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Abstract

Previous studies have shown that a random-matching model with divisible fiat money and without constraint on agents' money inventories possesses a continuum of stationary single-price equilibria. Wallace [10] conjectures that the indeterminacy can be eliminated by the use of commodity money. Instead, I find that in a similar random-matching model with dividend-yielding commodity money, a continuum of stationary single-price equilibria exists when the utility of dividend is not too high. This result casts doubt on the conventional belief that the indeterminacy of monetary equilibrium is caused only by the nominal nature of money.

Keywords: Commodity money, random matching, indeterminacy of equilibrium.

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1. INTRODUCTION

Green and Zhou [3] show the existence of a continuum of steady state single-price equilibria in a Kiyotaki and Wright type of random-matching model where agents can hold arbitrary amounts of divisible money, and production of indivisible good is costless. The continuum of steady states is indexed by the price at which all trades occur, and each has a distinct equilibrium allocation. The lower is the price, the higher is the welfare. Zhou [11] shows that such indeterminacy of steady state equilibrium persists in a similar environment with costly production. Wallace [10], in commenting on this indeterminacy result, speculates:

> "The multiplicity is almost certainly not robust to departing from the assumption that the money is a fiat object. That is, if nominal holdings of the fiat object give utility (can be used as paper weights, decoration, or burned as fuel), then the kind of multiplicity that has people treating x units of a fiat asset as a new fiat object disappears."

Such a conjecture may stem from the conventional belief that a gold standard, or a commodity-money standard in general, is somehow less arbitrary than a fiat-money standard, and that theoretically, an economy with commodity money possesses only one or few equilibria. Economists view indeterminacy of equilibrium in a fiat-money model economy as a formalization of arbitrariness. Such a belief is well supported by the analysis of other monetary models, such as overlapping-generations model with fiat money (see McCandless and Wallace [5], Chapters 6 and 7) and money-in-utility-function model (see Obstfeld and Rogoff [7], [8]). In these models, there are usually two steady state equilibria, one monetary and one non-monetary, and a continuum of dynamic monetary equilibria, each of which follows an inflationary path and converges to the non-monetary steady state. In this context, a government's commitment that it would convert to a commodity money system if inflation gets too bad can eliminate the non-monetary steady state and dynamic inflationary monetary equilibria converging to it. In contrast to these models, there is a continuum of steady state monetary equilibria in Green and Zhou [3]. It is unlikely that the same argument of eliminating equilibria where money has no value or is losing value by substituting it with a commodity money system would work here, given that in each of the continuum of steady state equilibria, money is valued and the value is constant.[1]

Indeed, this note shows that in the same environment as Green and Zhou [3], substituting dividend-bearing commodity money for fiat money does not eliminate the existence of a continuum of steady state single-price equilibria, as long as the utility of dividend is not too high. The intuition for this result is that commodity money functions as a medium of exchange, hence its value is derived partly from the consumption value of the dividend, and partly from its transaction value as money. The transaction value of the commodity money is determined endogenously, and a continuum of such values can be supported as steady state transaction value. Therefore, a continuum of prices can be supported as steady state equilibrium prices. By the same intuition, had commodity money been modeled as something that depreciates once utility is derived from it (such as paper burned as fuel), the result should be the same. In that environment, also, the value of money can be decomposed into consumption value, which is fixed, and transactional value, which is endogenously determined at equilibrium.

Therefore, the indeterminacy result is robust to the fiat money assumption.

2. THE MODEL

Time is discrete, $t = 0, 1, 2, \ldots$, and continues indefinitely. There are k types of indivisible, immediately perishable produced goods, where $k \geq 3$. There are k types of infinitely lived agent. Each type has a mass $1/k$ in the population, hence, the total mass of agents is of measure 1. Agents' production and consumption are assumed in such a way so that there is no double coincidence of wants in the produced goods: an agent of type i can produce one unit of good $i+1$ (modulo k) instantaneously and costlessly at each date. He consumes only good i, from which he derives instantaneous utility $u > 0$. In addition, there is a divisible, perfectly durable object that yields a utility dividend return $\varepsilon > 0$ util per unit at the beginning of each date. The total amount of this dividend-bearing asset

[1] Green and Zhou [4] takes up the question whether dynamic equilibrium from an initial state is also indeterminate. In a limit environment without discounting, with overtaking preference, the answer is yes.

is a constant M throughout time. There is no restriction on the amount of the asset an agent can hold. Each agent maximizes the discounted expected utility of his consumption stream, with discount factor β.

The sequence of events occurs at each date as follows. In the beginning of a date, an agent first enjoys the dividend yielded from the asset in his possession. Then, he is randomly paired with another agent in the economy. The distribution of partners' characteristics from which an agent's meeting is drawn matches the demographic distribution of characteristics in the entire population of the economy. Within a pairwise meeting, each agent observes the other's type, but not the trading partner's asset holdings and trading history. The matched trading partners will attempt to trade. If they succeed, production and consumption take place. Nothing happens otherwise. Both agents then proceed to next period.

Since all agents like to have the dividend-bearing assets, and there is no double coincidence of wants between any pair of agents in any produced goods[2], trade must involve the exchange of the asset for the produced consumption goods. From now on, I refer the dividend-bearing asset as commodity money, or money for short. Note that there will be no credit transaction since agents' trading histories are private information. I assume that transactions occur according to a simple trading mechanism, which I model as a simultaneous-move game. A potential trade occurs between a type-i agent who possesses commodity money (buyer) and a type-(i-1) agent who can produce the buyer's desired good (seller). In a trade meeting, the seller posts an offer that specifies the amount of commodity money the buyer has to pay in exchange for a unit of his consumption good, and the buyer submits a bid below which he is willing to pay to acquire his consumption goods. Trade occurs if and only if the bid is at least as high as the offer. In that case the buyer pays exactly the seller's offer price with the commodity money, and the seller produces a unit of his production goods on the spot.

As in Green and Zhou [3], I will consider only steady state symmetric equilibrium in the trading environment described above. An equilibrium is symmetric if all agents with identical characteristics (type and money holdings) act alike, and if agents of different types with the same money holdings act alike except that they are trading with their respective

[2]Produced goods cannot be used as medium of exchange because they are perishable.

buyers and sellers. With a symmetric equilibrium, the equilibrium analysis is conducted through the analysis of the decision of a generic agent of an arbitrary type.

Consider a generic agent of type i. His money holdings may be any nonnegative real number, which I will denote by η. The agent's trading strategy is a pair of real-valued functions of his current money holdings η: an offer strategy $o(\eta)$ specifies the offer he will make as a seller when he meets a type-(i+1) agent, and a bid strategy $b(\eta)$ specifies the bid he will make as a buyer when he meets a type-(i-1) agent. A buyer must always be able to pay his bid, so the bid should satisfy the feasibility constraint $b(\eta) \leq \eta$.

Let the stationary distribution of money holdings among agents be H. For any arbitrary $x \in \mathbb{R}_+$, $H(x)$ denote the proportion of agents whose money holdings are no more than x.

Given the stationary money holdings distribution H, and the time invariant offer and bid strategy o and b, the economywide offer distribution O and the bid distribution B are also time invariant and well defined. More specifically, for any arbitrary $x \in \mathbb{R}_+$, $O(x)$ is the proportion of agents whose offers are no more than x, $O(x) = H\{\eta \,|\, o(\eta) \leq x\}$, and $B(x)$ is the proportion of agents whose bids are below x, $B(x) = H\{\eta \,|\, b(\eta) < x\}$.[3]

The value function of an agent at the beginning of each date, denote it by V, is defined as the expected discounted utility that the agent will receive which depends only on his current money holdings, if he adopts an optimal trading strategy. More specifically, if the agent holds η units of commodity money at beginning of date t, he first enjoys the dividend $\eta\varepsilon$. One of the following three scenarios will then happen. (1) With $1/k$ probability, he meets a producer of his consumption good. In such a case, he may or may not be able to trade and consume depending on whether his bid b is higher than the random partner's offer (drawn from the offer distribution O), and his expected payoff of the meeting is $\int_0^b (u + \beta V(\eta - x))\, dO(x) + (1 - O(b))\beta V(\eta)$. (2) With $1/k$ probability, he meets a consumer of his production good. In such a case, he may or may not be able to trade and obtain more money depending on whether his offer o is below the random trading partner's bid (drawn from the bid distribution B), and his expected payoff of this meeting is $B(o)\beta V(\eta) + (1 - B(o))\beta V(\eta + o)$. (3) With the remaining $1 - 2/k$ probability, he meets

[3]For convenience, B is defined to be continuous from the left, rather than from the right as would be conventional.

one of those agents with whom there is no gain to trade, and the expected payoff is $\beta V(\eta)$. Note that agents will not swap commodity money since all commodity money are identical. Depending on the outcome of his date-t trading, the agent will have a different expected discounted utility from $t+1$ on. Formally, the value function V can be expressed in terms of the following Bellman equation that incorporates all the possibilities above,

$$\begin{aligned} V(\eta) &= \eta\varepsilon + \frac{1}{k}\max_{b\in[0,\eta]}\Big\{\int_0^b \Big(u+\beta V(\eta-x)\Big)\,dO(x) + \Big(1-O(b)\Big)\beta V(\eta)\Big\} \\ &\quad + \frac{1}{k}\max_{o\in\mathbb{R}_+}\Big\{B(o)\beta V(\eta) + \Big(1-B(o)\Big)\beta V(\eta+o)\Big\} + (1-\frac{2}{k})\beta V(\eta). \qquad (1) \end{aligned}$$

With standard arguments, it can be shown that, given the offer and bid distributions O and B, this Bellman equation has a unique solution in the space of bounded measurable functions and that the solution indeed specifies the optimal expected discounted value at each possible level of money holdings.

The state of the environment is summarized by the distribution H of money holdings. Implicitly, the money holdings of each agent is a Markov process on the state space $\mathbb{R}_+$. The transition probabilities are the probabilities of transactions occurring, induced by the optimal strategies (o, b). The environment is stationary if the measure H is a stationary initial distribution of this process. The equilibrium concept adopted here is stationary Bayesian Nash equilibrium. I will refer to this simply as steady state equilibrium.

Definition. A *steady state equilibrium* consists of a time-invariant profile $\langle H, O, B, o, b, V\rangle$ that satisfies

(i) Given that all agents play trading strategy (o, b), the money holdings distribution H, offer distribution O and bid distribution B are stationary.

(ii) Given the stationary distributions for money holdings H, offers O, and bids B, it is optimal for any agent to play strategy (o, b). That is, the trading strategy (o, b) and the value function V jointly solve the Bellman equation (1).

There are many stationary equilibria as defined above. In particular, the extreme case where the commodity money is never used as a medium of exchange, agents hold on to their assets for its dividend and give each other their production goods for free, is an equilibrium. However, the existence of this efficient non-monetary equilibrium is an artifact of the costless production technology. If production is costly, no matter how small the cost is, this equilibrium disappears. Zhou [11] shows that assuming costly production in a fiat-money environment does not change the equilibrium-indeterminacy result, but it complicates the equilibrium analysis significantly. For simplicity, I will maintain the costless-production assumption in the main body of the paper, and address the modification of the analysis for the costly production case in the Appendix. In the next section, I will focus on the topic of this note—the single-price equilibrium.

3. SINGLE-PRICE STEADY STATE EQUILIBRIUM

A single-price equilibrium is one at which all trades occur at the same price. In the following, I will first conjecture a potential single-price equilibrium, and characterize the corresponding equilibrium profile (stationary distributions of money holdings, offers and bids, offer and bid strategy, and the value function). Then use this profile to find sufficient conditions under which the conjectured equilibrium satisfies the definition of a steady state equilibrium given above. The set of sufficient conditions will directly deliver the indeterminacy result.

3.1. *A Conjectured Single-Price-p Steady State Equilibrium*

Consider a potential single-price steady state equilibrium where all trades occur at some arbitrary but fixed price $p > 0$. Given that the trading mechanism specifies that trades occur at sellers' offer prices, it has to be the case that all agents offer to sell at p.[4] That is,

$$\forall \eta \in \mathbb{R}_+ \qquad o(\eta) = p. \tag{2}$$

If an agent can afford price p, given that this is the only price at which he can purchase his consumption goods in any future dates, it is not optimal to bid below p which effectively delays consumption till a future date. Hence, I will conjecture that at the equilibrium, all

[4] The willingness of all traders—particularly those with high money balances—to sell at p depends on production being costless. See Appendix for the corresponding analysis for the costly-production case.

agents who have at least p units of money bid to buy at price no less than p. For buyers have money holdings less than p, because they can not afford to pay the price p, their bid strategy does not affect the equilibrium outcome. Without loss of generality, I assume that they bid all their money holdings. That is,

$$\forall \eta < p \quad b(\eta) = \eta \quad \text{and} \quad \forall \eta \geq p \quad b(\eta) \geq p. \tag{3}$$

Furthermore, since "loose change" will never enter transactions, for simplicity, I assume that all agents' money holdings are integer multiples of p. That is, the support of H will be the discrete set $p\mathbb{N} = \{0, p, 2p, 3p, \ldots\}$.

Consider the implied stationary money holdings distribution H. Given that the support of H is the discrete set of points $p\mathbb{N}$, it is convenient to drop the reference of p when referring to the distribution. For every $n \in \mathbb{N}$, define $h(n) \equiv H(\{np\})$. That is, $h(n)$ is the proportion of the agents who hold precisely np units of commodity money. For the rest of the paper, I will work with distribution h defined on $\mathbb{N}$. I will describe an agent as being in state n when his money holdings are np. The proportion of agents who hold positive money holdings is defined to be $m \equiv \sum_{n=1}^{\infty} h(n)$. Note that $h(0) = 1 - m$.

Given the conjectured trading strategy, an agent moves into state n (the state of having money holdings np) only by making either a sale from state $n-1$ or making a purchase from state $n+1$. He moves out of state n by either making a purchase or a sale. Similarly, an agent of type i will make a sale whenever he meets an agent of type $i+1$ whose money holdings are positive, and he will make a purchase whenever he meets an agent of type $i-1$ if his own money holdings are positive. For the distribution to be stationary requires that the population flow into state n from all other states equals the population flow out of state n, that is,

$$h(1) = mh(0) \tag{4}$$

$$\forall n > 0 \qquad h(n+1) + mh(n-1) = (m+1)h(n). \tag{5}$$

The only distributions that satisfy these two equations are of the form

$$\forall n \in \mathbb{N} \qquad h(n) = m^n(1-m) \tag{6}$$

for some $m \in (0,1)$, that is, the distribution has to be geometric. Furthermore, all agents' holdings of the commodity money should add up to the constant stock of commodity money M in the economy. Applying the geometric functional form specified by (6) to this adding-up relationship, I have

$$p = \frac{1-m}{m} M\,. \tag{7}$$

Eq. (6) and (7) characterize the stationary condition required by the equilibrium.

The offer and bid distributions are implied by the money holdings distribution and the trading strategy. The presumed optimal offer strategy, that all agents offer to sell at p, implies a degenerate offer distribution: $O(p) = 1$ and for any $z \in [0, p)$, $O(z) = 0$. Similarly, the presumed optimal bid strategy, every agent who holds at least p units of money is willing to purchase at p, implies that there is a fraction $1-m$ of bids below price p, that is, for any $z \in (0, p]$, $B(z) = 1-m$.

Given the offer and bid distributions and the offer and bid strategy, next, I solve Eq. (1) for the steady state value function. Consider the value function evaluated at integer multiples of p. Eq. (1) is simplified to the following,

$$V(0) = \frac{m}{k}\beta V(p) + (1 - \frac{m}{k})\beta V(0) \tag{8}$$

and for $n \geq 1$,

$$V(np) = np\varepsilon + \frac{1}{k}\Big(u + \beta V(np-p)\Big) + \frac{m}{k}\beta V(np+p) + (1 - \frac{1}{k} - \frac{m}{k})\beta V(np). \tag{9}$$

For all $n \geq 0$, define $\psi_n \equiv V(np+p) - V(np)$. Note that ψ_n is bounded, since it is the sum of dividend value derived from p units of commodity money and the transaction value brought about by p units of commodity money. The latter is bounded by $u/(1-\beta)$, the discounted value of an agent who were to consume his consumption goods everyday without having to pay for it. The system of equations (8) and (9) can be written as a system of equations of the first differences $\{\psi_n\}_{n=0}^{\infty}$ which has the following solution:

$$\begin{aligned} \psi_0 &= \frac{k\varepsilon p + u + (1-\lambda)\varepsilon pm/(1-\beta)}{k(1-\beta) - m(\lambda - \beta) + \beta} \\ \forall n \geq 1 \quad \psi_n &= \psi_0 \lambda^n + \frac{\varepsilon p}{1-\beta}(1-\lambda^n) \end{aligned}$$

where λ is the root to $\lambda^2 - \lambda(k/\beta - k + 1 + m)/m + 1/m = 0$ that is in $(0, 1)$ (the other root is greater than 1). Then the solution to the system of equations (8) and (9) can be recursively expressed in terms of $\{\psi_n\}_{n=0}^{\infty}$,

$$
\begin{aligned}
V(0) &= \frac{m}{k}\frac{\beta}{1-\beta}\psi_0 && (10)\\
\forall n \geq 1 \quad V(np) &= V(np-p) + \psi_n. && (11)
\end{aligned}
$$

The value function in between integer multiples of p is also well defined. Recall that by the presumed optimal strategy, only integer multiples of p will enter into transaction, any "change" in between would be held forever and its' dividend enjoyed by its holders. Formally, for any arbitrary money holdings η, there exists a unique $n \geq 0$ and a unique $\xi \in [0, p)$ such that $\eta = np + \xi$. Then, for any $n \geq 0$ and $\xi \in (0, p)$,

$$V(np + \xi) = \frac{\xi\varepsilon}{1-\beta} + V(np) \tag{12}$$

where $V(np)$ is given by (10) and (11).

The compound parameter ψ_0 in (10) measures the difference between holding p units of commodity money and holding no money at all, $V(p) - V(0)$. In order to have an active trading equilibrium, this value must exceed the discounted value of holding the p units of money forever and consuming all its future dividend, $p\varepsilon/(1-\beta)$, so that an agent with p units of money will not take it out of circulation and consume only its dividend. This necessary condition for the equilibrium can be written as

$$\frac{u}{\varepsilon M} > \frac{1-m}{m}\Big(\frac{\beta}{1-\beta} - m\Big). \tag{13}$$

It is easy to show that this necessary condition implies the following result.

LEMMA 1. *If condition (13) is satisfied, then the value function V is strictly increasing: for any $n \geq 0$ and $\xi, \xi' \in (0, p)$ such that $\xi < \xi'$, $V(np) < V(np + \xi) < V(np + \xi') < V(np + p)$, and furthermore, V is strictly concave on the integer multiples of p: for all $n \geq 0$, $V(np + p) - V(np) > V(np + 2p) - V(np + p)$.*

That is, at the presumed steady state equilibrium, the value function V has all the nice features. It is strictly increasing, continuous from the right, linear with a slope $\varepsilon/(1-\beta)$ in the open intervals between integer multiples of p, and jumps discretely at integer multiples of p with the jumps decreasing. The value of each additional p units of money diminishes as one's money holdings increase. Because agents discount more the transaction value for the p units of money that is going to be used further into the future, and the increase in the dividend value for keeping the p units of money longer is not as high as the decrease in the transaction value. This completes the description of the conjectured steady state equilibrium.

3.2. *Confirmation of the Steady State Equilibrium*

Since I impose stationarity while deriving the money holdings distribution, the distribution characterized by (6) and (7) satisfies the first criterion of the steady state equilibrium. Therefore, to confirm the conjectured equilibrium is an equilibrium, I need to show only that the presumed strategy given by (2) and (3) is indeed optimal.

The optimal bid reflects the agent's maximum willingness to pay to obtain a unit of his consumption good, which is the bid that solves the first maximization problem in Bellman equation (1), for any $\eta \in \mathbb{R}_+$,

$$b(\eta) = \max\Big\{ z \in [0, \eta] \mid u + \beta V(\eta - z) \geq \beta V(\eta) \Big\}. \tag{14}$$

According to this criterion, the optimality of the presumed bid strategy is given by the following lemma.

LEMMA 2. *If all sellers almost surely offer to sell at price p, and if $u + \beta V(0) \geq \beta V(p)$, or equivalently,*

$$\frac{u}{\varepsilon M} \geq \frac{1-m}{m} \frac{k + m(1-\lambda)/(1-\beta)}{k(1-\beta)/\beta + m(\beta - \lambda)/\beta} \equiv T(m) \tag{15}$$

then it is optimal for an agent with money holdings at least p to bid no less than p, i.e., $b(\eta) \geq p$ for all $\eta \geq p$.

Proof. It is easy to show that condition (15) implies condition (13). Hence, given that (15) holds, by Lemma 1, the value function computed is strictly increasing and concave on integer multiples of p. By (14) and the feasibility condition $b(\eta) \leq \eta$, $u + \beta V(0) \geq \beta V(p)$

(or condition (15))is necessary and sufficient for $b(p) = p$. For any $\eta > p$, suppose that $\eta = np+\xi$ where $n > 1$ and $\xi \in [0, p)$, by (12), $V(np+\xi)-V(np-p+\xi) = V(np)-V(np-p)$. Furthermore, by the concavity of V and condition (15), $V(np) - V(np-p) < V(p) - V(0) \leq u/\beta$. That is, $u > \beta(V(np+\xi) - V(np-p+\xi))$, which, by (14), implies that $b(\eta) = b(np+\xi) \geq p$. ∎

Next, I examine the offer strategy. The optimal offer should maximize the expected net gain from a sale, which is the offer that solves the second maximization problem in Bellman equation (1), for any $\eta \in \mathbb{R}_+$,

$$o(\eta) \in \arg\max_{z \in \mathbb{R}_+} \Big\{ \Big(1 - B(z)\Big)\Big(V(\eta + z) - V(\eta)\Big) \Big\}. \tag{16}$$

The presumed offer strategy is that all offers are made at price p. The following lemma establishes a sufficient condition for this strategy to be optimal.

LEMMA 3. *Given that condition (13) holds, if the proportion of agents with positive money holdings in a stationary distribution of the form (6) is $m \leq 1/2$, and if all agents with money holdings at least p bid to purchase at prices at least p, then it is optimal for an agent always to offer to sell at p, that is, $o(\eta) = p$ for all $\eta \in \mathbb{R}_+$.*

Proof. Consider an agent with money holdings $\eta \in \mathbb{R}_+$. Denote the net expected value of the agent offering $z \in \mathbb{R}_+$ by $W(\eta, z)$,

$$W(\eta, z) = \Big(1 - B(z)\Big)\Big(V(\eta + z) - V(\eta)\Big).$$

Note that $W(\eta, p) = m\Big(V(\eta + p) - V(\eta)\Big)$. For $z = 0$, $W(\eta, z) = 0 < W(\eta, p)$. For $z \in (0, p)$, $1 - B(z) = m$, hence $W(\eta, z) = m\Big(V(\eta + z) - V(\eta)\Big) < W(\eta, p)$ since V is strictly increasing by Lemma 1. For $z \in (jp - p, jp]$ where $j \geq 2$, the feasibility condition $b(\eta) \leq \eta$ and the geometric money holdings distribution (6) imply that $1 - B(z) \leq m^j$. Hence, for such a z, $W(\eta, z) \leq m^j\Big(V(\eta + jp) - V(\eta)\Big) < jm^j\Big(V(\eta + p) - V(\eta)\Big)$ by the concavity of V on integer multiples of p. If $m \leq 1/2$, then $jm^j \leq m$ for all $j \geq 2$, so $W(\eta, z) < m\Big(V(\eta + p) - V(\eta)\Big) = W(\eta, p)$. To summarize all above cases, for any $z \neq p$, $W(\eta, z) < W(\eta, p)$, hence by (16), $o(\eta) = p$. ∎

The intuition for this lemma is simple. An offer of zero would not bring any gain for a seller. Any offer that is strictly positive but less than p would have the same chance to succeed as the offer p but would bring in less money. Any offer strictly higher than p, say $2p$, would reduce the probability of successful trading at least by half in comparison to offering p when $m \leq 1/2$, since buyers holding p cannot afford to pay $2p$. However, the value of getting $2p$ over p units of money in one transaction is less than doubled since the value function is concave. Thus, regardless of one's money holdings, offering $2p$ cannot be optimal. Similarly, other higher offers, including those that are not integer multiples of p, are not optimal either.

Given that condition (15) is stronger than condition (13), and that the function $T(m)$ defined in (15) is a decreasing function of m, the following theorem summarizes the results of Lemma 2 and Lemma 3.

THEOREM. *In every trading environment with parameters k, β, u, and M, for every proportion of agents with positive money holdings $\hat{m} \leq 1/2$, there is $\hat{\varepsilon}$ such that $u/\hat{\varepsilon}M = T(\hat{m})$, for any $\varepsilon \leq \hat{\varepsilon}$, there is a continuum of distinct steady state equilibria indexed by $m \in [\hat{m}, 1/2]$ or by price $p \in [M, (1-\hat{m})/\hat{m}M]$. At each of such an equilibrium, all transactions occur at price p and all agents' money holdings are integer multiples of p.*

The condition on low dividend value (determined by (15)) is to guarantee active trading at equilibrium. If ε is too high, agents will all hold on to their commodity money and consume its dividend. For a given dividend value ε, the set of m that give rise to the single-price-p steady state equilibrium may be larger than the interval $[\hat{m}, 1/2]$. Since $m \leq 1/2$ is only a sufficient condition for Lemma 3, and if $\varepsilon < \hat{\varepsilon}$, there are $m < \hat{m}$ that satisfies condition (15). There may not be a single-price-p steady state if there are too many agents having money or too few agents having money. When there are too few agents holding money (m is too low), it might take too long to get some money back once they are traded away. Hence the discounted consumption value of the dividend over this long period of time may be too high in comparison with the utility gain of one unit of consumption good u. On the other hand, if there are too many agents holding money, it might be optimal for sellers to ask for a price higher than p, which is out of the range of the type of equilibrium

I study here.

The intuition for the coexistence of a continuum of single-price equilibria here is the same as that in the fiat money environment. For any $m_1 < m_2$, $m_1, m_2 \in [\hat{m}, 1/2]$, trades take place at a faster pace and a lower price level at the equilibrium where m_2 proportion of agents holding money than the equilibrium where m_1 proportion of agents holding money, since the only agents who can not trade are those without money. Thus, money is better utilized and has a higher transaction value in the equilibrium indexed by m_2 than that indexed by m_1. There is a different belief what the price will be in each economy, and both beliefs can be consistent with equilibrium beliefs.

The continuum of steady state single-price equilibria corresponds to a continuum of distinct equilibrium allocations. Among the continuum of equilibria, the higher the aggregate real money balance is, the higher welfare the equilibrium provides. Formally, let the welfare level of an equilibrium at which m proportion of agents holding money, denoted it by $U(m)$, as the sum of all agents' utility levels,

$$U(m) = \sum_{n=0}^{\infty} h(n)V(n) = \frac{1}{1-\beta}\Big(\frac{m}{k}u + M\varepsilon\Big). \tag{17}$$

That is, the welfare level is higher if the equilibrium proportion of agents holding money is higher. This is a restatement of the idea mentioned above; the fewer agents there are without money, the fewer trading opportunities will be foregone, and therefore the higher welfare will be.

4. CONCLUSION

I have shown that, unlike the overlapping-generations model or money-in-utility-function model, the use of commodity money instead of fiat money does not overturn the equilibrium indeterminacy result in the steady state random-matching model studied in Green and Zhou [3], although the set of equilibria may have changed. Agents may treat x units of commodity money as a new commodity money because the transactional value of the commodity money is endogenously determined by the distribution of the money holdings, which varies from equilibrium to equilibrium, despite the constant dividend value across equilibria. That is, the transactional value of one unit of commodity money in one environment and that of x units bundled together as one unit in another environment can both be consistent with the presumed equilibrium behavior, namely, always buy and sell at the same price whenever one is able to.

We are back with the original question asked in the beginning of the paper: is indeterminacy of equilibrium presented in this model a generic result of the random-matching model of money or is it a phenomenon caused by the some specific features of the model? The analysis of the costly production case in the Appendix demonstrates that the costless production assumption is very unlikely to be the culprit for equilibrium indeterminacy. There are a number of other alternative model economies that might be examined for the robustness of the indeterminacy result to re-specification.

One alternative model is to assume that all agents' money holdings are publicly observable. Intuitively, such an assumption may eliminate single-price equilibrium altogether, because then agents will make their bids and offers (hence trading prices) depending on trading partner's money holdings in order to extract maximum share of the trade surplus. Indeed, Camera and Corbae [1] show that price dispersion characterizes equilibrium in a model economy that resembles the present one in many respects, but agents' money holdings are observable and money is indivisible. Had money been divisible in that environment, price-dispersion equilibrium might also be indeterminate, since the intuition that the transactional value of money depends on the money holdings distribution, and that what agents view as "one" unit of perfectly divisible money affects the equilibrium distribution, should still hold. In general, assuming money holdings observable is to make agents less anonymous, and lack of anonymity, as Rubinstein and Wolinsky [9] emphasize, is conducive to the existence of many equilibria in which agents can be treated disparately.

Another important assumption is the double auction trading mechanism (simultaneous-move game), which prescribes that transaction occurs if and only if the bid is higher than the offer and it occurs at the offer price. I expect the indeterminacy result to hold in the family of such mechanisms in which trade occurs under the same condition, but the price may be different, e.g., the midpoint of the bid and offer prices.[5] An alternative is

[5]This mechanism was studied by Chatterjee and Samuelson [2] and was shown to be efficient for sale of

to substitute Stahl-Rubinstein strategic bargaining for a double auction mechanism as a representation of strategic price determination for transactions. In an unpublished paper, Ishihara studies a special case of a bargaining game, supposing a take-it-or-leave-it offer by buyers, in an environment similar to the one studied here (divisible fiat money, and agents' money holdings unobservable, but divisible goods), and shows that the equilibrium indeterminacy result holds.[6] There are many very different non-Walrasian trading mechanisms, for example, multilateral trading arrangement instead of bilateral trading arrangement. It is not intuitively obvious, ex ante, whether any of these particular specifications will overturn the indeterminacy result. Obviously, the double auction mechanism in the abstract model economy studied here is not to be taken literally as *the* representative strategic mechanism. There is a wide spectrum of trading mechanisms that are actually used to conduct various transactions. At the very least, the result presented here warrants the interpretation that some transaction mechanisms may be susceptible to indeterminacy of equilibrium if they are predominantly used in an economy. It is hoped that the development of other models that pay close attention to the micro-structure of transactions will provide deeper understanding of what is required for equilibrium to be determinate.

At this point, the conclusion of this paper suggests that commodity-money standard or other commodity-money-like system may also possess the arbitrariness that is often reserved for describing the fiat money system. If one takes such a model seriously, policy such as implementing fractionally backed fiat money or some kind of commodity standard may not resolve the underlying economic uncertainty.

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an indivisible good in a static setting by Myerson and Satterthwaite [6].

[6] H. Ishihara, "Price indeterminacy in a monetary random-matching model," Senshu University, Japan, 2000.

APPENDIX: THE CASE OF COSTLY PRODUCTION

To model costly production, I assume that a producer suffers c unit of utility loss when producing one unit of his production goods. Then, the Bellman equation corresponding to eq. (1) can be written as follow,

$$\begin{aligned} V(\eta) &= \eta\varepsilon + \frac{1}{k}\max_{b\in[0,\eta]}\Big\{\int_0^b (u+\beta V(\eta-x))\, dO(x) + (1-O(b))\beta V(\eta)\Big\} \\ &\quad + \frac{1}{k}\max_{o\in\mathbb{R}_+}\Big\{B(o)\beta V(\eta) + (1-B(o))\beta(V(\eta+o)-c)\Big\} + (1-\frac{2}{k})\beta V(\eta). \qquad (1') \end{aligned}$$

Following Zhou [11], consider the kind of single-price equilibrium at which agents' money holdings are endogenously bounded by Np. At such an equilibrium,

(i) All buyers with money holdings of at least p accept offer p: $\forall\eta \geq p \quad b(\eta) \geq p$.

(ii) All sellers with money holdings of less than Np offer price p: $\forall\eta < Np \quad o(\eta) = p$.

(iii) Agents with money holdings of greater than or equal to Np offer price above p: $\forall\eta \geq Np \quad o(\eta) > p$.

(iv) Offers made by agents with money holdings of Np or more are not accepted by any buyer: $\forall\eta \geq Np \quad o(\eta) > \max_{\eta\leq Np} b(\eta)$.

(v) Sellers with money holdings Np offer to sell at some Jp: $o(Np) = Jp$, $J \geq 2$.

(vi) There exists a least-money balance Kp $(K \geq N)$ such that buyers with money holdings above which are willing to accept offer Jp: $b(\eta) \geq Jp$ if and only if $\eta > Kp$.

I call an equilibrium where agents follow this strategy profile a N-J-p equilibrium. The implied money holdings distribution takes the following form,

$$\forall n \in \mathbb{N} \quad h(n) = \Big(\frac{m}{1-h(N)}\Big)^n (1-m) \qquad (6')$$

where $h(N)$ and m jointly satisfy

$$\begin{aligned} h(N)(1-h(N))^N &= (1-m)m^N \\ h(N) &= \frac{m-(1-m)M/p}{m(N+1)-M/p} \end{aligned}$$

and $0 \leq m \leq 1$ and $0 \leq h(N) \leq 1$. It can be shown that such a distribution always exists.

The Bellman equation ($1'$) can also be written as a system of equations in the first difference of the value function, $\psi_n \equiv V(np+p) - V(np)$, with

$$V(0) = \frac{1}{1-\beta}\frac{m}{k}(\beta\psi_0 - c). \tag{10'}$$

The solution for $\{\psi_i\}_{i=0}^{N-1}$ is

$$\forall\, n = 0, \ldots, N-1 \qquad \psi_n = \theta_0/(1-\beta) + \theta_1\lambda_1^n + \theta_2\lambda_2^n$$

where λ_1 and λ_2 are solutions to $\lambda^2 - \lambda(k/(\beta m) - k/m + 1 + (1-h(N)/m) + (1-h(N))/m = 0$, and θ_0, θ_1 and θ_2 are functions of parameters of the model $\beta, k, \varepsilon, c, u$ and endogenous parameters N, p, m. Given $\{\psi_i\}_{i=0}^{N-1}$, $\{\psi_i\}_{i=N}^{\infty}$ can be recursively obtained as follows,

$$\begin{aligned}
\forall\, n = N, \ldots K-1 \qquad & (1 - \beta + \beta\frac{1-h(N)}{k})\psi_n = p\varepsilon + \beta\frac{1-h(N)}{k}\psi_{n-1} \\
& (1 - \beta + \frac{\beta}{k})\psi_K = p\varepsilon + \frac{h(N)}{k}u + \beta\frac{1-h(N)}{k}\psi_{K-1} + \beta\frac{h(N)}{k}\sum_{i=1}^{J-1}\psi_{K-i} \\
\forall\, n \geq K+1 \qquad & (1 - \beta + \frac{\beta}{k})\psi_n = p\varepsilon + \beta\frac{1-h(N)}{k}\psi_{n-1} + \beta\frac{h(N)}{k}\psi_{n-J}.
\end{aligned}$$

Then, the value function at integer multiples of p, $\{V(np)\}_{n=0}^{\infty}$, is given by ($10'$) and (11) using the solution for $\{\psi_i\}_{i=0}^{\infty}$, and that at non-integer multiples of p is defined by (12).

The analysis of the equilibrium conditions is more complicated than the costless production case. It can be done following Zhou [11] (the fiat money case). The following is an illustration of the conditions for a particular equilibrium, 1-2-p as defined above. At this equilibrium, all trades occur at price p, agents' money holdings do not exceed p units, and agents holding p units of money offer to sell at price $2p$ but are unable to sell at $2p$. I conjecture that at such an equilibrium, the $2p$ offer made by agents with p units of money will be accepted by an agent holding $2p$ units of money, i.e., $K = 1$. (There are other equilibrium at which $K > 1$.) Then the value function can be computed as described above.

The following conditions are sufficient for the conjectured 1-2-p equilibrium to be a valid one.

(a) Agents with no money sell at price p; $\beta V(p) - c \geq \beta V(0)$.

(b) Agents with p units of money do not sell at price p; $\beta V(2p) - c < \beta V(p)$.

(c) Agents with p units of money sell at price $2p$; $\beta V(3p) - c \geq \beta V(p)$.

(d) Agents with no money do not sell at any price below p; $c/\beta > p\,\varepsilon/(1-\beta)$.

(e) If an agent has $2p$ units of money, he is willing to buy at price $2p$; $u + \beta V(0) \geq \beta V(2p)$.

These five conditions together ensure that all agents behave according to the strategy profile (i)—(vi).

Generically, if one of such equilibrium exists, there is a continuum of price p (or equivalently, a continuum of m) also satisfies all five conditions. That is, there is a continuum of such equilibria. As an example, for the environment with $u = 1$, $c = 0.5$, $\varepsilon = 0.01$, $k = 3$, $\beta = 0.95$, and $M = .4$, any price p in interval $[0.46, 0.577]$ (or $m \in [0.693, 0.869]$) corresponds to a 1-2-p equilibrium.

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