# FEDERAL RESERVE BANK of CHICAGO FINANCIAL MARKETS GROUP

**Private Takings** 

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#### Abstract

This paper considers the implications associated with a recent Supreme Court ruling that can be interpreted as supporting the use of eminent domain in transferring the property rights from one private agent—a landowner—to another private agent—a developer. A potential benefit of the ruling is that it effectively eliminates a hold-out problem, i.e., the problem associated with potential sellers withholding their property in an attempt to obtain a larger surplus. But, when property rights are transferred via eminent domain, landowners' investments in their properties become more inefficient and the level of redevelopment may actually fall. Compared to voluntary exchange, when property rights are transferred via eminent domain, social welfare will increase only if the hold-out problem is significant; otherwise, it will fall.

### 1 Introduction

A recent Supreme Court decision (*Kelo v. New London*) has effectively given communities the green light for *private takings*, where local and state governments have the authority to condemn private property for other *private* use. The ruling is controversial. Some observers believe the use of eminent domain in the *Kelo* case does not fulfil the public-use criterion as stated in the fifth amendment to the U.S. constitution: "nor shall private property be taken for *public* use, without just compensation" (my emphasis). They also might point out that the recent ruling seems to contradict an earlier Court ruling (*Hawaii Housing Authority v. Midkiff*): "A purely private taking could not withstand the scrutiny of the public use requirement; it would serve no legitimate purpose of government and would thus be void ... The court's cases

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have repeatedly stated that 'one person's property may not be taken for the benefit of another private person without a justifying public purpose, even though compensation be paid,' ." Other observers, however, believe that *Kelo* ruling is fully consistent with both the constitution and previous Court decisions. The Holmes Court ruled in 1925 what constitutes public use should be determined by state legislatures; so if legislatures deem that economic development of private property by private agents provides benefits to the community at large—due to, say, higher employment levels and tax base—then such a taking fulfils the public-use criterion as stated in the constitution. Proponents of the *Kelo* ruling would point out that the Supreme Court had no choice but to rule for the City of New London, since the taking was motivated by the public benefits associated with economic development.

From a public policy perspective, attempting to assess whether a taking is "appropriate," by determining whether or not it fulfils the public use criterion in the constitution, seems almost pointless: given the way the Court has interpreted public use, "almost anything goes" in the sense that if a state government *claims* that there is a public benefit associated with a taking, then there is a public benefit associated with the taking. A more meaningful and relevant exercise would be assess the social welfare implications associated with takings along the lines of Kelo, i.e., where the government takes property from one private agent and gives it to another. In this paper, I present a model where a developer has the opportunity to redevelop private property that is currently owned by another private party, a landowner. The developer can obtain the property rights from the landowner directly—by voluntary exchange—or indirectly—by having the government take the landowner's property rights and provide the landowner with just compensation. When the developer purchases property rights directly from the landowner, the price of the property rights is determined as an outcome to a bargaining problem.<sup>1</sup> Under voluntary exchange, the developer may face the so-called "hold-out problem." The problem here is that a seller recognizes that the if buyer must assemble multiple plots of land, then he may be able to extract a higher surplus for his property, by threatening to withhold or delay its sale. It has been suggested that the hold-out problem may have negative implications for redevelopment. If, however, the government takes the property from the landowner, then, by construction, the hold-out problem does not arise. In this case, the landowner receives "just compensation" for his taken property, which is paid for by the developer.

I consider two alternative bargaining scenarios when the developer and landowners voluntarily exchange property rights. In one scenario, the developer has a fixed bargaining weight, independent of how many landowners he bargains with. I interpret this as a situation as one where there is no hold-out problem, since the developer's share of the surplus is independent of the number of landowners he bargains with. Here, the use of eminent domain will always lower social welfare compared to vol-

<sup>&</sup>lt;sup>1</sup>In practice, there does not exist a well-defined, competitive market for real estate. The selling price for real estate is typically determined by some bargaining process between a buyer and a seller.

untary exchange. In the other bargaining scenario, the developer has a bargaining weight that declines with the number of landowners he bargains with. And, for any level of redevelopment, each landowner receives a bigger share of the surplus, compared to the "no hold-out" bargaining scenario. I interpret this as a situation where the developer faces a hold-out problem. When the hold-out problem exists, the use of eminent domain may increase or decrease social welfare compared to voluntary exchange depending on the importance of he hold-out problem. Social welfare will only increase if the hold-out problem is "significant."

The seminal economics paper on eminent domain and takings is Blume, Rubinfeld and Shapiro (1984). This paper, and the rather sizeable literature that follows it e.g., Blume and Rubinfeld (1984), Fischel and Shapiro (1989), Miceli and Segerson (1994), Innes (2000) and Nosal (2001)—focuses on issues related to optimal compensation and its effect on private investment, when condemned property is converted into a *public* good. The takings literature that addresses the condemnation of private property for private use is quite small. Rolnik and Davies (2006) and Garrett and Rothstein (2007), relying on solid economic reasoning, point out that when governments interfere with the market place, bad outcomes usually follow. Both of these discussions, however, are not conducted within the context of an explicit model.

The remainder of the paper is as follows. The next section considers a simple redevelopment and takings environment with one developer and one landowner. Since there is only one landowner, a hold-out problem does not arise. In section 3, the environment is extended so that there are many landowners and, as a result, a holdout problem may arise. The final section summarizes and concludes.

## 2 A Simple Model with One Landowner

There is a single landowner who is endowed with capital  $K_{\ell}$  and property rights to a tract of land. The landowner can invest his capital in two ways. He can invest in a safe asset that provides a gross rate of return R and he can make an irreversible investment  $x \leq K_{\ell}$  on his land, which gives a payoff of f(x), where f' > 0, f'' < 0and  $f'(0) = \infty$ .

A developer is endowed with capital  $K_d$ . He can redevelop the landowner's land and invest in the safe asset. If land is redeveloped, then the investment x and its potential payoff, f(x), are destroyed. There are two states of the world for redevelopment: good and bad. The probability that the state is good is  $1 - \theta$ . In the good state, redevelopment spending by the developer  $y \ge \bar{y}$  generates a payoff of Y and redevelopment spending  $y < \bar{y}$  gives a zero payoff. In the bad state, any redevelopment expenditure  $y \ge 0$  gives a zero payoff. Since there is only one landowner, there is no hold-out problem.

In order to redevelop land, the developer must acquire the property rights from the landowner. If the developer gets the landowner's property rights through voluntary exchange, then the purchase price, p, is determined by generalized Nash bargaining.

The developer's generalized Nash bargaining weight is denoted by  $\beta$ , where  $0 < \beta < 1$ , and the landowner's is  $1 - \beta$ . Since  $0 < \beta < 1$ , both the landowner and the developer have some bargaining power.

There exists a government that can condemn and expropriate, i.e., *take*, the landowner's property. When the government revokes the landowner's property rights, it sells them to the developer. The law requires the government provide "just compensation" to the landowner in the event that his land is taken. In this article, just compensation will be defined as f(x)—the value of the property to the landowner in the event that his property is not taken. The government must balance its budget. Hence, if land is taken, the government sells the property rights to the developer for f(x).

The timing of events is as follows. At date 0, the landowner is born; he invests x in his property and  $K_{\ell} - x$  in the safe asset. In between dates 0 and 1, the state of world—good or bad—is revealed. At date 1, the developer is born; he decides whether or not to redevelop the landowner's property. In the event that the redevelopment takes place, the developer either bargains with the landowner or the government takes the landowner's property rights and sells them to the developer. The developer spends either p or f(x) (at date 2) to acquire the property rights, where he spends p if the he bargains with the landowner and f(x) if he gets the property rights, via eminent domain, from the government. The developer invests  $(K_d - y)$  in the safe asset and spends y on redevelopment if he obtains the property rights to the land; otherwise he invests  $K_d$  in the safe asset. At date 2 all investments pay off, payments are exchanged, and the landowner and developer consume.

The objective of the landowner and the developer is to maximize their expected payoffs. The timing of the births of a landowner and the developer prevent them from interacting before the landowner makes his investment decision. This timing assumption is designed to reflect the real world fact that developers enter the scene long after initial investments are undertaken.

#### 2.1 Social Optimum

At date 0, the landowner invests x in his property. With probability  $\theta$ , the state of the world is bad and it is not efficient to redevelop the land. In this situation the payoff to the landowner is  $f(x) + (K_{\ell} - x) R$  and the payoff to the developer is  $K_d R$ . With probability  $(1 - \theta)$ , the state is good and it will be efficient to redevelop the land only if the total payoff to redevelopment,  $Y + (K_d - \bar{y}) R + (K_\ell - x) R$ , exceeds the payoff to not redeveloping the land,  $f(x) + K_d R + (K_\ell - x) R$ ; or if  $Y - \bar{y}R \ge f(x)$ . Define a critical level of investment in land,  $x_c$ , as

$$Y - \bar{y}R = f\left(x_c\right).$$

If  $x \leq x_c$ , then it is efficient to redevelop the land in the good state; if  $x > x_c$ , then it is not. To make the redevelopment problem interesting, I will assume the efficient level of investment in land—described below—is strictly less than  $x_c$ . If the efficient level of investment is greater than  $x_c$ , then redevelopment is never socially optimal.

The efficient level of investment in land is determined by the solution to

$$\max_{x} \theta \left( f(x) + K_{d}R \right) + (1 - \theta) \left( Y + (K_{d} - \bar{y})R \right) + (K_{\ell} - x)R, \tag{1}$$

i.e., the efficient level of investment in land maximizes total expected payoff to society. The efficient level of investment in land, denoted  $x^*$ , is given by the first-order condition to (1), which is

$$\theta f'(x^*) = R. \tag{2}$$

The social optimum is characterized by:

- 1. the landowner invests  $x^*$  at date 0;
- 2. the developer spends  $\bar{y}$  on redevelopment in the good state; and
- 3. no redevelopment occurs in the bad state.

### 2.2 Redevelopment Under Voluntary Exchange

The developer has no incentive to obtain property rights in the bad state. In the good state, the (date 2) payoff to the developer when he does obtain property rights is  $Y - \bar{y}R - p$  and (date 2) the payoff to the landowner is p - f. Note that the developer spends  $y = \bar{y}$  for redevelopment. At the time of bargaining, the investment x is sunk for the landowner, but  $\bar{y}$  is not; this is why the term " $\bar{y}R$ " shows up in the developer's payoff function and there is no comparable term in the landowner's payoff function. Let S represent the total surplus associated with redevelopment in the good state, where

$$S\left(x\right) = Y - f\left(x\right) - \bar{y}R,$$

S'(x) < 0 for all x > 0 and  $S(x_c) = 0$ . If S > 0—or equivalently  $x < x_c$ —then the developer will want to redevelop the land; if  $S \leq 0$ —or equivalently  $x \geq x_c$ —then he will not. If S > 0, then the price that the developer pays for the landowner's property rights, p, is given by the solution to

$$\max_{p} \left( Y - \bar{y}R - p \right)^{\beta} \left( p - f \right)^{1-\beta},$$

which is

$$p = f + (1 - \beta) S.$$

Suppose first that the landowner believes (correctly) that  $S \ge 0$  for his choice of x; then the optimal level of investment in his property is given by the solution to

$$\max_{x} \theta f(x) + (1 - \theta) \left[ f(x) + (1 - \beta) S(x) \right] + (K_{\ell} - x) R.$$
(3)

The solution,  $x_B$ , is characterized by

$$(\theta + (1 - \theta)\beta) f'(x_B) = R, \qquad (4)$$

where "B" stands for "bargaining." Comparing (4) with (2), note that  $x_B > x^*$ ; the landowner's investment in his property is larger than what is socially optimal.

Suppose now the landowner believes (correctly) that S < 0 for his choice of x; then his optimal level of investment is given by the solution to

$$\max_{x} f(x) + (K_{\ell} - x) R.$$
 (5)

The solution of (5) is characterized by

$$f'(x_N) = R, (6)$$

where the "N" stands for "no redevelopment." Comparing (6) with (4), note that  $x_N > x_B > x^*$ . Since S'(x) < 0 for all x > 0,

$$S(x^*) > S(x_B) > S(x_N).$$

If  $S(x_N) > 0$ —or  $x_N < x_c$ —then the developer will always redevelop the landowner's land in the good state. Hence, if  $S(x_N) > 0$ , then the relevant problem for landowner is (3) and he invests  $x_B$ .

If  $S(x_B) < 0$ —  $x_B > x_c$ —then developer will never redevelop the landowner's property and relevant problem is given by (5). Hence, the landowner invests  $x_N$  and no redevelopment takes place.

Finally, if  $S(x_B) > 0$  and  $S(x_N) < 0$ , then landowner's level of investment will determine whether or not there will be redevelopment in the good state. That is, if the landowner believes that redevelopment will occur, then he will invest  $x_B$ and redevelopment will occur in the good state. If he believes that redevelopment will not occur, then he will invest  $x_N$  and in the good state, the developer will have no incentive to redevelop the landowner's land. How much will the landowner invest at date 0,  $x_B$  or  $x_N$ ? While one might be tempted to conjecture that the landowner will invest  $x_B$ , since the surplus associated with investment  $x_B$  is greater than investment  $x_N$ , the landowner may actually invest  $x_N$ . This possible outcome is a feature associated with generalized Nash bargaining: the generalized Nash bargaining solution is not monotonic, which implies that although total surplus increases, the landowner's expected share (and payoff) may decrease. So although  $S(x_B) > 0$  and  $S(x_N) < 0$ , it may be the case that

$$f(x_N) + (K_{\ell} - x_N) R > f(x_B) + (1 - \theta) (1 - \beta) S(x_B) + (K_{\ell} - x_B) R.$$
(7)

Therefore, if  $S(x_B) > 0$  and  $S(x_N) < 0$  and condition (7) holds, then the landowner will invest  $x_N$ ; if condition (7) does not hold, then he will invest  $x_B$ .

Note that there are inefficiencies associated with redevelopment under voluntary exchange. First, the landowner always "overinvests" in his property since  $x_N > x_B > x^*$ . Second there may be too little redevelopment. In the social optimum, it is always efficient to redevelop in the good state. If either  $S(x_B) < 0$  or  $S(x_B) > 0$ ,  $S(x_N) < 0$  and condition (7), then redevelopment will not occur under voluntary exchange.

### 2.3 Redevelopment Under Eminent Domain

Under eminent domain, a government can take the landowner's property but must provide "just compensation," f(x), to the landowner. The government then sells the property rights to the developer for f(x). In terms of the model, eminent domain can be interpreted as giving all of the bargaining power to the developer since the landowner does not receive any of the surplus associated with redevelopment.

**Proposition 1** Using eminent domain to transfer property rights never increases, but can decrease, social welfare compared to voluntary exchange.

**Proof.** If  $x < x_c$ , then social welfare can be written as

$$W(x) = f(x) + (1 - \theta) S(x) - xR + (K_d + K_\ell) R$$

If  $x \ge x_c$ , then social welfare is given by

$$\tilde{W}(x) = f(x) - xR + (K_d + K_\ell)R.$$

Note that if the landowner optimally chooses  $x = x_B$  under voluntary exchange, then necessarily, redevelopment occurs under voluntary exchange in the good state. Under eminent domain, the landowner will always receive a payoff of  $f(x) + (K_{\ell} - x)R$ , independent of the redevelopment outcome, which means that he always invests  $x_N$ . When  $x = x_N$ , redevelopment may or may not occur under eminent domain depending on whether  $x_N < x_c$  or  $x_N \ge x_c$ , respectively. Hence, if  $x_N < x_c$ , we have

$$W(x^*) > W(x_B) > W(x_N) > W(x_c)$$

since W(x) is strictly concave for all  $x \in [0, x_c]$ . If  $x_B < x_c < x_N$ , we have

$$W(x^*) > W(x_B) > \tilde{W}(x_N) > W(x_c) = \tilde{W}(x_c).$$

There are a number of cases to consider:

1. If  $S(x_B) < 0$ , then redevelopment does not take place under voluntary exchange and the landowner invests  $x_N$ ; in this case, eminent domain and voluntary exchange generate the same level of welfare.

- 2. If  $S(x_N) > 0$ , then under voluntary exchange the landowner invests  $x_B$  and redevelopment occurs in the good state. Under eminent domain, the landowner invests  $x_N$  and the developer redevelops in the good state. In this case, the overinvestment problem is exacerbated under eminent domain, compared to bargaining—since  $x_N > x_B > x^*$ ; here social welfare is lower under eminent domain compared to the voluntary exchange since  $W(x_B) > W(x_N)$ .
- 3. If  $S(x_B) > 0$  and  $S(x_N) < 0$  and condition (7) does not hold, then under voluntary exchange, the landowner will invest  $x_B$  and redevelopment will occur in the good state. In this case, welfare is  $W(x_B)$ . Under eminent domain, the landowner will invest  $x_N > x_c > x_B$ ; there will be too much investment and too little redevelopment. In this case welfare is  $\tilde{W}(x_N)$ . Since  $\tilde{W}(x_N) < W(x_B)$ , the use of eminent domain strictly lowers welfare. If, however,  $S(x_B) > 0$  and  $S(x_N) < 0$  and condition (7) holds, then the landowner will invest  $x_N$  under both eminent domain and voluntary exchange. Social welfare will be the same under voluntary exchange and eminent domain.

From all of this we can conclude that the use of eminent domain will never increase social welfare, but may lower it.  $\blacksquare$ 

### 2.4 Discussion

Up to this point, the analysis seems to indicate that the recent Supreme Court decision on *Kelo v. New London* is wrong-headed: eminent domain, in conjunction with just compensation, can never increase social welfare and can only lower it. Eminent domain, along with just compensation, effectively gives all of the bargaining power to the developer, i.e., eminent domain and just compensation effectively sets  $\beta = 1$ in the landowner's investment problem (3), making it equivalent to (5). It is true that, absent eminent domain, the level of landowner investment x is too high, but the use of eminent domain, along with just compensation, can only exacerbate the overinvestment problem. Not only may there be too much investment, but the higher level of landowner investment may result in no redevelopment at all in the good state.

The results here are reminiscent of those from the property rights and nuisance literature, especially Pitchford and Snyder (2003). Transferring property rights by voluntary exchange is equivalent to a first-party injunctive rights regime. A firstparty injunctive rights regime means that after making his investment decision, the landowner gets to choose whether or not to sell his property; hence, he must receive at least f(x) if he is to sell. And transferring property rights by eminent domain is equivalent to a first-party damage rights regime. In this regime, the landlord is compensated for exactly what he loses, f(x), if he chooses to sell to the developer. Pitchford and Snyder (2003) demonstrate that both regimes are characterized by over-investment but there is less over-investment in a first-party injunctive rights regime, which are precisely the results above. In addition, both parties will make the same ex post decision,<sup>2</sup> and this decision is ex post optimal. As we shall see below, however, the nice equivalence between my results and the property rights and nuisance literature results break down when a hold-out problem exists; redevelopment decisions will no longer be efficient, and landowners and the developer will disagree on the level of redevelopment.

There are (at least) two ways to restore social efficiency under government takings.<sup>3</sup> One way has the government giving "more than just compensation" to the landowner. To see this, suppose that government compensates the landowner by transferring the entire net surplus to him; this is the solution suggested by Hermalin (1985). Such a scheme is equivalent to the landowner having all of the bargaining power.<sup>4</sup> When the landowner has all of the bargaining power, then the landowner's decision problem is given by (3) with  $\beta = 0$ ; the solution to this problem is  $x = x^*$ . A second way to restore efficiency is to give a fixed payment *c*—perhaps equal to zero—to the landowner; this is the famous solution suggested by Blume, Rubinfeld and Shapiro (1984), and further investigated by Blume and Rubinfeld (1984). In this case, the landowner's optimal level of investment problem

$$\max_{x} \theta f(x) + (1 - \theta) c + (K_{\ell} - x) R,$$

which is  $x = x^*$ .

In practice, I would conjecture the neither one of these schemes would be implemented as a policy. In terms of the first policy—giving the entire surplus to the landowner—there would probably be insufficient support in a legislature to pass such a law. By appealing to eminent domain, the legislature wants developers to redevelop properties; but, giving them zero surplus might have the opposite effect. In terms of the second policy, an arbitrary fixed payment c would probably not pass a "just compensation" criterion. One might argue that compensation could be a fixed payment, but need not be arbitrary; for example, why not set compensation equal to  $c = f(x^*)$ . Such a compensation rule, however, requires intimate knowledge of tastes and technology in order to determine  $x^*$ . In addition, the world is populated with many different technologies, each requiring its own "just" fixed payment. If  $c = f(x^*)$ , then policy makers would have to determine first-best levels of investment across many heterogeneous projects. It is not at all obvious that they have the capacity or expertise to make these calculations. In the end, if compensation were required to be just, in the sense that  $c = f(x^*)$ , then policy makers would probably equate or correlate compensation with something they can actually observe, e.g., actual markets values. But this would not implement the efficient level of investment,  $x^*$ .

<sup>&</sup>lt;sup>2</sup>For this article, the *ex post* decision is to redevelop or not; in Pitchford and Snyder (2003) the *ex post* decision is regarding *ex post* investments.

<sup>&</sup>lt;sup>3</sup>Given that the landlord and developer cannot contract prior to the investment decision, x, it is not obvious how one can restore efficiency under voluntary exchange, while at the same time maintaining the notion that exchange is voluntary.

<sup>&</sup>lt;sup>4</sup>It is also equivalent to a second-party injunctive rights regime.

And finally, parties who have their property taken by the government often claim that they are under-compensated. In the context of the model, if a landowner is compensated f(x) for his property rights, then there is some legitimacy to the claim that he is under-compensated. In the real-world economy, the price for real estate is typically determined by bargaining between a seller and buyers. The purchase price of land via voluntary exchange, p, is always strictly greater than the level of just compensation, f, as long as both parties to the bargain have some bargaining power, i.e.,  $f + (1 - \beta) S > f$ . So, in fact, sellers (landowners) receive a lower price for their property via eminent domain than they could have obtained by dealing directly with buyers (developers).

### 3 A Model with Many Landowners

In the model presented in section 2, the landowner has an interesting decision problem, while the developer really doesn't. The developer simply decides whether or not to redevelop, since the level of redevelopment and the amount spent on redevelopment are effectively predetermined. In this section, the developer's problem is made more interesting *and realistic* by having him choose both the amount of land to redevelop and the amount to spend on redevelopment. In addition, a hold-out problem can now be introduced because redevelopment may involve more than one piece of property. This richer environment can be obtained by making only slight modifications to the model in section 2.

Now there are N landowners, each having property rights to their own tract of land. The total value associated with property redevelopment is given by Y = F(A, y), where A represents the tracts of land or properties used for redevelopment and y is the total amount spent on redevelopment. I assume the function F(A, y) is strictly concave and increasing in both of its arguments, with  $F_A(0, y) = F_y(A, 0) \rightarrow \infty$  for y, A > 0.5

The timing of events is as follows: At date 0, landowners are born and invest x in their property and  $K_{\ell} - x$  in the safe asset. At date 1, the developer is born; he decides how many properties he wishes to redevelop, A, and the total amount that he will spend on redevelopment, y. The developer either bargains with a set of A landowners or the government takes the landowners' property rights away from A landowners and sells them to the developer, and the developer pays either Ap or Af(x) (at date 2), respectively, to acquire the property rights. The developer invests  $(K_d - y)$  in the safe asset. At date 2, all investments pay off, payments are exchanged, and the landowners and the developer consume.

<sup>&</sup>lt;sup>5</sup>One can assume, as in Nosal (2001), that only a fraction of properties are suitable for redevelopment and this is revealed only after landowners make their investments. Qualitatively speaking, this would not affect any of the results.

### 3.1 Social Optimum

Let W(A, x, y) represent the total payoff to society. The efficient levels of property redevelopment, A, property investment, x and redevelopment spending, y, are given by the solution to

$$\max_{x,y,A} W(A,x,y) = \max_{x,y,A} (N-A) f(x) + F(A,y) + N(K_{\ell}-x) R + (K_d-y) R.$$
(8)

All N landowners invest x in their property and the remainder of their capital in the safe asset. Since A properties will be redeveloped, the total payoff to the landowners investment is (N - A)f(x). The developer spends y on redevelopment so the payoff to redevelopment is F(A, y). The developer invests  $(K_d - y)$  in the safe asset. The first-order conditions to problem (8) are,

$$\frac{N-A}{N}f'(x) = R, (9)$$

$$F_y(A,y) = R, (10)$$

and

$$F_A(A, y) = f. \tag{11}$$

Conditions (9) and (10) simply say that the expected returns to investment x and spending y equal the opportunity cost of capital, R. In terms of property redevelopment, condition (11), properties continue to be redeveloped until the value last redeveloped unit equals the (social) cost of redevelopment, which is the value of the destroyed investment, f. Let  $(A^*, x^*, y^*)$  represent the solution to (9)-(11).

For what follows, it will be useful to diagrammatically characterize the social optimum in (x, A) space. The slope of the locus of points described by (9) is negative and is given by

$$\frac{dA}{dx} = \frac{f''}{f'} \left(N - A\right) < 0.$$
(12)

Equation (9) is depicted in figure 1 as  $\ell x_N$ ; " $\ell$ " for landowner. Note that the allocation  $(x_N, 0)$  lies on locus  $\ell x_N$ , where  $x_N$  solves  $f'(x_N) = R$ , i.e., equation (6). As well, since  $A \to N$  as  $x \to 0$ , allocation (0, N) lies on locus  $\ell x_N$ . The slope of the locus of points described by equation in (11), conditional on efficient redevelopment spending y, is also negative and given by

$$\frac{dA}{dx} = \frac{f'}{F_{AA} - \frac{F_{Ay}^2}{F_{yy}}} < 0,$$
(13)

since, from (10),  $dy = -F_{Ay}/F_{yy}dA$  and F is strictly concave. Figure 1 depicts equations (10) and (11) as dd'; "d" for developer. For convenience, let  $A^e(x)$  represent the efficient amount of land redevelopment, conditional on x and assuming that spending on redevelopment, y, is efficient, i.e., equation (10) holds.



In figure 1,  $\ell x_N$  and dd' loci intersect twice; social welfare is maximized at point  $a^*$ , where the slope of  $\ell x_N$  curve is steeper than that of the dd' curve.<sup>6</sup> Moving away from the point of intersection  $a^* = (x^*, A^*)$  along either curve  $\ell x_N$  or dd' unambiguously lowers social welfare. Assuming that condition (10) holds, the slope of a social welfare indifference curve is given by

$$\frac{dA}{dx} = \frac{(N-A)f' - NR}{f - F_A}$$

For allocations on the  $\ell x_N$  curve, the slope of the social welfare indifference curve is zero and for allocations on the dd' curve, it is infinite. A typical social welfare indifference curve that intersects allocation  $\tilde{a}$  (where  $\tilde{A} < A^*$  and  $\tilde{x} > x^*$ ) is given by the ellipse denoted  $\widetilde{SW}$  in figure 1. Note that for allocations that are south-east of allocation  $a^* = (x^*, A^*)$  and that lie in between (but not on) the  $\ell x_N$  and dd'curves—such as allocation  $\tilde{a}$ —the slopes of the social welfare indifference curves are all strictly positive and finite. This implies that if two allocations lie in the cone given by  $x_N a^* d'$  and a line that connects the two allocations has a strictly negative slope—such as allocations  $\bar{a}$  and  $\tilde{a}$  in figure 1—then the allocation that has higher redevelopment and lower investment will generate a higher level of social welfare, i.e., the social welfare associated with allocation  $\bar{a}$  exceeds that of  $\tilde{a}$ .

<sup>&</sup>lt;sup>6</sup>The appendix constructs an example where the  $\ell \hat{x}$  and dd' curves intersect twice—as in figure 1—and demonstrates that  $a^*$  is the solution to problem (8).

### 3.2 Redevelopment Under Voluntary Exchange

I do not explicitly model the landowners' hold-out problem; for treatments on this topic, see, e.g., O'Flaherty (1994) and Mendezes and Pitchford (2004). In this paper, the hold-out problem—or the lack of it—will be implicitly modeled by the structure of the various parties' cooperative bargaining weights. For example, a no hold-out problem is modeled by assuming that the developer's share of the surplus—or bargaining weight—is independent of the number of landowners he bargains with; a hold-out problem is modeled by assuming that the developer's share declines with the number of landowners he bargains with. (I discuss the relationship between bargaining weights and the hold-out problem in more detail below.) Of course, a more complete treatment of the hold-out problem would generate these bargaining shares as an equilibrium outcome to a game where landowners can threaten to withhold or delay the sale of their properties.

Consider a generalized Nash Bargaining problem with n > 2 parties, see Osborne and Rubinstein (1990, p. 23). The developer's bargaining power or weight is denoted by  $\beta(A)$ , where  $\beta' \leq 0$ . Each landlord involved in the bargain is assumed to have the same bargaining power, which is  $(1 - \beta)/A$ . If each landowner receives p for transferring his property rights, then the net surplus received by the developer is F - yR - pA and by each of the A landowners is p - f. Hence, the total net surplus, S, that the developer and landowners split is equal to F - Af - yR. The compensation, p, that each landowner receives for transferring his property rights to the developer is given by the solution to the generalized Nash bargaining problem with A + 1 players, i.e.,

$$\max_{p} (F - yR - pA)^{\beta} (p - f)^{\frac{1 - \beta}{A}} \cdots (p - f)^{\frac{1 - \beta}{A}}$$
$$= \max_{p} (F - yR - pA)^{\beta} (p - f)^{1 - \beta},$$

which is

$$p = f + \frac{1 - \beta}{A}S.$$
(14)

Each landowner's investment choice is given by the solution to

$$\max_{x} \frac{N-A}{N} f(x) + \frac{A}{N} \left( f(x) + \frac{1-\beta}{A} S(A, x, y) \right) + (K_{\ell} - x) R,$$
(15)

which is given implicitly by

$$\left(\frac{N-A}{N} + \frac{A}{N}\beta\right)f'(x_B) = R.$$
(16)

The developer's choice of the level of property redevelopment, A, and spending on redevelopment, y, is given by the solution to

$$\max_{A,y} F(A, x, y) - pA + (K_d - y) R.$$

In light of (14), the developer's problem can be rewritten as

$$\max_{A,y} \beta(A) S(A, x, y) + K_d R.$$

The solution to the developer's problem is given by

$$F_y(A,y) = R, (17)$$

and

$$F_A(A, x, y) = f - \frac{\beta'}{\beta}S.$$
(18)

Note the following: Independent of the structure  $\beta(A)$ ,

- 1. for any given A, landowners invest an amount that is greater than what is socially desirable, i.e., compare (16) to (9),<sup>7</sup> and
- 2. for any given (A, x), developer's redevelopment spending, y, is always efficient, i.e., equation (17).

#### 3.2.1 No Hold-out Problem

Here, it is assumed that the developer's bargaining weight is fixed and independent of A, i.e.,  $\beta' = 0$ . When the developer has a fixed bargaining weight, one can think of landowners as being incapable of increasing their (collective) share of the surplus which is  $1 - \beta$ —by threatening to withhold their properties. In this sense, a fixed bargaining weight for the developer models a no-hold out problem. To reinforce the idea that the developer's bargaining weight is fixed, let  $\beta \equiv \overline{\beta}$ .

Note a few things. First, given the amount of investment undertaken by landowners, x, the amount of property redeveloped, A, is efficient since  $\beta' = 0$ , i.e., compare equations (18) with (11). Second, from equations (17) and (18), the developer's choice of y and A does not depend directly on his bargaining strength; the choices do depend *indirectly* on bargaining strength since the landowner's choice of x depends upon on the bargaining strengths and the developer's choice variables depend on x. Hence, the developer will choose A and y so as to maximize total surplus, since this also maximizes his share of the surplus. This result mirrors that of Pitchford and Synder (2003), i.e., conditional on the choice of x, the ex post decisions, A and y, are efficient. Diagrammatically speaking, the developer's decision is described by locus dd' in figure 2, which is identical to the locus dd' in figure 1.

<sup>&</sup>lt;sup>7</sup>Diagramatically speaking, the landowner's investment locus will lie everywhere above locus  $\ell x_N$  in figure 1, except at point  $(x_N, 0)$ .



The locus of points that describe the landowner's optimal investment decision, (16), in (x, A) space is depicted in figure 2 as  $\ell_1 x_N$ . From figure 2, the equilibrium outcome  $a_B = (x_B, A_B)$  is given by the lower intersection of the  $\ell_1 x_N$  and dd' curves. It is evident that when the developer and landowners bargain over the price of landowners' property rights, there will be, from a social perspective, too much investment in property,  $x_B > x^*$ , and, as a result, too little redevelopment activity,  $A_B < A^*$ . (The total amount of investment associated with redevelopment, y, can be higher than, lower than or equal to  $y^*$ , depending on the sign of  $F_{Ay}$ .)

The equilibrium outcome  $a_B = (x_B, A_B)$  differs from the socially efficient levels because landowners take account of the fact that if their property is purchased for redevelopment, then, through the bargaining process, the purchase price will depend positively on the amount of investment undertaken. Because of this, landowners will tend to overinvestment in their properties.

#### 3.2.2 Hold-out Problem

Here, it is assumed that the developer's bargaining power,  $\beta(A)$ , falls with the number of landowners, A, he bargains with, i.e.,  $\beta' < 0$ . Without loss of generality, I also assume that  $\beta(1) = \overline{\beta}$ . This bargaining scheme models the hold-out problem in a rather direct, but simple, way. When A = 1, there cannot be a hold-out problem, and both bargaining scenarios—the no-hold out and the hold out—give the developer and the landowner bargaining weights of  $\overline{\beta}$  and  $1 - \overline{\beta}$ , respectively. However, as Aincreases, the developer's bargaining power strictly decreases, while the landowner's



share of the surplus increases, compared to the no hold-out bargaining scenario. Here, one can think of landowners as successfully holding out for higher surpluses as the number of landowners involved in the bargain increases. It is in this sense that a bargaining scheme where a developer's weight  $\beta(A)$  declines in A models a hold-out problem.

The locus of points that describe the landowner's investment decision (16) when  $\beta' < 0$  is depicted by  $\ell_1 x_N$  in figures 3 and 4. Note that this locus lies below a "no-hold out" locus for all A > 1, which is depicted in figure 3 by  $\hat{\ell}_1 x_N$ . (The equilibrium outcome for the no hold-out problem is given by allocation  $\hat{a}_B$ .)

When  $\beta' < 0$ , the level of development, A, is no longer socially efficient. The developer now faces a trade off: although total surplus may increase with A, his share of that surplus decreases. The end result may be that the developer "under-redevelops." By under-redevelop, I mean that, for a given x, the amount of redevelopment, A, is less than what is efficient,  $A^e(x)$ . To see that there is, indeed, under-redevelopment, for a large number of specifications of  $\beta(A)$ , note that equation (18) implies that

$$F_A(A^e(x), y) < f(x) - \frac{\beta'}{\beta}S(A^e(x), x, y).$$
(19)

This is because the second term on the right-hand side of (19) is strictly positive and, by definition,  $F_A(A^e(x), y) = f(x)$ . Now note that a reduction in A always increases the left-hand side of (18) since

$$\frac{dF_A}{dA} = F_{AA} - F_{Ay}\frac{dy}{dA} = F_{AA} - \frac{F_{Ay}^2}{F_{yy}} < 0;$$
(20)

and does not increase the right-hand side of (18) if

$$\frac{d\left[f - \frac{\beta'}{\beta}S\left(A, x, y\right)\right]}{dA} = -\frac{\beta''\beta - \left(\beta'\right)^2}{\beta^2} - \frac{\beta'}{\beta}\left(F_A - f\right) \ge 0.$$
(21)

The equilibrium will be characterized by under-redevelopment if inequality (21) holds, (inequality (20) always holds). Inequality (21) will hold for sure when  $\beta'' \leq 0$ ; it also holds for some specifications of  $\beta$ , where  $\beta'' > 0$ . A particularly attractive specification for  $\beta$  along this dimension is  $\beta(A) = 1/(A+1)$ , which that each party to the bargain has an equal bargaining weight. When  $\beta(A) = 1/(A+1)$ , the lefthand side of inequality (21) is equal to zero and, hence, the equilibrium will be characterized by under-redevelopment. In what follows, I will assume that  $\beta(A)$ satisfies inequality (21), i.e., the developer under-redevelops.<sup>8</sup>

When  $\beta'(A) < 0$ , the developer's behavior—given by equations (17) and (18)—is described in figures 3 and 4 by locus  $d_1d'_1$ ; the planner's decision along these dimensions is represented by locus dd'. The equilibrium  $a_B = (x_B, A_B)$  is characterized by the lower intersection of the  $d_1d'_1$  and  $\ell_1x_N$  loci. Compared to the social optimum,  $a^* = (x^*, A^*)$ , landowners overinvest and the developer redevelops less than  $A^*$  properties, which, qualitatively speaking, parallels the results when there was no hold-out problem. But, when there is no hold-out problem, the developer's level of redevelopment is, conditional on x, efficient. Although the existence of the holdout problem introduces a new inefficiency, it is not necessarily the case that society is made worse off because of this.<sup>9</sup> When a hold-out problem exists, the amount of redevelopment is lower than an already inefficiently low level when the hold-out problem is absent. But the amount of landowner investment under the former can be lower; this may be a good outcome because landowner investment in a no hold-out environment is inefficiently too high. Diagrammatically speaking, it is quite possible to have social welfare higher at allocation  $a_B$ ,  $SW_B$ , than at  $\hat{a}_B$ , see figure 3. If, however, the hold-out problem is more severe—meaning that under-redevelopment is more pronounced—the "new"  $d'd'_1$  curve will shift down closer to the origin in figure 3. In this scenario, landowner investment when the hold-out problem exists can exceed that of when the hold-out problem does not exist. If this is the case, then, unambiguously, social welfare will be lower when the hold-out problem exists.

<sup>&</sup>lt;sup>8</sup>In O'Flaherty (1994), there exists an externality owing to the public good nature of land assembly, which results in an inefficiently low level of redevelopment. Here, there exists an externality that comes from the developer's declining share of surplus associated with increasing levels of redevelopment. As in O'Flaherty (1994), this externality generates inefficiently low levels of redevelopment.

<sup>&</sup>lt;sup>9</sup>The theory of the second-best, tells us that this observation is not surprising.

#### 3.3 Redevelopment Under Eminent Domain

Under eminent domain, landowners understand that their payoff will be f(x), independent of whether or not their property is taken. Therefore, the optimal behavior for the landowner is given by f'(x) = R or  $x = x_N$ . The locus of points that describe the landowner's optimal investment decision under eminent domain is given by  $\ell_2 x_N$  in both figures 2 and 4. The developer's decision problem is to choose the level of redevelopment, A, and spending on redevelopment, y, given that he must pay f for each property, i.e.,

$$\max_{A,y} F(A,y) - fA + (K_d - y) R \equiv \max_{A,y} S + K_d R.$$

The solution to this problem is

$$F_{y}\left(A,y\right) = R$$

and

$$F_A(A, y) = f,$$

which is identical to (10) and (11), respectively. When property is transferred by eminent domain, the developer's decision regarding the level of redevelopment is represented by then locus dd' in figures 2 and 4, (which also represents the planner's locus along these dimensions).

The following two propositions demonstrate that although it gets the redevelopment decisions "right," eminent domain may have some rather undesirable consequences.

**Proposition 2** When there is no hold-out problem, if landowners believe that redevelopment will occur under eminent domain, then social welfare is strictly lower under eminent domain than under voluntary exchange.

**Proof.** In figure 2, the equilibrium under voluntary exchange,  $a_B = (x_B, A_B)$ , is given by the intersection of the  $\ell_1 x_N$  and dd'. The equilibrium under eminent domain,  $a_{ED} = (x_N, A_{ED})$ , is given by the intersection of the  $\ell_2 x_N$  and dd' loci. Since social welfare unambiguously falls as one moves away from  $a^* = (x^*, A^*)$  along the dd' locus, the social welfare associated with allocation  $a_B$  is strictly higher than that of allocation  $a_{ED}$ .

**Proposition 3** When there is a hold-out problem, if landowners believe that redevelopment will occur under eminent domain, then social welfare is strictly lower under eminent domain than under voluntary exchange when the hold-out problem is not is "significant"; when the hold-out problem is significant, welfare under eminent domain exceeds that of voluntary exchange.



**Proof.** The equilibrium under eminent domain—whether or not a hold-out problem exists—is given by allocation  $a_{ED} = (x_N, A_{ED})$ , with associated social welfare  $SW_{ED}$ , see figure 4. When a hold-out problem exists, the equilibrium allocation under voluntary exchange can lie anywhere on the locus  $a'_B x_N$ . Suppose the hold-out problem is "not significant" in that the curve  $d_1d'_1$  intersects curve  $\ell_1 x_N$  locus above the lower intersection of  $\ell_1 x_N$  and the  $SW_{ED}$  indifference curve. In this situation, the level of social welfare associated with allocation  $a_B$  exceeds that associated with the eminent domain allocation,  $a_{ED}$ . If, however, the hold-out problem is "significant," meaning that curve  $d_1d'_1$  locus intersects curve  $\ell_1 x_N$  below the lower intersection of  $\ell_1 x_N$  and the  $SW_{ED}$  indifference curve—this is suppressed in figure 4—then the social welfare associated with the eminent domain allocation,  $a_{ED}$ .

#### **3.4** Discussion

When there is no hold-out problem, the conclusions regarding the use of eminent domain mirror those of the one landlord case and the property rights and nuisance literature (Pitchford and Synder (2003)). That is, compared to voluntary exchange, the use of eminent domain exacerbates the landowners' overinvestment problem and lowers social welfare. As well, the level redevelopment is lower than the socially efficient level,  $A^*$ ; but, given the level of landlord investment, redevelopment and redevelopment spending are efficient. When there is a hold-out problem, it is still the case that the use of eminent domain exacerbates the overinvestment problem. However, the amount of redevelopment is no longer efficient for the level of landlord investment; if inequality (21) holds, it is too low for any given x. If the hold-out problem is not too severe, then social welfare under eminent domain is lower than that associated with voluntary exchange. This observation is consistent with the result that when there is no hold-out problem, i.e., the extreme case where the holdout problem is not at all severe, voluntary exchange dominates eminent domain.

If, however, the hold-out problem is significant, then the use of eminent domain can actually increase social welfare, compared to voluntary exchange. In this situation—compared to voluntary exchange—although the use of eminent domain exacerbates the landowner's overinvestment problem, it does increase the amount of redevelopment from inefficiently low levels. Hence, social welfare increases because the latter (positive) effect dominates the former (negative) effect.

If policy makers could reliably and costlessly observe the magnitude of the holdout problem, then the selective use of eminent domain to transfer property rights for redevelopment could be a social welfare enhancing policy. That is, eminent domain would be used only when the hold-out problem is severe; otherwise, the developer would bargain directly with landowners. Of course, the problem here is that one cannot costlessly observe the magnitude of the hold-out problem. And one cannot rely on developers or landowners to inform policy makers of the extent of the hold-out problem. Developers have an incentive to claim that the hold-out problem is severe, since the use of eminent domain implies that property can be purchased cheaper than through direct bargaining. Landowners' would have an incentive to claim that there is no hold-out problem and that they would bargain for the same price for their parcel of land independent of how many parcels of land the buyer in interested in purchasing.

In defending the state's right to take property from one private agent and give it to another private agent, proponents of the *Kelo* decision—who are often local governments—point to the increased benefits associated with higher levels of redevelopment, such as more employment and higher taxes collected. Although it may be the case that the use of eminent domain will increase the level of redevelopment and other activities associated with it—it is not obvious that this translates into higher social welfare. For example, allocation  $a_{ED}$  in figure 4 has a higher level of redevelopment compared to allocation  $a_B$ , but a lower level of social welfare. If local governments equate higher levels employment and tax revenue—that usually accompany higher levels of redevelopment—with a higher level of social welfare, then allowing communities to use eminent domain to promote redevelopment can lead to bad outcomes. For example, if local governments will use their power of eminent domain when the hold-out problem is not particularly severe, then there will be a negative impact on social welfare.

And finally, it is not even obvious that the use of eminent domain will increase the level of redevelopment. Recall that if there is no hold-out problem, then eminent domain will have a lower level of redevelopment (and social welfare) than voluntary exchange. By continuity, if the hold-up problem is not significant, then the level of redevelopment under eminent domain can be less than voluntary exchange.

## 4 Conclusion

One might think that a government policy that allows developers to purchase as many properties that they want for "just" compensation would promote redevelopment and enhance social welfare. If there is no hold-out problem, then such a policy will, in fact, reduce both redevelopment and welfare. When a hold-out problem does exists, such a policy need not promote redevelopment and increase social welfare. But even if the policy does promote a higher level of redevelopment activity, it need not enhance social welfare. It is only if the hold-up problem is significant that both redevelopment and social welfare will increase if local government's power of eminent domain is exercised.

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### 6 Appendix

The necessary conditions for a maximum to problem (8) are given by (9), (10) and (11). These conditions will identify a (local) maximum if the following second-order conditions are satisfied,

$$F_{yy} < 0 \tag{22}$$

$$F_{yy}F_{AA} - F_{Ay}^2 > 0 (23)$$

$$(N-A) f''(x) \left[ F_{yy}F_{AA} - F_{Ay}^2 \right] - F_{yy}f'(x)^2 < 0$$
(24)

Conditions (22) and (23) are satisfied since F(A, y) is strictly concave. Condition (24) can be rewritten as

$$\frac{f'(x)}{F_{AA} - \frac{F_{Ay}^2}{F_{yy}}} > \frac{f''(x)}{f'(x)} (N - A)$$

This condition has a nice interpretation; the first order conditions identify a (local) maximum if the  $\ell \hat{x}$  curve is steeper than that of the dd' curve, i.e., compare (12) and (13).

To show that an equilibrium exists and is characterized by what is identified in the text, consider the following example. Let  $f(x) = x^{.5}$  and  $Y(A, y) = A^{.4}y^{.4}$ . For these functional forms, the first-order conditions for problem (8) are

$$\frac{N-A}{2N}x^{-0.5} = R (25)$$

$$.4A^{.4}y^{-0.6} = R (26)$$

$$.4A^{-0.6}y^{.4} = x^{.5} (27)$$

The decision of a landlord is given by (25) and can be rewritten as

$$\frac{N-A}{2NR} = x^{.5};$$

this function is represented as  $\ell \hat{x}$  in figure 1. The decision of the developer is given by equations (26) and (27), which can be combined to read

$$.4^{\frac{10}{6}}R^{-\frac{2}{3}}A^{-1} = x^{.5}.$$

Note that  $A \to \infty$  as  $x \to 0$  and  $x \to \infty$  as  $A \to 0$ . This function is represented as dd' in figure 1. Together, the first-order conditions (25), (26) and (27) can be rewritten as the quadratic equation

$$-A^2 + AN - KN = 0$$

where  $K = (0.8) (0.4)^{\frac{2}{3}} R^{\frac{1}{3}}$ . The solution to this quadratic is

$$A = N \pm \frac{\sqrt{N\left(N - 4K\right)}}{2}.$$

If N > 4K, there will be two solutions, as depicted in figure 1. If N > 4K, then the maximum is given by allocation given by point  $a^*$ , since at point x the slope of the  $\ell \hat{x}$  curve is steeper that that of the dd' curve, (at point b the slope of the dd' curve is steeper that that of the  $\ell \hat{x}$  curve.) If  $N \leq 4K$ , then the solution is A = N. In the text, the more interesting case of N > 4K is assumed.