary wealth share is decreased (going from the solid line to the dashed line) and as the funding constraint is tightened (going from the blue to the red line). Data on broker-dealer leverage come from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics and Barclays.

based capital constraint becomes tighter, the financial intermediaries cannot take on as much leverage, making the system more resilient to the transmission of shock to household liquidity preference. This makes intermediary debt less risky, reducing the compensation for the risk associated with holding intermediary debt. Since households are less able to insure against liquidity shocks, capital becomes more risky from their viewpoint, increasing the expected excess return to holding capital.

In equilibrium, expected excess returns on both capital and risky debt are (generally) negatively related to the growth rate of intermediary leverage. The left panels of Figure 6 plot the model-implied expected excess returns as a function of intermediary leverage, for different levels of intermediary wealth and different levels of α. In particular, we see that the excess return to intermediary debt and capital increase as the growth rate of intermediary leverage decreases. Furthermore, a decrease in the intermediary's wealth share in the economy, ω_t, makes the relationship between the growth rate of leverage and the excess return to intermediary debt steeper. As a result, the excess return declines faster with an increased rate of leverage growth, while a tightening of the risk-based capital constraint does not mitigate the severity of the trade-off. The right panels of Figure 6 show that this negative relationship between returns and lagged broker-dealer leverage growth holds empirically. In fact, Adrian et al. (2010) document that broker-dealer leverage growth is a good empirical proxy for the time variation of expected returns for a variety of stock and bond portfolios.

When the economy is unconstrained, the equilibrium pricing kernel is given by

$$\begin{aligned}\frac{d\Lambda_t}{\Lambda_t} &= -r_{ft}dt - \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{ka,t} + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\sigma_{ba,t} + \sigma_\xi \rho_{\xi,a}\right) dZ_{at} \\ &\quad - \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{k\xi,t} + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\sigma_{b\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right) dZ_{\xi t}.\end{aligned}$$

While it is natural to express the pricing kernel as a function of the fundamental shocks ξ and a, these are not readily observable. Instead, we follow the empirical literature and express the pricing kernel in terms of shocks to output and leverage. Define the innovation to (log) output as

$$d\hat{y}_t = \sigma_a^{-1}\left(d\log Y_t - \mathbb{E}_t\left[d\log Y_t\right]\right) = dZ_{at}$$

and the innovation to the growth rate of leverage of the intermediaries as

$$\begin{aligned}d\hat{\theta}_t &= \left(\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2\right)^{-\frac{1}{2}}\left(\frac{d\theta_t}{\theta_t} - \mathbb{E}_t\left[\frac{d\theta_t}{\theta_t}\right]\right) \\ &= \frac{\sigma_{\theta a,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{at} + \frac{\sigma_{\theta\xi,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{\xi t}.\end{aligned}$$

Thus, we can express the pricing kernel as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt - \eta_{\theta t} d\hat{\theta}_t - \eta_{yt} d\hat{y}_t,$$

where the price of risk associated with shocks to the growth rate of intermediary leverage is

$$\eta_{\theta t} = \sqrt{1 + \frac{\sigma^2_{\theta a,t}}{\sigma^2_{\theta\xi,t}}} \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{k\xi,t} + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\sigma_{b\xi,t} + \sigma_\xi\sqrt{1-\rho^2_{\xi,a}} \right),$$

and the price of risk associated with shocks to output is

$$\begin{aligned}\eta_{yt} = & \frac{1-\theta_t\omega_t}{1-\omega_t}\left(\sigma_{ka,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{k\xi,t}\right) + \frac{\omega_t\left(\theta_t-1\right)}{1-\omega_t}\left(\sigma_{ba,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{b\xi,t}\right) \\ & + \sigma_\xi\left(\rho_{\xi,a} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sqrt{1-\rho^2_{\xi,a}}\right).\end{aligned}$$

Thus, while the households are unconstrained, a two-factor Merton (1973) ICAPM holds, with shocks to intermediary leverage driving the uncertainty about future investment opportunities.

Finally, notice that

$$\frac{dw_{ht}}{w_{ht}} = \frac{d\left(\left(1-\omega_t\right)p_{kt}A_tK_t\right)}{\left(1-\omega_t\right)p_{kt}A_tK_t}.$$

Thus, the expected rate of change in the financial intermediaries' wealth share in the economy is given by

$$\begin{aligned}\mu_{\omega t} = & \underbrace{\left(\theta_t - 1\right)\left(\mu_{Rkt} - \mu_{Rb,t}\right)}_{\text{expected porfolio return}} - \underbrace{\left(\sigma_{ka,t}\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right)}_{\text{compensation for portfolio risk}} \\ & \underbrace{+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma^2_\xi}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right]}_{\text{consumption provision to households}},\end{aligned}$$

and the loadings of the financial intermediaries' wealth share in the economy on the two sources of fundamental risk are given by

$$\sigma_{\omega a,t} = (\theta_t - 1)(\sigma_{ka,t} - \sigma_{ba,t})$$
$$\sigma_{\omega \xi,t} = (\theta_t - 1)(\sigma_{k\xi,t} - \sigma_{b\xi,t}).$$

That is, the risk loadings of the financial intermediaries' relative wealth reflect the ability of the financial intermediaries to absorb shocks to their balance sheets. The negative sign on the volatility of bond returns reflects the fact that losses in the value of the bonds benefit the intermediaries by reducing the liabilities side of their balance sheets.

3.3 Goods market clearing and price of capital

Recall that goods market clearing implies the households consume all output, except that used for investment

$$c_t = A_t (K_t - i_t k_t).$$

Recall further that the only real choice the intermediary has to make (since financing is restricted by the risk-based capital constraint) is in its optimal investment, given by

$$\frac{1}{p_{kt}} = \Phi'(i_t),$$

such that the equilibrium rate of investment is given by

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1\right).$$

As the price of capital increases, the book value of intermediary assets increases and the intermediaries are able to invest at a higher rate. Thus, in equilibrium, we must have

$$\underbrace{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) p_{kt}(1-\omega_t)}_{\text{household demand}} = \underbrace{1 - \frac{\theta_t \omega_t}{\phi_1}\left(\frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1\right)}_{\text{total supply}}.$$

The households' demand for the consumption good is driven by the households' wealth share in the economy, $1-\omega_t$, and the capital price p_{kt}. The supply of the consumption good, on the other hand, is determined by the financial intermediaries' wealth share in the economy, ω_t, financial intermediaries' leverage, θ_t, and the capital price. Denoting

$$\beta = \left(\frac{4}{\phi_0^2 \phi_1} \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) \right),$$

the price of capital solves

$$0 = p_{kt}^2 \theta_t \omega_t + \beta p_{kt} \left(1-\omega_t\right) - \frac{4}{\phi_0^2 \phi_1} - \frac{4\theta_t \omega_t}{\phi_0^2 \phi_1^2},$$

or

$$p_{kt} = \frac{-\beta\left(1-\omega_t\right) + \sqrt{\beta^2 \left(1-\omega_t\right)^2 + \frac{16}{\phi_0^2 \phi_1^2} \theta_t \omega_t \left(\phi_1 + \theta_t \omega_t\right)}}{2\theta_t \omega_t}. \tag{7}$$

As an aside, notice that, for the intermediary to disinvest, we must have

$$\left(1-\omega_t\right) \geq \frac{\phi_0 \phi_1}{2\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}.$$

Thus, the intermediary disinvests when the household is a large fraction of the economy—that is, when the intermediary has a relatively low value of equity. Applying Itô's lemma and equating coefficients, we obtain

$$\begin{aligned}
\left[dZ_{at}\right]: \quad & \beta \omega_t \sigma_{\omega a,t} = \left(2\theta_t \omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\sigma_{ka,t} - \sigma_a\right) \\
\left[dZ_{\xi t}\right]: \quad & \beta \omega_t \sigma_{\omega \xi,t} = \left(2\theta_t \omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_{k\xi,t} \\
\left[dt\right]: \quad & 0 = \left(p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2}\left(1-\theta_t \omega_t\right)\right)\theta_t \omega_t \Phi\left(i_t\right)\left(1-\theta_t \omega_t\right) \\
& + \left(2\theta_t \omega_t p_{kt} + \beta\left(1-\omega_t\right)\right) p_{kt} \left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a \sigma_{ka,t}\right) \\
& - \beta p_{kt} \omega_t \mu_{\omega t} + \theta_t \omega_t p_{kt}^2 \left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
& - \beta p_{kt} \omega_t \left(\left(\sigma_{ka,t} - \sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega \xi,t}\right).
\end{aligned}$$

Thus, in equilibrium, the financial intermediaries' wealth ratio in the economy reacts to shocks in the households' beliefs in the same direction as the return to capital.

3.4 Equilibrium

We summarize the resulting equilibrium outcomes in the following lemma.

Lemma 2. *In equilibrium, the expected excess return on capital and risky intermediary debt, as well as the expected return on intermediary equity, the risk-free rate, and the volatility of intermediary equity and intermediary debt, depends linearly on the volatility of the return to holding capital. In particular, we can express the endogenous variables as*

$$
\begin{aligned}
\mu_{Rk,t} &= \mathcal{K}_0\left(\omega_t,\theta_t\right) + \mathcal{K}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t} \\
\mu_{Rb,t} &= \mathcal{B}_0\left(\omega_t,\theta_t\right) + \mathcal{B}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{B}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
\mu_{\omega t} &= \mathcal{O}_0\left(\omega_t,\theta_t\right) + \mathcal{O}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{O}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
\mu_{\theta t} &= \mathcal{S}_0\left(\omega_t,\theta_t\right) + \mathcal{S}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} - \mathcal{O}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
r_{ft} &= \mathcal{R}_0\left(\omega_t,\theta_t\right) + \mathcal{R}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} \\
\sigma_{ba,t} &= \frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_a - \frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\theta_t\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_{ka,t} \\
\sigma_{b\xi,t} &= -\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\theta_t\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_{k\xi,t} \\
\sigma_{\theta a,t} &= -\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t}\left(\sigma_{ka,t}-\sigma_a\right) \\
\sigma_{\theta\xi,t} &= -\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t}\sigma_{k\xi,t},
\end{aligned}
$$

where the coefficients $\left(\mathcal{K}_0,\mathcal{K}_a,\mathcal{B}_0,\mathcal{B}_a,\mathcal{B}_\xi,\mathcal{O}_0,\mathcal{O}_a,\mathcal{O}_\xi,\mathcal{S}_0,\mathcal{S}_a,\mathcal{R}_0,\mathcal{R}_a\right)$ *are non linear functions of the state variables* (ω_t,θ_t)*, given by 8-21. The loadings of the return to holding capital on the shock to household beliefs,* $\sigma_{k\xi,t}$*, and on the shock to productivity,* $\sigma_{ka,t}$*, are given,* *respectively, by*

$$\sigma_{k\xi,t} = -\sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \sigma_{ka,t}^2}$$
$$\sigma_{ka,t} = \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \left(1 + \frac{1-\omega_t}{\omega_t \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)}\right).$$

Proof. See Appendix A.2. □

Notice that we pick the negative root in determining the exposure of capital to the household liquidity shocks, $\sigma_{k\xi,t}$. Intuitively, when the household experiences a negative liquidity shock, such that $dZ_{\xi t} < 0$, the household discount rate is increased, making households more impatient and decreasing the return to holding capital.

Recall that the price of the risk associated with shocks to intermediary leverage is given by

$$\eta_{\theta t} = \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta \xi,t}^2}} \left(\frac{1-\theta_t \omega_t}{1-\omega_t} \sigma_{k\xi,t} + \frac{\omega_t \left(\theta_t - 1\right)}{1-\omega_t} \sigma_{b\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2}\right).$$

Substituting the equilibrium expressions for $\sigma_{\theta a,t}$, $\sigma_{\theta\xi,t}$ and $\sigma_{b\xi,t}$, we obtain

$$\eta_{\theta t} = \sqrt{1 + \frac{\left(\sigma_{ka,t} - \sigma_a\right)^2}{\sigma_{k\xi,t}^2}} \left(-\frac{2\theta_t \omega_t p_{kt}}{\beta \left(1-\omega_t\right)} \sigma_{k\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2}\right).$$

Since capital has a negative exposure to the households' preference shocks, the price of risk associated with shocks to intermediary leverage is positive, so leverage risk commands a positive risk premium. While the sign of the risk premium is always positive, the dependence of the price of leverage risk on the leverage growth rate is nonmonotonic. The empirical literature strongly favors the positive price of leverage risk for stock and bond returns (see Adrian et al. (2011b)) and a negative relationship between the price of risk and the growth rate of leverage (see Adrian et al. (2010)). The left panel Figure 7 plots the price of risk of shocks to leverage as a function of leverage growth, showing a negative relationship between the two. Notice that tightening the risk-based capital constraint does not change the shape of the trade-off but rather reduces the possible range of growth rates of leverage.

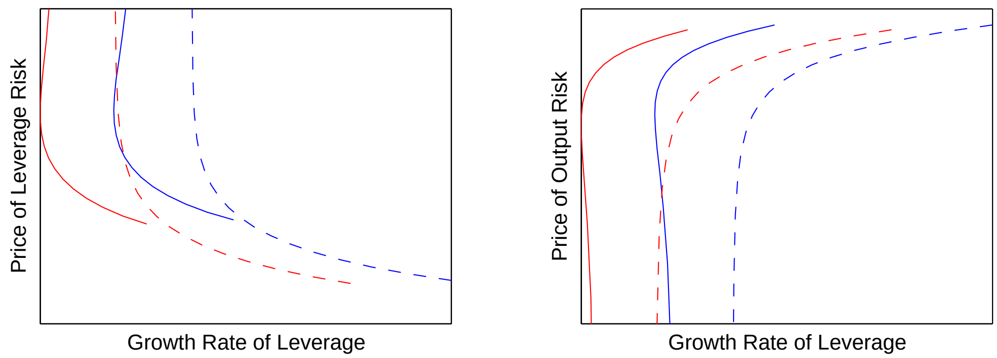


Figure 7: Equilibrium risk prices. Left panel: the price of risk associated with shocks to leverage ($\eta_{\theta t}$) as a function of leverage growth. Right panel: the price of risk associated with shocks to output (η_{yt}) as a function of leverage growth. The figure investigates the change in equilibrium as intermediary wealth share is reduced (going from the solid line to the dashed line) and as the risk-based capital constraint is tightened (going from the blue to the red line).

Similarly, the price of risk associated with shocks to output is given, in equilibrium, by

$$\begin{aligned}\eta_{yt} &= \frac{1-\theta_t\omega_t}{1-\omega_t}\left(\sigma_{ka,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{k\xi,t}\right) + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\left(\sigma_{ba,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{b\xi,t}\right) \\ &+ \sigma_\xi\left(\rho_{\xi,a} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right) \\ &= \sigma_a + \sigma_\xi\left(\rho_{\xi,a} - \frac{\sigma_{ka,t}-\sigma_a}{\sigma_{k\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right).\end{aligned}$$

Unlike the price of leverage risk, the price of risk associated with shocks to output changes signs, depending on whether the equilibrium sensitivity of the return to holding capital to output shocks is lower or higher than the fundamental volatility. The time-varying nature of the direction of the risk premium for output shocks makes it difficult to detect in observed returns, suggesting an explanation for the poor performance of the production CAPM in the data.

4 Financial Stability and Household Welfare

In this section, we describe the term structure of the distress probability, $\delta_t(T)$, and, in particular the effect of a tightening of the risk-based capital constraint. We then compare

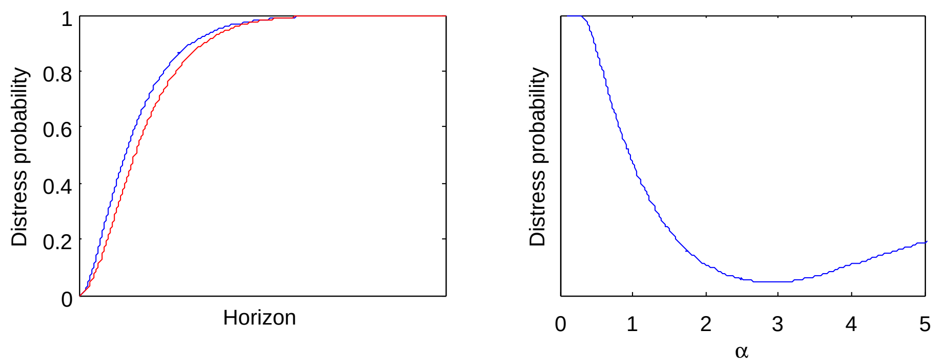


Figure 8: Probability of distress of financial intermediaries. Left panel: the term structure of the cumulative probability of distress, for two different levels of α. The movement from the blue term structure to the red term structure corresponds to a tightening of the risk-based capital constraint (higher α). Right panel: the probability of the intermediary becoming distressed within six months as a function of the tightness of the risk-based capital constraint. Probabilities of distress are calculated using 10000 simulations of sample paths of the economy.

the equilibrium outcomes in our model to the equilibrium outcomes in one with constant leverage. Finally, we discuss some implications of the risk-based capital constraint for the welfare of the households in the economy.

4.1 Intermediary distress

We begin by considering the term structure of the probability of distress. Figure 8 plots the cumulative probability of the intermediary becoming distressed before a given horizon.[4] As the capital constraint tightens, the whole term structure of cumulative distress probabilities shifts down. That is, as the capital constraint becomes more stringent, the expected time to distress increases. Intuitively, as the capital constraint tightens, the intermediaries in the economy are less able to increase their leverage. Thus, they borrow less from the households, decreasing the probability that the intermediaries become distressed. As the risk-based capital constraint becomes too tight, the intermediaries are unable to build enough inside equity during periods of positive output growth, making them more likely to become distressed.

[4]Although this probability cannot be computed analytically, we can easily compute it using Monte Carlo simulations. For the computation of the probability of distress, as well as for the expected discounted present value of household utility, we simulated 10000 paths of the economy.

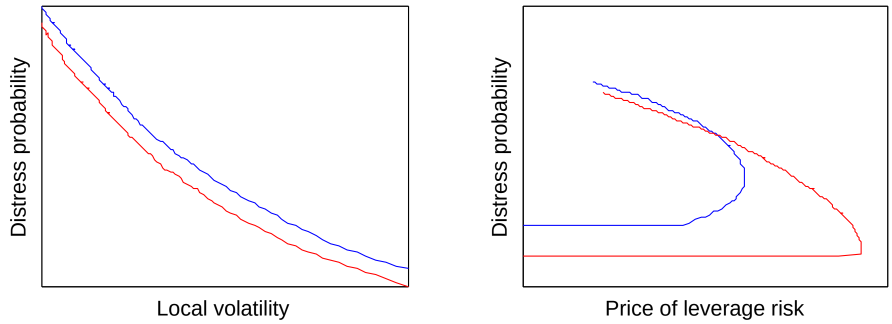


Figure 9: Volatility paradox. Left panel: the equilibrium trade-off between local volatility (x-axis) and the probability of distress of financial intermediaries (y-axis), for two different levels of α. Right panel: the equilibrium trade-off between the price of risk associated with shocks to the growth rate of intermediary leverage (x-axis) and the probability of distress of financial intermediaries (y-axis), for two different levels of α. Local volatility is computed as the instantaneous volatility of the return to holding capital. The movement from the blue equilibrium outcomes to the red equilibrium outcomes corresponds to a tightening of the risk-based capital constraint (higher α). Probabilities of distress are calculated using 10000 simulations of sample paths of the economy.

The right panel of Figure 8 plots the probability of the intermediary becoming distressed within six months as a function of the tightness of the constraint. We will return to the issue of nonlinearity of equilibrium outcomes with respect to α later in this section, when we discuss implications for household welfare.

Consider now the trade-off between the instantaneous riskiness of capital investment and the long-run fragilities in the economy. The left panel of Figure 9 plots the six month distress probability as a function of the current instantaneous volatility of the return to holding capital. We see that the model-implied quantities have the negative relationship observed in the run-up to the 2007-2009 financial crisis. In the context of the model, local volatility is inversely proportional to leverage. As leverage increases, the intermediaries issue more risky debt, making distress more likely. This leads to the negative relationship between the probability of distress and current period return volatility. Notice that tightening the risk-based capital constraint does not change the shape of the trade-off but rather limits the admissible range of current period volatility. The right panel of Figure 8 plots the trade-off between the six month distress probability and the price of risk associated with shocks to the growth rate of intermediary leverage. As the intermediary becomes more likely to be

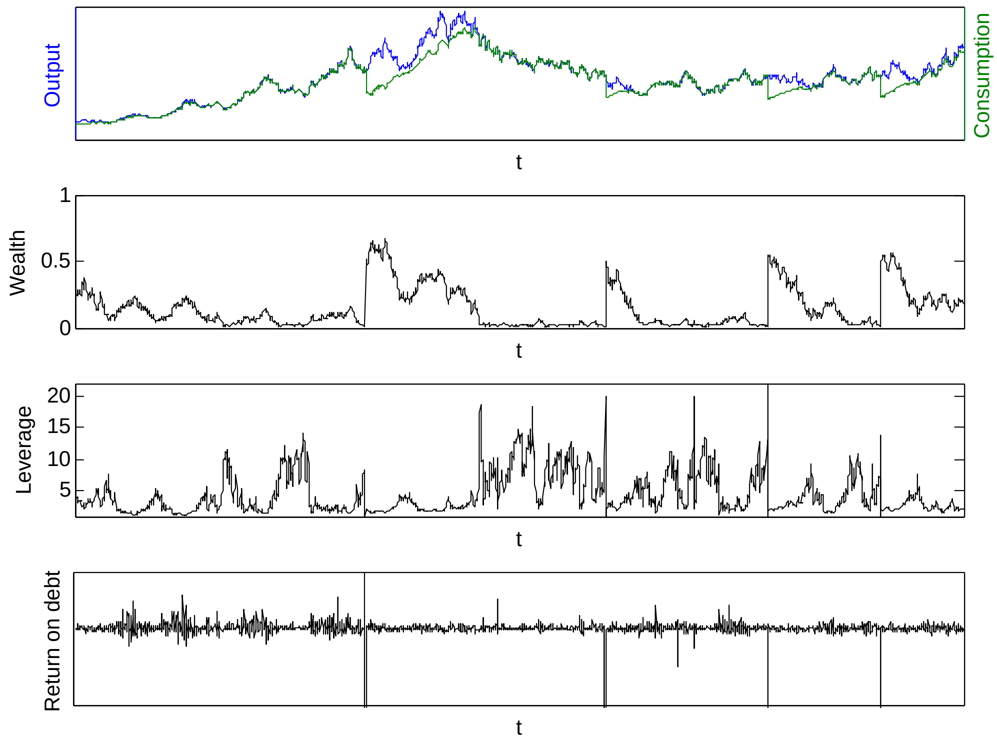


Figure 10: Sample evolution of the economy. Upper panel: evolution of output (left scale) and consumption (right scale). Middle two panels: evolution of the intermediary's wealth share (ω_t) in the economy and evolution of intermediary leverage, respectively. Lower panel: realized return on risky bank debt.

distressed, the households demand a higher compensation for the risk of being exposed to leverage shocks. Moreover, the sensitivity of the price of risk to changes in distress risk becomes higher as the probability of distress increases.

Intermediary distress is costly (in consumption terms) for the households. In Figure 10, we plot a sample evolution of the economy, focusing on the evolution of output and consumption (upper panel), intermediary wealth share in the economy and intermediary leverage (middle panels), and of the realized return to intermediary debt (lower panel). Notice first that, while intermediary distress is usually preceded by high intermediary leverage, distress can occur even when intermediary leverage is relatively low. Moreover, intermediaries can maintain high levels of leverage without becoming distressed. Thus, high leverage is not a foolproof indicator of distress risk. The recapitalization of intermediaries comes at the cost of a consumption drop for the households, which can be quite significant. Since the

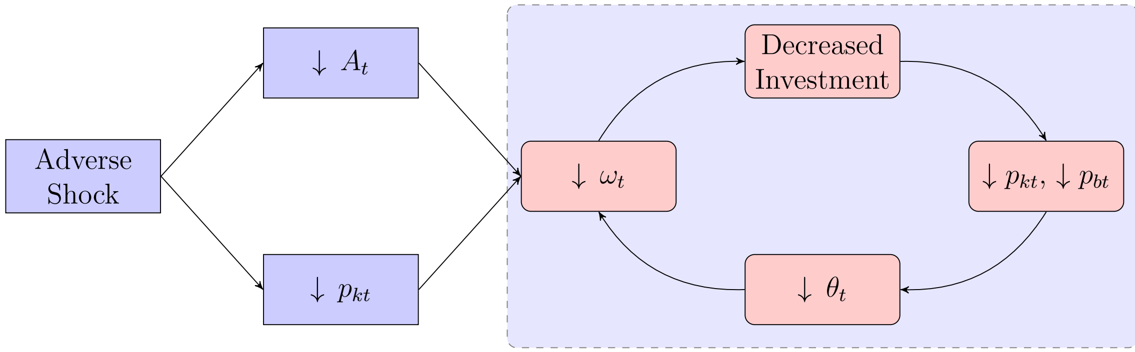


Figure 11: Shock amplification

restructuring of the intermediary is done through default on debt, household wealth (and, hence, consumption) exhibits sharp declines when the intermediary becomes distressed.

4.2 Distortions and amplifications

The simulated path of the economy in Figure 10 illustrates the negative implications of intermediary distress for the households in the economy. The risk-based capital constraint faced by the intermediaries in our economy amplifies the fundamental shocks in the economy and distorts equilibrium outcomes. This amplification mechanism is illustrated in Figure 11: A shock to the relative wealth of the intermediaries reduces the equilibrium level of investment, reducing the price of capital, which makes the risk-based capital constraint bind more, reducing further the financial intermediaries' relative wealth. The amplification mechanism acts through the time-varying leverage constraint that is induced by the risk-sensitive capital constraint. To understand the mechanism better, we describe the equilibrium outcomes in an economy with constant leverage, and contrast the resulting dynamics with those in the full model.

In particular, consider an economy in which, instead of facing the risk-based capital constraint, the intermediaries face a constant leverage constraint, such that

$$\frac{p_{kt} A_t k_t}{w_t} = \bar{\theta},$$

where $\bar{\theta}$ is a constant set by the prudential regulator. The equilibrium outcomes are summarized in the following lemma.

Lemma 3. *The economy with constant leverage converges to an economy with a constant wealth share of the intermediary in the economy*

$$\omega_t = \bar{\theta}^{-1}.$$

In the steady state, the intermediary sector owns all the capital in the economy, with the expected excess return to holding capital given by

$$\mu_{Rk,t} - r_{ft} = \frac{1}{p_k} + \sigma_a^2 - \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \Phi\left(i_t\right),$$

and the expected excess return to holding bank debt given by

$$\mu_{Rb,t} - r_{ft} = \sigma_a^2,$$

with the riskiness of the returns equal to the riskiness of the productivity growth

$$\sigma_{ka,t} = \sigma_{ba,t} = \sigma_a$$
$$\sigma_{k\xi,t} = \sigma_{b\xi,t} = 0.$$

Proof. See Appendix A.3. □

Thus, when the financial intermediaries face a constant leverage constraint, the intermediary sector does not amplify the fundamental shocks in the economy. Furthermore, since the intermediaries represent a constant fraction of the wealth of the economy with constant leverage, there is no risk of intermediary distress. Notice, however, that the excess return to holding capital compensates investors for the cost of capital adjustment. Thus, the financial system provides a channel through which market participants can share the cost of capital investment.

The benefit of having a financial system with a flexible leverage constraint is, then, increased output growth and more valuable capital, albeit at the cost of global stability. Since the rate of investment and the capital price are constant in this benchmark, the volatility of consumption growth equals the volatility of productivity growth, and the expected consumption

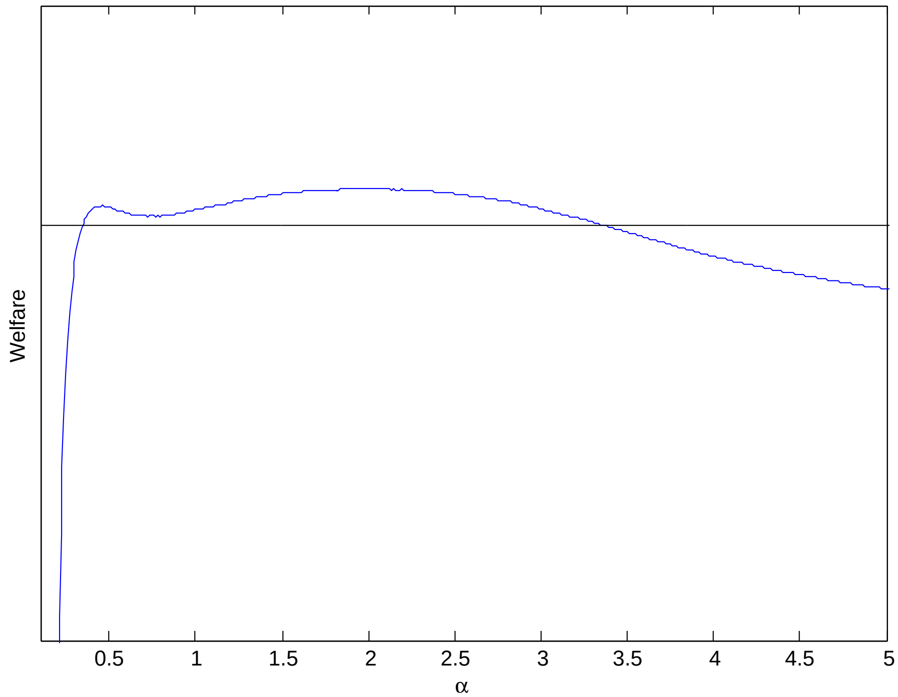


Figure 12: Household welfare. The expected discounted present value of household utility in the economy with pro-cyclical intermediary leverage (blue) and in the economy with constant intermediary leverage (black) as a function of the tightness of the risk-based capital constraint (x-axis). Expected present values are computed using 10000 simulations of the economy.

growth rate equals the expected productivity growth rate. In our model, the financial intermediary sector allows households to smooth consumption, reducing the instantaneous volatility of consumption during good times, but at the cost of higher consumption growth volatility during times of financial distress. In particular, notice that, in the full model, volatility of consumption growth is given by

$$\left\langle \frac{dc_t}{c_t} \right\rangle^2 = \left(-\frac{2\theta_t \omega_t}{\beta \left(1-\omega_t\right)} p_{kt} \left(\sigma_{ka,t} - \sigma_a\right) + \sigma_a \right)^2 + \left(\frac{2\theta_t \omega_t}{\beta \left(1-\omega_t\right)} p_{kt} \sigma_{k\xi,t} \right)^2,$$

which is lower than the fundamental volatility σ_a^2 when $\sigma_{ka,t}$ is bigger than σ_a.

More formally, consider the trade-off in terms of the expected discounted present value of household utility. In Figure 12, we plot the household welfare in the economy with pro-cyclical intermediary leverage as a function of the tightness of the risk-based capital constraint, as well as the the household welfare in the economy with constant leverage. Notice first that household welfare is not monotone in α: Initially, as the risk-based capital constraint becomes tighter, household welfare increases as distress risk decreases. For high enough levels of α, however, the household welfare decreases as the risk-based capital constraint becomes tighter. Intuitively, for low values of α, periods of financial distress (which are accompanied by sharp drops in consumption) are more frequent and the households become better off as the constraint becomes tighter. As α increases, the intermediaries become more stable, increasing household welfare. As α becomes too large, however, the risk-sharing function of the intermediaries is impeded, leading to higher probabilities of financial sector distress (see Figure 8) and lower household utility. Notice finally that household welfare in the economy with pro-cyclical leverage can be higher than that in the economy with constant leverage, even when a suboptimal α is chosen.

4.3 Stress tests

By introducing preferences for the financial intermediary, we can extend our model to study the impact of the use of stress tests as a macroprudential tool. By further introducing preferences for the prudential regulator, the model also provides implications for the optimal design of stress tests. We leave the formal treatment of these extensions for future work and provide here a sketch of how stress tests can be incorporated in the current setting.

Recall that, in our model, intermediary debt is subject to the risk-based capital constraint, which is a constraint on the local volatility of the asset side of the intermediary balance sheet

$$\theta_t^{-1} \geq \alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Stress tests, on the other hand, can be interpreted as a constraint on the total volatility of the asset side of the balance sheet over a fixed time interval

$$\theta_t^{-1} \geq \vartheta\sqrt{\mathbb{E}_t\left[\int_t^T \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2\right) ds\right]}.$$

Thus, in effect, stress tests can be thought of as a Stackelberg game between the policymaker and the financial intermediary, with the policymaker moving first to choose the maximal allowable level of volatility over a time interval, and the intermediary moving second to allocate the volatility allowance between different periods. Under the assumption that the prudential regulator designs stress tests to minimize total volatility, while the intermediary maximizes the expected discounted value of equity, the optimization problem for the intermediary resembles the optimal robust control problem under model misspecification studied by Hansen et al. (2006); Hansen and Sargent (2001); Hansen et al. (1999); Hansen and Sargent (2007), among others

$$V_t(\vartheta) = \max_{\{i,\beta,k\}} \min_{q \in \mathcal{Q}(\vartheta)} \int \int_t^{\tau_D} e^{-\rho(s-t)} w_t(i,\beta,k)\, ds dq$$

subject to

$$\theta_t^{-1} \geq \vartheta \sqrt{\int_t^T \int \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2\right) dq_s ds}.$$

Notice that, in the limit at $T \to t + dt$, this reduces to the risk-based capital constraint described above. In the language of Hansen et al. (2006), this is a *nonsequential* problem since the constraint is over a non-infinitesimal time horizon. The density function q is a density over the future realizations of the fundamental shocks $(dZ_{at}, dZ_{\xi t})$ in the economy, and $\mathcal{Q}$ is the set of densities that satisfies the stress-test constraint. Hansen et al. (2006) show how to move from the nonsequential robust controls problems to sequential problems. In particular, for the constraint formulation, they augment the state-space to include the continuation value of entropy and solve for the optimal value function that also depends on this continuation entropy.

In our setting, we can reformulate the optimization problem of the financial intermediary as

$$V_t(\vartheta) = \max_{\{i,\beta,k,\alpha_s\}} \mathbb{E}_t \left[\int_t^{\tau_D} e^{-\rho(s-t)} w_t(i,\beta,k)\, ds \right]$$

subject to

$$\frac{\theta_s^{-1}}{\alpha_s} \geq \sqrt{\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2}$$
$$\theta_t^{-1} \geq \vartheta \sqrt{\mathbb{E}_t \left[\int_t^T \frac{\theta_s^{-2}}{\alpha_s^2} ds \right]}.$$

That is, the intermediary chooses an optimal capital plan at the time of the stress test to maximize the discounted present value of equity subject to satisfying the intertemporal volatility constraint imposed by the stress test. Locally, the portfolio allocation decision of the intermediary satisfies a risk-based capital constraint, albeit with a time-varying α. However, along a given capital plan, the optimal decisions of both the households and the intermediary are as described above. Stress tests are hence a natural but technically challenging extension of the current setup and are left for future exploration.

5 Conclusion

We present a dynamic, general equilibrium theory of financial intermediaries' leverage cycle as a conceptual basis for policies geared toward financial stability. In this setup, any change in prudential policies has general equilibrium effects that impact the pricing of financial and nonfinancial credit, the equilibrium volatilities of financial and real assets, and the allocation of consumption and investment goods. From a normative point of view, such effects are important to understand, as they ultimately determine the effectiveness of prudential policies.

The assumptions of our model are empirically motivated, and our theory captures many important stylized facts about financial intermediary dynamics that have been documented in the literature. There is both direct and intermediated credit by households, giving rise to a substitution from intermediated credit to directly granted credit in times of tighter intermediary constraints. The risk-based funding constraint leads to procyclical intermediary leverage, matching empirical observations. Our theory generates the volatility paradox: times of low contemporaneous volatility allow high intermediary leverage, increases in forward-looking systemic risk. Finally, the time variation in pricing of risk is a function of leverage growth and the price of risk of asset exposure is positive, two additional features that are strongly borne out in the data.

The most important contribution of the paper is to directly study the impact of prudential policies on the likelihood of systemic liquidity and solvency risks. We uncover a systemic risk-return trade-off: Tighter intermediary capital requirements tend to shift the term structure of systemic downward, at the cost of increased risk pricing today.

References

Tobias Adrian and Markus K. Brunnermeier. CoVaR. Federal Reserve Bank of New York Staff Reports No. 348, 2010.

Tobias Adrian and Hyun Song Shin. Procyclical Leverage and Value-at-Risk. Federal Reserve Bank of New York Staff Reports No. 338, 2010a.

Tobias Adrian and Hyun Song Shin. Liquidity and leverage. *Journal of Financial Intermediation*, 19(3):418 – 437, 2010b.

Tobias Adrian, Emanuel Moench, and Hyun Song Shin. Financial Intermediation, Asset Prices, and Macroeconomic Dynamics. Federal Reserve Bank of New York Staff Reports No. 442, 2010.

Tobias Adrian, Paolo Colla, and Hyun Song Shin. Which Financial Frictions? Parsing the Evidence from the Financial Crisis of 2007-09. Federal Reserve Bank of New York Staff Reports No. 528, 2011a.

Tobias Adrian, Erkko Etula, and Tyler Muir. Financial Intermediaries and the Cross-Section of Asset Returns. Federal Reserve Bank of New York Staff Reports No. 464, 2011b.

Franklin Allen and Douglas Gale. Limited market participation and volatility of asset prices. *American Economic Review*, 84:933–955, 1994.

Paolo Angelini, Stefano Neri, and Fabio Panetta. Monetary and Macroprudential Policies. Bank of Italy Staff Report Number 801, 2011.

Ben Bernanke and Mark Gertler. Agency Costs, Net Worth, and Business Fluctuations. *American Economic Review*, 79(1):14–31, 1989.

Harjoat S. Bhamra, Lars-Alexander Kuehn, and Ilya A. Strebulaev. The Levered Equity Risk Premium and Credit Spreads: A Unified Approach. *Review of Financial Studies*, 23 (2):645–703, 2010a.

Harjoat S. Bhamra, Lars-Alexander Kuehn, and Ilya A. Strebulaev. The Aggregate Dynamics of Capital Structure and Macroeconomic Risk. *Review of Financial Studies*, 23(12):4175–4241, 2010b.

Javier Bianchi and Enrique Mendoza. Overborrowing, Financial Crises and Macro-prudential Policy. IMF Working Paper 11/24, 2011.

Markus K. Brunnermeier and Lasse Heje Pedersen. Market Liquidity and Funding Liquidity. *Review of Financial Studies*, 22(6):2201–2238, 2009.

Markus K. Brunnermeier and Yuliy Sannikov. The I Theory of Money. Unpublished working paper, Princeton University, 2011.

Markus K. Brunnermeier and Yuliy Sannikov. A Macroeconomic Model with a Financial Sector. Unpublished working paper, Princeton University, 2012.

John Cox and Chi–fu Huang. Optimal Consumption and Portfolio Policies when Asset Prices follow a Diffusion Process. *Journal of Economic Theory*, 49:33–83, 1989.

Jakša Cvitanić and Ioannis Karatzas. Convex Duality in Constrained Portfolio Optimization. *The Annals of Applied Probability*, 2(4):767–818, 1992.

Jon Danielsson, Hyun Song Shin, and Jean-Pierre Zigrand. Balance sheet capacity and endogenous risk. Working Paper, 2011.

Douglas W. Diamond and Philip H. Dybvig. Bank runs, deposit insurance and liquidity. *Journal of Political Economy*, 93(1):401–419, 1983.

Ana Fostel and John Geanakoplos. Leverage Cycles and the Anxious Economy. *American Economic Review*, 98(4):1211–1244, 2008.

John Geanakoplos. Liquidity, Default, and Crashes: Endogenous Contracts in General Equilibrium. In M. Dewatripont, L.P. Hansen, and S.J. Turnovsky, editors, *Advances in Economics and Econometrics II*, pages 107–205. Econometric Society, 2003.

Mark Gertler and Nobuhiro Kiyotaki. Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy. Unpublished working papers, Princeton University, 2012.

Mark Gertler, Nobuhiro Kiyotaki, and Albert Queralto. Financial Crises, Bank Risk Exposure, and Government Financial Policy. Unpublished working papers, Princeton University, 2011.

Charles A.E. Goodhart, Anil K. Kashyap, Dimitrios P. Tsomocos, and Alexandros P. Vardoulakis. Financial Regulation in General Equilibrium. NBER Working Paper No. 17909, 2012.

Lars Peter Hansen and Thomas J. Sargent. Robust control and model uncertainty. *American Economic Review*, 91:60–66, 2001.

Lars Peter Hansen and Thomas J. Sargent. Recursive robust estimation and control without commitment. *Journal of Economic Theory*, 136:1–27, 2007.

Lars Peter Hansen, Thomas J. Sargent, and Jr. Thomas D. Tallarini. Robust permanent income and pricing. *Review of Economic Studies*, 66:873–907, 1999.

Lars Peter Hansen, Thomas J. Sargent, Gauhar A. Turmuhambetova, and Noah Williams. Robust control, min-max expected utility, and model misspecification. *Journal of Economic Theory*, 128:45–90, 2006.

Zhiguo He and Arvind Krishnamurthy. A model of capital and crises. *Review of Economic Studies*, 79(2):735–777, 2012a.

Zhiguo He and Arvind Krishnamurthy. Intermediary asset pricing. *American Economic Review*, 2012b.

Zhiguo He and Arvind Krishnamurthy. A Macroeconomic Framework for Quantifying Systemic Risk. Unpublished working paper, 2012c.

Bengt Holmström and Jean Tirole. Private and public supply of liquidity. *Journal of Political Economy*, 106(1):1–40, 1998.

Nobuhiro Kiyotaki and John Moore. Credit Cycles. *Journal of Political Economy*, 105(2): 211–248, 1997.

Hayne E. Leland and Klaus Bjerre Toft. Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. *Journal of Finance*, 51(3):987–1019, 1996.

Robert C. Merton. An Intertemporal Capital Asset Pricing Model. *Econometrica*, 41(5): 867–887, 1973.

Adriano A. Rampini and S. Viswanathan. Financial Intermediary Capital. Duke University Working Paper, 2012.

A Online Appendix

A.1 Household's optimization (Proof of Lemma 1)

Recall that the household solves the portfolio optimization problem:

$$\max_{\{c_t,\pi_{kt},\pi_{bt}\}} \quad \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the wealth evolution equation:

$$\begin{aligned} dw_{ht} &= r_{ft}w_{ht}dt + w_{ht}\pi_{kt}\left\{\left(\mu_{Rk,t}-r_{ft}\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t}\right\} \\ &+ w_{ht}\pi_{bt}\left\{\left(\mu_{Rb,t}-r_{ft}\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}\right\} - c_t dt, \end{aligned}$$

and the no-shorting constraints:

$$\pi_{kt},\ \pi_{bt} \geq 0.$$

Instead of solving the dynamic optimization problem, we follow Cvitanić and Karatzas (1992) and rewrite the household problem in terms of a static optimization. Cvitanić and Karatzas (1992) extend the Cox and Huang (1989) martingale method approach to constrained optimization problems, such as the one that the households face in our economy.

Define $K = \mathbb{R}^2_+$ to be the convex set of admissible portfolio strategies and introduce the support function of the set $-K$ to be

$$\begin{aligned} \delta\left(x\right) = \delta\left(x\right| K\right) &\equiv \sup_{\vec{\pi}\in K}\left(-\vec{\pi}'x\right) \\ &= \begin{cases} 0, & x\in K \\ +\infty, & x\notin K \end{cases}. \end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$r_{ft}^{v} = r_{ft} + \delta\left(\vec{v}_t\right)$$
$$dR_{kt}^{v} = \left(\mu_{Rk,t} + v_{1t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t}$$
$$dR_{bt}^{v} = \left(\mu_{Rb,t} + v_{2t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t},$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V(K)$ of square-integrable, progressively measurable processes taking values in K. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta\left(\vec{v}_t\right)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma_{Rt}'\right)^{-1} d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t}\ \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t}\log c_t dt\right]$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_t dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves the dual problem

$$\min_{v \in V(K)}\ \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t}\tilde{u}\left(\lambda \eta_t^v\right)dt\right],$$

where $\tilde{u}(x)$ is the convex conjugate of $-u(-x)$

$$\tilde{u}(x) \equiv \sup_{z>0}\left[\log(zx) - zx\right] = -\left(1 + \log x\right)$$

and λ is the Lagrange multiplier of the static budget constraint. Cvitanić and Karatzas (1992) show that, for the case of logarithmic utility, the optimal choice of v satisfies

$$\begin{aligned} v_t^* &= \arg\min_{x\in K} \left\{ 2\delta(x) + \left|\left| (\vec{\mu}_{Rt} - r_{ft} + x)' \, \sigma_{Rt}^{-1} \right|\right|^2 \right\} \\ &= \arg\min_{x\in K} \left|\left| (\vec{\mu}_{Rt} - r_{ft} + x)' \, \sigma_{Rt}^{-1} \right|\right|^2 . \end{aligned}$$

Thus,

$$\begin{aligned} v_{1t} &= \begin{cases} 0, & \mu_{Rk,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rk,t}, & \mu_{Rk,t} - r_{ft} < 0 \end{cases} \\ v_{2t} &= \begin{cases} 0, & \mu_{Rb,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rb,t}, & \mu_{Rb,t} - r_{ft} < 0 \end{cases} \end{aligned} .$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\xi_t - \rho_h t}}{c_t} - \lambda \eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t - \rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t \left[\int_t^{+\infty} \eta_s^v c_s ds \right] = \mathbb{E}_t \left[\int_t^{+\infty} \frac{e^{-\xi_s - \rho_h s}}{\lambda} ds \right] = \frac{e^{-\xi_t - \rho_h t}}{\lambda \left(\rho_h - \sigma_\xi^2/2 \right)}.$$

Thus

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$
\begin{aligned}
\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} &= -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 \\
&= \left(-\rho_h + \frac{1}{2}\sigma_\xi^2\right) dt - \sigma_\xi \rho_{\xi,a} dZ_{at} - \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} dZ_{\xi t}.
\end{aligned}
$$

On the other hand, applying Itô's lemma, we obtain

$$
\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}} \frac{d\eta_t^v}{\eta_t^v}.
$$

Equating the coefficients on the stochastic terms, we obtain

$$
\vec{\pi}_t' = \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)' \left(\sigma_{Rt}' \sigma_{Rt}\right)^{-1} - \sigma_\xi \begin{bmatrix} \rho_{\xi a} & \sqrt{1-\rho_{\xi a}^2} \end{bmatrix} \sigma_{Rt}^{-1}.
$$

A.2 Equilibrium outcomes (Proof of Lemma 2)

To summarize, in equilibrium, we must have

$$
\begin{aligned}
\mu_{\theta t} &= \Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2 \\
\mu_{Rk,t} - r_{ft} &= \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right)\frac{1-\theta_t\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t-\omega_t}{1-\omega_t} \\
&\quad + \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
\mu_{Rb,t} - r_{ft} &= \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)\frac{\theta_t\omega_t-\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t} \\
&\quad + \sigma_\xi\left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
\mu_{\omega t} &= \left(\theta_t - 1\right)\left(\mu_{Rkt} - \mu_{Rb,t}\right) + \left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) \\
&\quad + \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right] \\
\sigma_{\theta a,t} &= -\left(\theta_t - 1\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right) \\
\sigma_{\theta\xi,t} &= -\left(\theta_t - 1\right)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) \\
\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{ba,t} &= -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{ka,t} + \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_a \\
\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{b\xi,t} &= -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{k\xi,t} \\
\alpha^{-2}\theta_t^{-2} &= \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 \\
0 &= \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\left(1-\theta_t\omega_t\right)\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) \\
&\quad + \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)p_{kt}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\sigma_{ka,t}\right) \\
&\quad - \beta p_{kt}\omega_t\mu_{\omega t} + \theta_t\omega_t p_{kt}^2\left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
&\quad - \beta p_{kt}\omega_t\left(\left(\sigma_{ka,t} - \sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right).
\end{aligned}
$$

Notice that the first eight equations describe the evolutions of $\theta_t\omega_t$, ω_t, the return of risky intermediary debt R_{bt}, and the expected excess return to direct capital holding in terms of the two state variables, $(\theta_t\omega_t, \omega_t)$ and the loadings, $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$, of the return to direct capital holding on the two fundamental shocks in the economy.[5] The final two equations, then, express $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$ in terms of the state variables.

[5]Recall from Equation 7 that we have also expressed the price of capital in terms of the state variables.

Before solving the final two equations, we simplify the equilibrium conditions. Notice first that

$$\left(\sigma_{ka,t}\sigma_{\theta a,t}+\sigma_{k\xi,t}\sigma_{\theta\xi,t}\right)=-\left(\theta_t-1\right)\left(\sigma_{ka,t}\left(\sigma_{ka,t}-\sigma_{ba,t}\right)+\sigma_{k\xi,t}\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)\right),$$

and

$$\begin{aligned}\mu_{Rkt}-\mu_{Rb,t}=&\left(\sigma_{ka,t}^2+\sigma_{k\xi,t}^2-\sigma_{ka,t}\sigma_{ba,t}-\sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t}\\&-\left(\sigma_{ba,t}^2+\sigma_{b\xi,t}^2-\sigma_{ka,t}\sigma_{ba,t}-\sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\\&+\sigma_\xi\left(\left(\sigma_{ka,t}-\sigma_{ba,t}\right)\rho_{\xi,a}+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\left(\mu_{Rkt}-\mu_{Rb,t}\right)+\frac{1}{\theta_t-1}\left(\sigma_{ka,t}\sigma_{\theta a,t}+\sigma_{k\xi,t}\sigma_{\theta\xi,t}\right)=&-\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\left(\left(\sigma_{ka,t}-\sigma_{ba,t}\right)^2+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)^2\right)\\&+\sigma_\xi\left(\left(\sigma_{ka,t}-\sigma_{ba,t}\right)\rho_{\xi,a}+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).\end{aligned}$$

Using

$$\begin{aligned}\beta\left(\theta_t\omega_t-\omega_t\right)\left(\sigma_{ka,t}-\sigma_{ba,t}\right)&=\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\sigma_{ka,t}-\sigma_a\right)\\\beta\left(\theta_t\omega_t-\omega_t\right)\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)&=\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\sigma_{k\xi,t}\end{aligned}$$

we can thus express the drift of ω_t as

$$\begin{aligned}\mu_{\omega t}=&-\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2\left[\left(\sigma_{ka,t}-\sigma_a\right)^2+\sigma_{k\xi,t}^2\right]\\&+\frac{\sigma_\xi}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\left(\sigma_{ka,t}-\sigma_a\right)\rho_{\xi,a}+\sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)\\&+\frac{1-\omega_t}{\omega_t}\left[\left(\rho_h-\frac{\sigma_\xi^2}{2}\right)-\frac{1}{p_{kt}}+\Phi\left(i_t\right)\theta_t\omega_t\right].\end{aligned}$$

Substituting the risk-based capital constraint, this becomes

$$\begin{aligned}\mu_{\omega t} &= -\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left[\sigma_a^2 - 2\sigma_a\sigma_{ka,t} + \frac{\theta_t^{-2}}{\alpha^2}\right] \\ &+ \frac{\sigma_\xi}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\left(\sigma_{ka,t}-\sigma_a\right)\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\ &+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right] \\ &\equiv \mathcal{O}_0\left(\omega_t,\theta_t\right) + \mathcal{O}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{O}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t},\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}_0\left(\omega_t,\theta_t\right) &= -\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left[\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2}\right] \\ &- \frac{\sigma_\xi\sigma_a\rho_{\xi,a}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right) \\ &+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right]\end{aligned} \tag{8}$$

$$\mathcal{O}_a\left(\omega_t,\theta_t\right) = \frac{2\sigma_a}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 + \frac{\sigma_\xi\rho_{\xi,a}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right) \tag{9}$$

$$\mathcal{O}_\xi\left(\omega_t,\theta_t\right) = \frac{\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right). \tag{10}$$

Substituting into the drift rate of intermediary leverage

$$\begin{aligned}\mu_{\theta t} &= \Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2 \\ &= \mathcal{S}_0\left(\omega_t,\theta_t\right) + \mathcal{S}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{S}_\xi\left(\omega_t,\kappa_t\right)\sigma_{k\xi,t},\end{aligned}$$

where

$$\mathcal{S}_0\left(\omega_t,\theta_t\right) = \Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) - \mathcal{O}_0\left(\omega_t,\theta_t\right) + \left(\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \tag{11}$$

$$\mathcal{S}_a\left(\omega_t,\theta_t\right) = -2\sigma_a\left(\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t}\right) - \mathcal{O}_a\left(\omega_t,\theta_t\right) \tag{12}$$

$$\mathcal{S}_\xi\left(\omega_t,\theta_t\right) = -\mathcal{O}_\xi\left(\omega_t,\theta_t\right). \tag{13}$$

Similarly, the excess return on capital is given by

$$
\begin{aligned}
\mu_{Rk,t} - r_{ft} &= \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right) \frac{1-\theta_t\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \\
&+ \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \frac{\theta_t^{-2}}{\alpha^2} \frac{1-\theta_t\omega_t}{1-\omega_t} - \frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \frac{\theta_t^{-2}}{\alpha^2} \\
&+ \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \sigma_a\sigma_{ka,t} + \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \frac{-2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)} \frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right).
\end{aligned}
$$

The excess return on intermediary debt is given by

$$
\begin{aligned}
\mu_{Rb,t} - r_{ft} &= \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{1-\theta_t\omega_t}{1-\omega_t} \\
&+ \sigma_\xi \left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right) \left(\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
&+ \left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right) \left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)^2 \sigma_a^2 \\
&- 2\left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right) \left(\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right) \left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right) \sigma_{ka,t}\sigma_a \\
&- \left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t - \omega_t}\right) \frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \frac{\theta_t^{-2}}{\alpha^2} + \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right) \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)} \sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi\rho_{\xi,a} \left[-\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)} \sigma_{ka,t} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)} \sigma_a\right] \\
&- \sigma_\xi\sqrt{1-\rho_{\xi,a}^2} \frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)} \sigma_{k\xi,t}.
\end{aligned}
$$

Notice also that we can now derive the risk-free rate. Recall that, in the unconstrained region, the risk-free rate satisfies the household Euler equation

$$
r_{ft} = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}\left[\frac{\langle dc_t\rangle^2}{c_t^2} + \frac{\langle dc_t, d\xi_t\rangle^2}{c_t}\right].
$$

Applying Itô's lemma to the goods clearing condition, we obtain

$$\begin{aligned} dc_t &= d\left(K_t A_t - i_t k_t A_t\right) \\ &= A_t dK_t + \left(K_t - i_t k_t\right) dA_t - A_t k_t di_t - A_t i_t dk_t - k_t \left\langle di_t, dA_t \right\rangle. \end{aligned}$$

From the financial intermediaries' optimal investment choice, we have

$$\begin{aligned} di_t = & \frac{\phi_0^2 \phi_1 p_{kt}^2}{4}\left(2\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\left(\sigma_{ka,t} - \sigma_a\right)\right) + \left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) dt \\ & + \frac{\phi_0^2 \phi_1 p_{kt}^2}{2}\left(\sigma_{ka,t} - \sigma_a\right) dZ_{at} + \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} dZ_{\xi t}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] &= \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k + \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \Phi\left(i_t\right)\left(1 - i_t\right) \\ & \quad - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{4}\left(2\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\right) + \left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\ \frac{1}{dt}\mathbb{E}\left[\frac{\left\langle dc_t \right\rangle^2}{c_t^2}\right] &= \left(\sigma_a - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2}\left(\sigma_{ka,t} - \sigma_a\right)\right)^2 + \left(\left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t}\right)^2 \\ \frac{1}{dt}\mathbb{E}\left[\frac{\left\langle dc_t d\xi_t \right\rangle^2}{c_t}\right] &= \sigma_\xi\left(\sigma_a - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{ka,t}\right) \rho_{\xi,a} \\ & \quad - \sigma_\xi\left(\left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t}\right) \sqrt{1 - \rho_{\xi,a}^2}. \end{aligned}$$

Figure 13 plots the three components of the risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). We see that the precautionary savings motive dominates the consumption growth component, and even more so as leverage increases.

Recall that, in equilibrium, we have

$$1 - i_t \theta_t \omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) p_{kt}\left(1 - \omega_t\right),$$

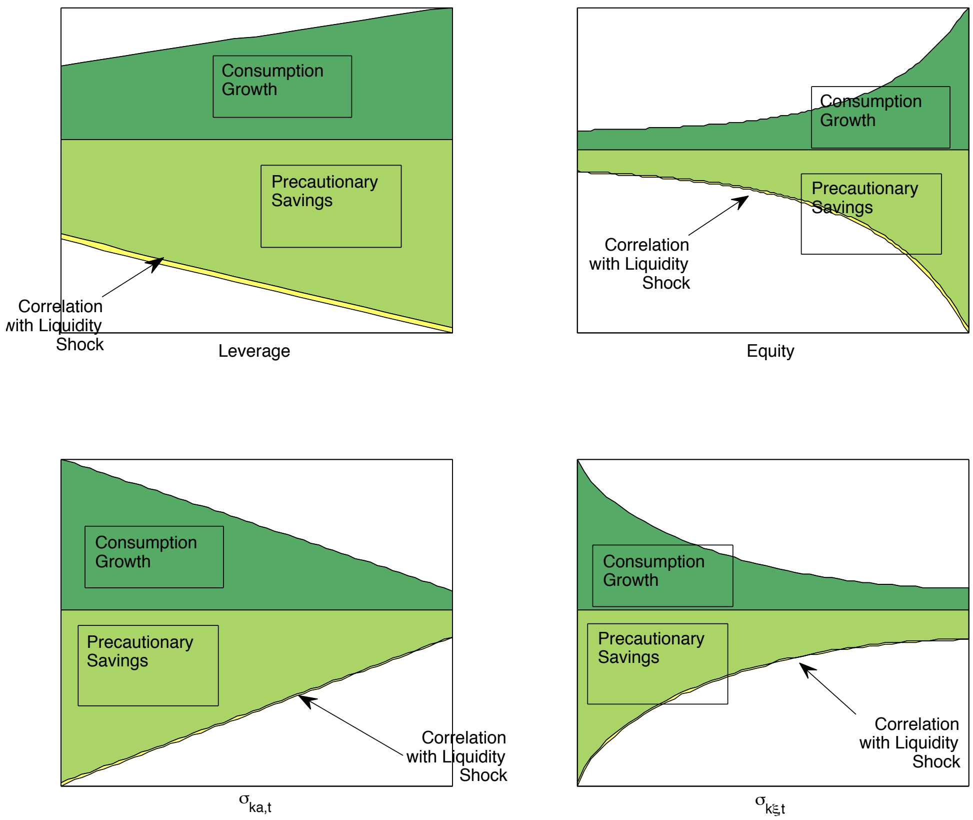


Figure 13: Equilibrium risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). "Consumption growth" refers to the component of the risk-free rate that's due to the expected consumption growth rate. "Precautionary saving" is measured as the contribution to the risk-free of the variance of the consumption growth rate. "Correlation with Liquidity Shock" is measured as the contribution to the risk-free of the correlation between the consumption growth rate and the liquidity shock $d\xi_t$. The equilibrium risk-free rate is then given as the lower boundary of the shaded region in each graph.

so that

$$\frac{1}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \left(\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1-\omega_t\right)\right)^{-1}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \frac{p_{kt}}{\beta\left(1-\omega_t\right)}$$

and

$$1 + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}.$$

Substituting into the expression for the risk-free rate, we obtain

$$
\begin{aligned}
r_{ft} = {} & \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right) \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} + \frac{\theta_t\omega_t}{1 - i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) \\
& - \frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}}\right) - \frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} - 2\sigma_{ka,t}\sigma_a\right) \\
& - \sigma_a^2\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2 + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
& - \sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right) - \left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \\
& + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t} + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}
$$

We can now solve for the return on capital. In particular, we have

$$
\begin{aligned}
\mu_{Rk,t} & = r_{ft} - \frac{2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} + \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
& = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} + \frac{\theta_t\omega_t}{1 - i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) \\
& - \frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}}\right) - \frac{3\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 4\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
& - \sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} + \left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2\right) + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
& - \sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right) - \left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \\
& + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}
$$

Solving for $\mu_{Rk,t}$, we obtain

$$
\mu_{Rk,t} = \mathcal{K}_0\left(\omega_t,\theta_t\right) + \mathcal{K}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{K}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t},
$$

where

$$\mathcal{K}_0(\omega_t,\theta_t) = \frac{\beta(1-\omega_t)}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\left(\rho_h - \frac{\sigma_\xi^2}{2} + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi(i_t)(1-i_t) + \frac{2\theta_t\omega_t}{\beta(1-\omega_t)}\right) \quad (14)$$
$$+\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\left(3 + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)}\right)\frac{\theta_t^{-2}}{\alpha^2}$$
$$-\sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}} + \left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\right)\right) - \sigma_a\sigma_\xi\rho_{\xi,a}$$

$$\mathcal{K}_a(\omega_t,\theta_t) = \sigma_\xi\rho_{\xi,a} + \frac{\beta(1-\omega_t)+4\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\sigma_a + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)}\sigma_a \quad (15)$$

$$\mathcal{K}_\xi(\omega_t,\theta_t) = \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}. \quad (16)$$

We can now express the risk-free rate in the economy as

$$r_{ft} = \mu_{Rk,t} + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2} - \frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\sigma_a\sigma_{ka,t} - \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)$$
$$\equiv \mathcal{R}_0(\omega_t,\theta_t) + \mathcal{R}_a(\omega_t,\theta_t)\sigma_{ka,t},$$

where

$$\mathcal{R}_0(\omega_t,\theta_t) = \mathcal{K}_0(\omega_t,\theta_t) + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2} \quad (17)$$

$$\mathcal{R}_a(\omega_t,\theta_t) = \mathcal{K}_a(\omega_t,\theta_t) - \frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\sigma_a - \sigma_\xi\rho_{\xi,a}. \quad (18)$$

Substituting into the excess return on holding intermediary debt, we obtain

$$\mu_{Rb,t} = \mathcal{B}_0(\omega_t,\theta_t) + \mathcal{B}_a(\omega_t,\theta_t)\sigma_{ka,t} + \mathcal{B}_\xi(\omega_t,\theta_t)\sigma_{k\xi,t},$$

where

$$\mathcal{B}_0(\omega_t,\theta_t) = \mathcal{R}_0(\omega_t,\theta_t) + \left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \tag{19}$$
$$+\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\sigma_a^2+\sigma_\xi\rho_{\xi,a}\sigma_a\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}$$
$$-\left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t-\omega_t}\right)\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2}$$
$$\mathcal{B}_a(\omega_t,\theta_t) = \mathcal{R}_a(\omega_t,\theta_t) + \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right)\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\sigma_a \tag{20}$$
$$-2\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\sigma_a$$
$$-\sigma_\xi\rho_{\xi,a}\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)$$
$$\mathcal{B}_\xi(\omega_t,\theta_t) = \mathcal{R}_\xi(\omega_t,\theta_t) - \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}. \tag{21}$$

Notice that

$$p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\mathcal{K}_\xi(\omega_t,\theta_t)-\beta p_{kt}\omega_t\mathcal{O}_\xi(\omega_t,\theta_t)=0.$$

Using these results and the risk-based capital constraint, we can rewrite

$$0=\theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2-\frac{4}{\phi_0^2\phi_1^2}\right)$$
$$+p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\mu_{Rk,t}-\frac{1}{p_{kt}}-\bar{a}-\frac{\sigma_a^2}{2}+\lambda_k-\sigma_a(\sigma_{ka,t}-\sigma_a)\right)$$
$$-\beta p_{kt}\omega_t\mu_{\omega_t}-p_{kt}(\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left((\sigma_{ka,t}-\sigma_a)^2+\sigma_{k\xi,t}^2\right)$$

as

$$
\begin{aligned}
0 &= \theta_t \omega_t \left(1 - \theta_t \omega_t\right) \Phi\left(i_t\right) \left(p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2}\right) \\
&+ p_{kt} \left(2\theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t\right)\right) \left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a \left(\sigma_{ka,t} - \sigma_a\right)\right) \\
&- \beta p_{kt} \omega_t \mu_{\omega_t} - p_{kt} \left(\theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t\right)\right) \left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 - 2\sigma_{ka,t} \sigma_a\right) \\
&\equiv \mathcal{C}_0 \left(\omega_t, \theta_t\right) + \mathcal{C}_a \left(\omega_t, \theta_t\right) \sigma_{ka,t},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{C}_0 \left(\omega_t, \theta_t\right) &= \theta_t \omega_t \left(1 - \theta_t \omega_t\right) \Phi\left(i_t\right) \left(p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2}\right) - p_{kt} \left(\theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t\right)\right) \left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right) \\
&+ p_{kt} \left(2\theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t\right)\right) \left(\mathcal{K}_0 \left(\omega_t, \theta_t\right) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k\right) - \beta p_{kt} \omega_t \mathcal{O}_0 \left(\omega_t, \theta_t\right)
\end{aligned} \tag{22}
$$

$$
\mathcal{C}_a \left(\omega_t, \theta_t\right) = p_{kt} \left(2\theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t\right)\right) \mathcal{K}_a \left(\omega_t, \theta_t\right) - \beta p_{kt} \omega_t \mathcal{O}_a \left(\omega_t, \theta_t\right) + p_{kt} \beta \left(1 - \omega_t\right) \sigma_a. \tag{23}
$$

Solving for $\sigma_{ka,t}$, we obtain

$$
\sigma_{ka,t} = -\frac{\mathcal{C}_0 \left(\omega_t, \theta_t\right)}{\mathcal{C}_a \left(\omega_t, \theta_t\right)}.
$$

Substituting into the risk-based capital constraint, we obtain

$$
\frac{\theta_t^{-2}}{\alpha^2} = \sigma_{k\xi,t}^2 + \left(\frac{\mathcal{C}_0 \left(\omega_t, \theta_t\right)}{\mathcal{C}_a \left(\omega_t, \theta_t\right)}\right)^2,
$$

so that

$$
\sigma_{k\xi,t} = \sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \left(\frac{\mathcal{C}_0 \left(\omega_t, \theta_t\right)}{\mathcal{C}_a \left(\omega_t, \theta_t\right)}\right)^2}.
$$

We can further simplify the above expressions by substituting for $\mathcal{O}_0$, $\mathcal{O}_a$, $\mathcal{K}_0$, and $\mathcal{K}_a$. Notice first that

$$\begin{aligned}\mathcal{C}_a(\omega_t,\theta_t) &= p_{kt}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\mathcal{K}_a\left(\omega_t,\theta_t\right)-\beta p_{kt}\omega_t\mathcal{O}_a\left(\omega_t,\theta_t\right)+p_{kt}\beta\left(1-\omega_t\right)\sigma_a\\
&= p_{kt}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\mathcal{K}_a\left(\omega_t,\theta_t\right)-\sigma_\xi\rho_{\xi,a}\right)\\
&+ p_{kt}\left\{-\frac{2\sigma_a}{\beta}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2+\beta\left(1-\omega_t\right)\sigma_a\right\}\\
&= p_{kt}\left(\beta\left(1-\omega_t\right)+4\theta_t\omega_t p_{kt}+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\right)\sigma_a\\
&+ p_{kt}\left\{-\frac{2\sigma_a}{\beta}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2+\beta\left(1-\omega_t\right)\sigma_a\right\}\\
&= \frac{2\sigma_a p_{kt}}{\beta}\left(\frac{\omega_t}{1-\omega_t}\right)\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{C}_0(\omega_t,\theta_t) &= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2-\frac{4}{\phi_0^2\phi_1^2}\right)-p_{kt}\left(\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2}+\sigma_a^2\right)\\
&+ p_{kt}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\mathcal{K}_0\left(\omega_t,\theta_t\right)-\frac{1}{p_{kt}}-\bar{a}+\frac{\sigma_a^2}{2}+\lambda_k\right)-\beta p_{kt}\omega_t\mathcal{O}_0\left(\omega_t,\theta_t\right)\\
&= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2-\frac{4}{\phi_0^2\phi_1^2}\right)-p_{kt}\left(\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2}+\sigma_a^2\right)\\
&+ \beta p_{kt}\left(1-\omega_t\right)\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right)\\
&- \theta_t\omega_t p_{kt}^2\left(3+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)\frac{\theta_t^{-2}}{\alpha^2}-\sigma_a^2 p_{kt}\left(\theta_t\omega_t p_{kt}+\frac{\left(\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}\right)^2}{\beta\left(1-\omega_t\right)}\right)\\
&+ \frac{p_{kt}}{\beta}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2\left(\sigma_a^2+\frac{\theta_t^{-2}}{\alpha^2}\right)-\beta p_{kt}\left(1-\omega_t\right)\Phi\left(i_t\right)\theta_t\omega_t\end{aligned}$$

Collecting like terms, we obtain

$$
\begin{aligned}
\mathcal{C}_0(\omega_t,\theta_t) &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2-\frac{4}{\phi_0^2\phi_1^2}\right)-\beta p_{kt}(1-\omega_t)\theta_t\omega_t\Phi(i_t)\left(1-\frac{1-i_t}{1-i_t\theta_t\omega_t}\right)\\
&\quad+\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\left\{(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))^2-\beta(\theta_t\omega_t p_{kt}+\beta(1-\omega_t))-\theta_t\omega_t p_{kt}\left(3\beta+\frac{4p_{kt}\theta_t\omega_t}{1-\omega_t}\right)\right\}\\
&\quad+\sigma_a^2\frac{p_{kt}}{\beta}\left\{(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))^2-\beta(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))-\frac{(\beta(1-\omega_t)+2\theta_t\omega_t p_{kt})^2}{(1-\omega_t)}\right\}\\
&= -\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\frac{\omega_t}{1-\omega_t}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))^2\\
&\quad-\sigma_a^2\frac{p_{kt}}{\beta}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\frac{\omega_t(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))+1-\omega_t}{1-\omega_t}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sigma_{ka,t} &= -\frac{\mathcal{C}_0(\omega_t,\theta_t)}{\mathcal{C}_a(\omega_t,\theta_t)}\\
&= \frac{\theta_t^{-2}}{\alpha^2}+\sigma_a^2\left(1+\frac{1-\omega_t}{\omega_t(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))}\right).
\end{aligned}
$$

Figure 14 plots the equilibrium sensitivities of the excess return to capital (top panels), the excess return to intermediary debt (middle panels), and the growth rate of the intermediary wealth (lower panels), as a function of intermediary's wealth share in the economy ω_t and intermediary leverage θ_t. The sensitivity of the return to holding capital to both the productivity and the household beliefs shock decreases in the leverage of the intermediary sector, while an increase in the intermediary's wealth share in the economy has little impact. Intuitively, since total volatility of the return to holding capital is inversely proportional to intermediary leverage, an increase in leverage has to lead to a decrease in volatility. The sensitivity of the return to intermediary debt, on the other hand, has an inverse relationship to the intermediary's wealth share in the economy, with the total volatility of the excess return to intermediary debt increasing as the intermediary owns a smaller fraction of wealth in the economy. Intuitively, as the intermediary's wealth share in the economy decreases, shocks impact intermediary equity more, increasing the volatility of the return to holding intermediary debt.

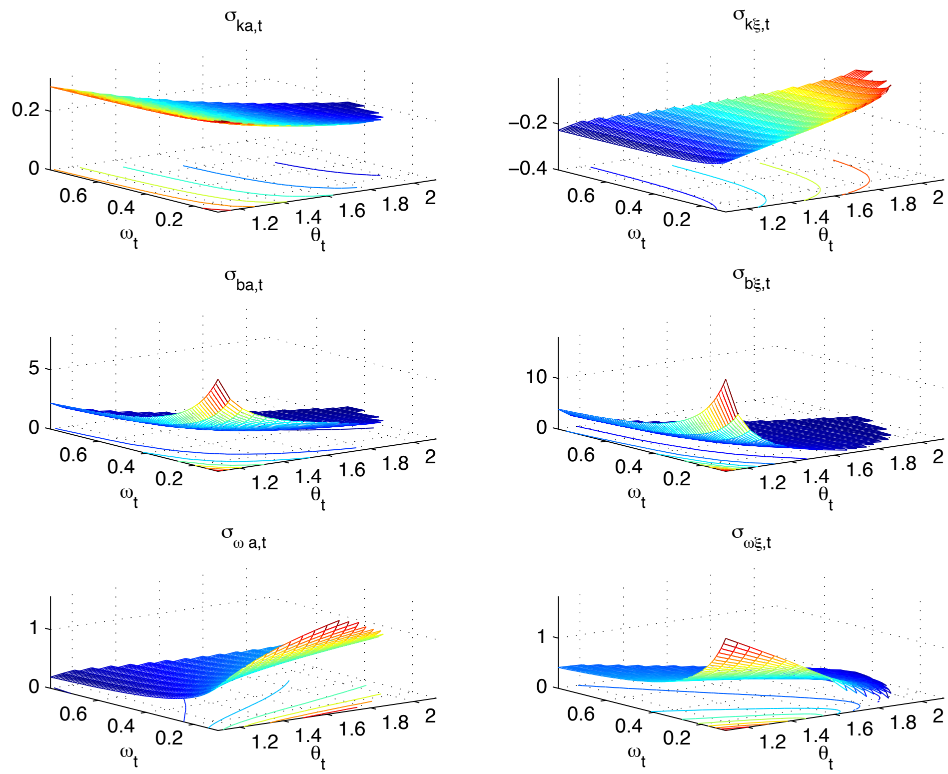


Figure 14: Loadings on the shock to productivity (left panels) and the shock to beliefs (right panels) as a function of intermediary leverage θ_t (x-axis) and financial intermediaries' wealth share in the economy ω_t (y-axis). Upper panels: loadings $(\sigma_{ka,t}, \sigma_{k\xi,t})$ of the return to capital; middle panels: loadings $(\sigma_{ba,t}, \sigma_{b\xi,t})$ of the return to intermediary debt; lower panels: loadings $(\sigma_{\omega a,t}, \sigma_{\omega\xi,t})$ of the growth rate of intermediary equity.

A.3 Constant leverage benchmark (Proof of Lemma 3)

We begin by solving for the equilibrium dynamics of the intermediaries' wealth share in the economy. Recall that the capital held by the intermediaries is given by

$$k_t = \theta \omega_t K_t.$$

Applying Itô's lemma, we obtain

$$\begin{aligned}\frac{dk_t}{k_t} &= \frac{d\omega_t}{\omega_t} + \frac{dK_t}{K_t} \\ &= \left(\mu_{\omega t} + \Phi\left(i_t\right)\theta\omega_t - \lambda_k\right)dt + \sigma_{\omega a,t} dZ_{at} + \sigma_{\omega\xi,t} dZ_{\xi,t}.\end{aligned}$$

Recall, on the other hand, that intermediary capital evolves as

$$\frac{dk_t}{k_t} = (\Phi(i_t) - \lambda_k)\, dt.$$

Thus, equating coefficients, we obtain

$$\begin{aligned} \sigma_{\omega a,t} &= 0 \\ \sigma_{\omega \xi,t} &= 0 \\ \mu_{\omega t} &= \Phi(i_t)(1 - \theta\omega_t). \end{aligned}$$

Consider now the wealth evolution of the representative household. From the households' budget constraint, we have

$$\frac{dw_{ht}}{w_{ht}} = \left(r_{ft} - \rho_h + \frac{\sigma_\xi^2}{2}\right) dt + \frac{1 - \bar{\theta}\omega_t}{1 - \omega_t}(dR_{kt} - r_{ft}dt) + \frac{\omega_t\left(\bar{\theta} - 1\right)}{1 - \omega_t}(dR_{bt} - r_{ft}dt).$$

On the other hand, from the definition of ω_t, we obtain

$$\begin{aligned} \frac{dw_{ht}}{w_{ht}} &= \frac{d\left((1 - \omega_t)\, p_{kt} A_t K_t\right)}{(1 - \omega_t)\, p_{kt} A_t K_t} \\ &= \frac{dp_{kt}}{p_{kt}} + \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\omega_t}{1 - \omega_t}\frac{d\omega_t}{\omega_t} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t} \right\rangle. \end{aligned}$$

Equating coefficients once again and simplifying, we obtain

$$\begin{aligned} \sigma_{ba,t} &= \sigma_{ka,t} \\ \sigma_{b\xi,t} &= \sigma_{k\xi,t} \\ \mu_{Rb,t} &= \mu_{Rk,t} + \frac{1 - \omega_t}{\omega_t\left(\bar{\theta} - 1\right)}\left(\rho_h - \frac{\sigma_\xi^2}{2} - \frac{1}{p_{kt}}\right) + \Phi(i_t). \end{aligned}$$

We now turn to solving for the equilibrium price of capital. The goods clearing condition in this economy reduces to

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1 - \omega_t)\, p_{kt} = 1 - i_t\bar{\theta}\omega_t.$$

Substituting the optimal level of investment

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right),$$

we obtain that the price of capital satisfies

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1-\omega_t)\, p_{kt} = 1 - \frac{\bar{\theta}\omega_t}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

Then the price of capital satisfies

$$0 = \bar{\theta}\omega_t p_{kt}^2 + \beta(1-\omega_t)\, p_{kt} - \frac{4}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right),$$

or:

$$p_{kt} = \frac{-\beta(1-\omega_t) + \sqrt{\beta^2(1-\omega_t)^2 + \frac{16\bar{\theta}\omega_t}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right)}}{2\bar{\theta}\omega_t}.$$

Applying Itô's lemma, we obtain

$$0 = \bar{\theta}\omega_t p_{kt}^2\left(2\frac{dp_{kt}}{p_{kt}} + \left\langle\frac{dp_{kt}}{p_{kt}}\right\rangle^2 + \frac{d\omega_t}{\omega_t}\right) + \beta(1-\omega_t)\, p_{kt}\frac{dp_{kt}}{p_{kt}} - \frac{\omega_t}{1-\omega_t}\beta p_{kt}\frac{d\omega_t}{\omega_t} + \frac{4}{\phi_0^2\phi_1^2}\bar{\theta}\frac{d\omega_t}{\omega_t}.$$

Equating coefficients and simplifying, we obtain

$$\begin{aligned}
\sigma_{ka,t} &= \sigma_a \\
\sigma_{k\xi,t} &= 0 \\
\mu_{Rk,t} &= \frac{1}{p_{kt}} + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\Phi(i_t)\left(1-\bar{\theta}\omega_t\right)}{p_{kt}\left(2\bar{\theta}\omega_t p_{kt} + \beta(1-\omega_t)\right)}\left(\frac{4\bar{\theta}}{\phi_0^2\phi_1^2} - \frac{\omega_t}{1-\omega_t}\beta p_{kt} + \bar{\theta}\omega_t p_{kt}^2\right).
\end{aligned}$$

Finally, consider the equilibrium risk-free rate. Notice that

$$\begin{aligned}
\frac{dc_t}{c_t} &= \frac{d\left(\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t\right)}{\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t} \\
&= \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}di_t - \frac{i_t\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}\frac{d\omega_t}{\omega_t} - \frac{\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}\left\langle di_t, \frac{dA_t}{A_t}\right\rangle,
\end{aligned}$$

and:

$$di_t = d\left(\frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta} p_{kt}^2 - \frac{1}{\phi_1}\right) = \frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta} p_{kt}^2 \left(2\frac{dp_{kt}}{p_{kt}} + \left\langle \frac{dp_{kt}}{p_{kt}} \right\rangle^2\right).$$

Using:

$$1 - i_t\bar{\theta}\omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1-\omega_t)\,p_{kt},$$

the risk-free rate is thus given by

$$\begin{aligned} r_{ft} &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}_t\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}_t\left[\left\langle\frac{dc_t}{c_t}\right\rangle\right] \\ &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \bar{a} - \frac{\sigma_a^2}{2} + \Phi\left(i_t\right)\bar{\theta}\omega_t - \lambda_k - \frac{2\bar{\theta}\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2}\right) \\ &- \frac{i_t\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t}\mu_{\omega t}. \end{aligned}$$