Federal Reserve Bank of Chicago

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WP 2000-01

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First version: February, 2000
Current version: April, 2001

Abstract

This article concerns an infinite horizon economy where trade must occur pairwise, using a double auction mechanism, and where fiat money overcomes lack of double coincidence of wants. Traders are anonymous and lack market power. Goods are divisible and perishable, and are consumed at every date. Preferences are defined by utility-stream overtaking. Money is divisible and not subject to inventory constraints. The evolution of individual and economy-wide money holdings distributions is characterized. There is a welfare-ordered continuum of single price equilibria, reflecting indeterminacy of the price level rather than of relative prices.

J.E.L. Classification: C78, D51, E40

[*]Ruilin Zhou thanks the National Science Foundation for financial support. Part of this research was done while both authors held appointments at the Federal Reserve Bank of Minneapolis. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Chicago, Federal Reserve Bank of Minneapolis or the Federal Reserve System.

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1. Introduction

This article contributes to a research program, both classical and contemporary, concerning the relationship between decentralized trading and allocative efficiency. Classical economists observed that chains of transactions link traders who do not deal directly with one another or even have specific knowledge of one another's existence. They understood that such a pattern of sequential trade within overlapping coalitions reflects environmental constraints (e.g., geographic separation) that rule out having an economywide market (as envisioned by Walras' parable of an auction) in which all traders would participate directly. Those classical economists, then, regarded fragmentation of trading coalitions as an important fact for economic theory to take explicitly into account. In fact, their view about the efficiency of competitive trade was based on their assessment that traders largely succeed in overcoming this obstacle, at least insofar as achieving economywide uniformity of relative prices is concerned.

A closely similar perspective, that a credible welfare characterization of market equilibrium needs to come to terms with how traders interact strategically in an environment that enforces fragmented market participation, motivates the model of pairwise trading to be studied here. The main results establish the existence of equilibrium, display indeterminacy by looking at single price equilibrium, and characterize the long run behavior of trade and money holdings in these equilibria.

Analysis of this model supports the classical economists' conjecture that, despite the absence of a central market, it is possible for equilibrium relative prices to remain constant across trading pairs and throughout time. This is a nontrivial finding about the model economy because exchange and consumption occur in real time and result in endogenous heterogeneity of wealth among both buyers and sellers. Such heterogeneity might have been suspected to induce disparity of terms of trade across trading pairs, but is shown not necessarily to do so.

While equilibrium in the model economy is consistent with economywide uniformity of relative prices, nevertheless the equilibrium price level is indeterminate. Given a fixed, nominal stock of money, which is taken to be a parameter of the economy, a higher price level implies a stochastically dominated distribution of real balances among traders in the economy. The lower level of real balances results in a lower incidence of completed transactions among trading pairs, hence in economic inefficiency. The model economy possesses a continuum of distinct, Pareto-ranked, equilibrium allocations. It remains an open question whether this indeterminacy of equilibrium reflects a fundamental fact about decentralized competition or whether it may be an accidental consequence of specific, fragile, modeling assumptions. This question is discussed further in the conclusion.

An informal description of some main features of the model economy, and a brief comparison with formulation of three prior models of decentralized exchange may be a helpful prelude to technical exposition. The model economy comprises a continuum of infinitely lived traders who populate a discrete time environment. At each date, every trader receives an endowment of a perishable, divisible, differentiated good and enjoys consumption of his own endowment good and of the endowment goods of some, but not all, other agents. Agents' preferences between random consumption streams are determined according to a von Neumann-Morgenstern utility function for "temporary utility" at each date, and by an overtaking criterion to compare infinite, expected temporary utility sequences. Indeed, there are gains to economywide trade because each trader receives higher marginal utility from consumption of others' endowment goods than from consumption of the good with which he is endowed.

At each date, the population of agents is randomly partitioned into pairs who are able to trade with one another. All trading pairs satisfy the condition that exactly one agent can obtain utility from consumption of the other's endowment good. Trading is anonymous, in the sense that no pair meets more than once and also that each agent knows the variety of good with which his partner is endowed but nothing else about the partner. A particular double auction mechanism governs trade within all pairs. The equilibrium concept for the economy is a version of Bayesian Nash equilibrium.

The model just described most closely resembles that of Green and Zhou (1998). The present model differs from its predecessor in three respects. Equilibrium is defined in terms of the evolution of the economy from an initial state, rather than the analysis being concerned exclusively with steady state analysis, in order to investigate indeterminacy of equilibrium in its strict sense. The utility-overtaking criterion is adopted, rather than a discounted utility formulation, in order to facilitate analysis of dynamic equilibrium without sacrificing essentiality of money.[1] Goods, as well as money, are modeled as being divisible, in order to remove indivisibility and nonconvexity as possible causes of price level indeterminacy.

Further perspective on the model is obtained by comparing its formulation with those of Gale (1986a,b) and of Shi (1995) and Trejos and Wright (1995). As in Gale's model, traders transfer endowments of divisible goods in random, pairwise meetings that take place in discrete time, and assumptions of anonymity and absence of time preference prevent monopoly power from

[1] Money is inessential in the model of Green and Zhou (1998) because agents in that model are unable to consume their own endowments. Using a combination of analytical and numerical approaches, Zhou (1999a) studies steady state equilibrium of a model economy in which money is essential and in which agents maximize expected discounted utility.

being exercised in these meetings. As in the Shi-Trejos-Wright model, economic activity (i.e., the process of receiving endowments, trading, and consumption) takes place repeatedly through time, rather than each trader consuming only once and then leaving the market as in Gale's model. Also as Shi-Trejos-Wright, the insufficiency of trading pairs to garner directly the full gain to economywide trade takes the particularly stark form of a complete absence of double coincidence of wants. This assumption gives money an essential role as medium of exchange, as Gale emphasized would be desirable in an extension of his model. In particular, the presence of money (which is absent in Gale's model economy) enables the price level, as well as relative prices of various goods, to be considered explicitly. However, unlike the Shi-Trejos-Wright model, money here is modeled as being divisible and not subject to inventory constraints.[2] Finally, as in both the Gale and Shi-Trejos-Wright models, exchange within each trade meeting is assumed to be governed by a strategic form mechanism broadly resembling an auction protocol.

2. The Environment

Economic activity occurs at dates $0, 1, 2, \ldots$. Agents are infinitely lived, and they are nonatomic. For convenience, we assume that the measure of the set of all agents is one. Each agent has a type in $(0, 1]$. The mapping from the agents to their types is a uniformly distributed random variable, independent of all other random variables in the model. Similarly, there is a continuum of differentiated goods, each indexed by a number $j \in (0, 1]$. These goods are perfectly divisible but nonstorable. Each agent of type i receives an endowment of one unit of "brand" i good in each period. An agent can consume his own endowment and half of the other brands in the economy; agent i consumes goods $j \in [i, i + \frac{1}{2}] (\text{mod } 1)$ (for example, agent 0.3 consumes goods $j \in [0.3, 0.8]$, and agent 0.7 consumes goods $j \in [0.7, 1] \cup (0, 0.2])$. He prefers other goods in his consumption range to his endowment good; while consumption of his endowment yields utility c per unit, consumption of any other good in his feasible range yields utility u per unit, and $u > c > 0$.[3] In addition to the consumption goods, there is a fiat money.[4] Money is perfectly

[2] The Shi-Trejos-Wright model amends the Kiyotaki-Wright (1989) model by making goods divisible, while retaining assumptions of money indivisibility and a one-unit inventory constraint on money holdings, to model non-par exchange of money for goods. See Green and Zhou (1998) for a discussion of why the indivisibility and inventory-constraint assumptions are undesirable. Models along the lines of Shi-Trejos-Wright that partially relax those ad hoc constraints (by posting a finite bound, greater than 1, on the amount of money carried into a trade meeting) include Camera and Corbae (1999), Hendry (1993), Molico (1997), and Wallace (1996).

[3] In principle, a consumption bundle could be defined to be a finite measure μ on [0,1) and the utility of μ to an agent i could be defined to be $c\mu(\{i\}) + u\mu((i, i + \frac{1}{2}] (\text{mod } 1))$. In practice, at any date an agent can only consume his and his trading partner's endowment goods.

[4] Logically, fiat money is an economywide accounting system that satisfies restrictions such as we now describe. It is customary in the money/search literature, but not logically necessary, to interpret fiat money as some physical object.

divisible, and an agent can costlessly hold any quantity of it. The total nominal stock of money remains constant at M^* units per capita. We assume that agents do not discount future utility. Their preferences are characterized by an overtaking criterion with respect to expected utility, which will be formalized below.

Agents randomly meet pairwise each period. By the assumed pattern of endowments and consumption sets, there is no double coincidence of wants in any pairwise meeting.[5] Each agent meets a partner endowed with one of his consumption goods with probability one half, and a partner who can consume his endowment good with probability one half. So, in every meeting, one partner is a potential buyer and the other is a potential seller.

Consumption goods cannot be used as commodity money because they are nonstorable, so money is the only medium of exchange available. An agent is characterized by his type and the amount of money he holds. Each agent has an initial money holdings, which, like the agent's type, is exogenously and deterministically given. Within the population, types and initial money holdings are independently distributed. The economywide initial money holdings distribution is common knowledge.

Within a pairwise meeting, each agent observes the other's type, but not the trading partner's money holdings and trading history. They cannot communicate about this information either. For simplicity, we assume that each transaction occurs according to the following simultaneous move game. The potential buyer submits a bid specifying a maximum price and also a quantity that he is willing to buy at any price weakly below that maximum price. And the potential seller submit an offer specifying the price at which she is willing to sell and the maximum quantity she will sell at that price. Trade occurs if and only if the bid price is at least as high as the offer price, but the bid quantity is no higher than the offer quantity. In that case, the seller transfers the quantity of his endowment good prescribed by the bid to the buyer, and the buyer pays with money at the seller's offer price.

This particular double auction mechanism is closely related to a family of such mechanisms in which trade occurs if and only if the bid price is at least as high as the offer price, the quantity transferred from the seller to the buyer is then the minimum of the bid and the offer, and the transaction price is weakly between the offer price and the bid price.[6] The mechanism defined here fails to belong to the family only because trade is specified not to take place if the offer quantity

[5] Strictly speaking, there is a double coincidence of wants only when types i and j are matched, with $i \equiv j + 1/2 \pmod 1$. Such a match occurs with probability zero. Hence, we ignore this possibility.

[6] Examples of such mechanisms are (1) a short side mechanism in which trade occurs at the offer price if the bid quantity is less than the offer quantity and at the bid price otherwise, and (2) a mechanism in which trade occurs at the midpoint of the bid and offer prices. The latter mechanism was studied by Chatterjee and Samuelson (1983) and was shown to be efficient for sale of an indivisible good in a static setting by Myerson and Satterthwaite (1983).

is smaller than the bid quantity—a situation that intuitively would not occur in an equilibrium of an auction type mechanism in this economy because the seller (who has linear temporary utility) should be willing to sell his entire endowment if he is willing to sell any of it. By enforcing the bid quantity to be no greater than the offer quantity when trade occurs, some algebraic expressions in the definitions and proofs below are made simpler than if explicit reference to the minimum of the bid and offer quantities were necessary. However the proofs remain sound, with inessential modifications, for any mechanism in the related family just characterized.

3. The Definition of Equilibrium

The domain of agents' money holdings is $\mathbb{R}_+$. Let Δ be the space of countably additive probability measures on $\mathbb{R}_+$. Suppose that the initial money holdings distribution is given by μ_0.

At each date, the set of agents is randomly partitioned into pairs. Within each pair, one of the agents desires to consume the other's endowment. Thus, a bid and offer are associated with each pair.

Now we provide an intuitive discussion of the distributions of bids and offers, and we state some formal assumptions about those distributions. Our assumptions are in the spirit of a "continuum law of large numbers."[7] For each random partition π of the agents into pairs at date t, there is a sample distribution B_t^π of bids and a sample distribution O_t^π of offers. We assume that these sample distributions do not depend on the partition. That is, there are bid and offer distributions B_t and O_t such that for all partitions π, $B_t^\pi = B_t$ and $O_t^\pi = O_t$. Moreover, because each agent has a trading partner assigned at random, the probability distribution of the trading partner's bid and offer should be identical to the sample distribution. That is, B_t and O_t are the probability distributions of bid and offer respectively that are received at date t by each individual agent, as well as being the sample distribution in each random pairing of the population of agents.

Now let the probability space $(\Omega, \mathcal{B}, \mathcal{P})$ represent the stochastic process of encounters faced by a generic agent. This agent faces a sequence ω of random encounters, one at each date. Agent i's date-t encounter, with some agent of type j, is characterized by agent j's trading type (buyer or seller) in the meeting and her bid/offer price and quantity. Denote the trading partner's type by $\omega_t = (\omega_{t1}, \omega_{t2}, \omega_{t3})$, which is interpreted as follows.

If the trading partner is a buyer, $\omega_{t1} = b$, ω_{t2} is the bid price, ω_{t3} is the bid quantity

If the trading partner is a seller, $\omega_{t1} = s$, ω_{t2} is the offer price, ω_{t3} is the offer quantity.

[7]That is, we believe that they are logically consistent with the results from probability theory that we will apply in our analysis, although they cannot be derived from those results. See Green (1994) and Gilboa and Matsui (1992) for further discussion.

The encounters $\{\omega_t\}_{t=0}^{\infty} \equiv \omega$ are independent across time. Ω is the set of all possible sequences of encounters that a generic agent in the economy faces.

At each date t, pairwise meetings are independent across the population. That is, for each agent, ω_{t1} follows a Bernoulli distribution, a potential buyer's bid $(\omega_{t2},\, \omega_{t3})$ is drawn from the bid distribution having c.d.f. B_t, and a potential seller's offer $(\omega_{t2},\, \omega_{t3})$ is drawn from the offer distribution having c.d.f. O_t. For $t \geq 1$, let $\mathcal{B}_t$ be the smallest σ-algebra on Ω that makes the vector of the first t coordinates, $\omega^t = (\omega_0, \omega_1, \ldots, \omega_{t-1})$, measurable, and $\mathcal{B}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{P}_t$ be the probability measure defined on $\mathcal{B}_t$. Then, for all $t \geq 0$, $x \in \mathbb{R}_+$, and $y \in [0,1]$,

$$\mathcal{P}_t\{\omega_{t1} = b\} = \mathcal{P}_t\{\omega_{t1} = s\} = \frac{1}{2} \tag{1}$$

$$\mathcal{P}_t\Big\{\omega_{t2} \leq x,\, \omega_{t3} \leq y \,|\, \omega_{t1} = b\Big\} = B_t(x,y) \tag{2}$$

$$\mathcal{P}_t\Big\{\omega_{t2} \leq x,\, \omega_{t3} \leq y \,|\, \omega_{t1} = s\Big\} = O_t(x,y). \tag{3}$$

Define $\mathcal{B} = \mathcal{B}_\infty$ and $\mathcal{P} = \mathcal{P}_\infty$.

We focus on symmetric equilibrium, that is, equilibrium in which agents are anonymous, an agent's strategy is a function of only his own trading history and initial money holdings, and strategy is symmetric with respect to agents' types. Let σ be the trading strategy of a generic agent with initial money holdings η_0. His date-t strategy $\sigma_t \equiv (\sigma_{t1},\, \sigma_{t2}, \sigma_{t3},\, \sigma_{t4})$ specifies his bid and offer as a function of his initial money holdings and his encounter history ω, and it is measurable with respect to $\mathcal{B}_t$. The bid $(\sigma_{t1},\, \sigma_{t2})$ is the maximum price σ_{t1} at which the agent is willing to buy and the quantity σ_{t2} that he is willing to purchase (at price no higher than σ_{t1}) if he is paired with a seller of his consumption goods. The offer $(\sigma_{t3},\, \sigma_{t4})$ represents the price σ_{t3} at which he is willing to sell and the maximum quantity σ_{t4} that he is willing to sell at price σ_{t3} if he meets a consumer of his endowment good. Because of the restriction on endowment, $\sigma_{t4} \leq 1$. As a buyer, the agent has to be able to pay his bid. Let η_t^σ denote the agent's money holdings at the beg inning of date t by adopting strategy σ. Then

$$\sigma_{t1}(\eta_0, \omega)\, \sigma_{t2}(\eta_0, \omega) \leq \eta_t^\sigma(\eta_0, \omega). \tag{4}$$

Given the agent's initial money holdings η_0, encounter history ω, and strategy $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, his money holdings evolves recursively as follows: $\eta_0^\sigma(\eta_0, \omega) = \eta_0$ and, for $t \geq 0$,

$$\eta_{t+1}^\sigma(\eta_0, \omega) = \begin{cases} \eta_t^\sigma(\eta_0, \omega) + \sigma_{t3}(\eta_0, \omega)\omega_{t3} & \text{if } \omega_{t1} = b,\ \sigma_{t3}(\eta_0, \omega) \leq \omega_{t2},\ \sigma_{t4}(\eta_0, \omega) \geq \omega_{t3} \\ \eta_t^\sigma(\eta_0, \omega) - \omega_{t2}\sigma_{t2}(\eta_0, \omega) & \text{if } \omega_{t1} = s,\ \sigma_{t1}(\eta_0, \omega) \geq \omega_{t2},\ \sigma_{t2}(\eta_0, \omega) \leq \omega_{t3} \\ \eta_t^\sigma(\eta_0, \omega) & \text{otherwise} \end{cases} \tag{5}$$

Let $v_t^\sigma(\eta_0, \omega)$ denote the agent's utility achieved at date t by adopting strategy σ. Then

$$v_t^\sigma(\eta_0,\omega) = \begin{cases} -c\,\omega_{t3} & \text{if } \omega_{t1} = b,\ \sigma_{t3}(\eta_0,\omega) \le \omega_{t2},\ \sigma_{t4}(\eta_0,\omega) \ge \omega_{t3} \\ u\,\sigma_{t2}(\eta_0,\omega) & \text{if } \omega_{t1} = s,\ \sigma_{t1}(\eta_0,\omega) \ge \omega_{t2},\ \sigma_{t2}(\eta_0,\omega) \le \omega_{t3} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Then, strategy σ *overtakes* another strategy $\hat{\sigma}$ if for all $\eta_0 \in \mathbb{R}_+$,

$$\liminf_{t\to\infty}\ \mathsf{E}\Big[\sum_{\tau=0}^{t} v_\tau^\sigma(\eta_0,\omega) - \sum_{\tau=0}^{t} v_\tau^{\hat{\sigma}}(\eta_0,\omega)\Big] > 0 \tag{7}$$

where E is the expectation operator with respect to the probability measure $\mathcal{P}$.

At the beginning of date t, given all agents' trading strategy σ_t and the initial money holdings distribution μ_0, rational expectation requires that agents' beliefs regarding the c.d.f. of the bid distribution B_t and the c.d.f. of the offer distribution O_t that prevail during date-t trading coincide with the actual distributions implied by the strategy. That is, for all x, $y \in \mathbb{R}_+$,

$$B_t(x,y) = \int_0^\infty \mathcal{P}_t\Big\{\omega \mid \sigma_{t1}(z,\omega) \le x,\ \sigma_{t2}(z,\omega) \le y\Big\} d\,\mu_0(z) \tag{8}$$

$$O_t(x,y) = \int_0^\infty \mathcal{P}_t\Big\{\omega \mid \sigma_{t3}(z,\omega) \le x,\ \sigma_{t4}(z,\omega) \le y\Big\} d\,\mu_0(z). \tag{9}$$

Similarly, the money holdings distribution at the beginning of the of date t is defined as follows, for any set $A \in \mathcal{B}_t$,

$$\mu_t(A) = \int_0^\infty \mathcal{P}_t\Big\{\omega \mid \eta_t(z,\omega) \in A\Big\} d\,\mu_0(z) \tag{10}$$

The equilibrium concept we adopt is Bayesian Nash equilibrium with respect to the overtaking criterion.

Definition. A Bayesian Nash equilibrium is a four tuple $\langle \sigma, \mu_0, \{B_t\}_{t=0}^\infty, \{O_t\}_{t=0}^\infty \rangle$ that satisfies

(i) μ_0 is the initial money holdings distribution in the environment.

(ii) No strategy overtakes σ, given that $\{B_t\}_{t=0}^\infty$ and $\{O_t\}_{t=0}^\infty$ characterize trading partners' decisions.

(iii) For each $t \ge 0$, B_t and O_t satisfy equations (8) and (9). That is, these distributions reflect the adoption of strategy σ by all agents.

We are going to study one particular example of equilibrium, single price equilibrium. In such an equilibrium, all trades occur at the same price, say p, at all dates, $p > 0$. That is, all traders bid to buy one unit or as much as they can afford of their desired consumption goods at price p, and offer to sell one unit of their endowment goods at price p. We call this a price-p

equilibrium. Price-p equilibrium is markovian in the sense that the dependence of agents' strategy on time and trading history is only through their own current money holdings, despite the dynamic environment. Formally, define the strategy $\tilde{\sigma}^p$ as follows, for all $\eta_0 \in \mathbb{R}_+$, encounter history $\omega \in \Omega$, and $t \geq 0$,

$$\tilde{\sigma}^p_{t1}(\eta_0, \omega) = p, \quad \tilde{\sigma}^p_{t2}(\eta_0, \omega) = \min\{\tilde{\eta}_t(\eta_0, \omega)/p,\ 1\} \tag{11}$$

$$\tilde{\sigma}^p_{t3}(\eta_0, \omega) = p, \quad \tilde{\sigma}^p_{t4}(\eta_0, \omega) = 1 \tag{12}$$

where $\tilde{\eta}_t(\eta_0, \omega) = \eta^{\tilde{\sigma}^p}_t(\eta_0, \omega)$. Let $\tilde{\mu}_t \in \Delta$ denote the money holdings distribution at the beginning of date t induced by strategy $\tilde{\sigma}^p$. The bid distribution implied by strategy $\tilde{\sigma}^p$ are as follows: for any x, $y \in \mathbb{R}_+$,

$$\tilde{B}_t(x, y) = \begin{cases} 0 & \text{if } x < p \\ \tilde{\mu}_t([0, py]) & \text{if } x \geq p \text{ and } y < 1 \\ 1 & \text{if } x \geq p \text{ and } y \geq 1 \end{cases} \tag{13}$$

The offer distribution implied by $\tilde{\sigma}^p$ is stationary and degenerate with mass at price p and quantity 1. That is, for any x, $y \in \mathbb{R}_+$,

$$\tilde{O}_t(x, y) = \begin{cases} 0 & \text{if } x \leq p \\ 0 & \text{if } x \geq p \text{ and } y < 1 \\ 1 & \text{if } x \geq p \text{ and } y \geq 1 \end{cases} \tag{14}$$

The evolution of the money holdings distribution $\tilde{\mu}_t$ is specified in the next section.

In order to show that $\langle \tilde{\sigma}^p, \mu_0, \{\tilde{B}_t\}_{t=0}^{\infty}, \{\tilde{O}_t\}_{t=0}^{\infty} \rangle$ is an equilibrium, we first investigate the properties of the dynamic path of the two distributions relevant to the equilibrium at hand: the economywide money holdings distribution which determines the bid distribution, and the distribution of an arbitrary individual's money holdings which helps to define the optimality of the strategy. In the next section, we show both distributions converge asymptotically under a mild condition on the distribution of initial money holdings. In section 5, we show that the strategy $\tilde{\sigma}^p$ is optimal.

4. The Convergence of Money Holdings Distributions at Price-p Equilibrium

Given that there is a continuum of nonatomic agents, if all agents adopt strategy $\tilde{\sigma}^p$, the convergence path of the economywide money holdings distribution over time is deterministic. However, the trading path of a single agent in the economy is random. The probability structure introduced in section 3 is defined in terms of the stochastic process of encounters faced by such an agent. For a generic agent with initial money holdings η_0, the distribution for his possible money holdings at date t is not necessarily given by the economywide money holdings distribution $\tilde{\mu}_t$, which is the money holdings distribution of a potential trading partner at date t. In order to study the optimality of strategy $\tilde{\sigma}^p$, we need to know the evolution of the money holdings distribution for a single agent with arbitrary initial money holdings η_0.

In this section, we show that if all agents adopt the strategy $\tilde{\sigma}^p$, and if the initial money holdings distribution μ_0 satisfies a certain condition, then the economywide money holdings distribution converges weakly to a unique geometric distribution at which the economy is stationary. Furthermore, we show that given the economywide money holdings distribution converges, the distribution of a generic agent with an arbitrary initial money holdings converges to the aggregate limit distribution, and the mean of his money holdings converges to the per capita money holdings in the economy M^*.

To show the convergence of the economywide money holdings distribution, as a technical intermediate step, we first show the convergence of the economywide distribution and the distribution of an individual agent if money is indivisible with unit p. That is, all agents' money holdings are in multiple units of p. We then use the result to prove the convergence of money holdings distributions when money is divisible.

We now introduce a notation that will be used throughout the paper. In the trading environment introduced above, the aggregate nominal quantity of money does not change over time. That is, in an economy with aggregate nominal quantity of money M, the probability measure μ that represents the economywide money holdings distribution satisfies the aggregation condition, $\int_0^\infty \eta d\mu = M$. Let Δ^M be the restricted probability space where the aggregation condition holds. That is, $\Delta^M = \{\mu \,|\, \mu \in \Delta,\ \int_0^\infty \eta d\mu = M\}$.

4.1. *The Indivisible Money Case*

Consider an economy where the aggregate nominal quantity of money is M, $M > 0$, and where money is indivisible with unit p. In this economy, the support of money holdings distribution is in the lattice $p\mathbb{N}$ of multiples of p. ($p\mathbb{N} \equiv \{0, p, 2p, \ldots\}$.) Denote the set of these probability measures with support in $p\mathbb{N}$ by $\Delta_{p\mathbb{N}}$, and the subset of probability measures in $\Delta_{p\mathbb{N}}$ with mean M to be $\Delta^M_{p\mathbb{N}}$. Given that all agents adopt strategy $\tilde{\sigma}^p$, and given the initial distribution μ_0, the economywide money holdings distribution evolves deterministically as follows: for any $t \geq 0$,

$$\mu_{t+1}(\{0\}) = (1 - \frac{m(\mu_t)}{2})\mu_t(\{0\}) + \frac{1}{2}\mu_t(\{p\}) \tag{15}$$

and for all $k \geq 1$,

$$\mu_{t+1}(\{kp\}) = (1 - \frac{m(\mu_t)}{2} - \frac{1}{2})\mu_t(\{kp\}) + \frac{1}{2}\mu_t(\{(k+1)p\}) + \frac{m(\mu_t)}{2}\mu_t(\{(k-1)p\}) \tag{16}$$

where $m(\mu_t) = \sum_{k=1}^{\infty} \mu_t(\{kp\})$ is the measure of agents who have money. The sequence $\{\mu_t\}_{t=0}^{\infty}$ of money holdings distributions can be obtained by applying (15) and (16) recursively. It is easy to check that if $\mu_0 \in \Delta_{p\mathbb{N}}^{M}$, then at any point of time $t \geq 1$, $\mu_t \in \Delta_{p\mathbb{N}}^{M}$.

For technical convenience, we work with a transformation of the probability measure μ instead of μ itself. Define a mapping $L\colon \Delta_{p\mathbb{N}} \to [0,1]^{\infty}$ as follows,

$$\forall \mu \in \Delta_{p\mathbb{N}} \quad \forall k \in \mathbb{N} \qquad L_k(\mu) = \mu([kp, \infty)) = \sum_{j \geq k} \mu(\{jp\}). \tag{17}$$

Obviously, $L_0(\mu) = 1$, $L_k(\mu) \in [0,1]$ and $L_k(\mu) \geq L_{k+1}(\mu)$ for all $k \in \mathbb{N}$. Let $\Gamma = L(\Delta_{p\mathbb{N}})$. Then, for any $x \in \Gamma$, x satisfies that $x_0 = 1$, $x_k \in [0,1]$ and $x_k \geq x_{k+1}$ for all $k \in \mathbb{N}$. By definition, L is a one-to-one affine mapping from $\Delta_{p\mathbb{N}}$ to Γ. The aggregate real money balance corresponding to the distribution $\mu \in \Delta_{p\mathbb{N}}^{M}$ can be written as $\sum_{k=1}^{\infty} k\Big(L_k(\mu) - L_{k+1}(\mu)\Big) = \sum_{k=1}^{\infty} L_k(\mu) = M/p$. Define $S = L(\Delta_{p\mathbb{N}}^{M})$. Clearly S is a subset of Γ.

$$S = \Big\{x \,|\, x \in \Gamma,\ \sum_{k=1}^{\infty} x_k = M/p\Big\}. \tag{18}$$

The set S is the space we are going to work with primarily in the first half of this section. It is easy to show the following (the proof is omitted).

LEMMA 1. *Both S and Γ are convex. That is, for $X = S$ or $X = \Gamma$, $\forall x, y \in X$ and $\forall \alpha \in [0,1]$, $\alpha x + (1-\alpha) y \in X$.*

By equations (15) and (16), the law of motion of the transformation of money holdings distribution $L(\mu)$ is a mapping $T\colon S \to S$ such that for all $x \in S$,

$$\forall\, k \geq 1 \qquad T_k(x) = \frac{1 - x_1}{2} x_k + \frac{1}{2} x_{k+1} + \frac{x_1}{2} x_{k-1}. \tag{19}$$

It is easy to show that $T_0(x) = 1$ and $T(x) \in S$. For any given $\mu_0 \in \Delta_{p\mathbb{N}}^{M}$, $x^0 = L(\mu_0)$, $x^t = T(x^{t-1}) = L(\mu_t)$ for all $t \geq 1$. The following lemma states that the mapping T has a unique fixed point.

LEMMA 2. *The mapping T has a unique fixed point $\bar{x} \in S$ such that $\bar{x} = T(\bar{x})$:*

$$\forall k \in \mathbb{N} \quad \bar{x}_k = \bar{m}^k, \qquad \textit{where} \quad \bar{m} = \frac{M/p}{1 + M/p}. \tag{20}$$

Proof. For all $x \in S$, by equation (19), $T(x) = x$ requires that for all $k \geq 1$,

$$x_k = (\frac{1 - x_1}{2})x_k + \frac{1}{2}x_{k+1} + \frac{x_1}{2}x_{k-1}$$

and $x_0 = 1$. This system of equations has a unique solution $\bar{x}$ that satisfies, for all $k \in \mathbb{N}$, $\bar{x}_k = (\bar{x}_1)^k$. Since $\bar{x} \in S$, $\sum_{k=1}^{\infty} \bar{x}_k = \sum_{k=1}^{\infty} (\bar{x}_1)^k = M/p$, which implies that $\bar{x}_1 = \frac{M/p}{1+M/p} = \bar{m}$. ∎

The unique fixed point $\bar{x}$ of T given in Lemma 2 corresponds to a geometric money holdings distribution with parameter $\bar{m}$: $\bar{\mu} = L^{-1}(\bar{x}) \in \Delta^M_{p\mathbb{N}}$. In particular, for all $k \in \mathbb{N}$, $\bar{\mu}(\{kp\}) = (1 - \bar{m})\bar{m}^k$. We want to show that starting from a given initial state x^0, the economy as a dynamic system evolving according to mapping T converges asymptotically to the steady state characterized by $\bar{x}$. Toward this objective, we construct a Liapunov function that is a function of the state of the dynamic system. We show that the Liapunov function decreases over time and asymptotically approaches its minimum. Therefore, by a standard argument of dynamical systems theory, the economy asymptotically approaches a steady state, which is represented by the unique fixed point of T, $\bar{x}$, which does not depend on the initial state.

The Liapunov function we choose to use can be interpreted as the expected hazard rate for the corresponding distribution. Define $Z: \Gamma \to \mathbb{R}_+$, for all $x \in \Gamma$,

$$Z(x) = \sum_{k=0}^{\infty} \frac{(x_k - x_{k+1})^2}{x_k}. \tag{21}$$

For technical reasons, we define the function Z on the larger space Γ instead of on S. For Z to be a Liapunov function, it should be continuous in some metric, it should be decreasing along the trajectory of the system defined by T, and it should have a unique minimum on S where it is applied. We show that Z has these properties one by one. Many of the technical proofs are given in the appendix.

LEMMA 3. *The function Z is strictly convex on Γ.*

Using Lemma 3, the following lemma shows that the function Z is strictly decreasing along the trajectory defined by T.

LEMMA 4. *For all $x \in S$, $Z(T(x)) < Z(x)$, unless $x = T(x)$.*

Proof. Define mappings $\lambda: S \to \Gamma$ and $\rho: S \to \Gamma$ as follows: for all $x \in S$, $k \in \mathbb{N}$,

$$\lambda_k(x) = \frac{x_{k+1}}{x_1}, \qquad \rho_0(x) = 1, \quad \rho_{k+1}(x) = x_1 x_k.$$

The measure λ is a normalized left shift of x, and ρ is a normalized right shift of x. It is easy to check that $\lambda(x) \in \Gamma$ and $\rho(x) \in \Gamma$, but neither is necessarily an element of S. Then by (19), $T(x)$ can be rewritten as a convex combination of x, $\lambda(x)$ and $\rho(x)$,

$$T(x) = \frac{x_1}{2}\lambda(x) + \frac{1}{2}\rho(x) + \frac{1-x_1}{2}x.$$

Since Z is strictly convex on Γ by Lemma 3, unless $\lambda(x) = \rho(x) = x$,

$$\begin{aligned}
Z(T(x)) &< \frac{x_1}{2}Z(\lambda(x)) + \frac{1}{2}Z(\rho(x)) + \frac{1-x_1}{2}Z(x) \\
&= \frac{x_1}{2}\frac{1}{x_1}\Big(Z(x) - (1-x_1)^2\Big) + \frac{1}{2}\Big((1-x_1)^2 + x_1 Z(x)\Big) + \frac{1-x_1}{2}Z(x) \\
&= Z(x).
\end{aligned}$$

It is easy to verify that $\lambda(x) = x$ if and only if $x = T(x)$. Therefore, unless $x = T(x)$, we have $Z(T(x)) < Z(x)$. ∎

Because of the aggregation condition ($\sum_{k=1}^{\infty} x_k = M/p$), S is a subset of the complete metric space $(\mathcal{X}, d)$, where $\mathcal{X} = \{x \in [0,1]^\infty \mid \sum_{k=0}^{\infty} x_k < \infty\}$ and d is the usual ℓ_1-metric associated with $\mathcal{X}$, for any $x, y \in \mathcal{X}$,

$$d(x,y) = \sum_{k=0}^{\infty} |x_k - y_k|. \tag{22}$$

By standard argument, (S,d) is a complete subspace of $(\mathcal{X}, d)$. The following two lemmas state that both Z and T are continuous mappings in metric d.

LEMMA 5. *The function Z is continuous on S.*

LEMMA 6. *The mapping T is continuous on S.*

The set S we have been working with is unfortunately not compact. To insure the convergence of the system from some initial state, we introduce a subset of S that is compact and closed under mapping T.

Define an ordering relation between two vectors x and y: y dominates x (denoted by $x \preceq_d y$) if and only if for all $k \in \mathbb{N}$, $x_k \le y_k$. For a given $\pi \in \mathcal{X}$, let S_π be the set of vectors in S that are dominated by π,

$$S_\pi = \{x \in S \mid x \preceq_d \pi\}. \tag{23}$$

LEMMA 7. *For every vector $\pi \in \mathcal{X}$, the set S_π is compact.*

The vector π can be any element of $\mathcal{X}$. In particular, let π^θ denote the geometric vector defined by some $\theta \in (0,1)$: for all $k \in \mathbb{N}$,

$$\pi_k^\theta = \theta^k. \tag{24}$$

The vector π^θ as defined above is an element of $\mathcal{X}$ as well as Γ. Also, it is a fixed point of T, i.e., $T(\pi^\theta) = \pi^\theta$. The following lemma states that for π^θ, S_{π^θ} is closed under T.

LEMMA 8. *For any $x \in S$ and any $\theta \in (0,1)$, if $x \preceq_d \pi^\theta$, then $T(x) \preceq_d \pi^\theta$.*

By Lemma 8, if state x^0 satisfies the following condition on the initial distribution of money holdings,

(ID$_1$) *there exist $\theta \in (0,1)$ and $t \geq 0$ such that $T^t(x^0) \preceq_d \pi^\theta$*

then all the subsequent states of the dynamic system $T^n(x^0)$, for all $n \geq t$, are dominated by π^θ as well, hence, they are elements of S_{π^θ}.

PROPOSITION 1. *Consider an environment where money is indivisible with unit p. Suppose that the initial money holdings distribution μ_0 is such that $x^0 = L(\mu_0)$ satisfies condition ID$_1$. As a dynamic system evolving from x^0 according to mapping T, this economy converges asymptotically to the steady state characterized by distribution $\bar{x}$, which uniquely satisfies $T(\bar{x}) = \bar{x}$ and $d(\bar{x}, 0) = d(x^0, 0)$.*

Proof. Suppose that condition ID$_1$ holds, that is, there exist $\theta \in (0,1)$ and $t \geq 0$ such that $T^t(x^0) \preceq_d \pi^\theta$. Then $T^n(x^0) \in S_{\pi^\theta}$ for all $n \geq t$. By Lemma 7, S_{π^θ} is a compact set, and by Lemma 5, the function Z is continuous on S, hence on S_{π^θ}, so Z achieves its minimum on S_{π^θ}. Furthermore, by Lemma 3, Z is strictly convex on S, hence on S_{π^θ}, Z has a unique minimum on S_{π^θ}. Last, Z is strictly decreasing along the trajectory of the system defined by T by Lemma 4. Therefore, Z is a Liapunov function. With this Liapunov function, we show next the convergence of the system from the initial state x^0.

From the given x^0, construct a sequence $\{x^n\}_{n=1}^\infty$ by applying T recursively, $x^n = T^n(x^0)$. Consider the sequence excluding the first t elements, $\{x^n\}_{n=t}^\infty$, which has just been shown to lie within S_{π^θ}. By Lemma 4, the corresponding sequence $\{Z(x^n)\}_{n=t}^\infty$ is monotonically decreasing. Since S_{π^θ} is compact, there exist a subsequence $\{x^{n_k}\}$ that converges to some $\hat{x} \in S_{\pi^\theta}$. Suppose that $\hat{x}$ is not a fixed point of T. Then by Lemma 4, $Z(T(\hat{x})) < Z(\hat{x})$. Since Z is continuous and T is continuous, there exists $\delta > 0$ such that for all y satisfying $d(\hat{x}, y) < \delta$, $Z(T(y)) < Z(\hat{x})$. Since

$\{x^{n_k}\}$ converges to $\hat{x}$, there exists K such that for all $k \geq K$, $d(\hat{x}, x^{n_k}) < \delta$, hence, $Z(T(x^{n_k})) < Z(\hat{x})$, or,

$$Z(x^{n_k+1}) < Z(\hat{x}). \tag{25}$$

But since $\{Z(x^n)\}_{n=t}^{\infty}$ is monotonically decreasing, and since $\hat{x}$ is the limit of x^{n_k},

$$Z(x^{n_k+1}) \geq Z(\hat{x}) \tag{26}$$

which contradicts (25). Therefore, the limit $\hat{x}$ has to be a fixed point of T. Since T has a unique fixed point $\bar{x}$ in S by Lemma 2, $\bar{x} = \hat{x} \in S_{\pi^\theta}$. Hence, for the given initial state x^0, $T^n(x^0) \to \bar{x}$ as $n \to \infty$. This strengthened statement, that the entire sequence (rather than only the subsequence selected above) converges to $\bar{x}$, follows from a standard argument involving the Liapunov function Z. ∎

The convergence of $T^t(x^0)$ to $\bar{x}$ as $t \to \infty$ in ℓ_1-metric implies that for each k, $T_k^t(x^0) \to \bar{x}_k$ as $t \to \infty$, which by definition, implies weak convergence of the corresponding sequence of probability measures $\mu_t \Rightarrow \bar{\mu}$ as $t \to \infty$, where $\bar{\mu}$ is the probability measure corresponding to the fixed point $\bar{x}$.

Corollary 1.1. *Consider an environment where money is indivisible with unit p. Suppose that the initial money holdings distribution μ_0 is such that $x^0 = L(\mu_0)$ satisfies condition ID$_1$, and that all agents adopt strategy $\tilde{\sigma}^p$. Then the economywide money holdings distribution $\{\mu_t\}_{t=0}^{\infty}$ converges weakly to the unique geometric distribution $\bar{\mu}$.*

Next we show that the money holdings distribution of a generic agent with an arbitrary initial money holdings converges, given that the economywide money holdings distribution converges.

Suppose that all agents adopt strategy $\tilde{\sigma}^p$, and that the initial state $x^0 = L(\mu_0)$ satisfies condition ID$_1$, that is, the economywide money holdings distribution converges to a geometric distribution defined on $p\mathbb{N}$. Consider an agent with initial money holdings $\eta_0 = lp$, $l \in \mathbb{N}$. Let $\varphi_t^l \in \Delta_{p\mathbb{N}}$ represent the probability distribution of the agent's date-t money holdings $\tilde{\eta}_t(lp, \omega)$, that is, for any set $D \in \mathcal{B}_t$, $\varphi_t^l(D) = \mathcal{P}_t\{\omega \,|\, \tilde{\eta}_t(lp, \omega) \in D\}$. Then, $\varphi_0^l(\{lp\}) = 1$ and $\varphi_0^l(\{kp\}) = 0$ for all $k \neq l$. For any $\varphi_t^l \in \Delta_{p\mathbb{N}}$, define $y^{lt} = L(\varphi_t^l)$, hence, $y^{lt} \in \Gamma$. Obviously, φ_t^l and y^{lt} uniquely determine each other. As x^t represents the date-t aggregate state of the economy, y^{lt} represents the distribution of the date-t personal state for the agent with initial money holdings lp. Given that the distribution of money holdings in the population follows the path of $\{x^t\}_{t=0}^{\infty}$, the distribution of the agent's personal state from one date to the next is a mapping $U : \Gamma \times \Gamma \to \Gamma$

such that for any arbitrary individual state $y \in \Gamma$, for any aggregate state $x \in \Gamma$, and for all $k \geq 1$,

$$U_k(y, x) = \frac{1 - x_1}{2} y_k + \frac{1}{2} y_{k+1} + \frac{x_1}{2} y_{k-1} \tag{27}$$

and $U_0(y, x) = 1$, where x_1 is the measure of agents holding money at the date (hence able to purchase) as defined above, which is taken as given by each agent. That is, if y^{lt} represents the distribution of the agent's date-t state and x^t represents the aggregate date-t state, then the distribution of his date-$(t+1)$ state is given by $y^{l(t+1)} = U(y^{lt}, x^t)$. The following proposition states that each agent's money holdings probability distribution converges to the same geometric distribution as the economywide money holdings sample distribution does. The initial money holdings of an agent do not matter in the limit.

Proposition 2. *Consider an environment where money is indivisible with unit p. Suppose that the initial money holdings distribution μ_0 is such that $x^0 = L(\mu_0)$ satisfies condition ID_1, and that all agents adopt strategy $\tilde{\sigma}^p$. Then the money holdings distribution of a generic agent with initial money holdings $\eta_0 = lp$, $\{\varphi_t^l\}_{t=0}^{\infty}$, converges weakly to the same aggregate limit regardless of η_0.*

Proof. Consider a trader with initial money holdings $\eta_0 = lp$, $l \in \mathbb{N}$ who, as everyone else in the economy, adopts strategy $\tilde{\sigma}^p$. To prove that the trader's money holdings $\tilde{\eta}_t(\eta_0, \omega)$ converges weakly to the aggregate limit, we need to show that for all k, $|y_k^{lt} - \bar{x}_k| \to 0$ as $t \to \infty$. Given that the aggregate state x^t converges to $\bar{x}$, i.e., for any $k \geq 1$, $x_k^t \to \bar{x}_k$ as $t \to \infty$, it is sufficient to show that for all k, $|y_k^{lt} - x_k^t| \to 0$ as $t \to \infty$. We show this by induction on k.

For any $t \geq 0$, given that the date-t aggregate state x^t and the distribution of the agent's personal state y^{lt}, by equations (19) and (27), the corresponding date-$(t+1)$ states are defined as follows, for any $k \geq 1$,

$$\begin{aligned} x_k^{t+1} &= \frac{1 - x_1^t}{2} x_k^t + \frac{1}{2} x_{k+1}^t + \frac{x_1^t}{2} x_{k-1}^t \\ y_k^{l(t+1)} &= \frac{1 - x_1^t}{2} y_k^{lt} + \frac{1}{2} y_{k+1}^{lt} + \frac{x_1^t}{2} y_{k-1}^{lt}. \end{aligned}$$

The difference of the above two equations is, for any $k \geq 1$,

$$y_k^{l(t+1)} - x_k^{t+1} = \frac{1 - x_1^t}{2} (y_k^{lt} - x_k^t) + \frac{1}{2} (y_{k+1}^{lt} - x_{k+1}^t) + \frac{x_1^t}{2} (y_{k-1}^{lt} - x_{k-1}^t). \tag{28}$$

Then, given that $y_0^{lt} = x_0^t = 1$,

$$d(y^{l(t+1)}, x^{t+1}) = \sum_{k=1}^{\infty} |y_k^{l(t+1)} - x_k^{t+1}| \leq \frac{1}{2} \Big(\sum_{k=1}^{\infty} |y_k^{lt} - x_k^t| + \sum_{k=2}^{\infty} |y_k^{lt} - x_k^t| \Big)$$

or

$$d(y^{l(t+1)},\, x^{t+1}) \leq d(y^{lt},\, x^t) - \frac{1}{2}|y_1^{lt} - x_1^t|. \tag{29}$$

That is, $\{d(y^{lt},\, x^t)\}_{t=0}^{\infty}$ is a weakly decreasing sequence, and since it is bounded below by zero, it has a limit α. Then, equation (29) implies that $|y_1^{lt} - x_1^t| \to 0$ as $t \to \infty$.

Now suppose that for all $j \leq k$, $|y_j^{lt} - x_j^t| \to 0$. We want to show that $|y_{k+1}^{lt} - x_{k+1}^t| \to 0$. By induction hypothesis, $\sum_{j=1}^{k} |y_j^{lt} - x_j^t| \to 0$. Since $d(y^{lt},\, x^t) = \sum_{j=1}^{\infty} |y_j^{lt} - x_j^t| \to \alpha$, we have $\sum_{j=k+1}^{\infty} |y_j^{lt} - x_j^t| \to \alpha$. Then,

$$\Big| \sum_{j=k+1}^{\infty} |y_j^{l(t+1)} - x_j^{t+1}| - \sum_{j=k+1}^{\infty} |y_j^{lt} - x_j^t| \Big| \to 0 \qquad as\;\; t \to \infty. \tag{30}$$

By equation (28),

$$\sum_{j=k+1}^{\infty} |y_j^{l(t+1)} - x_j^{t+1}| \leq \sum_{j=k+1}^{\infty} |y_j^{lt} - x_j^t| - \frac{1}{2}|y_{k+1}^{lt} - x_{k+1}^t| + \frac{x_1^t}{2}|y_k^{lt} - x_k^t|$$

or

$$\frac{1}{2}|y_{k+1}^{lt} - x_{k+1}^t| \leq \Big| \sum_{j=k+1}^{\infty} |y_j^{l(t+1)} - x_j^{t+1}| - \sum_{j=k+1}^{\infty} |y_j^{lt} - x_j^t| \Big| + |y_k^{lt} - x_k^t|. \tag{31}$$

Applying (30) and induction hypothesis to (31), we have $|y_{k+1}^{lt} - x_{k+1}^t| \to 0$ as $t \to \infty$. Hence, by induction, for all $k \geq 1$, $|y_k^{lt} - x_k^t| \to 0$ as $t \to \infty$. ∎

We have studied an economy with the following features: money is indivisible with unit p, the aggregate nominal quantity of money is M, the initial money holdings distribution μ_0 is such that $x^0 = L(\mu_0)$ satisfies condition ID_1, and all agents adopt strategy $\tilde{\sigma}^p$. We have shown that, in such an environment, both the economywide money holdings sample distribution and the money holdings probability distribution of a generic agent with an arbitrary initial money holdings converge weakly to the unique geometric distribution defined on support $p\mathbb{N}$ with mean M.

4.2. *The Divisible Money Case*

Now, we return to the environment introduced in section 2 where aggregate nominal quantity of money is M^*, and where money is perfectly divisible. We first define the law of motion of the economywide money holdings distribution given that all agents adopt strategy $\tilde{\sigma}^p$.

Suppose that the economywide money holdings distribution is μ at the beginning of a date. Parallel to the L transformation made in the indivisible money case, we need to know only the evolution of the money holdings distribution defined on sets of the form $[y, \infty)$ rather than on any arbitrary set $D \subseteq \mathbb{R}_+$. Let $\rho(\mu)[y, \infty)$ for any $y \in \mathbb{R}_+$ be the measure of agents whose after-trade money holdings are at least y conditional on being a seller. To characterize $\rho(\mu)[y, \infty)$ for any $y \geq p$, note that a seller's higher-than-y money holdings after trade can result from either of two types of transaction. The seller can begin with a before-trade money holdings no lower than $y - \eta_b$, and acquire $\eta_b \in [0, p)$ units from a buyer with only η_b units of money by selling η_b/p units of his endowment. He can also start with at least $y - p$ units of money before trade, and acquire p units from a buyer with at least p units of money by selling 1 unit of his endowment. That is,

$$\forall y \geq p \quad \rho(\mu)[y, \infty) = \int_{[0,p)} \mu[y - \eta_b, \infty) d\mu(\eta_b) + \mu[p, \infty)\mu[y - p, \infty). \tag{32}$$

If $y < p$, then a seller may have more than y units of money after trade from either of two kinds of transaction: starting with at least $y - \eta_b$ units before trade, and acquiring $\eta_b \in [0, y)$ units from a buyer with only η_b units of money; or trading with a buyer with at least y units of money. That is,

$$\forall y \in (0, p) \quad \rho(\mu)[y, \infty) = \int_{[0,y)} \mu[y - \eta_b, \infty) d\mu(\eta_b) + \mu[y, \infty). \tag{33}$$

Similarly, for any $y \in \mathbb{R}_+$, let $\lambda(\mu)[y, \infty)$ be the measure of agents whose after-trade money holdings are at least y conditional on being a buyer. In particular, a buyer's after-trade money holdings is always nonnegative, that is,

$$\lambda(\mu)[0, \infty) = 1. \tag{34}$$

For any $y > 0$, a buyer can only have a higher-than-y money holdings after trade if he has more than $y + p$ units of money before trade and spends p units to buy consumption goods. If he has less than p units, he will spend all of it and reduce his money holdings to 0. That is,

$$\forall y > 0 \quad \lambda(\mu)[y, \infty) = \mu[y + p, \infty). \tag{35}$$

Then, since half of the agents are buyers and half are sellers, the evolution of the money holdings distribution, $T : \Delta \to \Delta$, is given by the following, for all $\mu \in \Delta$,

$$\forall y \geq p \qquad T(\mu)[y, \infty) = \frac{1}{2}\Big[\int_{[0,p)} \mu[y - \eta_b, \infty) d\mu(\eta_b) + \mu[p, \infty)\mu[y - p, \infty) + \mu[y + p, \infty)\Big] \tag{36}$$

$$\forall y \in (0, p) \quad T(\mu)[y, \infty) = \frac{1}{2}\Big[\int_{[0,y)} \mu[y - \eta_b, \infty) d\mu(\eta_b) + \mu[y, \infty) + \mu[y + p, \infty)\Big]. \tag{37}$$

Obviously, $T(\mu)[0, \infty) = 1$. It is easy to check that if $\mu \in \Delta^{M^*}$, then $T(\mu) \in \Delta^{M^*}$. Equations (36) and (37) together define the evolution of the economywide money holdings distribution, which is $\mu_t = T^t(\mu_0)$ at date t. Note that when the support of μ is the p-lattice $p\mathbb{N}$, for any $k \in \mathbb{N}$, $T(\mu)[kp, \infty)$ defined by (36) and (37) coincides with $T_k(L(\mu))$ defined by (19) in the indivisible money case.

The dominance relationship $\preceq_d$ defined on Γ in last subsection is the restriction to $\Delta_{p\mathbb{N}}$ of the relationship of stochastic dominance defined on Δ: for any μ_1, $\mu_2 \in \Delta$, $\mu_1 \preceq \mu_2$ if and only if for any $y \in \mathbb{R}_+$, $\mu_1[y, \infty) \leq \mu_2[y, \infty)$. The following lemma states that the mapping T preserves the relationship of stochastic dominance.

LEMMA 9. *For any μ_1, $\mu_2 \in \Delta$, if $\mu_1 \preceq \mu_2$, then $T(\mu_1) \preceq T(\mu_2)$.*

Divide the agents according to their money holdings into two groups: those whose money holdings are integer multiples of p (on-lattice) and those whose money holdings are between integer multiples of p (off-lattice). It can be shown that over the dynamic path of economy, the measure of the on-lattice group is increasing in time. Intuitively, in most pairwise trades, buyers and sellers either keep their status (on- or off-lattice) or exchange their status. These trades do not affect the size of either group. But when a buyer and a seller, both with money holdings strictly between 0 and p (off-lattice), meet, the buyer spends all his money on the preferred consumption good, and hence reduces his after-trade money holdings to 0. In other words, trading among this group of agents results half of the group moving from off-lattice status to on-lattice status. The following lemma summarizes this intuition and states a quantitative implication.

LEMMA 10. *For any $\mu \in \Delta$, $\{T^n(\mu)(p\mathbb{N})\}_0^\infty$ is a nondecreasing sequence, and satisfies the following relationship: for all $n \geq 0$,*

$$T^{n+1}(\mu)(p\mathbb{N}) \geq \mu(p\mathbb{N}) + 2^{-(2n+1)} \Big[\mu(np,\, (n+1)p)\Big]^2. \tag{38}$$

Proof. Consider all the possible pairwise meetings between a buyer with η_b units of money and a seller with η_s units of money where trade occurs. Let η_b' and η_s' denote the corresponding after-trade money holdings.

(i) If $\eta_b \in p\mathbb{N}$ and $\eta_s \in p\mathbb{N}$, then, $\eta_b' \in p\mathbb{N}$ and $\eta_s' \in p\mathbb{N}$.

(ii) If $\eta_b \geq p$, then $\eta_z' \in p\mathbb{N}$ iff $\eta_z \in p\mathbb{N}$ where $z = b,\, s$.

(iii) If $\eta_b \in (0, p)$ and $\eta_s \in p\mathbb{N}$, then, $\eta_b' = 0 \in p\mathbb{N}$ and $\eta_s' \notin p\mathbb{N}$.

(iv) If $\eta_b \in (0, p)$ and $\eta_s \notin p\mathbb{N}$, then, $\eta_b' = 0 \in p\mathbb{N}$ and $\eta_s' \notin p\mathbb{N}$ almost surely.

Among these four types of active trade, the first three do not change the measure of agents with money holdings on the lattice $p\mathbb{N}$. The last type of trade moves the buyer's money holdings onto the lattice. A subset of the last type of trades occur among agents (both buyers and sellers) with money holdings in $(0,\, p)$. Therefore,

$$T(\mu)(p\mathbb{N}) - \mu(p\mathbb{N}) \geq \frac{1}{2}\Big[\mu(0,\, p)\Big]^2 \geq 0 \tag{39}$$

that is, $T(\mu)(p\mathbb{N}) \geq \mu(p\mathbb{N})$. Hence, for any $\mu \in \Delta$, $\{T^n(\mu)(p\mathbb{N})\}_0^\infty$ is nondecreasing in n.

To prove the second part of the lemma, we first prove the following claim: For any $\mu \in \Delta$, $n \geq 1$ and $\tau \leq n$,

$$T^\tau(\mu)\Big((n-\tau)p,\, (n-\tau+1)p\Big) \geq 2^{-\tau}\mu(np,\, (n+1)p). \tag{40}$$

We prove the claim by induction on τ. When $\tau = 0$, (40) holds with equality. Suppose the claim holds for $\tau = k < n$, consider the case when $\tau = k+1$. Given the matching technology and the strategy, half of the agents with money holdings in $((n-k)p, (n-k+1)p)$ are buyers (note that $n - k \geq 1$) whose after-trade money holdings will be in $((n-k-1)p,\, (n-k)p) = ((n-(k+1))p,\, (n-(k+1)+1)p)$. Hence,

$$\begin{aligned} T^{k+1}(\mu)\Big((n-(k+1))p, (n-(k+1)+1)p\Big) &\geq \frac{1}{2}\Big[T^k(\mu)\Big((n-k)p,\, (n-k+1)p\Big)\Big] \\ &\geq \frac{1}{2}\Big[2^{-k}\mu(np,\, (n+1)p)\Big] = 2^{-(k+1)}\mu(np,\, (n+1)p) \end{aligned}$$

where the second inequality is due to the induction hypothesis. That is, (40) holds for $\tau = k+1$. By induction, (40) holds for all $\tau \leq n$.

By the above claim, for all $n \geq 0$,

$$T^n(\mu)(0,\, p) \geq 2^{-n}\mu(np,\, (n+1)p). \tag{41}$$

By (39), (41) and the fact that $T^n(\mu)(p\mathbb{N})$ is nondecreasing in n, for all $n \geq 0$, for all $\mu \in \Delta$,

$$T^{n+1}(\mu)(p\mathbb{N}) \geq T^n(\mu)(p\mathbb{N}) + \frac{1}{2}\Big[T^n(\mu)(0,\, p)\Big]^2 \geq \mu(p\mathbb{N}) + 2^{-(2n+1)}\Big[\mu(np,\, (n+1)p)\Big]^2.$$

That is, (38) holds. ∎

Applying Lemma 10, we show an even stronger result: asymptotically, all agents' money holdings will be integer multiples of p.

Lemma 11. *For any $\mu \in \Delta^{M^*}$, $T^n(\mu)(p\mathbb{N}) \to 1$ as $n \to \infty$.*

For an arbitrary probability measure $\mu \in \Delta$, define μ^+ be the probability measure resulting from right shifting all the probability mass in interval $((n-1)p,\, np)$ to the point np on the p-lattice for all $n \geq 1$, that is,

$$\mu^+\{0\} = \mu\{0\}, \qquad \forall n \geq 1 \quad \mu^+\{np\} = \mu\Big((n-1)p,\, np\Big]. \tag{42}$$

Similarly, let μ^- be the probability measure resulting from left shifting all the probability mass in interval $(np,\, (n+1)p)$ to the point np on the p-lattice for all $n \geq 0$, that is,

$$\forall n \geq 0 \qquad \mu^-\{np\} = \mu\Big[np,\, (n+1)p\Big). \tag{43}$$

The supports of μ^+ and μ^- are within $p\mathbb{N}$, and μ^+, $\mu^- \in \Delta$. By definition,

$$\mu^- \preceq \mu \preceq \mu^+. \tag{44}$$

Let g^θ denote the probability measure defined on $\mathbb{R}_+$ but with $g^\theta(p\mathbb{N}) = 1$ that corresponds to the geometric vector π^θ for some $\theta \in (0,1)$ introduced in last subsection, that is, for any $k \geq 1$ and $x \in [0, p)$,

$$g^\theta[kp - x,\, \infty) = g^\theta[kp,\, \infty) = \theta^k$$

$g^\theta \in \Delta$. It is easy to check that g^θ is a fixed point of T, that is, $T(g^\theta) = g^\theta$. If μ^+ is stochastically dominated by some g^θ where $\theta \in (0,1)$, given that the mapping T preserves stochastic dominance (Lemma 9), we have

$$T(\mu^-) \preceq T(\mu) \preceq T(\mu^+) \preceq g^\theta. \tag{45}$$

LEMMA 12. *For an arbitrary $\mu \in \Delta$, if there exists $\theta \in (0,1)$ such that $\mu^+ \preceq g^\theta$, then for any $n \geq 0$, $[T^n(\mu)]^+ \preceq g^\theta$.*

Now we are ready to show the convergence of the economywide money holdings distribution. The condition on the initial money holdings distribution η_0 that guarantees the convergence of the distribution is formally very similar to condition ID_1.[8]

(ID_2) *There exist $\theta \in (0,1)$ and $t > 0$ such that $[T^t(\mu_0)]^+ \preceq g^\theta$.*

The following proposition states that, under this condition, the limit distribution is the probability measure corresponding to the geometric distribution with mass on the p-lattice $p\mathbb{N}$ and mean M^*. That is, the limit distribution is g^{m^*}, where $m^* = \frac{M^*/p}{1+M^*/p}$.

[8]However the mapping T is defined in terms of p, which is an exogenous parameter of the indivisible money economy but an endogenous price in the divisible money economy. Thus, despite their formal similarity, condition ID_2 does not relate the initial money holdings distribution to exogenous parameters of the economy as ID_1 does. A sufficient condition ID_3 for ID_2, to be formulated below, will avoid reference to the price or other endogenous quantities.

Proposition 3. *Suppose that the initial money holdings distribution $\mu_0 \in \Delta^{M^*}$ satisfies condition ID_2. Then the sequence of money holdings distributions evolving from μ_0 according to mapping T converges weakly to the steady state characterized by geometric distribution g^{m^*}. That is, $T^n(\mu_0) \Rightarrow g^{m^*}$ as $n \to \infty$.*

Proof. Take an arbitrary initial distribution $\mu_0 \in \Delta^{M^*}$ such that there exist $\theta \in (0,1)$ and $t \geq 0$, satisfying $[T^t(\mu_0)]^+ \preceq g^\theta$. Without loss of generality, assume that $\mu_0^+ \preceq g^\theta$. Let $l_1 = 0$ and $n_1 = 0$. We construct a subsequence $\{T^{l_k+n_k}(\mu_0)\}_{n=0}^\infty$ of the sequence $\{T^n(\mu_0)\}_{n=0}^\infty$ by repeating the following, three-step procedure for all $k \geq 2$.

(i) By Lemma 11, there exists $n_k \geq n_{k-1} + l_{k-1}$ such that

$$T^{n_k}(\mu_0)(p\mathbb{N}) > 1 - \frac{1}{kp}.$$

(ii) Let $\mu_k^+ \equiv [T^{n_k}(\mu_0)]^+$, and $\mu_k^- \equiv [T^{n_k}(\mu_0)]^-$. By Lemma 12, given that $\mu_0^+ \preceq g^\theta$, $\mu_k^+ \preceq g^\theta$. Hence,

$$\mu_k^- \preceq T^{n_k}(\mu_0) \preceq \mu_k^+ \preceq g^\theta. \tag{46}$$

Furthermore, define $M_k^+ = \int_0^\infty \eta d\mu_k^+$, and $M_k^- = \int_0^\infty \eta d\mu_k^-$. Since $T^{n_k}(\mu_0)(\overline{p\mathbb{N}}) < \frac{1}{kp}$ ($\overline{p\mathbb{N}}$ denotes the complement of $p\mathbb{N}$), and $\int_0^\infty \eta dT^{n_k}(\mu_0) = M^*$, we have

$$M_k^+ < M^* + 1/k \quad \text{and} \quad M_k^- > M^* - 1/k. \tag{47}$$

Define

$$m_k^+ \equiv \frac{M_k^+/p}{1 + M_k^+/p} \quad \text{and} \quad m_k^- \equiv \frac{M_k^-/p}{1 + M_k^-/p}.$$

Consider μ_k^+ and μ_k^- as the initial distributions of an economy where money is indivisible with unit p. Then, given that $\mu_k^+ \preceq g^\theta$ and $\mu_k^- \preceq g^\theta$, both $L(\mu_k^+)$ and $L(\mu_k^-)$ satisfy condition ID_1. By Corollary 1.1, $T^n(\mu_k^+)$ weakly converges to $g^{m_k^+}$ and $T^n(\mu_k^-)$ weakly converges to $g^{m_k^-}$ as $n \to \infty$. Hence, for $\nu = 6/(kp)$, for all $j \geq 0$,

$$\exists\, l_{kj}^+ \quad \forall n \geq l_{kj}^+ \quad \left| T^n(\mu_k^+)[jp, \infty) - [m_k^+]^j \right| < \nu/6 = 1/(kp).$$

Since the mean of $T^n(\mu_k^+)$ is M_k^+ and $m_k^+ < 1$, there exists J such that for all $j \geq J$, $T^n(\mu_k^+)[jp, \infty) < \nu/12$ and $(m_k^+)^j < \nu/12$. Take $l_k^+ = \max\{l_{k0}, \ldots, l_{kJ}\}$. Then,

$$\forall n \geq l_k^+ \quad \forall j \geq 0 \quad \left| T^n(\mu_k^+)[jp, \infty) - [m_k^+]^j \right| < \nu/6 = 1/(kp).$$

By the same argument, there exists l_k^- such that

$$\forall n \geq l_k^- \quad \forall j \geq 0 \quad \left| T^n(\mu_k^-)[jp, \infty) - [m_k^-]^j \right| < \nu/6 = 1/(kp).$$

Take $l_k = \max\{l_k^+, l_k^-\}$. Then, by (46),

$$T^{l_k}(\mu_k^-) \preceq T^{l_k+n_k}(\mu_0) \preceq T^{l_k}(\mu_k^+) \tag{48}$$

and for all $j \geq 0$,

$$\left| T^{l_k}(\mu_k^+)[jp, \infty) - [m_k^+]^j \right| < 1/(kp) \quad \text{and} \quad \left| T^{l_k}(\mu_k^-)[jp, \infty) - [m_k^-]^j \right| < 1/(kp). \tag{49}$$

(iii) Increase k by 1. Go back to step (i).

Now we have a subsequence $\{T^{l_k+n_k}(\mu_0)\}_{n=0}^\infty$ of the sequence $\{T^n(\mu_0)\}_{n=0}^\infty$ that satisfies (48) and (49). We want to show that this subsequence weakly converges to g^{m^*}. By (48), for all $y \in \mathbb{R}_+$,

$$T^{l_k}(\mu_k^-)[y, \infty) \leq T^{l_k+n_k}(\mu_0)[y, \infty) \leq T^{l_k}(\mu_k^+)[y, \infty) \tag{50}$$

By (47), it is easy to check that for any $j \geq 0$,

$$\left| (m_k^+)^j - (m^*)^j \right| < 1/(kp) \quad \text{and} \quad \left| (m_k^-)^j - (m^*)^j \right| < 1/(kp). \tag{51}$$

Therefore, for any $\varepsilon > 0$, $k \geq 6/(\varepsilon p)$, for any $y > 0$, write $y = jp - x$ with $j \geq 1$, and $x \in [0, p)$,

$$\begin{aligned}
& \left| T^{l_k+n_k}(\mu_0)[y, \infty) - g^{m^*}[y, \infty) \right| = \left| T^{l_k+n_k}(\mu_0)[y, \infty) - (m^*)^j \right| \\
\leq\ & \left| T^{l_k+n_k}(\mu_0)[y, \infty) - T^{l_k}(\mu_k^+)[y, \infty) \right| + \left| T^{l_k}(\mu_k^+)[y, \infty) - (m^*)^j \right| \\
\leq\ & \left| T^{l_k}(\mu_k^-)[y, \infty) - T^{l_k}(\mu_k^+)[y, \infty) \right| + \left| T^{l_k}(\mu_k^+)[y, \infty) - (m_k^+)^j \right| + \left| (m_k^+)^j - (m^*)^j \right| \\
\leq\ & \left| T^{l_k}(\mu_k^-)[y, \infty) - (m_k^-)^j \right| + 2\left| T^{l_k}(\mu_k^+)[y, \infty) - (m_k^+)^j \right| + \left| (m_k^-)^j - (m^*)^j \right| + 2\left| (m_k^+)^j - (m^*)^j \right| \\
\leq\ & 6/(kp) < \varepsilon.
\end{aligned}$$

The second inequality is due to (50) and the last is due to (49) and (51) given that $T^{l_k}(\mu_k^-)[y, \infty) = T^{l_k}(\mu_k^-)[jp, \infty)$ and $T^{l_k}(\mu_k^+)[y, \infty) = T^{l_k}(\mu_k^+)[jp, \infty)$. That is, the subsequence $\{T^{l_k+n_k}(\mu_0)\}_{n=0}^\infty$ weakly converges to g^{m^*}.

The foregoing argument can be generalized as follows. Consider an arbitrary number sequence $i_0 < i_1 < i_2 < \ldots$ and consider the sequence $\{T^{i_j}(\mu_0)\}_{j=0}^\infty$. This sequence has a subsequence that converges weakly to g^{m^*}. Since every subsequence of $\{T^n(\mu_0)\}_{n=0}^\infty$ has a subsequence that converges weakly to g^{m^*}, the entire sequence must converge weakly to g^{m^*}. ∎

The requirement that the initial distribution μ_0 satisfies condition ID_2 is fairly weak. The following lemma gives a class of initial distributions that satisfy the condition.

LEMMA 13. *For any given equilibrium price $p > 0$, if there exist $J_p > 0$ and $\alpha_p \in (0,1)$ such that for all $j > J_p$, $\mu_0(jp,\,\infty) \le \alpha_p^{j+1}$, then μ_0 satisfies condition ID$_2$.*

Proof. To prove condition ID$_2$ holds, we need to show that there exist $\theta \in (0,1)$ and $t > 0$ such that for all $j > 0$, $T^t(\mu_0)(jp,\infty) \le \theta^{j+1}$.

Let $K = 0$ if $\mu_0\{0\} > 0$, and $K = \min\{j \mid \mu_0((j-1)p, jp] \neq 0\}$ otherwise. Then, $\mu_0(jp,\,\infty) = 1$ for all $j < K$. We first show by induction on K a claim: that $T^K(\mu_0)\{0\} > 0$. For K=0, the claim holds automatically. Suppose that it holds for $K = n$, that is, $\mu_0((n-1)p, np] \neq 0$ and $\mu_0(jp,\,\infty) = 1$ for all $j < n$ implies that $T^n(\mu_0)\{0\} > 0$. Consider the case when $K = n+1$. Given that $\mu_0(np,\,(n+1)p] \neq 0$ and $\mu_0(jp,\,\infty) = 1$ for all $j < n+1$, by the definition of T given in equations (36) and (37),

$$T(\mu_0)\Big((n-1)p,\,np\Big] = \frac{1}{2}\mu_0\Big(np,\,(n+1)p\Big] > 0.$$

Applying the induction hypothesis, we have $T^{n+1}(\mu_0)\{0\} > 0$. That is, the claim holds.

Given that $\mu_0(jp,\,\infty) \le \alpha_p^{j+1}$ for all $j > J_p$, applying equations (36) recursively, we have $T^K(\mu_0)(jp,\infty) < \alpha_p^{j+1}$ for all $\forall j > J_p + K$.

By the above claim, $T^K(\mu_0)\{0\} > 0$, hence $T^K(\mu_0)(0,\,\infty) < 1$. Then, there exists $\theta \in [\alpha_p, 1)$ such that $T^K(\mu_0)(0,\;\infty) \le \theta^{J_p+K+1}$. For $j \le J_p + K$, $T^K(\mu_0)(jp,\infty) \le T^K(\mu_0)(0,\,\infty) \le \theta^{J_p+K+1} \le \theta^{j+1}$. For $j > J_p + K$, $T^K(\mu_0)(jp,\infty) \le \alpha_p^{j+1} \le \theta^{j+1}$ since $\alpha_p \le \theta$. That is, for all $j \in \mathbb{N}$, $T^K(\mu_0)(jp,\infty) \le \theta^{j+1}$. Therefore, $[T^K(\mu_0)]^+ \preceq g^\theta$, or equivalently, μ_0 satisfies condition ID$_2$. ∎

Condition in Lemma 13 is more transparent than ID$_2$ given that it is expressed directly in terms of the initial distribution μ_0. However, it is still cumbersome since it depends on equilibrium price p, which is endogenous. In fact, if the condition holds for one particular price, it holds for any other price. The following proposition exploits this feature and gives a sufficient condition for ID$_2$ that depends only on exogenous parameters.

PROPOSITION 4. *If the initial money holdings distribution μ_0 has a tail that is stochastically dominated by the tail of a geometric distribution, that is, if*

(*ID$_3$*) *there exist $J > 0$ and $\alpha \in (0,1)$ such that for all $j > J$,* $\mu_0(j,\,\infty) \le \alpha^{j+1}$

then μ_0 satisfies condition ID$_2$.

Proof. We need to show only that if μ_0 satisfies condition ID_3, then for any $p > 0$, there exist $J_p > 0$ and $\alpha_p \in (0, 1)$ such that for all $j > J_p$, $\mu_0(jp, \infty) \leq \alpha_p^{j+1}$. Then, by Lemma 13, condition ID_2 holds.

Take an arbitrary $p > 0$. First, note that if $p \geq 1$, take $J_p = J$ and $\alpha_p = \alpha$. Then by ID_3, for all $j > J_p$, $\mu_0(jp, \infty) \leq \mu_0(j, \infty) \leq \alpha^{j+1} = \alpha_p^{j+1}$.

Next, consider the nontrivial case $0 < p < 1$. Define J_p to be any integer such that

$$J_p \geq \frac{J+1}{p} \quad \text{and} \quad \alpha_p = \alpha^{1/x} \;\text{ where }\; x = \frac{J_p + 1}{pJ_p}.$$

Obviously, $\alpha_p \in (\alpha, 1)$ since $x > 1$. For any $j > J_p$, take k_j to the integer such that $k_j = \max\{k \,|\, jp \geq k\}$. That is, $k_j \leq jp < k_j + 1$. The first inequality implies that $\mu_0(jp, \infty) \leq \mu_0(k_j, \infty)$. The second inequality leads to

$$k_j + 1 > jp > J_p p \geq \frac{J+1}{p} p = J + 1$$

that is, $k_j > J$. Then, by condition ID_3,

$$\mu_0(k_j, \infty) \leq \alpha^{k_j+1} = (\alpha_p^x)^{k_j+1} \leq (\alpha_p^x)^{jp} = \alpha_p^{\frac{J_p+1}{pJ_p} jp} = \alpha_p^{\frac{J_p+1}{J_p} j}.$$

Since $j > J_p$, $\frac{J_p+1}{J_p} > \frac{j+1}{j}$. Therefore, given that $\alpha_p < 1$,

$$\mu_0(jp, \infty) \leq \mu_0(k_j, \infty) \leq \alpha_p^{\frac{J_p+1}{J_p} j} < \alpha_p^{\frac{j+1}{j} j} = \alpha_p^{j+1}.$$

That is, the claim holds: for all $j > J_p$, $\mu_0(jp, \infty) \leq \alpha_p^{j+1}$. ∎

As a practical matter, economists are not likely to find condition ID_3 restrictive. There are at least two classes of initial money holdings distributions μ_0 satisfy the condition. One class of distributions consists of those with finite support (that is, there is a $Y > 0$ such that for all $y \geq Y$, $\mu_0[y, \infty) = 0$). Probability measures with finite support are dense in the space of probability simplex Δ. The other class of distributions includes those that are results of injecting money into a steady state economy with a geometric distribution of money holdings by giving uniformly bounded amount to agents whose money holdings are below some particular level (i.e., "poor" people).

We can conclude now that if the initial money holdings distribution satisfies condition ID_3 and if all agents adopt strategy $\tilde{\sigma}^p$, then by Proposition 3, the economywide money holdings distribution converges weakly to a unique geometric distribution at which the environment is stationary.

Given that the economywide money holdings sample distribution converges, in what follows, we apply the result of the indivisible money case and show that the probability distribution of money holdings of a generic agent with an arbitrary initial money holdings converges to the aggregate limit distribution g^{m^*}.

Consider an agent with an initial money holdings η_0. Let $\varphi_t^{\eta_0} \in \Delta$ denote the probability distribution of the agent's date-t money holdings $\tilde{\eta}_t(\eta_0, \omega)$, that is, for any set $D \in \mathcal{B}_t$, $\varphi_t^{\eta_0}(D) = \mathcal{P}_t\{\omega \mid \tilde{\eta}_t(\eta_0, \omega) \in D\}$. Then $\varphi_0^{\eta_0}$ is degenerate and satisfies that for all $y \leq \eta_0$, $\varphi_0^{\eta_0}[y, \infty) = 1$, and for all $y > \eta_0$, $\varphi_0^{\eta_0}[y, \infty) = 0$. Similar to the evolution of the economywide distribution (which follows the path of $\{\mu_t\}_{t=0}^{\infty}$ given by (36) and (37)), the evolution of the individual agent's money holdings distribution from one date to the next is a mapping $U : \Delta \times \Delta \to \Delta$ such that for any individual distribution $\varphi \in \Delta$ and any economywide distribution $\mu \in \Delta$, for all $y \geq p$,

$$U(\varphi, \mu)[y, \infty) = \frac{1}{2}\Big[\int_{[0,p)} \varphi[y - \eta_b, \infty) d\mu(\eta_b) + \mu[p, \infty)\varphi[y - p, \infty) + \varphi[y + p, \infty)\Big] \tag{52}$$

and for all $y \in (0, p)$,

$$U(\varphi, \mu)[y, \infty) = \frac{1}{2}\Big[\int_{[0,y)} \varphi[y - \eta_b, \infty) d\mu(\eta_b) + \mu[y, \infty) + \varphi[y + p, \infty)\Big]. \tag{53}$$

Note that if the support of μ is the p-lattice $p\mathbb{N}$, then for any $k \in \mathbb{N}$, $U(\varphi, \mu)[kp, \infty)$ defined in (52) and (53) coincides with $U_k(L(\varphi), L(\mu))$ defined by (27) in the indivisible money case. Then, given initial individual money holdings distribution $\varphi_0^{\eta_0}$ and initial economywide distribution μ_0, the sequence $\{\varphi_t^{\eta_0}\}_{t=0}^{\infty}$ is recursively defined: for all $t \geq 0$, $\varphi_{t+1}^{\eta_0} = U(\varphi_t^{\eta_0}, \mu_t)$ where $\mu_t = T^t(\mu_0)$ given by (36) and (37).

The following proposition shows that in a divisible money environment, an individual agent's money holdings distribution, regardless of his initial money holdings, converges to the same geometric distribution as the economywide money holdings distribution does. Similar to the proof of Proposition 3, we bound the money holdings distribution by two distributions in indivisible money environments, and then applying Proposition 2.

PROPOSITION 5. *Consider any $p > 0$. Suppose that the economywide initial money holdings distribution $\mu_0 \in \Delta^{M^*}$ satisfies condition ID$_3$, and that all agents adopt strategy $\tilde{\sigma}^p$. Then the money holdings distribution of a generic agent with initial money holdings η_0, $\{\varphi_t^{\eta_0}\}_{t=0}^{\infty}$, converges weakly to the same aggregate limit distribution g^{m^*} regardless of his initial money holdings η_0.*

Proof. Consider an agent with an arbitrary initial money holdings η_0. By Proposition 3, the economywide money holdings distribution converges weakly to the geometric distribution g^{m^*}.

Let $\mu_n^+ = [T^n(\mu_0)]^+$, $\mu_n^- = [T^n(\mu_0)]^-$, and let M_n^+, M_n^-, m_n^+ and m_n^- be defined the same way as in the proof of Proposition 3. Then for any $\varepsilon > 0$, there exists $t \geq 0$ such that $|m_t^+ - m^*| < \varepsilon/6$ and $|m_t^- - m^*| < \varepsilon/6$, which implies that for any $j \geq 1$,

$$|(m_t^+)^j - (m^*)^j| < \varepsilon/6 \quad \text{and} \quad |(m_t^-)^j - (m^*)^j| < \varepsilon/6. \tag{54}$$

By the evolution of the individual's money holdings distribution (52) and (53), it is easy to check the following claim holds: For any φ_1, $\varphi_2 \in \Delta$ and $\mu_1, \mu_2 \in \Delta$, if $\varphi_1 \preceq \varphi_2$ and $\mu_1 \preceq \mu_2$, then for all $n \geq 0$, $U^n(\varphi_1, \mu_1) \preceq U^n(\varphi_2, \mu_2)$. For the t chosen above, we have

$$[\varphi_t^{\eta_0}]^- \preceq \varphi_t^{\eta_0} \preceq [\varphi_t^{\eta_0}]^+ \quad \text{and} \quad \mu_t^- \preceq \mu_t \preceq \mu_t^+$$

where $[\,\cdot\,]^+$ and $[\,\cdot\,]^-$ are defined as in (42) and (43). Applying the above claim, for all $n \geq 0$,

$$U^n([\varphi_t^{\eta_0}]^-, \mu_t^-) \preceq U^{n+t}(\varphi_0^{\eta_0}, \mu_0) \preceq U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)$$

which implies that for all $y \in \mathbb{R}_+$,

$$U^n([\varphi_t^{\eta_0}]^-, \mu_t^-)[y, \infty) \leq U^{n+t}(\varphi_0^{\eta_0}, \mu_0)[y, \infty) \leq U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)[y, \infty) \tag{55}$$

As noted earlier, both $U([\varphi_t^{\eta_0}]^-, \mu_t^-)$ and $U([\varphi_t^{\eta_0}]^+, \mu_t^+)$ can be interpreted as the law of motion of the individual agent's money holdings distribution given by (27) in an indivisible money environment, where the individual agent's initial money holdings distribution is given by $[\varphi_t^{\eta_0}]^-$ and $[\varphi_t^{\eta_0}]^+$ respectively, and the economywide initial distribution is given by μ_t^- and μ_t^+ respectively. It is easy to verify the hypothesis that the economywide initial money holdings distribution satisfies condition ID$_3$, hence it satisfies condition ID$_2$, implies that $x_t^- = L(\mu_t^-)$ and $x_t^+ = L(\mu_t^+)$ satisfy condition ID$_1$. Then, by applying Proposition 2, $U^n([\varphi_t^{\eta_0}]^-, \mu_t^-) \Rightarrow g^{m_t^-}$ and $U^n([\varphi_t^{\eta_0}]^+, \mu_t^+) \Rightarrow g^{m_t^+}$ as $n \to \infty$. (Note that, the proof of Proposition 2 does not require the individual agent's initial distribution to be degenerate). With the same argument as in the proof of Proposition 3, then, there exists $\hat{n} \geq 0$ such that for all $n \geq \hat{n}$, for all $y = jp - x$ with $j \geq 1$ and $x \in [0, p)$,

$$\left| U^n([\varphi_t^{\eta_0}]^-, \mu_t^-)[y, \infty) - (m_t^-)^j \right| < \varepsilon/6 \quad \text{and} \quad \left| U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)[y, \infty) - (m_t^+)^j \right| < \varepsilon/6. \tag{56}$$

This is because $U^n([\varphi_t^{\eta_0}]^-, \mu_t^-)[y, \infty) = U^n([\varphi_t^{\eta_0}]^-, \mu_t^-)[jp, \infty)$ and $U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)[y, \infty) = U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)[jp, \infty)$. By (54) and (56), for all $n \geq \hat{n}$,

$$\left| U^n([\varphi_t^{\eta_0}]^-, \mu_t^-)[y, \infty) - (m^*)^j \right| < \varepsilon/3 \quad \text{and} \quad \left| U^n([\varphi_t^{\eta_0}]^+, \mu_t^+)[y, \infty) - (m^*)^j \right| < \varepsilon/3 \tag{57}$$

Therefore, for any $n \geq \hat{n}$, for $y = jp - x$ with $j \geq 1$ and $x \in [0, p)$,

$$\begin{aligned}
& \left| U^{n+t}(\varphi_0^{\eta_0},\, \mu_0)[y,\, \infty) - g^{m^*}[y,\, \infty) \right| = \left| U^{n+t}(\varphi_0^{\eta_0},\, \mu_0)[y,\, \infty) - (m^*)^j \right| \\
\leq \;& \left| U^{n+t}(\varphi_0^{\eta_0},\, \mu_0)[y,\, \infty) - U^n([\varphi_t^{\eta_0}]^+,\, \mu_t^+)[y,\, \infty) \right| + \left| U^n([\varphi_t^{\eta_0}]^+,\, \mu_t^+)[y,\, \infty) - (m^*)^j \right| \\
\leq \;& \left| U^n([\varphi_t^{\eta_0}]^-,\, \mu_t^-)[y,\, \infty) - U^n([\varphi_t^{\eta_0}]^+,\, \mu_t^+)[y,\, \infty) \right| + \left| U^n([\varphi_t^{\eta_0}]^+,\, \mu_t^+)[y,\, \infty) - (m^*)^j \right| \\
\leq \;& \left| U^n([\varphi_t^{\eta_0}]^-,\, \mu_t^-)[y,\, \infty) - (m^*)^j \right| + 2 \left| U^n([\varphi_t^{\eta_0}]^+,\, \mu_t^+)[y,\, \infty) - (m^*)^j \right| \\
< \;& \varepsilon.
\end{aligned}$$

The second inequality is due to (55) and the last one is due to (57). Hence, regardless of the agent's initial money holdings η_0, $U^n(\varphi_0^{\eta_0},\, \mu_0)[y,\, \infty) \Rightarrow g^{m^*}$ as $n \to \infty$. ∎

The weak convergence of the random variable $\tilde{\eta}_t(\eta_0,\, \omega)$ established in Proposition 5 is now shown to imply the convergence of $\mathsf{E}[\tilde{\eta}_t(\eta_0,\, \omega)]$ to the aggregate mean money holdings M^*.

Proposition 6. *Consider any $p > 0$. Suppose that the economywide initial money holdings distribution $\mu_0 \in \Delta^{M^*}$ satisfies condition ID$_3$, and that all agents adopt strategy $\tilde{\sigma}^p$. Then the expected money holdings of a generic agent adopting strategy $\tilde{\sigma}^p$ converges to the per capita money holdings M^* regardless of his initial money holdings η_0. That is, for any $\eta_0 \in \mathbb{N}$,*

$$\lim_{t \to \infty} \mathsf{E}[\tilde{\eta}_t(\eta_0,\, \omega)] = M^*. \tag{58}$$

Proof. Consider a generic agent with initial money holdings $\eta_0 = lp + x$, $l \in \mathbb{N}$ and $x \in [0, p)$, who, as everyone else in the economy, adopts strategy $\tilde{\sigma}^p$. Suppose the economywide initial distribution μ_0 satisfies condition ID$_3$, hence it satisfies condition ID$_2$. We first show that there exists a date t_l such that the sequence of distributions of the trader's money holdings from t_l on, $\{\varphi_t^{\eta_0}(\eta_0,\, \omega)\}_{t=t_l}^{\infty}$, is dominated by a geometric distribution. Note that since condition ID$_2$ is satisfied, by Lemma 12, there exist t' and $\theta' \in (0, 1)$ such that for all $t \geq t'$, $[T^t(\mu_0)]^+ \preceq g^{\theta'}$.

Given that the agent's initial money holdings is η_0, for all $y \leq \eta_0$, $\varphi_0^{\eta_0}[y, \infty) = 1$, and for all $y > \eta_0$, $\varphi_0^{\eta_0}[y, \infty) = 0$. By the law of motion (52) and (53), after $l + 1$ repeated operations of U on $\varphi_0^{\eta_0}$, $\varphi_{l+1}^{\eta_0}\{0\} = U^{l+1}(\varphi_0^{\eta_0},\, \mu_0)\{0\} > 0$, and for all $y > (2l + 1)p + x$, $\varphi_{l+1}^{\eta_0}[y, \infty) = 0$. Then, for all $t > l + 1$, $\varphi_t^{\eta_0}\{0\} = U^t(\varphi_0^{\eta_0},\, \mu_0)\{0\} > 0$, and for all $y > (l + t)p + x$, $\varphi_t^{\eta_0}[y,\, \infty) = U^t(\varphi_0^{\eta_0},\, \mu_0)[y,\, \infty) = 0$. Or equivalently, $[\varphi_t^{\eta_0}]^+[p, \infty) < 1$, and for all $j > l + 1 + t$, $[\varphi_t^{\eta_0}]^+[jp, \infty) = 0$. Take $t_l = \max\{l + 1, t'\}$. Choose $\theta_l \in [\theta', 1)$ such that $[\varphi_t^{\eta_0}]^+[p, \infty) < (\theta_l)^{l+1+t_l}$. Since $[\varphi_t^{\eta_0}]^+[jp, \infty)$ is decreasing in j, for $j \leq l + 1 + t_l$, $[\varphi_t^{\eta_0}]^+[jp, \infty) \leq [\varphi_t^{\eta_0}]^+[p, \infty) \leq (\theta_l)^{l+1+t_l} \leq (\theta_l)^j$.

For $j > l+1+t_l$, $[\varphi_t^{\eta_0}]^+[jp,\infty) = 0 \leq (\theta_l)^j$. Therefore, $[\varphi_t^{\eta_0}]^+ \preceq g^{\theta_l}$. Given that $\theta' \leq \theta_l$ and $t_l \geq t'$, $[\mu_{t_l}]^+ = [T^{t_l}(\mu_0)]^+ \preceq g^{\theta'} \preceq g^{\theta_l}$. That is, both $[\mu_{t_l}]^+$ and $[\varphi_{t_l}^{\eta_0}]^+$ are dominated by g^{θ_l}. Then, by a similar argument as in the proof of Proposition 5, for all $t \geq t_l$,

$$\varphi_t^{\eta_0} = U^t(\varphi_0^{\eta_0}, \mu_0) = U^{t-t_l}(\varphi_{t_l}^{\eta_0}, \mu_{t_l}) \preceq U^{t-t_l}([\varphi_{t_l}^{\eta_0}]^+, [\mu_{t_l}]^+) \preceq g^{\theta_l}.$$

A random variable with geometric distribution corresponding to g^{θ_l} is integrable. Because the distributions of the sequence of trader's money holdings from t_l on, $\{\tilde{\eta}_t(\eta_0, \omega)\}_{t=t_l}^{\infty}$, is dominated by the same geometric distribution, $\tilde{\eta}_t(\eta_0, \omega)$ is uniformly integrable for $t \geq t_l$. Then, by *Theorem 25.12* (Billingsley 1995), weak convergence of $\varphi_t^{\eta_0}$ to the aggregate limit distribution which is geometric with mean M^* by Proposition 5, implies that the expectation of $\tilde{\eta}_t(\eta_0, \omega)$ converges to the same mean M^*. ∎

This concludes our investigation of the distributions in the economy, given the all agents adopt the presumed optimal strategy $\tilde{\sigma}^p$. To summarize, if the economywide initial money holdings distribution μ_0 satisfies condition ID_3, then the economywide money holdings sample distribution as well as each individual agent's money holdings probability distribution converges. Furthermore, the mean money holdings of each individual agent converges to the per capita money holdings M^* in the economy.

5. The Existence of Price-p Equilibrium

In this section, we show that if the initial distribution μ_0 satisfies condition ID_3, the price-p equilibrium defined in Section 3 is a Bayesian Nash equilibrium. In particular, we show that for an arbitrary agent, given that all other agents in the economy adopt the strategy $\tilde{\sigma}^p$ defined in (11) and (12),[9] it is optimal for the agent to adopt strategy $\tilde{\sigma}^p$ as well. That is, no strategy overtakes $\tilde{\sigma}^p$.

Consider a generic agent of any type. Suppose that the agent's initial money holdings is η_0. Since η_0 is fixed and is taken as given when we compare different strategies, for notational convenience, we will suppress η_0 as an argument of all functions such as σ and η_t^σ in the rest of the section, and write them as functions of ω alone. Also note that given all the other agents adopt strategy $\tilde{\sigma}^p$ and the agent in question has measure 0, although his trading history will be determined by his strategy σ, his encounter history ω is independent of the strategy he adopts.

Let $\eta_t^\sigma(\omega)$ denote the agent's money holdings at the beginning of date t with encounter history ω if he adopts strategy σ, $\eta_0^\sigma(\omega) = \eta_0$. Define the agent's *achievement function* at the beginning

[9]Hence, the bid and offer distributions are given by $\{\tilde{B}_t\}_{t=0}^{\infty}$ and $\{\tilde{O}_t\}_{t=0}^{\infty}$ defined in (13) and (14).

of date t, $A_t^\sigma : \Omega \to \mathbb{R}_+$, to be the sum of his total utility up to date t and the future utility that will be bought by the money accumulated up to date t, η_t^σ, given that the agent buys his future consumption goods at price p. That is, for any encounter history $\omega \in \Omega$,

$$A_t^\sigma(\omega) = \sum_{\tau=0}^{t-1} v_\tau^\sigma(\omega) + \frac{\eta_t^\sigma(\omega)}{p} u \tag{59}$$

where $v_\tau^\sigma(\omega)$ is defined in (6), and $\eta_t^\sigma(\omega)$ is defined recursively by (5). For notational convenience, define for all $t \geq 0$,

$$\tilde{A}_t = A_t^{\tilde{\sigma}^p}, \qquad \tilde{v}_t = v_t^{\tilde{\sigma}^p}, \qquad \tilde{\eta}_t = \eta_t^{\tilde{\sigma}^p}. \tag{60}$$

By the definition of the overtaking criterion (7), given that all other agents adopt strategy $\tilde{\sigma}^p$, any strategy that specifies at any point of time to offer at a price lower than p is obviously overtaken by the corresponding strategy which replaces the lower offer price by p. This is because any transaction that would occur at price lower than p would have occurred at price p. Hence, the seller in transaction would have been better off by obtaining more money while suffering the same endowment loss or by obtaining the same amount of money while suffering the less endowment loss. In the rest of the paper when we compare strategies with $\tilde{\sigma}^p$, we exclude those strategies with offer price lower than p at any point of time. The following lemma shows that strategy $\tilde{\sigma}^p$ is associated with the highest achievement function of any strategy.

LEMMA 14. *Consider any $p > 0$. If all other agents adopt strategy $\tilde{\sigma}^p$, then for an arbitrary agent facing any encounter history $\omega \in \Omega$, adopting a strategy σ, for all $t \geq 0$, $A_t^\sigma(\omega) \leq \tilde{A}_t(\omega)$.*

Proof. We prove the lemma by induction. Consider an agent of type i with a history $\omega \in \Omega$. Obviously, $A_0^\sigma(\omega) = \tilde{A}_0(\omega) = \eta_0 u/p$. Assume that the lemma holds up to date t, we compare an arbitrary strategy σ with $\tilde{\sigma}^p$ at the beginning of date $t+1$, $t \geq 0$.

(i) $\omega_{t1} = s$, $\omega_{t2} = p$, and $\omega_{t3} = 1$. In this case, regardless of the agent's strategy (including $\tilde{\sigma}^p$), $A_{t+1}^\sigma(\omega) = A_t^\sigma(\omega)$.

(ii) $\omega_{t1} = b$, $\omega_{t2} = p$, and $\omega_{t3} = 0$. This is a case that the buyer encountered has no money. So regardless of the strategy (including $\tilde{\sigma}^p$), no trade can take place. Consequently $A_{t+1}^\sigma(\omega) = A_t^\sigma(\omega)$.

(iii) $\omega_{t1} = b$, $\omega_{t2} = p$, and $\omega_{t3} = y \in (0, 1]$. Adopting strategy $\tilde{\sigma}^p$ implies $\tilde{A}_{t+1}(\omega) - \tilde{A}_t(\omega) = y(u - c) > 0$. Depending on the strategy σ, there are three subcases.

(a) $\sigma_{t3}(\omega) > p$. Since the seller's offer price is higher than the buyer's bid price p, no trade will take place if strategy σ is adopted. Hence, $A^{\sigma}_{t+1}(\omega) - A^{\sigma}_{t}(\omega) = 0 < \tilde{A}_{t+1}(\omega) - \tilde{A}_{t}(\omega)$.

(b) $\sigma_{t4}(\omega) < y$. In this case, the seller's offer quantity is smaller than the amount buyer wants to buy, no trade will take place if strategy σ is adopted. Hence, $A^{\sigma}_{t+1}(\omega) - A^{\sigma}_{t}(\omega) = 0 < \tilde{A}_{t+1}(\omega) - \tilde{A}_{t}(\omega)$.

(c) $\sigma_{t3}(\omega) = p$ and $\sigma_{t4}(\omega) \geq y$. In this case, trade yields the same net gain as by adopting strategy $\tilde{\sigma}^{p}$, $A^{\sigma}_{t+1}(\omega) - A^{\sigma}_{t}(\omega) = y(u-c) = \tilde{A}_{t+1}(\omega) - \tilde{A}_{t}(\omega)$.

In all three cases, $A^{\sigma}_{t+1}(\omega) - \tilde{A}_{t+1}(\omega) \leq A^{\sigma}_{t}(\omega) - \tilde{A}_{t}(\omega) \leq 0$. The last inequality is implied by the induction hypothesis.

Therefore, we conclude that for any strategy σ, for all history $\omega \in \Omega$, $A^{\sigma}_{0}(\omega) - \tilde{A}_{0}(\omega) = 0$, and for all $t \geq 0$, $A^{\sigma}_{t+1}(\omega) - \tilde{A}_{t+1}(\omega) \leq 0$. By induction, for all $t \geq 0$, $A^{\sigma}_{t}(\omega) \leq \tilde{A}_{t}(\omega)$. ∎

Proposition 6 states that if all other agents adopt strategy $\tilde{\sigma}^{p}$, and if the economywide initial distribution μ_0 satisfies condition ID_3, the expected money holdings of an agent adopting strategy $\tilde{\sigma}^{p}$ converges to M^*. The next lemma shows that if an agent adopts some other strategy σ, then in the limit, he may end up with more money on average.

LEMMA 15. *Consider any $p > 0$. Suppose that the economywide initial money holdings distribution μ_0 satisfies condition ID_3, and that an arbitrary agent adopts strategy σ while all other agents adopt strategy $\tilde{\sigma}^{p}$. If $\mathsf{E}[A^{\sigma}_{t}(\omega) - \tilde{A}_{t}(\omega)] \not\to -\infty$ as $t \to \infty$, then $\liminf_{t\to\infty} \mathsf{E}[\eta^{\sigma}_{t}(\omega)] \geq M^*$.*

Proof. For strategy σ, for all $\omega \in \Omega$, define $\delta^{\sigma}(\omega)$ to be the set of dates at which the agent who adopts strategy σ meets a buyer with money, and either his offer price is above p or his offer quantity is below the buyer's bid quantity,[10]

$$\delta^{\sigma}(\omega) = \Big\{ t \,|\, \omega_{t1} = b \text{ and } \omega_{t3} > 0 \text{ and } \left(\sigma_{t3}(\omega) > p \text{ or } (\sigma_{t3}(\omega) = p \text{ and } \sigma_{t4}(\omega) < \omega_{t3})\right) \Big\}.$$

On these occasions, trade will not occur which would have occurred had the agent adopted strategy $\tilde{\sigma}^{p}$.

Claim 1. If $\mathsf{E}[A^{\sigma}_{t}(\omega) - \tilde{A}_{t}(\omega)] \not\to -\infty$ as $t \to \infty$, then $\sum_{t\in\delta^{\sigma}} \omega_{t3} < \infty$ a.s.

[10] These are the relevant deviations of offer strategy, since they block trade that would have occurred otherwise. As stated earlier, we do not explicitly consider the deviation of offering at price below p since it is obviously suboptimal. Also, we need not consider the case in which an agent deviates by offering to sell at price p and quantity below 1, but the quantity happens to be above the buyer's bid quantity. In that case, trade will occur at the buyer's bid quantity.

To prove this, consider an arbitrary encounter sequence $\omega \in \Omega$. For any date $t \in \delta^\sigma(\omega)$, given that the agent adopts strategy σ, no trade takes place, hence, $A^\sigma_{t+1}(\omega) - A^\sigma_t(\omega) = 0$. On the other hand, if the agent adopts strategy $\tilde{\sigma}^p$, then trade takes place at price p, $\tilde{A}_{t+1}(\omega) - \tilde{A}_t(\omega) = \omega_{t3}(u-c) > 0$. Therefore,

$$\forall t \in \delta^\sigma(\omega) \quad A^\sigma_{t+1}(\omega) - \tilde{A}_{t+1}(\omega) = A^\sigma_t(\omega) - \tilde{A}_t(\omega) - \omega_{t3}(u-c). \tag{61}$$

For $t \notin \delta^\sigma(\omega)$, it is easy to check, as for all the cases considered in the proof of Lemma 14, that $A^\sigma_{t+1}(\omega) - \tilde{A}_{t+1}(\omega) = A^\sigma_t(\omega) - \tilde{A}_t(\omega)$. Hence, by (61) and the fact that $A^\sigma_0(\omega) = \tilde{A}_0(\omega)$, if $\sum_{t \in \delta^\sigma(\omega)} \omega_{t3} = \infty$, then

$$\lim_{t\to\infty} [A^\sigma_t(\omega) - \tilde{A}_t(\omega)] = \lim_{t\to\infty} \sum_{t \in \delta^\sigma(\omega)} -\omega_{t3}(u-c) = -\infty.$$

Therefore, if $\mathcal{P}\{\omega \mid \sum_{t\in\delta^\sigma(\omega)} \omega_{t3} = \infty\} > 0$, then $\lim_{t\to\infty} \mathsf{E}[A^\sigma_t(\omega) - \tilde{A}_t(\omega)] = -\infty$, which contradicts to the assumption. Thus, the claim holds.

Given claim 1, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\mathcal{P}\{\omega \mid \sum_{t\in\delta^\sigma(\omega),\, t>t_\varepsilon} \omega_{t3} < \varepsilon\} > 1 - \varepsilon/2$. Recall that for all $t \in \delta^\sigma(\omega)$, $\tilde{\sigma}^p_{t2}(\omega) = p$, $\tilde{\sigma}^p_{t3}(\omega) = 1$. Define $\zeta_\varepsilon(\omega) \equiv \min\{t \mid t \geq t_\varepsilon,\, \tilde{\eta}_t(\omega) = 0\}$.

Claim 2. $\zeta_\varepsilon < \infty$ *a.s.*

Suppose to the contrary. Let $A = \{\omega \mid \forall t\ \tilde{\eta}_t(\omega) > 0\}$ and suppose that $\mathcal{P}(A) > 0$. Take an arbitrary date t. For all $n \in \mathbb{N}$, define $D_n \equiv \{\omega \mid \tilde{\eta}_t(\omega) \in [np,\, (n+1)p)\}$. Then $A = \bigcup_{n\in\mathbb{N}}(A \cap D_n)$. Since $\mathcal{P}(A) > 0$, there exists $j \in \mathbb{N}$ such that $\mathcal{P}(A \cap D_j) > 0$. Given that all other agents play strategy $\tilde{\sigma}^p$ and the random matching at each date is independent of those at other dates, the probability that an agent with money holdings $\tilde{\eta}_t(\omega) \in D_j$ spends all of his money in next j consecutive dates is $(1/2)^j$. That is,

$$\mathcal{P}\Big\{\omega \mid \tilde{\eta}_{t+j}(\omega) = 0 \text{ and } \omega \in A \cap D_j\Big\} = \Big(\frac{1}{2}\Big)^j \mathcal{P}(A \cap D_j) > 0$$

which contradicts the definition of the set A.

Claim 3. For all $\omega \in \Omega$, *for all* $t \geq \zeta_\varepsilon(\omega)$, $\tilde{\eta}_t(\omega) \leq \eta^\sigma_t(\omega) + p\sum_{\tau\in\delta^\sigma(\omega),\, t_\varepsilon<\tau<t} \omega_{\tau3}$.

This claim can be proved by induction. For $t = \zeta_\varepsilon(\omega)$, the claim holds automatically since $\tilde{\eta}_t(\omega) = 0 \leq \eta^\sigma_t(\omega) + p\sum_{\tau\in\delta^\sigma(\omega),\, t_\varepsilon<\tau<t} \omega_{\tau3}$. Suppose that it holds for some $t \geq \zeta_\varepsilon(\omega)$, consider the date-$(t+1)$ transaction.

(i) If $\omega_{t1} = b$, $\omega_{t2} = p$, $\omega_{t3} \in [0, 1]$, then $\tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) + p\,\omega_{t3}$.

(a) If $t \in \delta^\sigma(\omega)$, then trade does not occur if the agent adopts strategy σ, so $\eta_t^\sigma(\omega) = \eta_{t+1}^\sigma(\omega)$. By induction hypothesis,

$$\tilde{\eta}_{t+1}(\omega) \leq \eta_t^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t} \omega_{\tau 3} + p\,\omega_{t3} = \eta_{t+1}^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t+1} \omega_{\tau 3}.$$

(b) If $t \notin \delta^\sigma(\omega)$, trade occurs at price p and quantity ω_{t3} with both strategies σ and $\tilde{\sigma}$, then, by induction hypothesis,

$$\tilde{\eta}_{t+1}(\omega) \leq \eta_t^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t} \omega_{\tau 3} + p\,\omega_{t3} = \eta_{t+1}^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t+1} \omega_{\tau 3}.$$

(ii) If $\omega_{t1} = s$, $\omega_{t2} = p$, $\omega_{t3} = 1$, then $\tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) - p$, and $t \notin \delta^\sigma(\omega)$.

(a) If $\tilde{\eta}_t(\omega) < p$, then, $\tilde{\eta}_{t+1}(\omega) = 0 \leq \eta_{t+1}^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t+1} \omega_{\tau 3}$.

(b) If $\tilde{\eta}_t(\omega) \geq p$, then, given that $\sigma_{t3}(\omega) \leq 1$ and the induction hypothesis,

$$\tilde{\eta}_{t+1}(\omega) \leq \eta_t^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t} \omega_{\tau 3} - p\,\sigma_{t3}(\omega) = \eta_{t+1}^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t+1} \omega_{\tau 3}.$$

That is, $\tilde{\eta}_{t+1}(\omega) \leq \eta_{t+1}^\sigma(\omega) + p \sum_{\tau \in \delta^\sigma(\omega),\, t_\varepsilon < \tau < t+1} \omega_{\tau 3}$. Hence, the claim holds for all $t \geq \zeta_\varepsilon(\omega)$.

By Claim 3, for all $\omega \in \Omega$ such that $\sum_{t \in \delta^\sigma(\omega),\, t > t_\varepsilon} \omega_{t3} < \varepsilon$, for all $t \geq \zeta_\varepsilon(\omega)$,

$$\eta_t^\sigma(\omega) - \tilde{\eta}_t(\omega) \geq -p\,\varepsilon. \tag{62}$$

Since $\zeta_\varepsilon < \infty$ a.s., for the ε chosen above, there exists $\xi_\varepsilon > 0$ such that $\mathcal{P}\{\omega \,|\, \zeta_\varepsilon(\omega) \leq \xi_\varepsilon\} > 1 - \varepsilon/2$. Define

$$\Omega_1(\varepsilon) = \Big\{\omega \;\Big|\; \sum_{t \in \delta^\sigma(\omega),\, t > t_\varepsilon} \omega_{t3} < \varepsilon \;\text{ and }\; \zeta_\varepsilon(\omega) \leq \xi_\varepsilon\Big\}$$

$$\Omega_2(\varepsilon) = \Big\{\omega \;\Big|\; \sum_{t \in \delta^\sigma(\omega),\, t > t_\varepsilon} \omega_{t3} \geq \varepsilon \;\text{ or }\; \zeta_\varepsilon(\omega) > \xi_\varepsilon\Big\}.$$

Then $\Omega = \Omega_1(\varepsilon) \cup \Omega_2(\varepsilon)$, $\mathcal{P}(\Omega_1(\varepsilon)) > 1 - \varepsilon$ and $\mathcal{P}(\Omega_2(\varepsilon)) < \varepsilon$. Take $\varepsilon = 1/n^2$. For $\omega \in \Omega_1(1/n^2)$, $t_{1/n^2} \leq \zeta_{1/n^2}(\omega) \leq \xi_{1/n^2}$. For a fixed n, consider the sequence $\{\eta_t^\sigma(\omega) - \tilde{\eta}_t(\omega)\}$ for $t \geq \xi_{1/n^2}$. Let $\Omega_{1n} \equiv \Omega_1(1/n^2)$ and $\Omega_{2n} \equiv \Omega_2(1/n^2)$.

$$\begin{aligned} \liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega) - \tilde{\eta}_t(\omega)] \;\;\geq\;\; & \liminf_{t\to\infty} \int_{\Omega_{1n}} (\eta_t^\sigma(\omega) - \tilde{\eta}_t(\omega))\,d\mathcal{P}(\omega) \\ & + \liminf_{t\to\infty} \Big(\int_{\Omega_{2n}} \eta_t^\sigma(\omega)\,d\mathcal{P}(\omega) - \int_{\Omega_{2n}} \tilde{\eta}_t(\omega)\,d\mathcal{P}(\omega) \Big). \end{aligned} \tag{63}$$

The first term on the right hand side of (63) is greater than $(-p/n^2)\mathcal{P}(\Omega_{1n}) \geq -p/n^2$ because (62) holds for all $\omega \in \Omega_{1n}$. We will now prepare to consider the second term on the right side of (63). Note that by Proposition 6, the limit of $\mathsf{E}[\tilde{\eta}_t(\omega)]$ exists and equals M^*.

Claim 4. For any $n \geq 1$, $\lim_{t\to\infty} \int_{\Omega_{2n}} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega) = M^* \mathcal{P}(\Omega_{2n})$.

To prove this claim, for any $t \geq 0$, define

$$\Lambda_t \equiv \Big\{ \omega \,|\, \exists \hat{t} \, [t_{1/n^2} < \hat{t} \leq t \text{ and } \hat{t} \in \delta^\sigma(\omega)] \quad \text{or} \quad \zeta_{1/n^2}(\omega) > \xi_{1/n^2} \Big\}.$$

Then, for all $t \geq 0$, $\Lambda_t \subseteq \Lambda_{t+1}$, $\cup_{t=0}^{\infty} \Lambda_t = \Omega_{2n}$, and $\mathcal{P}(\Lambda_t) \to \mathcal{P}(\Omega_{2n})$ as $t \to \infty$. Hence, for any $\varepsilon > 0$, there exists $t_1 \geq 0$ such that for all $t \geq t_1$,

$$|\mathcal{P}(\Lambda_t) - \mathcal{P}(\Omega_{2n})| < \frac{\varepsilon}{3M^*}. \tag{64}$$

Let θ be the parameter such that the geometric distribution g^θ dominates both the aggregate money holdings distribution as well as the agent's money holdings distribution from some date onward. (See the proof of Proposition 6 for the construction of θ.) For the ε above, choose $k > 0$ such that

$$\sum_{j \geq k} j(1-\theta)\theta^j < \varepsilon/3.$$

By Proposition 5, the distribution of the agent's money holdings converges weakly to the aggregate money holdings distribution g^{m^*}. That is, for the k chosen above, for all $\nu \in (0, (m^*)^k)$, there exists $t_2 \geq t_1$ such that for all $t \geq t_2$, $|\mathcal{P}\{\tilde{\eta}_t(\omega) \geq kp\} - (m^*)^k| < \nu$, which implies $0 < (m^*)^k - \nu < \mathcal{P}\{\tilde{\eta}_t(\omega) \geq kp\}$. Since $\{\Omega_{2n} \backslash \Lambda_t\}_{t=0}^{\infty}$ is decreasing and $\mathcal{P}(\Omega_{2n} \backslash \Lambda_t) \to 0$ as $t \to \infty$, there exists $t_3 \geq t_2$ such that for all $t \geq t_3$, $\mathcal{P}(\Omega_{2n} \backslash \Lambda_t) \leq (m^*)^k - \nu < \mathcal{P}\{\tilde{\eta}_t(\omega) \geq kp\}$. Then, for all $t \geq t_3$,

$$\int_{\Omega_{2n} \backslash \Lambda_t} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega) \leq \int_{\{\omega : \tilde{\eta}_t(\omega) \geq kp\}} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega). \tag{65}$$

Furthermore, given that the agent's money holdings distribution is dominated by g^θ from some date on, there exists $t_4 \geq t_3$ such that for all $t \geq t_4$, $\mathcal{P}\{\tilde{\eta}_t(\omega) \geq kp\} \leq \theta^k$. Then, for all $t \geq t_4$,

$$\int_{\{\omega : \tilde{\eta}_t(\omega) \geq kp\}} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega) \leq \sum_{j \geq k} j(1-\theta)\theta^j < \varepsilon/3. \tag{66}$$

By inequalities (65) and (66), for all $t \geq t_4$,

$$\int_{\Omega_{2n} \backslash \Lambda_t} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega) \leq \varepsilon/3. \tag{67}$$

For any $t \geq \xi_{1/n^2}$, $\Lambda_t \in \mathcal{B}_t$. For any $\tau > t$, ω_τ is independent of $\mathcal{B}_t$, and in particular, the distribution of the trading partner's money holdings conditional on $\mathcal{B}_t$ is given by μ_τ, and the conditional probability that the trading partner is a potential seller is one half. Therefore, analogously to Proposition 6, $\lim_{\tau\to\infty} \mathsf{E}[\tilde{\eta}_\tau(\omega)|\Lambda_t] = M^*$. Then, for all $t \geq t_4$, there is a $\bar{\tau}_t$ such that for all $\tau \geq \bar{\tau}_t$,

$$\Big| \int_{\Lambda_t} \tilde{\eta}_\tau(\omega) d\mathcal{P}(\omega) - M^* \mathcal{P}(\Lambda_t) \Big| < \varepsilon/3. \tag{68}$$

Combining the results given by (64), (67) and (68), for some $t \geq t_4$, for all $\tau \geq \bar{\tau}_t$, we have

$$
\begin{aligned}
&\Big|\int_{\Omega_{2n}} \tilde{\eta}_\tau(\omega) d\mathcal{P}(\omega) - M^*\mathcal{P}(\Omega_{2n})\Big| \\
&\leq \Big|\int_{\Lambda_t} \tilde{\eta}_\tau(\omega) d\mathcal{P}(\omega) - M^*\mathcal{P}(\Lambda_t)\Big| + M^*\Big|\mathcal{P}(\Lambda_t) - \mathcal{P}(\Omega_{2n})\Big| + \int_{\Omega_{2n}\setminus\Lambda_t} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega) \\
&< \varepsilon/3 + M^*\left(\varepsilon/3M^*\right) + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

That is, the claim holds.

By Claim 4, the second lim inf on the right hand side of (63) can be broken down to two terms,

$$
\liminf_{t\to\infty} \Big(\int_{\Omega_{2n}} \eta_t^\sigma(\omega) d\mathcal{P}(\omega) - \int_{\Omega_{2n}} \tilde{\eta}_t(\omega) d\mathcal{P}(\omega)\Big) = \liminf_{t\to\infty} \int_{\Omega_{2n}} \eta_t^\sigma(\omega) d\mathcal{P}(\omega) - M^*\mathcal{P}(\Omega_{2n}). \tag{69}
$$

The first term of the right hand side of (69) is nonnegative. Therefore,

$$
\liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega) - \tilde{\eta}_t(\omega)] \geq -p/n^2 - M^*\mathcal{P}(\Omega_{2n}) \geq -\frac{p+M^*}{n^2}. \tag{70}
$$

Take limits as $n \to \infty$ in inequality (70). The left hand side is unrelated to n, hence constant, and the right hand side goes to 0. Therefore, $\liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega)] \geq \lim_{t\to\infty} \mathsf{E}[\tilde{\eta}_t(\omega)] = M^*$ by Proposition 6. ∎

Now, we are ready to show the last step for the existence of the price-p equilibrium.

PROPOSITION 7. *Consider any $p > 0$. If the economywide initial money holdings distribution $\mu_0 \in \Delta^{M^*}$ satisfies condition ID$_3$, and if all other agents adopt strategy $\tilde{\sigma}^p$, then it is optimal for an arbitrary agent to take strategy $\tilde{\sigma}^p$ as well. That is, there is no strategy σ that overtakes $\tilde{\sigma}^p$.*

Proof. For an arbitrary strategy σ, consider the following two cases.

Case 1. $\limsup_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega)] \geq M^*$. Then, for any $\varepsilon > 0$, there exists an infinite set $G_\varepsilon^\sigma = \{t \,|\, \mathsf{E}[\eta_t^\sigma(\omega)] \geq M^* - \varepsilon/2\}$. Since $\lim_{t\to\infty} \mathsf{E}[\tilde{\eta}_t(\omega)] = M^*$ by Proposition 6, the set $J_\varepsilon^\sigma = \{t \,|\, \mathsf{E}[\tilde{\eta}_t(\omega)] < M^* + \varepsilon/2\}$ is also infinite. For all $t \in G_\varepsilon^\sigma \cap J_\varepsilon^\sigma$, by Lemma 14,

$$
0 \geq \mathsf{E}[A_t^\sigma - \tilde{A}_t] = \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] + \mathsf{E}[\frac{\eta_t^\sigma}{p} - \frac{\tilde{\eta}_t}{p}]u \geq \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] - \frac{u}{p}\varepsilon
$$

Since ε can be arbitrarily small, the above inequality implies that

$$
\liminf_{t\to\infty} \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] \leq 0.
$$

By definition of overtaking criterion (7), strategy σ does not overtake $\tilde{\sigma}^p$.

Case 2. $\limsup_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega)] < M^*$. By the proof of Lemma 14, for all $\omega \in \Omega$, $\{A_t^\sigma(\omega) - \tilde{A}_t(\omega)\}_{t=0}^\infty$ is a weakly decreasing sequence. If $\mathsf{E}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] \not\to -\infty$ as $t \to \infty$, by Lemma 15, $\liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega)] \geq M^*$, which contradicts to the assumption. If $\mathsf{E}[A_t^\sigma - \tilde{A}_t] \to -\infty$ as $t \to \infty$, and since

$$\mathsf{E}[A_t^\sigma - \tilde{A}_t] = \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] + \mathsf{E}[\frac{\eta_t^\sigma}{p} - \frac{\tilde{\eta}_t}{p}]u$$

then we have

$$\liminf_{t\to\infty} \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] \leq \liminf_{t\to\infty} \mathsf{E}[A_t^\sigma - \tilde{A}_t] + \frac{u}{p}\left(M^* - \liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma]\right) = -\infty.$$

Again by the definition (7), strategy σ does not overtake $\tilde{\sigma}^p$. ∎

By Proposition 7, strategy $\tilde{\sigma}^p$ is a Bayesian Nash strategy according to the overtaking criterion. This proves (ii) of the equilibrium definition, and (iii) is evidently satisfied. Hence, the price-p equilibrium always exists.

6. Conclusion

This article has provided an analysis of equilibrium in an infinite horizon economy where trade must occur pairwise rather than in a central market, and where the exchange of fiat money for goods overcomes a lack of double coincidence of wants. Although an agent can bargain with only one trading partner at a time, trade has the characteristics of anonymity and absence of market power. These characteristics are ensured by assumptions about the random matching process for pairwise trade, agents' lack of information about trading partners' money holdings and histories, and absence of time preference. (This last assumption deprives an agent's current trading partner of monopoly power because the agent considers consumption in the future to be a perfect substitute for current consumption.) Goods in the economy are divisible and perishable, with new endowments being received and consumption occurring at every date. Money is also divisible, and is not subject to inventory constraints. Exchange within each trading pair is governed by a double auction mechanism.

Two main results have been obtained. The first is a characterization of the evolution of individual agents' money holdings and of the economywide distribution of these holdings when trading occurs as extensively as possible at a specified, economywide price. This characterization is instrumental to deriving the second result: that for any price, and for any initial distribution of money holdings that is dominated in the tail by a geometric distribution, there is an equilibrium in which all trades occur at the specified price. This second result shows the existence and also the indeterminacy of single price equilibrium, a special class of Bayesian Nash equilibrium of the economy.

That single price equilibrium exists, despite the fragmentation of trading activity, confirms an expectation that economists have long held.[11] While not a foregone conclusion, this result is consistent with numerous earlier findings such as those cited in the introduction. Camera and Corbae (1999) show that price dispersion characterizes equilibrium in a model economy that resembles the present one in many respects, but trading in that economy is not completely anonymous because agents observe their trading partners' money holdings. As Rubinstein and Wolinsky (1990) emphasize, lack of anonymity is conducive to the existence of many equilibria in which agents can be treated disparately.

The indeterminacy of equilibrium in the model economy is a surprising result. The nature of this indeterminacy is rather different from that which is familiar from Walrasian general equilibrium models such as overlapping generations models with fiat money.[12] In those models, there is an indeterminacy of relative prices between various dated, location specific, state contingent commodities. In the present model, all such commodities would trade at par because they all have the same nominal price. It is the nominal price level, rather than the relative prices of various goods, that is indeterminate. This price level indeterminacy has a real effect through its influence on the distribution of money holdings, with agents not holding money being unable to participate in beneficial transactions. This effect is a further example of the existence of Pareto dominated equilibria in coordination game models, such as macroeconomic models due to Cooper and John (1988) and Diamond (1984), where an inefficiently low level of economywide search effort or other complementary investment results from agents' failure to take proper account of the positive external effects of their actions. The trading mechanism in the present model does not provide a coordination device to take advantage of the positive externality that sellers collectively could provide by offering their endowment at a lower nominal price. If the market price could be lowered in that way, then the aggregate real money balances M^*/p in the economy would increase and welfare would improve.

There are a number of alternative model economies that might be examined to investigate whether the indeterminacy of equilibrium is robust to re-specification. Among them would be to substitute Stahl-Rubinstein strategic bargaining for a double auction mechanism as a representation of strategic price/quantity determination for transactions; to impute discounted utility

[11] Indeed, we strongly suspect that all stationary Bayesian Nash equilibria in this model economy are single price equilibria. Green and Zhou (2001) prove this result for a virtually identical model.

[12] Cass (1992) and Werner (1990) show that indeterminacy of equilibrium is a generic feature of Walrasian economies with incomplete markets or other forms of restricted market participation.

preferences, rather than preferences characterized by an overtaking criterion, to agents; and to introduce a durable good alongside the perishable commodities currently traded. We consider the first of these alternatives, Stahl-Rubinstein bargaining, to be the one most likely to overturn the indeterminacy result by itself.

Indeterminacy of the steady state money holdings distribution, which is the key to the equilibrium indeterminacy result in the present model, has been exhibited in closely related model economies with discounted utility preferences and a durable, tradable good.[13] We conjecture that these economies exhibit indeterminacy of dynamic equilibrium from an initial state (under some assumptions such as that the initial money holdings distribution is nice and that the discount factor is close to 1), as well as indeterminacy of steady state distributions.

Regarding the issue of Stahl-Rubinstein bargaining versus double auction mechanisms, and more generally regarding a research program of investigating the robustness of the equilibrium indeterminacy result derived here, we would offer three methodological observations. First, neither of the two types of representation of strategic transactions is a literal model of actual economic activity or is the uniquely privileged representation in any other way, especially since a wide spectrum of mechanisms are actually used to conduct various transactions. At the very least, the result proved here warrants the interpretation that some transaction mechanisms may be susceptible to indeterminacy of equilibrium if they are predominantly used in an economy. We hope that the present result may stimulate the development of further models that pay closer attention to the micro-structure of transactions and that might provide deeper understanding of what is required for equilibrium to be determinate. Second, while indeterminacy of equilibrium may be a symptom that a model is incompletely specified (in a sense resembling the idea that when there are fewer equations than unknowns, a well motivated equation may have been ignored), indeterminacy does not necessarily show that a model is misspecified or implausible. While indeterminacy of equilibrium would be an inconvenient situation for policy analysis, the possibility that actual economies may exhibit this feature cannot be ruled out. Third, assumptions or features of specification that make the difference between a model economy having determinate or indeterminate equilibrium should be regarded as economically crucial. Once discovered to have such an effect, an assumption should not be regarded as merely a convenient formal simplification—even if that is why it was originally introduced. In view of the indeterminacy of equilibrium that results when indivisibility and finite-inventory-constraint assumptions about money are avoided in a random matching model, such determinacy-inducing assumptions should not casually be adopted.

[13] Cf. Green and Zhou (1998), Zhou (1999a,b).

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Appendix

The Proof of Lemma 3: The function Z is strictly convex on Γ.

Take arbitrary x, $y \in \Gamma$, $x \neq y$, and $\alpha \in (0,1)$. Let $w(\alpha) = (1-\alpha)x + \alpha y = x + \alpha(y-x)$. The set Γ is convex, hence $w(\alpha) \in \Gamma$. For all $k \in \mathbb{N}$, define $\delta_k \equiv y_k - x_k$ and

$$z_k(w(\alpha)) \equiv \frac{(w_k(\alpha) - w_{k+1}(\alpha))^2}{w_k(\alpha)} = \frac{(x_k - x_{k-1} + \alpha(\delta_k - \delta_{k+1}))^2}{x_k + \alpha\delta_k}.$$

Direct computation reveals that

$$\frac{d^2 Z(w(\alpha))}{d\alpha^2} = \sum_{k=0}^{\infty} \frac{d^2 z_k(w(\alpha))}{d\alpha^2} = \sum_{k=0}^{\infty} \frac{2}{x_k + \alpha\delta_k}\left(\delta_k - \delta_{k+1} - \frac{\delta_k(x_k - x_{k+1} + \alpha(\delta_k - \delta_{k+1}))}{x_k + \alpha\delta_k}\right)^2 \geq 0.$$

Moreover $\frac{d^2 Z(w(\alpha))}{d\alpha^2} = 0$ if an only if $\forall k \in \mathbb{N}$ $\frac{d^2 z_k(w(\alpha))}{d\alpha^2} = 0$, which is equivalent to $\forall k \in \mathbb{N}$ $y_{k+1}x_k = y_k x_{k+1}$, that is (since $x_0 = y_0 = 1$), $\forall k \in \mathbb{N}$ $x_k = y_k$. Given that $x \neq y$, $\frac{d^2 Z(w(\alpha))}{d\alpha^2} > 0$. Hence, Z is strictly convex on Γ. ∎

The Proof of Lemma 5: The function Z is continuous on S.

We need to show that for any given $\varepsilon > 0$, for any $x \in S$, there exists a δ-neighborhood of x such that for all y satisfying $d(x,y) < \delta$, $|Z(x) - Z(y)| < \varepsilon$.

Fix an arbitrary $\varepsilon > 0$, and an arbitrary $x \in S$. Since x_k is decreasing in k and $\sum_{k=1}^{\infty} x_k = M$, there exists $I \geq 1$ such that

$$x_I < \varepsilon/8. \tag{71}$$

Let $J = \max\{j \,|\, j \leq I,\, x_j > 0\}$. So $x_J > 0$. Without loss of generality, assume $J \geq I - 1$. Let $\delta = (\varepsilon/8)x_J > 0$. (Otherwise $x_{J+1} = 0$, so $J+1$ satisfies $x_{J+1} < \varepsilon/8$.) Then for any y such that $d(x,y) < \delta$,

$$y_I \leq |y_I - x_I| + x_I \leq d(x,y) + x_I < (\varepsilon/8)x_J + \varepsilon/8 \leq \varepsilon/4. \tag{72}$$

For all $k \in \mathbb{N}$, define $\xi_k(x) \equiv (x_k - x_{k+1})/x_k \leq 1$, and $\xi_k(y) \equiv (y_k - y_{k+1})/y_k \leq 1$. Then, for $k \leq I-1$, $x_k \geq x_J$,

$$|\xi_k(x) - \xi_k(y)| \leq \frac{1}{x_k}\left(|x_{k+1} - y_{k+1}| + |x_k - y_k|\frac{y_{k+1}}{y_k}\right) < \frac{2}{x_J}\frac{\varepsilon}{8}x_J = \frac{\varepsilon}{4}. \tag{73}$$

Now, applying (71)–(73), we have

$$|Z(x) - Z(y)|$$

$$
\begin{aligned}
&= \sum_{k=0}^{I-1}\Big|(x_k - x_{k+1})\xi_k(x) - (y_k - y_{k+1})\xi_k(y)\Big| + \sum_{k\ge I}(x_k - x_{k+1})\xi_k(x) + \sum_{k\ge I}(y_k - y_{k+1})\xi_k(y) \\
&\le \sum_{k=0}^{I-1}\Big((x_k - x_{k+1})|\xi_k(x) - \xi_k(y)| + |x_k - y_k|\xi_k(y) + |x_{k+1} - y_{k+1}|\xi_k(y)\Big) + x_I + y_I \\
&< \sum_{k=0}^{I-1}(x_k - x_{k+1})\frac{\varepsilon}{4} + \sum_{k=0}^{I-1}|x_k - y_k| + \sum_{k=0}^{I-1}|x_{k+1} - y_{k+1}| + \varepsilon/8 + \varepsilon/4 \\
&< \varepsilon/4 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/4 < \varepsilon.
\end{aligned}
$$

We have shown that for any given $\varepsilon > 0$, for any $x \in S$, there is $\delta > 0$ such that for all y satisfying $d(x,y) < \delta$, $|Z(y) - Z(x)| < \varepsilon$. Hence, Z is continuous on S. ∎

The Proof of Lemma 6: The mapping T is continuous on S.

We need to show that for any given $\varepsilon > 0$ and $x \in S$, there is a $\delta > 0$ such that for all y satisfying $d(x,y) < \delta$, $d(T(x), T(y)) < \varepsilon$.

Fix an arbitrary $\varepsilon > 0$ and an arbitrary $x \in S$. By (19), for all $y \in S$, for all $k \ge 1$,

$$T_k(y) = \frac{1 - y_1}{2} y_k + \frac{1}{2} y_{k+1} + \frac{y_1}{2} y_{k-1}.$$

Take $\delta = \varepsilon/3 > 0$, and let y be such that $d(y,x) < \delta$. Then,

$$
\begin{aligned}
d(T(x), T(y)) &= \sum_{k=1}^{\infty} |T_k(y) - T_k(x)| \\
&= \sum_{k=1}^{\infty}\frac{1}{2}\Big(|x_k - y_k| + |x_{k+1} - y_{k+1}| + x_1|x_k - y_k| + x_1|x_{k+1} - y_{k+1}| + (y_k + y_{k+1})|x_1 - y_1|\Big) \\
&< \frac{1}{2}\Big(\varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3\Big) = \varepsilon
\end{aligned}
$$

Therefore, T is continuous on S. ∎

The Proof of Lemma 7: For every vector $\pi \in \mathcal{X}$, the set S_π is compact.

To prove S_π is compact, we need to show that S_π is complete and totally bounded subset of $\mathcal{X}$. The completeness of S_π is trivial given that S is complete, and the proof is omitted here. To show that S_π is totally bounded, we need to show that there exist a finite ε-net for S_π in $\mathcal{X}$ for any $\varepsilon > 0$.

Fix an arbitrary $\varepsilon > 0$. Since $\pi \in \mathcal{X}$, $\sum_{k=0}^{\infty} \pi_k < \infty$. Hence, there exists $I > 0$ such that $\sum_{k>I} \pi_k < \varepsilon/2$. For any $x \in S_\pi$, let $\hat{x}$ be the vector of x truncated at I, $\hat{x} = (x_0, x_1, \ldots, x_I, 0, 0, \ldots)$. Then $d(x, \hat{x}) = \sum_{k>I} x_k \leq \sum_{k>I} \pi_k < \varepsilon/2$. Let $\hat{S}_\pi$ be the set of $\hat{x}$ associated with $x \in S_\pi$. The set $\hat{S}_\pi$ is a totally bounded in I-dimensional Euclidean space (with the usual metric). Let A be a finite $\varepsilon/2$-net for $\hat{S}_\pi$. Then A is a finite ε-net for S_π. ∎

The Proof of Lemma 8: For any $x \in S$ and any $\theta \in (0,1)$, if $x \preceq_d \pi^\theta$, then $T(x) \preceq_d \pi^\theta$.

Suppose that there exists $\theta \in (0,1)$ such that $x \preceq_d \pi^\theta$. By definition, $x \preceq_d \pi^\theta$ implies that $x_k \leq \theta^k$ for all $k \in \mathbb{N}$. By equation (19), for all $k \geq 1$,

$$T_k(x) = \frac{1}{2}\Big((1-x_1)x_k + x_{k+1} + x_1 x_{k-1}\Big) \leq \frac{1}{2}\Big((1-x_1)\theta^k + \theta^{k+1} + x_1\theta^{k-1}\Big).$$

Since the expression in the right hand side of the above inequality is an increasing function of x_1 and by assumption, $x_1 \leq \theta$,

$$T_k(x) \leq \frac{1}{2}\Big((1-\theta)\theta^k + \theta^{k+1} + \theta\theta^{k-1}\Big) = \theta^k.$$

By definition, $T_0(x) = 1 = \pi_0^\theta$. Therefore, $T(x) \preceq_d \pi^\theta$. ∎

The Proof of Lemma 9: For any μ_1, $\mu_2 \in \Delta$, if $\mu_1 \preceq \mu_2$, then $T(\mu_1) \preceq T(\mu_2)$.

For any μ_1, $\mu_2 \in \Delta$ such that $\mu_1 \preceq \mu_2$, for any nondecreasing function $f : \mathbb{R}_+ \to [0,1]$, we have[14]

$$\int_0^\infty f d\mu_1 \leq \int_0^\infty f d\mu_2. \tag{74}$$

For any $y \geq p$, define

$$f(x) = \begin{cases} \mu_1[y-x, \infty) & \text{if } x < p \\ \mu_1[y-p, \infty) & \text{if } x \geq p \end{cases}$$

Then, by (32) and (74),

$$\begin{aligned}\rho(\mu_1)[y,\infty) &= \int_0^\infty f(x)d\mu_1(x) \leq \int_0^\infty f(x)d\mu_2(x) \\ &= \int_{[0,p)} \mu_1[y-x,\, \infty)d\mu_2(x) + \mu_2[p,\infty)\mu_1[y-p,\infty) \\ &\leq \int_{[0,p)} \mu_2[y-x,\, \infty)d\mu_2(x) + \mu_2[p,\infty)\mu_2[y-p,\infty) = \rho(\mu_2)[y,\infty)\end{aligned}$$

[14] This is a standard result about stochastic dominance.

The last inequality holds because $\mu_1 \preceq \mu_2$. For any $y < p$, define

$$f(x) = \begin{cases} \mu_1[y-x,\infty) & \text{if } x < y \\ 1 & \text{if } x \geq y \end{cases}$$

Then, by (33) and (74),

$$\begin{aligned}\rho(\mu_1)[y,\infty) &= \int_0^\infty f(x)d\mu_1(x) \leq \int_0^\infty f(x)d\mu_2(x) \\ &= \int_{[0,y)} \mu_1[y-x,\,\infty)d\mu_2(x) + \mu_2[y,\infty) \\ &\leq \int_{[0,y)} \mu_2[y-x,\,\infty)d\mu_2(x) + \mu_2[y,\infty) = \rho(\mu_2)[y,\infty)\end{aligned}$$

That is, $\rho(\mu_1) \preceq \rho(\mu_2)$. Also, by equations (34) and (35), $\lambda(\mu_1)[0,\infty) = \lambda(\mu_2)[0,\infty) = 1$, and for any $y > 0$,

$$\lambda(\mu_1)[y,\infty) = \mu_1[y+p,\infty) \leq \mu_2[y+p,\infty) = \lambda(\mu_2)[y,\infty)$$

we have $\lambda(\mu_1) \preceq \lambda(\mu_2)$. Hence, by (36) and (37), $\mu_1 \preceq \mu_2$ implies $T(\mu_1) \preceq T(\mu_2)$. ∎

The Proof of Lemma 11: For any $\mu \in \Delta^{M^*}$, $T^n(\mu)(p\mathbb{N}) \to 1$ as $n \to \infty$.

By Lemma 10, for a given $\mu \in \Delta^{M^*}$, $T^n(\mu)(p\mathbb{N})$ is nondecreasing in n. Since $T^n(\mu)(p\mathbb{N})$ is bounded by 1, it has a limit $\alpha \leq 1$. We want to show that $\alpha = 1$. Suppose to the contrary, $\alpha < 1$. Let $n^* \equiv \min\{n \,|\, np > 2M^*/(1-\alpha)\,\}$. Since for any $n \geq 0$, $T^n(\mu) \in \Delta^{M^*}$, or $\int_0^\infty \eta dT^n(\mu) = M^*$, we have $T^n(\mu)[n^*p,\,\infty) < (1-\alpha)/2$, which implies

$$\sum_{l=n^*}^{\infty} T^n(\mu)(lp,(l+1)p) < \frac{1-\alpha}{2}.$$

Correspondingly,

$$\sum_{l=0}^{n^*-1} T^n(\mu)(lp,(l+1)p) > \frac{1-\alpha}{2}. \tag{75}$$

Therefore, there exists $l^* < n^*$ such that for an infinite sequence $n_0, n_1, \ldots$, for any $k \geq 0$,

$$T^{n_k}(\mu)(l^*p,(l^*+1)p) > \frac{1-\alpha}{2n^*} \tag{76}$$

(otherwise, a contradiction of (75)). Without loss of generality, assume that for any $k \geq 0$, $n_{k+1} \geq n_k + l^* + 1$. Then, by Lemma 10 and (76), for any $k \geq 0$,

$$\begin{aligned}T^{n_{k+1}}(\mu)(p\mathbb{N}) &\geq T^{n_k+l^*+1}(\mu)(p\mathbb{N}) \\ &\geq T^{n_k}(\mu)(p\mathbb{N}) + 2^{-(2l^*+1)}\Big[T^{n_k}(\mu)(np,(n+1)p)\Big]^2 \\ &\geq T^{n_k}(\mu)(p\mathbb{N}) + 2^{-(2l^*+1)}\left[\frac{1-\alpha}{2n^*}\right]^2\end{aligned}$$

The assumption $\alpha < 1$ then implies that $\lim_{k\to\infty} T^{n_{k+1}}(\mu)(p\mathbb{N}) = \infty$, which contradicts the fact that for any $k \geq 1$, $T^{n_{k+1}}(\mu)(p\mathbb{N}) \leq 1$. Hence, $\alpha = 1$. ∎

The Proof of Lemma 12: For an arbitrary $\mu \in \Delta$, if there exists $\theta \in (0,1)$ such that $\mu^+ \preceq g^\theta$, then for any $n \geq 0$, $[T^n(\mu)]^+ \preceq g^\theta$.

Take an arbitrary $\mu \in \Delta$ and some $\theta \in (0,1)$ such that $\mu^+ \preceq g^\theta$. We first show that

$$[T(\mu)]^+ \preceq T(\mu^+). \tag{77}$$

Since both $[T(\mu)]^+$ and $T(\mu^+)$ have all their probability mass concentrated on $p\mathbb{N}$, we need to check (77) only on $p\mathbb{N}$. For any $k \geq 1$, the only transactions that would result difference between $[T(\mu)]^+\{kp\}$ and $T(\mu^+)\{kp\}$ are those trades between buyers with money holdings $\eta_b \in (0,p)$ and sellers with money holdings $\eta_s \in ((k-1)p, kp)$ such that $\eta_b + \eta_s < kp$. Such a trade will add the seller to the measure $[T(\mu)]^+\{kp\}$, but for $T(\mu^+)$, the same trade occurs as if it is between a buyer with $\eta_b = p$ units of money and a seller with $\eta_s = kp$ units of money, which increases the seller's money holdings to $(k+1)p$ rather than kp. That is, for all $k \geq 1$, $[T(\mu)]^+[(k+1)p, \infty) \leq T(\mu^+)[(k+1)p, \infty)$. It is easy to check that $[T(\mu)]^+\{p\} = T(\mu^+)\{p\}$. Therefore, $[T(\mu)]^+ \preceq T(\mu^+)$. Repeatedly applying (77), we have for any $n \geq 0$,

$$[T^n(\mu)]^+ \preceq T([T^{n-1}(\mu)]^+) \preceq T^2([T^{n-2}(\mu)]^+) \preceq \ldots \preceq T^n(\mu^+) \preceq g^\theta.$$

The last inequality is because $\mu^+ \preceq g^\theta$. ∎