This appendix derives the LGD functions in Table 1 and proves that the model of Tasche can represent functions no steeper than Frye-Jacobs.

Throughout, it is assumed that cDR has a Vasicek Distribution. A Vasicek variable is a transformation of a normal variable:

(1) $$V \;=\; \Phi\left[\frac{\Phi^{-1}[\mu] + \sqrt{\rho}\, Z}{\sqrt{1-\rho}}\right] \;;\; Z \sim N\,[0,1]$$

where E[V] = μ is the mean. If the default rate has the Vasicek distribution μ is equal to PD, and if the loss rate has the Vasicek Distribution μ is equal to EL. Equation (1) leads directly to the inverse CDF:

(2) $$V \;=\; \Phi\left[\frac{\Phi^{-1}[\mu] + \sqrt{\rho}\,\Phi^{-1}[q]}{\sqrt{1-\rho}}\right];\; q \in (0,1)$$

Inverting Equation (2) gives the CDF:

(3) $$F_V[v] \;=\; \Phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}[v] - \Phi^{-1}[\mu]}{\sqrt{\rho}}\right]$$

The inverse of Equation (1) is used several times in what follows:

(4) $$Z \;=\; \frac{\sqrt{1-\rho}\,\Phi^{-1}[v] - \Phi^{-1}[\mu]}{\sqrt{\rho}}$$

Applying change-of-variable to (1) produces the Vasicek PDF, which is referred to in the section titled “Exact Regression”:

(5) $$f_V[v] \;=\; \frac{\sqrt{1-\rho}}{\sqrt{\rho}\;\phi[\Phi^{-1}[v]]}\;\phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}[v] - \Phi^{-1}[\mu]}{\sqrt{\rho}}\right]$$

Frye (2000)

Frye’s Equation (4) states the recovery of an individual loan as a function of the systematic risk factor designated X:

> Turning to the recovery side, the recovery equation is similar to asset equation (1). Recovery in default j depends on the systematic factor X and also on an idiosyncratic factor, Z_j, which affects only the recovery in default j:
>
> $$R_j = \mu_j + \sigma q X + \sigma\sqrt{1-q^2}\,Z_j \qquad (4)$$

The expectation conditioned on X = x eliminates the last term. Frye's X = -Z. Then,

(6) $$cLGD \;=\; 1-(\mu+\sigma q(-Z)) = 1-\mu+\sigma q(\sqrt{1-\rho}\,\Phi^{-1}[cDR]-\;\Phi^{-1}[PD])/\sqrt{\rho}$$

Pykhtin

Pykhtin's Equation (16) expresses cLoss as a function of the systematic factor Y:

$$E\left[L_i|Y\right]=N\left[\bar{X}_i(Y)\right] \left(N\left[\bar{\eta}_i(Y)\right]-\exp\left[\mu_i+\sigma_i\beta_i Y+\frac{\sigma_i^2}{2}\left(1-\beta_i^2\right)\right]N\left[\bar{\eta}_i(Y)-\sigma_i\sqrt{1-\beta_i^2}\right]\right) \quad (16)$$

where in his notation

$$\bar{X}_i(Y)=\frac{N^{-1}(p_i)-\alpha_i Y}{\sqrt{1-\alpha_i^2}} \qquad \bar{\eta}_i(Y)=\frac{-\mu_i/\sigma_i-\beta_i Y}{\sqrt{1-\beta_i^2}}$$

The first factor of Pykhtin (16) is cDR. Table 1 transcribes the second factor taking note that Pykhtin's Y = – Z.

Tasche

Tasche's Equation (2.6b) expresses cLoss as a function of the systematic risk factor:

$$\mathrm{E}[L\,|\,X=x] \;=\; \int_{\frac{\Phi^{-1}(1-p)-\sqrt{\rho}\,x}{\sqrt{1-\rho}}}^{\infty} \varphi(z)\,F_D^*\left(\frac{\Phi(\sqrt{\rho}\,x+\sqrt{1-\rho}\,z)-1+p}{p}\right)dz, \qquad (2.6b)$$

where p = PD, Tasche's X = Z, and Tasche's z is an idiosyncratic risk factor. Using Equation (4) to restate the lower limit of integration,

(7) $$\frac{\Phi^{-1}[1-p]-\sqrt{\rho}\,x}{\sqrt{1-\rho}} = \frac{-\,\Phi^{-1}[PD]-(\sqrt{1-\rho}\;\Phi^{-1}[cDR]-\;\Phi^{-1}[PD])}{\sqrt{1-\rho}} \;=\; -\,\Phi^{-1}[cDR]$$

Tasche specifies $F_D[\cdot]$ as the Beta Distribution with a = ELGD (1-v)/v and b = (1-ELGD) (1-v)/v. Making this substitution and again using Equation (4),

(8) $E[cLoss|cDR]$

$$= \int_{-\Phi^{-1}[cDR]}^{\infty} \phi[z]\, BetaCDF^{-1}\left[\frac{\Phi\left[\sqrt{1-\rho}\,\Phi^{-1}[cDR] - \Phi^{-1}[PD] + \sqrt{1-\rho}\, z\right] - 1 + PD}{PD}, a, b\right] dz$$

Tasche's LGD function is Equation (8) divided by cDR.

Giese

Giese's Equation (11) is a direct specification of cLGD as a function of Giese's p, which equals cDR:

$$\hat{\mu}_i(p) = 1 - a_0\left(1 - p^{a_1}\right)^{a_2} \qquad (11)$$

Hillebrand

Hillebrand's Equation (11) expresses a function of cLGD as a function of two systematic factors:

$$\Phi^{-1}\left(E\left(LGD|Z\right)\right) = a - \frac{bdc}{e} + \frac{bd}{e}\Phi^{-1}\left(PD\left(Y\right)\right) - b\sqrt{1-d^2}\,X \qquad (11)$$

where $\Phi^{-1}(\mathbb{E}(LGD|Z))$ is the response variable, $\Phi^{-1}(PD(Y))$ the predictor variable and X the residual.

The residual systematic factor is integrated out in Table 1.

Proof that Tasche[v = 1] = Frye-Jacobs

Tasche's Equation (2.5) says that loss on a loan depends on two random factors. The systematic factor is X = Z, and the idiosyncratic factor is ξ.

$$L = \begin{cases} 0, & \text{if } \sqrt{\rho}X + \sqrt{1-\rho}\,\xi \le \Phi^{-1}(1-p), \\ F_D^*\left(\frac{\Phi(\sqrt{\rho}X+\sqrt{1-\rho}\,\xi)-1+p}{p}\right), & \text{otherwise.} \end{cases} \quad (2.5)$$

(2.5) can be equivalently written as the product of the indicator function of the default event $\sqrt{\rho}X + \sqrt{1-\rho}\,\xi > \Phi^{-1}(1-p)$ and the factor $F_D^*\left(\frac{\Phi(\sqrt{\rho}X+\sqrt{1-\rho}\,\xi)-1+p}{p}\right)$.

When v → 1, the variance of the Beta Distribution approaches its maximum; then, a Beta Distribution approaches a Bernoulli Distribution. In the limit F_D and F_D* are step functions, and

(8) $$F_D^*[t, ELGD, v = 1] = I[t > 1 - ELGD]$$

Transcribing this,

(9) $$F_D^*\left[\frac{\Phi[\sqrt{\rho}\,Z + \sqrt{1-\rho}\,\xi] - 1 + PD}{PD}, ELGD, v = 1\right]$$

$$= I\left[\frac{\Phi[\sqrt{\rho}\,Z + \sqrt{1-\rho}\,\xi] - 1 + PD}{PD} > 1 - ELGD\right]$$

$$= I\left[\Phi[\sqrt{\rho}\,Z + \sqrt{1-\rho}\,\xi] > 1 - PD\ ELGD = 1 - EL\right]$$

This condition is never less restrictive than the first condition, because EL ≤ PD. Therefore the first condition can be dropped:

(10) $$L = I\left[\Phi[\sqrt{\rho}\,Z + \sqrt{1-\rho}\,\xi] > 1 - EL\right]$$

Using Equation (4) to introduce cDR,

(11) $$L = I\left[\sqrt{1-\rho}\,\Phi^{-1}[cDR] - \Phi^{-1}[PD] + \sqrt{1-\rho}\,\xi > \Phi^{-1}[1 - EL] = -\Phi^{-1}[EL]\right]$$

$$= I\left[\xi < \Phi^{-1}[cDR] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL]}{\sqrt{1-\rho}}\right]$$

Taking the expectation,

(12) $$cLoss = \int_{-\infty}^{\Phi^{-1}[cDR] - \frac{\Phi^{-1}[PD]-\Phi^{-1}[EL]}{\sqrt{1-\rho}}} \phi[z]\,dz = \Phi\left[\Phi^{-1}[cDR] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL]}{\sqrt{1-\rho}}\right]$$

Dividing (12) by cDR equals the Frye-Jacobs LGD function.