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Abstract

Economists have recently revived the notion that recessions play a useful role in fostering innovation and growth. But in practice, a major source of innovation, R&D, is procyclical. In fact, R&D is procyclical even for firms that do not appear to be financially constrained. This paper argues the reason R&D is procyclical is because of a dynamic externality inherent to R&D that makes entrepreneurs short-sighted and concentrate their innovation in booms even though it is optimal to concentrate it in recessions. Thus, what previous authors have argued is a desirable feature of fluctuations in the previous literature – creating opportunities for intertemporal substitution – turns out to be a social liability in equilibrium.

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Introduction

In recent years, economists have revived the Schumpeterian idea that recessions promote various activities that contribute to long-run productivity.[1] Modern reincarnations of this hypothesis emphasize the role of intertemporal substitution: since the opportunity cost of investing in growth – the forgone output or sales that could have been obtained instead – is lower in recessions, there is more incentive to undertake such activities in downturns. Recessions should therefore stimulate long-run productivity growth, and cyclical fluctuations may raise welfare by allowing the economy to grow at a lower overall resource cost.

Although some growth-enhancing activities do appear to be countercyclical, one of the major sources of long-run productivity growth – research and development (henceforth R&D) – is clearly procyclical. That is, spending on R&D falls rather than rises in recessions, despite the fact that R&D is relatively less costly in downturns. Griliches (1990) summarizes the evidence for procyclical R&D activity. More recent work by Fatas (2000) reaffirms Griliches' conclusions. Still more recently, Comin and Gertler (2004) find that R&D is strongly procyclical at frequencies of between 8 and 50 years, suggesting sustained low output growth is associated with low R&D activity.

How can we reconcile this apparent conflict between theory and data? Is procyclical R&D inefficient, and if so is there a role for policy intervention? One hypothesis is that recessions do not really create opportunities for intertemporal substitution, e.g. because R&D does not use resources that could otherwise be used in production. Indeed, Aghion and Saint Paul (1998) show that if productivity-improving activities use final goods rather than factor inputs, innovation will be procyclical.[2] But Aghion and Saint Paul are also quick to dismiss this explanation. As Griliches (1984) observes, the main input into R&D is labor, not produced goods. Moreover, productivity in the goods sector is procyclical, even after correcting for variable utilization as in Burnside, Eichenbaum, and Rebelo (1993) and Basu (1996), whereas Griliches (1990) concludes productivity in the R&D sector (with patents as a proxy for output) is acyclical. This suggests that the forgone output from employing resources in R&D is indeed lower in recessions, and procyclical R&D may be socially inefficient.

[1] See, for example, Hall (1991), Mortensen and Pissarides (1994), and Gomes, Greenwood, Rebelo (2001) on the effects of recessions on search; Cooper and Haltiwanger (1993), Aghion and Saint Paul (1998), and Canton and Uhlig (1999) on the effects of recessions on technical change; and DeJong and Ingram (2001), Dellas and Sakellaris (2003), and Barlevy and Tsiddon (2004) on the effects of recessions on human capital accumulation.

[2] Following Rivera-Batiz and Romer (1991), this is known as the lab-equipment model. Comin and Gertler (2004) also assume R&D uses final goods, and note that this assumption helps to generate procyclical R&D.

So why would firms fail to concentrate R&D in recessions when it is relatively less costly? One natural explanation is that firms are somehow constrained from doing so. For example, Aghion, Angeletos, Banerjee, and Manova (2005) argue that credit constraints may discourage firms from undertaking growth-enhancing activities in recessions. In their model, the opportunity cost of R&D is lower in recessions, but downturns also reduce the amount of internal funds that firms can use to finance ongoing R&D projects. Aghion *et al* show that if firms had unlimited access to credit, they would choose to concentrate growth-enhancing activities in recessions. But if firms anticipate they will be constrained from borrowing, they may focus their R&D efforts in booms. This logic would suggest that facilitating borrowing in recessions should serve to restore efficiently countercyclical R&D.

While credit constraints are undoubtedly responsible for part of the cyclical pattern in R&D, there are reasons to suspect they cannot account for all of it. First, as I document below, R&D remains procyclical even for firms that are relatively unconstrained. Second, other investments in productivity that are just as vulnerable to credit constraints do not appear to be procyclical. For example, Francois and Lloyd-Ellis (2003) cite various studies that show firms concentrate reorganization, retraining, and machine upgrading in periods of weak demand. Since these activities can be just as costly as R&D, binding credit constraints would presumably lead these activities to also turn procyclical, a point Aghion *et al* (2005) argue explicitly. Thus, there appears to be something about R&D that makes it particularly prone to being procyclical, independently of credit market conditions.

This paper argues that the distinguishing feature of R&D that helps to explain its procyclicality is a dynamic externality inherent to the research process. In particular, when an innovator comes up with a new idea, she allows others to build on and further profit from her insights, often at her own expense. This is distinct from a firm upgrading machines to newer existing models, which improves its productivity but confers no spillovers to other firms.

It is already well-known from the work of Grossman and Helpman (1991) and Aghion and Howitt (1992) that if innovators cannot appropriate the spillovers from their research, there may be too little steady-state R&D activity. But there is also an important temporal aspect to this externality that previous work has ignored, and which holds the key to the procyclical bias in R&D. Since rival innovators are more likely to succeed in improving on a new idea the more time they have to refine and work on it, the benefits from a new idea that pay off in the future are increasingly more likely to accrue to someone other than the original innovator. Hence, the incentives to engage in R&D depend on the short-term benefits of successful innovation. Since profits are procyclical, innovators chasing short-term profits will undertake more R&D in booms than is socially optimal. If profits are sufficiently procyclical, R&D will turn procyclical, even though booms are precisely when the cost of R&D is highest.[3]

This paper demonstrates the above intuition in a general equilibrium model. The main difficulty lies not with establishing the procyclical distortion in R&D, but in getting the model to deliver sufficiently volatile profits. The standard Schumpeterian model predicts equilibrium R&D will exhibit a procyclical bias, but it also implies that profits are no more volatile than other macroeconomic series. As a result, the incentive to shift R&D towards booms will not be enough to turn R&D procyclical. However, when I modify the model to accord with the fact that profits are highly procyclical, namely by introducing fixed costs and insuring that markups are not too countercyclical, the absolute level of R&D will be higher in booms, even though the optimal path still dictates it should be countercyclical.

I then use the model to explore the welfare implications of such inefficiently procyclical R&D. I show that the distorted timing of R&D increases the cost of achieving productivity growth. Thus, cyclical fluctuations impose a social cost. This cost is three times as large as what Lucas (1987) estimated for the cost of business cycles based only on risk aversion.

The paper is organized as follows. Section 1 reviews the evidence on the procyclicality of R&D and argues it cannot be entirely explained by credit constraints. Section 2 formalizes the intuition for why equilibrium innovation suffers from a procyclical bias. Section 3 extends these results to a more realistic environment that can be calibrated to assess the welfare implications of procyclical R&D. Section 4 conjectures whether the results would survive additional modifications. Section 5 concludes.

1. Empirical Evidence

I begin by briefly reviewing the evidence on the cyclicality of R&D. As noted in the Introduction, the classic reference on the cyclical properties of R&D is Griliches (1990). He argues that

[3] At first, this result appears to reiterate the insight in Shleifer (1986) that enterpreneurs introduce new technologies in booms to capture high profits. However, Shleifer examines when firms *implement* new technologies, not when they undertake R&D. When Francois and Lloyd-Ellis (2004) endogenize innovation in Shleifer's model, they find it is countercyclical: entrepreneurs undertake innovation in recessions, but wait to implement their ideas in booms. Here, firms implement new ideas immediately. However, it will still be socially optimal to concentrate R&D in recessions, which creates strong incentives to concentrate R&D during recessions. The contribution of this paper is to explain why private entrepreneurs may concentrate R&D in booms despite these incentives.

both R&D spending and its output, patents, are procyclical.[4] Figure 1 reproduces some of these findings. In particular, it plots the growth rate of real GDP against the growth rate of two distinct measures of R&D. The first measure is the growth rate of inflation-adjusted expenditures on R&D funded and performed by private industry, as reported by the National Science Foundation (NSF). The second is the growth rate of full-time equivalent R&D scientists and engineers employed in companies performing R&D. The latter only captures part of the inputs into R&D, but it has the advantage that it doesn't depend on the price of R&D inputs. This is important, since inflation adjustments to nominal R&D expenditures may not accurately reflect changes in the prices of R&D inputs.

As evident from the figure, the two series track each other relatively closely, and both tend to track real GDP growth. In particular, the growth rate of R&D declines in nearly all NBER recession years before 1980. However, R&D appears to be less synchronized with NBER recession years in the second half of the sample. For example, growth in R&D was essentially flat through the 1980 recession. While growth in R&D employment fell in the 1991 recession, growth in R&D expenditures only started to decline after the recession ended. Conversely, although R&D expenditures fell dramatically during the 2001 recession, employment growth in R&D began to decline several years earlier. But even if the exact timing does not correspond to NBER dating conventions between 1980 and 2002, R&D growth and GDP growth are positively correlated over this period.

Figure 1 uses aggregate data. To gain some insight on the role of credit constraints in accounting for the procyclical pattern, I turn to firm level data on R&D expenditures. In particular, I examine whether firms that are relatively less credit constrained have less procyclical R&D expenditures. In this regard, the Standard & Poor's Compustat database provides data on R&D expenditures for publicly traded companies. Although this sample does not capture all of the R&D activity in the NSF data, it turns out that large firms account for the vast majority of R&D activity, and thus tracks the aggregate time series quite well. Figure 2 reports the average growth rate of real R&D over the previous year among all Compustat firms reporting positive R&D spending, together with inflation adjusted private R&D as reported by the NSF and depicted in Figure 1. The two series again track each other quite closely; if anything, R&D growth is more systematically correlated with NBER recession dates among Compustat firms.

[4] The fact that patents are synchronized with the business cycle might seem surprising given the time it presumably takes to undertake research. However, Griliches argues that micro data shows R&D leads to patenting without significant delay: "[T]he evidence is quite strong that when a firm changes its R&D expenditures, parallel changes occur also in its patent numbers. The relationship is close to contemporaneous with some lag effects which are small and not well estimated (Hall, Griliches, and Hausman, 1986). This is consistent with the observation that patents tend to be taken out relatively early in the life of a research project." (p1674).

I consider two ways of identifying firms that are relatively unconstrained financially. One is based on the cash flow available to the firm, which would mitigate the need to borrow externally in order to finance expenditures. That is, I look at whether the growth rate of R&D expenditures for firms that report at least $50 million of cash (in 1996 dollars) in the year in which they undertake R&D appears less procyclical. This corresponds to the top quintile of the firms in my sample as ranked by their cash flow within each year. As an alternative indicator of constrainedness, I use the net worth of a firm, given that net worth can be used as collateral against which the firm can borrow. That is, I look at the cyclicality of the growth rate of R&D expenditures for firms reporting at least $150 million in net worth (in 1996 dollars) in the year they undertake R&D. Not surprisingly, there is a fair degree of overlap in the two samples. Figure 3 illustrates the growth rate of real R&D expenditures for these two groups. The results are striking: the growth rate of R&D among these relatively unconstrained firms is actually *more* synchronized with the business cycle than for the sample as a whole: the correlation with real GDP growth is higher, and the average growth rate of R&D falls in each NBER recession. Changing the cutoff levels does not change these qualitative results.

It would therefore appear that even firms that are relatively free to concentrate their R&D activity in downturns choose not to do so. But since the opportunity cost of innovation appears to be countercyclical, we need to explain why firms would deliberately pursue such a policy.

2. A Model of Schumpeterian Growth

For my model, I use a variation of the Grossman and Helpman (1991) quality-ladder model that allows for fluctuations in the relative productivity of the goods sector. To maintain tractability, I initially impose certain restrictions on preferences and technology that are unrealistic but convey the intuition more transparently. I then relax these in the subsequent section.

The economy consists of a representative agent whose instantaneous utility is given by

$$U(C_t) = C_t \tag{2.1}$$

I relax the assumption of risk neutrality in the next section. As will become clear, this assumption plays a role in the analysis, but is not essential. Utility is discounted at rate ρ.

The agent is endowed with a constant labor endowment L pet unit time and an initial capital stock normalized to one. For now, I assume capital is not accumulable and does not depreciate, i.e. it is a fixed factor (e.g. land). This assumption will also be relaxed in the next section.

Labor and capital can be converted into consumption goods according to a two-stage process. First, labor is converted into a series of intermediate goods indexed by $j \in [0,1]$. Second, intermediate goods are combined with capital to produce a non-storable consumption good.

At the second stage, I assume the intermediate goods x_{jt} must first be assembled into a composite good, whose quantity I denote by X_t, using a Cobb-Douglas technology

$$X_t = \exp\left[\int_0^1 \ln x_{jt} dj\right] \tag{2.2}$$

Given X_t units of this composite good and K_t units of capital, it is possible to produce Y_t units of the consumption good, where

$$Y_t = z_t K_t^{\alpha} X_t^{1-\alpha} \tag{2.3}$$

Here z_t reflects productivity in the final goods sector. To capture the fact that productivity in the goods sector varies over the business cycle, I assume z_t follows a Markov switching process between two states, $Z_1 \geq Z_0$, with a constant hazard rate μ. I treat these fluctuations as exogenous, although one could potentially derive them endogenously.[5]

Turning next to the production for intermediate goods, I assume each good j can be produced from labor according to a linear technology

$$x_{jt} = \lambda_{jt} L_{jt} \tag{2.4}$$

where L_{jt} denotes the amount of labor employed in the production of good j at date t. The coefficient λ_{jt} is given by

$$\lambda_{jt} = \lambda^{m_{jt}} \tag{2.5}$$

where $\lambda > 1$ is a constant and m_{jt} is an integer that denotes the generation of technology used for producing good j at date t. Each good j starts out at generation m_{j0}, respectively, but agents can advance to higher-generation technologies by engaging in research. That is, starting with generation m_j, devoting R_j units of labor to research on good j gives rise to a hazard ϕR_j of discovering generation $m_j + 1$ in the next instant, which will be more productive given $\lambda > 1$. Once a new generation is discovered, research can begin on the next generation. This last assumption captures the spillovers inherent to research: when one researcher succeeds in discovering a new generation, she allows others to build on her work and develop the next successive technology. In line with the evidence on the acyclicality of productivity in the research sector, I assume ϕ is fixed over time.

[5] For example, Benhabib and Farmer (1994) describe an economy with spillovers in which there are equilibria where the scale of production, and thus the productivity of individual producers, fluctuates over time.

To recap, labor in this economy has two uses: production and innovation. Agents must therefore choose between producing more consumption goods now and employing labor in research activities that allow for more consumption goods to be produced in the future.

To see this tradeoff formally, let $M_t = \int_0^1 m_{jt} dj$ denote the average generation across intermediate goods, and let $R_t = \int_0^1 R_{jt} dj$ denote aggregate employment in R&D. We can now express the output of consumption goods Y_t directly in terms of labor resources. In particular, suppose each sector uses the same amount of labor, i.e. $L_{jt} = L - R_t$, which is both optimal and holds in any equilibrium. Since the supply of capital is normalized to 1, it follows that

$$Y_t = z_t K^{\alpha} X_t^{1-\alpha} = z_t \left[\lambda^{M_t} (L - R_t)\right]^{1-\alpha} \tag{2.6}$$

The indirect productivity of labor *in terms of final goods* thus depends on both an exogenous term z_t and an endogenous term $\lambda^{(1-\alpha)M_t}$. We assume the law of large numbers holds across intermediate goods producers, which implies $\dot{M}_t = \phi R_t$. The growth rate of the endogenous component of labor productivity is therefore given by

$$\frac{d}{dt}\lambda^{(1-\alpha)M_t} = (1-\alpha)\dot{M}_t \ln\lambda = (1-\alpha)\phi R_t \ln\lambda \tag{2.7}$$

Equations (2.6) and (2.7) show the essential tradeoff: faster growth requires a higher R_t, which leaves fewer resources to produce goods in the current instant.[6] Note that with risk-neutrality, the utility of the agent is finite only if the growth rate does not exceed the discount rate ρ, so an optimal policy exists only if utility is bounded for any feasible innovation, i.e.

$$\rho > (1-\alpha)\phi L \ln\lambda \tag{2.8}$$

It will not be necessary to restrict ρ this way when I allow for curvature in the utility function in the next section.

In the next two subsections, I solve for how the optimal and equilibrium paths of R_t vary with z_t. In particular, I show that equilibrium R&D suffers from a procyclical bias. However, due to the model's counterfactually low profit volatility, equilibrium R&D remains countercyclical. In the last subsection, I modify the model to correct its counterfactual implication and show that when profits are sufficiently volatile, the optimal path remains countercyclical but equilibrium innovation will turn procyclical.

[6] One might ask whether specialized labor employed in R&D is really substitutable for production workers at high frequencies. While some R&D expenditures involve scientists and engineers who may not be easily shifted to production, NSF data suggests that on average 40% of wage payments in R&D is allocated to support staff.

2.1. The Social Planner's Problem

The neo-Schumpeterian view argues it is desirable to concentrate innovation in periods of low productivity in the goods sector, since it reduces the overall cost of achieving a given average rate of productivity growth. Solving the planning problem that maximizes the utility of the agent confirms this intuition. Formally, let Z_i for $i \in \{0,1\}$ denote the initial level of productivity, and recall that M_0 denotes the initial value of the average generation across all goods. The expected utility of the agent under the optimal path starting from $z_0 = Z_i$ is given by

$$V_i(M_0) = \max_{R_t} E\left[\int_0^\infty z_t \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} e^{-\rho t} dt \;\middle|\; z_0 = Z_i\right] \tag{2.9}$$

subject to the constraint

$$\dot{M}_t = \phi R_t$$

We can rewrite (2.9) recursively as

$$\rho V_i(M) = \max_{R \in [0,L]} \left\{ Z_i \left[\lambda^M (L - R)\right]^{1-\alpha} + \mu\left(V_{1-i}(M) - V_i(M)\right) + \frac{\partial V_i}{\partial M}\phi R \right\} \tag{2.10}$$

Given the stationarity of the environment, the planner will choose a constant level of employment R for a given Z_i. Thus, finding the optimal policy reduces to finding a pair of numbers (R_0, R_1). Note that this solution does not specify how innovation varies across intermediate goods j, so wlog we can assume $R_j = R$ for all $j \in [0,1]$. I now demonstrate the existence of an optimal path and argue that it undertakes more innovation when productivity in the final goods sector is low. It is a special case of the more general Proposition 3 below. The proof of that proposition, along with those of all remaining propositions, is contained in an Appendix.

Proposition 1: If (2.8) is satisfied, there exists a unique solution to the social planner's problem, and innovation is (weakly) countercyclical along the optimal path, i.e. $R_0 \geq R_1$.

Note that while a countercyclical policy allows the economy to achieve growth at a lower cost, it also makes output more volatile: fewer inputs will be allocated to production precisely when productivity is already low. This is irrelevant given the agent is assumed to be risk neutral. But it may make countercyclical R&D undesirable when the agent is risk averse. I will return to this issue in the next section.

2.2. Decentralized Equilibrium

I now turn to the decentralized equilibrium of this economy. All goods – both intermediate and final – are produced by profit-maximizing firms. The technology for producing final goods is freely available, so profits in this sector will equal zero in equilibrium. By contrast, intermediate goods producers enjoy some market power: the entrepreneur who discovers the m-th generation for producing good j earns a patent that grants him exclusive rights to use this technology. Since no firm would undertake innovation without patent protection, some monopoly power is necessary in this environment in order to sustain growth.

In what follows, I focus on equilibria where R_j is the same across all goods j, and where their common value only depends on the value of aggregate productivity Z_i. Formally, I restrict attention to symmetric Markov perfect equilibria. This is natural given these features are true for the optimal path. Solving for an equilibrium proceeds in several steps. Briefly, I first express the equilibrium profits of intermediate goods producers π in terms of aggregate R&D employment R. As an important aside, I observe that equilibrium profits are essentially as volatile as wages, an important but counterfactual implication of the model. I then use the expression for profits π to express the value of a successful innovation, v, strictly in terms of (R_0, R_1), aggregate employment in R&D for the two respective levels of aggregate productivity. The uninterested reader can skip ahead to Proposition 2.

I begin by solving for the price p_{jt} the producer of intermediate good j would charge. Given the Cobb-Douglas aggregator X, the demand of final goods producers for each intermediate good j will be unit elastic. Thus, each intermediate-goods producer would want to charge as high a price as possible: his revenue will be constant regardless of the price he charges, but at higher prices he can produce fewer goods and lower his costs. However, if he were to charge more than the marginal cost of his next most efficient competitor, the latter could steal away his business. Thus, in equilibrium, only the monopolist with the most productive technology will supply goods, at a price p_{jt} equal to the marginal cost of his most efficient competitor. As Grossman and Helpman observe, incumbent producers benefit less from extending their lead than new entrants do from overtaking the lead, so only entrants engage in innovation in equilibrium. Hence, the next most efficient producer will use the $(m_{jt} - 1)$-th generation technology.[7] Normalizing the wage to 1, the marginal cost of the next most efficient producer is $\lambda^{-(m_{jt}-1)}$, the number of labor units he requires to produce a unit of good j.

Let $e_{jt} = p_{jt} x_{jt}$ denote total expenditures by final goods producers on intermediate good j. Given the Cobb-Douglas specification for X, final goods producers will equalize expenditures

[7] As in Grossman and Helpman (1991), this requires that a firm's R&D expenditures on a particular intermediate good are unobservable, so an incumbent has no incentive to undertake R&D to discourage entry.

across intermediate good j, i.e.

$$e_{jt} = (1-\alpha) P_t Y_t \equiv e_t$$

where P_t denotes the price of the final good. With the wage normalized to 1, the cost of production is just the number of employed workers $\lambda^{-m_{jt}} x_{jt}$. Since $x_{jt} = e_t/p_{jt}$, this cost is equal to $\lambda^{-1} e_t$. Hence, the profits of the incumbent firm that supplies good j are given by

$$\pi_{jt} = e_t - \lambda^{-1} e_t = \left(1-\lambda^{-1}\right)(1-\alpha) P_t Y_t$$

Profits are thus the same for all goods j, i.e. $\pi_{jt} = \pi_t$ for all j. To express these profits in terms of R_t, we use the fact that total spending on consumption goods must equal the income of the representative agent in equilibrium. Thus, $P_t Y_t$ equals the sum of aggregate profits Π_t and payments to factors,

$$\begin{aligned} P_t Y_t &= \Pi_t + r_t K + L \\ &= \int_0^1 \pi_t dj - R_t + r_t K + L \end{aligned} \tag{2.11}$$

where r_t denotes the rental rate of capital at date t. Substituting in for π_t from above and using the fact that cost minimization by final goods producers implies $r_t K = \alpha P_t Y_t$ allows us to express equilibrium profits π_t in terms of research employment R_t:

$$\pi_t = (\lambda - 1)(L - R_t) \tag{2.12}$$

Note that nominal profits do not depend on z_t. Thus, for a fixed level of innovation R, profits are just as cyclically volatile as the numeraire good, labor. As noted above, this counterfactually implies that the volatility of profits is commensurate with the volatility of the cost of R&D. This implication will figure prominently below.

Entrepreneurs who succeed in innovation earn profits (2.12) as long as their technology is the most advanced. To calculate the value of a successful innovation, let $\mathbb{I}_{jt}$ denote an indicator which equals 1 if the entrepreneur is the leading-edge producer of good j and zero otherwise, and let v_j denote the value to a claim on the profits of a successful innovation at date 0. Since the representative agent owns all claims in equilibrium, the price v_j must leave him indifferent between buying and selling an additional claim. This indifference condition implies

$$\begin{aligned} v_j &= E\left[\int_0^\infty \mathbb{I}_{jt} \cdot \frac{U'(C_t)/P_t}{U'(C_0)/P_0} \pi_t e^{-\rho t} dt \;\middle|\; z_0\right] \\ &= E\left[\int_0^\infty \mathbb{I}_{jt} \cdot \frac{P_0}{P_t} \pi_t e^{-\rho t} dt \;\middle|\; z_0\right] \end{aligned} \tag{2.13}$$

where the expectation above is taken over all possible paths for z_t and $\mathbb{I}_{jt}$. Non-incumbent firms choose R_j to maximize the expected value from a successful innovation net of R&D costs

$\phi R_j v_j - R_j$. It follows that $\phi v_j \leq 1$ in equilibrium, with strict equality if $R_j > 0$. Recall that I restrict attention to equilibria in which $R_{jt} = R_t$ for all j. This implies the value of a successful innovation is the same for each intermediate good j, i.e. $v_j = v$ for all $j \in [0,1]$.

To express v in terms of R_0 and R_1, the values that R_t assumes when $z_t = Z_0$ and Z_1, respectively, I need to express the price of final goods P_t in terms of R_t. Since the production of final goods is competitive, the equilibrium price P_t equals the minimum cost to produce a single unit of the good in equilibrium, i.e.

$$P_t = \min_{x_{jt}, K_t} \left\{ \int_0^1 p_{jt} x_{jt} dj + r_t K_t \right\}$$
$$\text{s.t. } z_t K_t^{\alpha} \left(\exp \left[\int_0^1 \ln x_{jt} dj \right] \right)^{1-\alpha} = 1$$

Using the fact that $p_{jt} = \lambda^{-(m_{jt}-1)}$ and $r_t = \alpha P_t Y_t$, one can show that

$$P_t = \frac{\lambda (L - R_t)^{\alpha}}{(1-\alpha) z_t \lambda^{(1-\alpha)M_t}} \tag{2.14}$$

Let v_i denote the value of a successful innovation if initial productivity $z_0 = Z_i$. Substituting in the above expression P_t into (2.13) yields

$$v_i = E \left[\int_0^{\infty} \mathbb{I}_t \cdot \frac{z_t \lambda^{(1-\alpha)M_t}}{z_0 \lambda^{(1-\alpha)M_0}} \left(\frac{L - R_0}{L - R_t} \right)^{\alpha} (\lambda - 1) (L - R_t) e^{-\rho t} dt \,\middle|\, z_0 = Z_i \right] \tag{2.15}$$

Next, for any z_t-measurable function $X(\cdot)$, the value of the integral

$$W_i (M_0) = E \left[\int_0^{\infty} \mathbb{I}_t \cdot \lambda^{(1-\alpha)M_t} X(z_t) e^{-\rho t} dt \,\middle|\, z_0 = Z_i \right]$$

subject to $\dot{M}_t = \phi R_t$ can be characterized by the recursive equation

$$(\rho + \mu) W_i (M) = \lambda^{(1-\alpha)M} X(Z_i) + \mu W_{1-i}(M) + \left[\frac{\partial W_i}{\partial M} - W_i(M) \right] \phi R_i$$

The method of undetermined coefficients confirms that $W_i(M) = w_i \lambda^{(1-\alpha)M}$ where

$$w_i = \frac{\omega(R_{1-i}) X(Z_i) + \mu X(Z_{1-i})}{\omega(R_i)\omega(R_{1-i}) - \mu^2}$$

and

$$\omega(R) = \rho + \mu + (1 - (1-\alpha)\ln\lambda)\phi R$$

The value of a successful innovation v_i can thus be expressed in terms of R_0 and R_1:

$$v_i(R_i, R_{1-i}) = (\lambda - 1) \frac{\omega(R_{1-i})(L - R_i) + \mu \frac{Z_{1-i}}{Z_i} (L - R_{1-i})^{1-\alpha} (L - R_i)^{\alpha}}{\omega(R_i)\omega(R_{1-i}) - \mu^2} \tag{2.16}$$

A symmetric Markov-perfect equilibrium is any pair (R_0, R_1) in $[0, L]^2$ where $\phi v_i(R_i, R_{1-i}) \leq 1$, with strict equality if $R_i > 0$. The next proposition characterizes these equilibria, and provides a sufficient condition for a unique equilibrium to exist.[8]

Proposition 2: Innovation along the equilibrium path is weakly countercyclical in any symmetric Markov-perfect equilibrium, i.e. $R_0 \geq R_1$ along any equilibrium path. Moreover, if $\lambda < e^{\frac{1}{1-\alpha}}$, there exists a unique symmetric Markov-perfect equilibrium.

At first glance, these results seem to deny the intuition presented in the Introduction: despite the presence of spillover effects, equilibrium R&D is countercyclical. However, as I now argue, the problem lies not with the intuition but with the inability of the model to generate sufficiently procyclical profits. In particular, there is a precise sense in which the decentralized economy exhibits a procyclical bias in R&D. It is only because profits are not sufficiently procyclical in the model that this bias is not enough to turn R&D procyclical.

To appreciate why R&D in the market economy suffers from a procyclical bias, let us consider the ratio of the real value of a successful innovation in a boom to its real value in a recession. This ratio is of interest because the higher it is, the more incentive entrepreneurs have to undertake their R&D in booms than in recessions. To compute this ratio in equilibrium, we divide the value of a successful innovation v_i in (2.15) by the equilibrium price P of final goods in (2.14), evaluated at $z_t = Z_i$. This ratio reduces to

$$\frac{v_1/P_1}{v_0/P_0} = \frac{E\left[\int_0^\infty \mathbb{I}_t \cdot z_t \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} e^{-\rho t} dt \;\middle|\; z_0 = Z_1\right]}{E\left[\int_0^\infty \mathbb{I}_t \cdot z_t \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} e^{-\rho t} dt \;\middle|\; z_0 = Z_0\right]} \tag{2.17}$$

where $\mathbb{I}_t$ is an indicator that is equal to 1 if the entrepreneur at date 0 remains the leading edge producer by date t. For the social planner, the analogous value for a successful innovation when $z_t = Z_i$ is given by $\partial V_i/\partial M$, where $V_i(M)$ is given by (2.10). Using the solution to the planner's problem in the Appendix, the ratio of the two values is given by

$$\frac{\partial V_1/\partial M}{\partial V_0/\partial M} = \frac{E\left[\int_0^\infty z_t \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} e^{-\rho t} dt \;\middle|\; z_0 = Z_1\right]}{E\left[\int_0^\infty z_t \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} e^{-\rho t} dt \;\middle|\; z_0 = Z_0\right]} \tag{2.18}$$

[8] Canton and Uhlig (1999) show that equilibrium innovation is countercyclical in a similar model. Aghion and Saint Paul (1998) also show equilibrium innovation is countercyclical, but their model assumes only one firm undertakes all innovation, so there is no externality that could lead to procyclical innovation.

Now, suppose we tried to implement the optimal program and set $R_0 > R_1$ in the decentralized market. Under this program, more output will be produced in booms than in recessions, since

$$Z_1 \left[\lambda^{M_t} (L - R_1)\right]^{1-\alpha} > Z_0 \left[\lambda^{M_t} (L - R_0)\right]^{1-\alpha} \tag{2.19}$$

Since the probability that $\mathbb{I}_t = 1$ decreases with t, the integrals in (2.17) assign more weight to output at dates close to $t = 0$ than the integrals in (2.18). Combined with the fact that z_t is mean-reverting, it follows that the ratio in (2.17) is higher than the ratio in (2.18). Thus, if we tried to implement the optimal path in the decentralized economy, entrepreneurs would assign too much value to innovations in booms relative to recessions, and they will have incentive to deviate from the optimal path and instead concentrate more of their R&D in booms. This is the inherent procyclical bias of R&D in the decentralized economy.[9]

However, this bias is not enough to turn R&D procyclical. As noted above, this is because equilibrium profits in the model are only as volatile as the cost of R&D: in a recession, both the cost of R&D and profits fall in proportion to z_t. But the expected discounted value of *future* profits falls by less than today's profits, since z_t is stationary and future profits are expected to eventually revert to their unconditional mean. Hence, the value of a successful innovation falls by less than the cost of R&D, giving incentive to undertake more innovation in recessions. For R&D to turn procyclical, the value of a successful innovation must fall by more than the cost of R&D in recessions, which in turn requires profits to fall by more than the cost of R&D.

Empirically, of course, profits are far more volatile than the cost of R&D. Mansfield (1987) constructs R&D price indices from 1969 to 1983 and finds the R&D deflator is closely synchronized with the GDP deflator at high frequencies, implying real R&D costs are not very cyclical. This is not surprising given R&D is labor intensive and real wages are only mildly procyclical. The remainder of this section modifies the model so it can accord with the larger relative volatility of profits.

2.3. Fixed Costs and the Volatility of Profits

In analyzing the behavior of profits over the business cycle, Ramey (1991) argues an important piece of evidence is the strongly procyclical pattern in the ratio of aggregate profits to aggregate

[9] The inefficiency concerns relative rather than absolute values of R&D; that is, the tendency towards procyclical R&D in independent of whether the overall level of R&D is too high or too low relative to the socially optimal level. Hence, this inefficiency is distinct from the one emphasized by Grossman and Helpman (1991) and Aghion and Howitt (1992) concerning the potential inefficiency of steady-state growth.

sales. By contrast, in the model, this ratio is given by

$$\frac{\pi_t}{e_t} = 1 - \frac{1}{\lambda}$$

which does not vary with z_t. In principle, we could make profits more volatile than sales in the model by making the markup λ_t procyclical. However, empirical evidence summarized in Rotemberg and Woodford (1999) suggests markups are moderately countercyclical. This observation leads Ramey to conclude that the reason observed profits are so volatile over the business cycle is the presence of fixed costs of production. To see why, suppose producers of intermediate goods had to pay some fixed amount F to initiate production. In this case, the profit-to-sales ratio equals

$$\frac{\pi_t}{e_t} = \left(1 - \frac{1}{\lambda_t}\right) - \frac{F}{e_t} \tag{2.20}$$

As long as F is constant and the markup λ_t is not too countercyclical, the ratio of profits to sales will increase with sales e_t. Thus, to properly reconcile the model with evidence of highly volatile profits, we need to introduce fixed costs of production, and at the same time ensure that markups are not too countercyclical. The remainder of this section modifies the model to incorporate these two features, and confirms that equilibrium R&D will indeed turn procyclical when profits are sufficiently more volatile than the cost of R&D.

The first step is to introduce fixed costs of production. Suppose that in order to produce any intermediate good $j \in [0,1]$, a producer must first purchase F units of the final good. To insure that the economy doesn't outgrow this cost, we need to further scale this fixed cost so that it grows at the same rate as the economy. Let us therefore assume that at date t, the amount of the final good required to initiate production is equal to $\lambda^{(1-\alpha)M_t}F$. The notion that fixed costs grow with the rest of the economy seems plausible; for example, overhead labor will naturally become more expensive as overall labor productivity increases.[10]

The next step is to ensure markups are not too countercyclical. It turns out that introducing fixed costs of production leads to strongly countercyclical markups. To see why, note that since demand for each intermediate good is unit elastic, the markup an intermediate goods producer charges is the gap between his own cost and the price at which his next most efficient competitor

[10] This begs the question why I did not model the fixed cost directly in terms of labor. The reason is that in a frictionless model, the price of labor changes with z_t. As a result, the cost of overhead labor rises in booms, cutting into profits and preventing them from rising too much. Empirically, wages do not appear to respond much to short term variations over the business cycle, although they certainly grow over longer time horizons. Assuming the fixed cost is denominated in final goods is a way of capturing rigidity in the salaries of overhead labor without unnecessarily complicating the model. Since I assume the same fixed cost when I solve the planner's problem, the planner will take this rigidity into account when choosing R&D.

breaks even. In a boom, any competitor would earn proportionately higher gross profits at any given price. At the same time, the fixed cost the competitor faces would remain unchanged. To keep the rival from entering in a boom, then, an intermediate goods producer would have to lower his price. In the model, this fall in the markup is large enough so that profits would only be as volatile as sales, despite the presence of a fixed cost. We therefore need to modify the model to prevent markups from being so countercyclical. In other words, we need to separate between the price an incumbent charges and the price at which the producer using the previous generation technology breaks even.

One reason the two prices might differ is that new ideas can be partly imitated, and producers have to set prices to also deter entry from imitators. That is, suppose entrepreneurs can engineer knock-off versions of the latest generation of any technology that, while inferior to the leading-edge technology, are more profitable than the best version of the previous generation. As long as the price at which an imitator breaks even is not too countercyclical, profits will be more volatile than sales. The latter will be true if knock-off versions involve lower fixed costs but higher variable costs than the leading edge technology. This assumption is plausible; after all, imitators would not incur the costs of patent protection that a leading-edge producer incurs, and inferior knock-offs would presumably involve higher variable costs of production than the technology they imitate. In what follows, I assume the inferior version of each technology is such that imitators break even at a price of λ times the marginal cost of the leading technology, implying a constant markup.[11] This is a convenient simplification; we would obtain similar results with moderately countercyclical markups.

Before examining the effects of more volatile profits on equilibrium R&D, let us first consider how these modifications affect the planner's problem. Since the planner would always employ the leading technology, the presence of inferior knockoffs is irrelevant. However, the planner will react to the presence of fixed costs. The analog to equation (2.10) now corresponds to

$$\rho V_i(M) = \max_{R\in[0,L]} \left\{ \begin{array}{l} \lambda^{(1-\alpha)M}\left[Z_i(L-R)^{1-\alpha} - F\right] + \\ \qquad \mu\left(V_{1-i}(M) - V_i(M)\right) + \dfrac{\partial V_i}{\partial M}\phi R \end{array} \right\} \tag{2.21}$$

That is, the planner takes into account that a fixed amount of the output produced must be spent on overhead. While this changes the optimal *level* of innovation, there is no reason to expect it will affect the *timing* of innovation with respect to z_t. The next proposition confirms that the optimal path is indeed countercyclical, at least at an interior optimum.

[11] An example of such a knock-off technology is one where the variable cost is λ times the variable cost under the new technology and which requires no fixed cost of production. This technology is clearly more profitable than the leading-edge version of the previous generation.

Proposition 3: If (2.8) is satisfied, there exists a unique solution to the social planner's problem. If the optimal level of R&D is always positive, then innovation is countercyclical along the optimal path, i.e. $R_0 > R_1$. For small F, the optimal path is weakly countercyclical even if the optimal path involves periods of zero innovation, i.e. $R_0 \geq R_1$.

I now turn to the decentralized economy. Again, the uninterested reader can skip ahead to Proposition 4. Since my assumptions imply $p_{jt} = \lambda^{-(m_{jt}-1)}$, profits will equal

$$\pi_{jt} = \left(1 - \lambda^{-1}\right) e_t - \lambda^{(1-\alpha)M_t} P_t F \tag{2.22}$$

Once again, we can express profits directly in terms of R_t using the aggregate resource constraint. However, the original constraint in (2.11) must be revised to reflect the fact that the household only purchases those goods that are not used by intermediate goods producers, i.e.

$$P_t \left(Y_t - \lambda^{(1-\alpha)M_t} F\right) = \Pi_t + r_t K + L \tag{2.23}$$

Substituting this aggregate resource constraint into the expression for profits yields

$$\pi_t = (\lambda - 1)(L - R_t) - \lambda^{(1-\alpha)M_t} P_t F \tag{2.24}$$

instead of (2.12). Since P_t varies over the cycle, profits π_t will be more volatile than the price of the numeraire good. Solving for P_t, which turns out to be identical to the expression in (2.14), we obtain the following as the value of a successful innovation:

$$v_i(R_i, R_{1-i}) = (\lambda - 1) \frac{\omega(R_{1-i})(L - R_i) + \mu \frac{Z_{-i}}{Z_i}(L - R_{1-i})^{1-\alpha}(L - R_i)^{\alpha}}{\omega(R_i)\,\omega(R_{1-i}) - \mu^2} - \frac{\omega(R_{1-i}) + \mu}{\omega(R_i)\,\omega(R_{1-i}) - \mu^2} \frac{\lambda (L - R_i)^{\alpha} F}{(1-\alpha) Z_i} \tag{2.25}$$

Since profits are now more volatile than the cost of R&D, we would expect equilibrium R&D can turn procyclical. To confirm this conjecture, I begin with the following lemma:

Lemma: Suppose $\lambda < e^{\frac{1}{1-\alpha}}$. Then for any $F > 0$, there exists a unique $R^* < L$ such that $\phi v_0(R^*, R^*) = \phi v_1(R^*, R^*)$. Moreover, there exists a $F^* > 0$ such that $\phi v_i(R^*, R^*) < 1$ for $F < F^*$ and $\phi v_i(R^*, R^*) > 1$ for $F > F^*$.

The lemma above establishes that for any fixed cost F, there is a unique level of innovation R^* that leaves the nominal value of a successful innovation v constant over the cycle. From the proof of the lemma, one can show that this value $v(R^*, R^*)$ increases with F and ranges from zero to infinity, so there must be some F^* for which it equals $1/\phi$. As the next proposition establishes, if F is greater than F^*, we are assured of finding a pair (R_0, R_1) where $R_1 > R_0$ and which satisfies the condition that $\phi v_i (R_i, R_{1-i}) = 1$ for both $i \in \{0, 1\}$.

Proposition 4: Suppose $\lambda < e^{\frac{1}{1-\alpha}}$. If $F > F^*$, where F^* is defined in Lemma 2 , there exists a pair $R_0 < R_1$ such that $\phi v_i (R_i, R_{1-i}) = 1$ as required of an interior equilibrium.

Proposition 4 suggests that for sufficiently large fixed costs, equilibrium innovation will covary positively with productivity.[12] More precisely, it states that for large enough fixed costs we will always be able to find a solution for the system of equations that characterize an interior equilibrium such that $R_1 > R_0$. However, this does not guarantee that the solution (R_0, R_1) lies in $[0, L]^2$ as required of an interior equilibrium. Numerically, though, the solution does appear to lie in the interior of $[0, L]^2$ for large L, mirroring a result in Grossman and Helpman (1991) for the case of no fixed cost. As to whether this equilibrium is unique, the set $\{(R_0, R_1) \mid \phi v_i (R_i, R_{1-i}) = 1\}$ can have multiple solutions. However, these additional solutions do not appear to correspond to equilibria; rather, they appear to involve very high levels of innovation (close to L) for which the revenue of intermediate goods producers does not cover their fixed costs. Experimenting with several parameter values always led to a unique symmetric Markov equilibrium that was countercyclical if $F < F^*$ but procyclical if $F > F^*$.

In sum, when rivals are likely to steal away future profits, entrepreneurs act short-sightedly when they undertake R&D. They will therefore tend to undertake too much R&D in periods of high profits than is socially optimal. For larger fixed costs of production, which imply more volatile equilibrium profits, relatively more R&D will be shifted towards booms, until eventually R&D turns procyclical. However, even for large fixed costs, the optimal path continues to dictate that R&D be countercyclical.

An important practical question is whether empirically plausible fixed costs are enough to account for the procyclical pattern of R&D in the data, and if so how costly is this procyclicality of R&D. To address these questions, I need to relax some of the assumptions above to make the model more amenable to quantitative analysis. This is precisely what I do in the next section.

As an aside, it is worth noting here that although unrealistic, the assumption of risk neutrality does highlight some of the model's stark welfare implications. Consider the implied welfare cost of volatility in the model, i.e. the cost of moving from an economy with constant productivity

[12] One has to be careful about referring to this variation as procyclical, since it is possible that output will fall when productivity Z rises. In all of my numerical simulations, though, output comoves with productivity.

$\overline{Z}$ to one in which z_t fluctuates between Z_0 and Z_1 where $E[z_t] = \overline{Z}$. In the stable environment, the optimal path mandates a constant level of R. Under risk neutrality, the planner can always achieve the same expected utility in the stochastic environment by adopting the same R, since at any date t

$$\begin{aligned} E\left[\lambda^{(1-\alpha)M_t}\left(z_t(L-R)^{1-\alpha}-F\right)\right] &= \lambda^{(1-\alpha)M_t}\left(E[z_t](L-R)^{1-\alpha}-F\right) \\ &= \lambda^{(1-\alpha)M_t}\left(\overline{Z}(L-R)^{1-\alpha}-F\right) \end{aligned}$$

Since Proposition 3 tells us the optimal (interior) path will vary R with z_t, it follows that the planner can achieve an even higher utility. By contrast, in the decentralized economy, welfare can be lower in the volatile environment than in the stable one. Hence, cyclical fluctuations can reduce welfare *even when they allow a benevolent planner to achieve a higher utility.* This cost is due to the suboptimal use of resources in response to time-varying productivity, and is distinct from the cost of consumption volatility in Lucas (1987) due to risk aversion. It is also distinct from previous work that argues cycles are costly because they affect growth, e.g. Barlevy (2004), which involves the volatility of innovation rather than its timing. Once I allow for risk-aversion, fluctuations may no longer necessarily allow the planner to achieve a higher utility than in a stable environment. Nevertheless, fluctuations still allow the planner to achieve growth at a lower overall cost, which the inefficient timing of innovation in the decentralized equilibrium precludes.

3. Schumpeterian Growth with Concave Utility and Accumulable Capital

As noted in the previous section, concentrating innovation in recessions lowers the average cost of growth but increases the volatility of output. Under the assumption of risk neutrality, this volatility is inconsequential. However, when the utility function exhibits curvature, this volatility may make it undesirable to concentrate innovation in recessions. This section modifies the model to allow for concave utility, and examines whether procyclical innovation remains inefficient for empirically plausible assumptions.

In introducing risk aversion, it will be important to also relax the assumption that capital is not accumulable. Otherwise, the only way to smooth consumption over the cycle is to vary R&D with productivity, implying procyclical R&D may in fact be optimal. In practice, though, there are other options to smooth consumption such as inventories and capital accumulation, and since these activities do not occur at the expense of current production, they presumably dominate R&D for purposes of consumption smoothing.

Formally, I modify the model in two ways. First, I replace (2.1) with a more reasonable utility

$$U(C_t) = \ln C_t \tag{3.1}$$

Log utility also allows us to drop the restriction on ρ in (2.8). Second, I replace the assumption that $K_t \equiv 1$ for all t with the assumption that capital satisfies the law of motion

$$\dot{K}_t = I_t - \delta K_t \tag{3.2}$$

where I_t denotes investment, i.e. net output of final goods that is not consumed, and δ denotes the rate at which capital depreciates. Finally, since capital accumulation contributes to growth, we need to scale the fixed cost of production differently. Specifically, an intermediate goods producer will now require $\lambda^{M_t}F$ units of consumption good to initiate production at date t.

While these modifications are simple to describe, they greatly complicate the analysis by introducing additional state variables. In the next two subsections, I sketch out how to derive the optimal and equilibrium paths of R in this environment. I then solve for these paths numerically for parameter values meant to replicate certain features of U.S. data, and show that equilibrium innovation is indeed inefficiently procyclical at empirically plausible parameter values.

3.1. The Social Planner's Problem

I begin with the planner's problem. In lieu of (2.9), the planner's problem is now given by

$$V_i(K_0, M_0) = \max_{R_t, I_t} E\left[\int_0^\infty \ln\left(z_t K_t^\alpha \left[\lambda^{M_t}(L - R_t)\right]^{1-\alpha} - \lambda^{M_t}F - I_t\right) e^{-\rho t} dt \;\middle|\; z_0 = Z_i\right]$$

$$\text{s.t.} \quad \begin{array}{l} 1.\ \dot{M}_t = \phi R_t \\ 2.\ \dot{K}_t = I_t - \delta K_t \end{array}$$

To solve this problem, define $k = \lambda^{-M}K$ and $\iota = \lambda^{-M}I$. Using the law of motion for M, one can show that $V_i(K_0, M_0) = v_i(k) + M_0 \dfrac{\ln\lambda}{\rho}$, where $v_i(k)$ satisfies

$$\rho v_i(k) = \max_{\iota, R} \left\{ \begin{array}{c} \ln\left(Z_i k^\alpha (L - R)^{1-\alpha} - F - \iota\right) + \dfrac{\phi R \ln\lambda}{\rho} + \\ \dfrac{\partial v_i}{\partial k}\left(\iota - (\delta + \phi R \ln\lambda) k\right) + \mu\left(v_{1-i}(k) - v_i(k)\right) \end{array} \right\} \tag{3.3}$$

The planner can now control two variables, investment and R&D. The first-order conditions for the maximization problem with respect to these two variables are

$$\begin{aligned}\frac{1}{Z_i k^{\alpha}(L-R)^{1-\alpha}-F-\iota} &= \frac{\partial v_i}{\partial k}\\ \frac{(1-\alpha)Z_i k^{\alpha}(L-R)^{-\alpha}}{Z_i k^{\alpha}(L-R)^{1-\alpha}-F-\iota} &= \left(\frac{1}{\rho}-k\frac{\partial v_i}{\partial k}\right)\phi\ln\lambda\end{aligned}$$

Substituting the first equation into the second yields the following formula for R_i, the value of R&D when productivity is equal to Z_i:

$$R_i = L - \left[\left(\frac{1}{\rho(\partial v_i/\partial k)}-k\right)\frac{\phi\ln\lambda}{(1-\alpha)Z_i^{\alpha}k^{\alpha}}\right]^{-1/\alpha}$$

Rather than two numbers R_0 and R_1, an optimal plan now corresponds to two functions $R_0(k)$ and $R_1(k)$. I will refer to a policy as procyclical if it assigns $R_1(k) > R_0(k)$ for any k in the limiting set of capital-to-productivity ratios for this economy, i.e. for any level of k that occurs infinitely often along the optimal path with probability 1, and countercyclical if $R_1(k) < R_0(k)$ for all such k. Note that this definition may not correspond to the way R&D would appear to covary with z_t in data generated by this model, since changes in capital accumulation over the cycle may offset the response of R&D to changes in productivity for a fixed k.

To solve for $R_i(k)$, we need an expression for the value function $v_i(k)$. Since the system in (3.3) does not yield a closed-form solution, I need to solve it numerically. My implementation uses a collocation method whereby I approximate each $v_i(k)$ with a polynomial in k.[13]

3.2. Decentralized Equilibrium

Next, I characterize the equilibrium allocation. Certain features of the equilibrium remain unchanged from the previous section. For example, intermediate good producers will continue to charge the price at which their next most efficient rival breaks even. Once again, I assume new ideas can be imperfectly imitated, and the price at which an imitator breaks even is $p_{jt} = \lambda^{-(m_{jt}-1)}$. As such, nominal profits for each producer once again correspond to (2.24). The price of final goods P is analogous to but a little different from (2.14), namely

$$P = \frac{\lambda^{1-M}(L-R)^{\alpha}}{z(1-\alpha)k^{\alpha}} \tag{3.4}$$

[13] More precisely, I obtain an n-th order polynomial approximation of each function by choosing $n+1$ coefficients such that when I replace the true $v_i(k)$ with the approximate function, (3.3) is exactly satisfied at $n+1$ particular values of k. These particular values correspond to the roots of the first $n+1$ Chevyshev polynomials, adapted to the limiting interval for k. The results reported in the paper are based on fourth-order polynomials.

As in the previous section, I only consider symmetric Markov equilibria. In equilibrium, the path for R must satisfy the free entry condition $\phi v = 1$, where recall v denotes the expected present discount value of profits to the leading producer,

$$v = E\left[\int_0^\infty \mathbb{I}_t \cdot \frac{U'(C_t)/P_t}{U'(C_0)/P_0} \pi_t e^{-\rho t} dt \;\middle|\; z_0 \right]$$

With risk aversion, $U'(C_t)$ is no longer a constant. Hence, the value of a successful innovation depends on the consumption of the household. In order to solve for an equilibrium, then, we would need to solve the household problem.

The household problem can be characterized as follows. At any point in time, a household must decide how to divide its wealth between physical capital and claims on entrepreneurial profits. It does this taking as given the distribution of future prices, e.g. the price of capital, the rate of return on capital, the profits it expects entrepreneurs to earn, and so on. In addition, the household must choose how much of its wealth to use to finance consumption.

Since the household must own all of the claims to entrepreneurial profits in equilibrium, it will be convenient to pretend as if there was a mutual fund company that pooled all entrepreneurs into a single portfolio on behalf of the household. Arbitrage requires the value of this portfolio to be the same as the cost of buying up all firms, which is just $\int_0^1 v dj = v$. To insure the fund continues to own all incumbents, it must pay the research expenses of any potential innovator in exchange for the rights to the patent if the innovator is successful, i.e. the fund deducts an operating expense R out of dividends. Thus, as far as the household is concerned, it can either allocate its wealth to physical capital or to an asset whose price is v and which yields a dividend of $\Pi = \pi - R$ per unit time, where π is given by (2.22).

Let w denote the household's nominal wealth and σ denote the fraction of this wealth the household allocates to capital. The instantaneous change in the household's nominal wealth from its investments in physical capital derives from rental income and capital gains. As long as aggregate productivity remains constant over the next instant, the return per unit of capital is $r + \dot{P}$, and the number of units of capital it holds is $\sigma w/P$. Similarly, as long as aggregate productivity z_t remains constant, the return per share of the mutual fund it owns is $\Pi + \dot{v}$, and the number of shares it owns in the mutual fund is $(1-\sigma) w/v$. Note that the free entry condition implies $\dot{v} = 0$. In addition to the returns to its assets, the household also earns labor income and spends some of its resources on consumption. As long as z_t is constant, then, nominal wealth w would evolve continuously according to the law of motion

$$\dot{w} = \left[\left(\frac{r}{P} + \frac{\dot{P}}{P} \right) \sigma + \frac{\Pi}{v} (1-\sigma) \right] w + L - \lambda^M Pc \tag{3.5}$$

where $c = \lambda^{-M}C$. If instead productivity z_t changes in the next instant, the nominal value of the physical capital the household owns will jump as the price of capital P itself jumps. The nominal value of wealth held in the mutual fund does not change, however, since the value of the mutual fund $v = \phi^{-1}$ independently of aggregate productivity. Hence, the wealth of the household will jump from w to w^* where

$$w^* = \left[\frac{P_{1-i}}{P_i}\sigma + (1-\sigma)\right] w \tag{3.6}$$

Let W denote the aggregate wealth of the economy. In equilibrium, of course, $w = W$. However, since individual households act as price takers, they treat the path of W as given and assume it determines the values of all relevant economic variables. Let $R_i(W)$ denote the equilibrium employment in R&D when $z_t = Z_i$ and aggregate wealth is W_i. If we knew $R_i(W)$, we would be able to derive all remaining equilibrium quantities. For example, using the fact that $W = PK + v = \lambda^M Pk + \phi^{-1}$ and the expression for P in (3.4), we can solve for the capital-to-productivity ratio $k_i(W)$:

$$k_i(W) = \left[\frac{\left(W - \phi^{-1}\right) Z_i (1-\alpha)}{\lambda (L - R_i(W))^{\alpha}}\right]^{\frac{1}{1-\alpha}}$$

We can similarly express the nominal quantities r, P, and Π as functions of W. This implies we can express the household problem recursively in terms of two state variables, w and W:

$$\rho V_i(w, W) = \max_{\sigma, c} \left\{ \begin{array}{c} \ln c + \frac{\phi R \ln \lambda}{\rho} + \frac{\partial V_i}{\partial w}\dot{w} + \frac{\partial V_i}{\partial W}\dot{W} + \\ \mu \left(V_{1-i}(w^*, W^*) - V_i(w, W)\right) \end{array} \right\} \tag{3.7}$$

subject to (3.5) and (3.6), the free entry condition $\phi v = 1$, and the laws of motion for W, i.e. if z_t remains constant over the next instant, then

$$\dot{W} = \left(r + \dot{P}\right) \lambda^M k + \Pi + L - \lambda^M P c(W, W) \tag{3.8}$$

while if z_t changes over the next instant, W will jump to W^*, where

$$W^* = \lambda^M P_{1-i} k_i(W) + \phi^{-1}$$

The first order conditions for the household problem with respect to σ and c are given by

$$\left(\frac{r}{P} + \frac{\dot{P}}{P}\right) - \phi\Pi = \mu \frac{\partial V_{1-i}(w^*, W^*)/\partial w^*}{\partial V_i(w, W)/\partial w}\left[1 - \frac{P_{1-i}}{P_i}\right] \tag{3.9}$$

$$\frac{1}{Pc(w, W)} = \frac{\partial V_i}{\partial w} \tag{3.10}$$

An equilibrium therefore corresponds to a set of functions $w_i^*(w)$, $V_i(w,W)$, and $R_i(W)$ which satisfy equations (3.6), (3.7), and (3.9).

Again, I can only solve this equilibrium numerically. In particular, I approximate $R_i(W)$ and $w_i^*(w)$ using n-th degree polynomials in W and w, respectively, and I approximate the function $V_i(w,W)$ with the polynomial $\sum_{k=0}^{n}\sum_{\ell=0}^{n-k} a_{k\ell}w^kW^\ell$. The coefficients of these polynomials are chosen so that the equations above hold exactly at particular values of (w,W).[14]

3.3. Calibration and Results

I now proceed to solve the model numerically for particular parameter values. Since this version of the model corresponds to a standard real business cycle model, only with an endogenously determined growth rate, I can build on previous literature in assigning many of its parameters. The particular values I use are summarized in Table 1.

Table 1

ρ	0.05	Z_0	0.94	μ	0.20	λ	1.20	F	3.6
α	0.33	Z_1	1.06	δ	0.08	ϕ	0.10	L	30.8

Normalizing a unit of time in the model to correspond to a year, the discount rate ρ is set to 5%. The share of capital in the production of final goods is set to one third. To accord with an unconditional standard deviation of detrended productivity growth of 6%, I set Z_0 to 0.94 and Z_1 to 1.06. The transition rate μ is set so that the average length of a complete cycle is 10 years, slightly longer than the 8 year frequency often used to identify business cycle fluctuations. The depreciation rate is set to 8% per year. For λ, I follow Rotemberg and Woodford (1999) in calibrating the markup to 20%. The productivity term ϕ turns out to be a pure scaling parameter, so I normalize it to 0.10.

The remaining two parameters, F and L, are chosen to yield a growth rate of 2% per year and a share of R&D in output of 2%, in accordance with the average share of total R&D expenditures (private and public, since both can contribute to long-run growth) in GDP. It is not obvious whether the model counterpart to the latter statistic is the share of R&D in gross

[14] In particular, the set of points w that I use correspond to the roots of the Chebyshev polynomials, adapted to the limiting interval for w. The set of values for W are the same as for w, but I focus on the triangular array $\{w_i, W_j\}_{1\leq i\leq j\leq n+1}$. Note that I need to approximate $V(w,W)$ on and off the equilibrium path (in which $w=W$) to approximate both $\partial V_i/\partial w$ and $\partial V_i/\partial W$.

output or output net of the amount used by intermediate goods producers to cover fixed cost. However, at $F = 3.6$, R&D accounts for 2.0% of gross output and 2.2% of net output, which are virtually indistinguishable. Either way, R&D accounts for a tiny share of GDP, as it does in the data. Simulating the model for these parameter values reveals that the model also generates reasonable time variation in R&D: the standard deviation of the log R&D share over time is 0.139 and 0.136 for gross and net output respectively, compared to 0.137 for the log share of total R&D between 1953 and 2002.

Measured against output, the value of F implies that fixed costs account for 8.1% of gross output (and 8.8% of net output). By comparison, Basu (1996), following Ramey (1991), suggests using non-production workers as a proxy for overhead labor. Non-production workers account for about 20% of the labor force during the post-War period. Since labor accounts for two-thirds of total output, this suggests an even larger overhead cost of 13% of output.

Figure 4 plots both optimal and equilibrium $R_i(k)$ for the parameter values in Table 1. In each case, only the limiting set for k is depicted. For the parameter values in Table 1, optimal R&D policy is countercyclical, despite the fact that the representative household is risk-averse. This is consistent with the notion that capital accumulation is a much more efficient way to smooth consumption that varying R&D. Equilibrium R&D, by contrast, is unambiguously procyclical, even after accounting for more rapid capital accumulation in booms.

Comparing the axes of the two panels in Figure 4 reveals that optimal R&D is an order of magnitude larger than equilibrium R&D. This result mirrors the findings of Jones and Williams (2000), although the wedge between the optimal and equilibrium levels of R&D is larger than in their analysis. This is because Jones and Williams allow for diminishing returns to R&D; with linear returns to R&D, the planner would want to devote almost all available labor resources to innovation. One should therefore be skeptical about this model's predictions for optimal R&D. However, as long as we limit attention to small perturbations around the non-stochastic equilibrium, we can use this model to analyze the efficiency of observed R&D. This is because the equilibrium of the model is calibrated to match average long-run growth in the data, and curvature in the production function for R&D is irrelevant when shocks are small enough since the production function is locally linear. Thus, the calibrated model should be informative as whether the procyclical R&D we observe in the data is socially inefficient.

Formally, suppose aggregate productivity fluctuates between $Z_0 = 1-\varepsilon$ and $Z_1 = 1+\varepsilon$, where ε is small. Let R denote non-stochastic steady state equilibrium R&D when $Z_0 = Z_1 = 1$. For ε small, we can approximate the equilibrium of the stochastic economy by $R_0 = (1-\varepsilon')R$

and $R_1 = (1+\varepsilon')\,R$ for some $\varepsilon' > 0$. We can then check numerically the effect of changing ε' on welfare. For the parameter values in Table 1, the welfare of the representative household proves to be decreasing in ε'. Thus, independently of what our calibration implies for the global optimum, the representative household would be better off if the timing of R&D were countercyclical as opposed to procyclical, to a degree that depends on the exact process for z_t.

Note the cost of procyclical R&D in the model is fundamentally a cost of macroeconomic volatility; in a stable environment where $Z_0 = Z_1 = 1$, there is no opportunity to distort the timing of R&D in a way that would make growth more costly. Thus, the implied cost of procyclical R&D is related to the implied cost of macroeconomic volatility in the model. Let us therefore compare the utility of the household when $Z_0 = Z_1 = 1$ with the utility of the same household when Z_0 and Z_1 are as in Table 1. For concreteness, I evaluate both utilities starting at the deterministic steady-state level of k, and for the stochastic environment I assume z_0 is distributed according to the invariant distribution for z_t. Following Lucas (1987), we can express the change in utility in terms of the fraction of lifetime consumption agents would be willing to give up to remain in the stable environment. By a standard Taylor approximation argument, the cost of moving to a volatile environment is proportional to the variance of aggregate productivity σ_z^2. For the parameter values in Table 1, I calculate this constant of proportionality to be approximately 1. We can then decompose this cost into a cost due to volatility in z_t holding R&D fixed at its deterministic equilibrium level (but allowing agents to accumulate capital), and a cost due to changes in equilibrium R&D. The former cost reflects the cost of volatility due to risk aversion, while the latter reflects the cost due to the effect of fluctuations on growth. For the parameter values in Table 1, I calculate the cost of volatility holding R&D fixed to be approximately $\frac{1}{3}\sigma_z^2$. Taking into account the inefficient response of R&D to aggregate fluctuations fully triples the cost of business cycles relative to models in which the growth rate is treated as exogenous and fixed.

Of course, since σ_z^2 is small empirically, the implied welfare cost of procyclical R&D is small. For example, when $\sigma_z = 6\%$ as in Table 1, the cost of fluctuations is 0.36% of lifetime consumption. This is indeed three times as large as Lucas' estimated cost of business cycles using direct evidence on consumption (as opposed to calibrating to productivity shocks). That said, this estimate probably understates the true cost of the inefficient timing of R&D over the cycle. This is because the model substantially underpredicts the volatility of trend growth $\phi R \ln \lambda$. The standard deviation of trend growth in the model is about 0.2 percentage points. By contrast, when Barlevy (2004) estimates the volatility of trend growth, he finds a standard deviation of 1.8 percentage points. To be sure, not all of this variation is due to R&D. However, as is well-known, the preferences I use fail to generate asset prices that are as volatile as in the data.

This is important, since asset prices affect the incentives of firms to engage in innovation, and more volatile asset prices would result in more volatile trend growth. Recent work has argued that allowing for time non-separable preferences can help to reconcile RBC models with asset prices. Analyzing the model with this class of preferences would be an important next step for getting a better sense of the cost of the inefficient timing of R&D.

Finally, since the fixed cost F plays an essential role, some discussion of the robustness of the results to changes in this parameter are in order. In particular, how do the results change as we vary F while adjusting L to maintain a steady-state growth rate of 2% per year? As F is driven to zero, equilibrium innovation turns countercyclical. How large does F have to be for R&D to be procyclical? For the parameters values in Table 1, the cutoff level above which R&D is procyclical is approximately 3.3. This corresponds to a fixed cost of about 8.6% of net output, implying that fixed costs cannot be much smaller than those I calibrated. For larger values of F than in Table 1, equilibrium R&D is procyclical. Moreover, a similar perturbation analysis to the one reported above reveals that this procyclical pattern remains inefficient for much larger values of F. Interestingly, the optimal path for R&D – which recall may be suspect given the absence of diminishing returns – actually turns procyclical for high values of F. Intuitively, increasing F acts in a similar way to increasing the curvature of the utility function. Since along the optimal path the planner uses almost all labor resources for R&D rather than production, fluctuations in consumption are large in a proportional sense. Thus, a very risk-averse planner would resort to using even R&D to smooth consumption, especially given that the amount of resources devoted to R&D at the optimal path is so large. But since the equilibrium R&D share of GDP is only 2%, there is very little to gain from using R&D to smooth consumption in practice, and the procyclical pattern in equilibrium R&D remains suboptimal even when the optimal path turns procyclical.

4. Extensions

While the previous section goes some lengths to render the model more plausible, it still abstracts from certain features that may be relevant for its conclusions. This section briefly raises some of these considerations and speculates on whether the results above are likely to survive the introduction of further modifications.

One feature the model abstracts from is that innovations take time to bring to fruition, so firms should not expect to be able to bring their innovations on-line as soon as aggregate conditions improve, which they conceivably can under a Poisson technology for innovation.

Introducing diffusion lags might make entrepreneurs reluctant to undertake more innovation in booms if they are inherently temporary. Formally, suppose an idea discovered at date t can only be used for production at date $t+T$. By continuity, inefficient timing would continue to arise even for very small T. For larger T, we would have to explicitly solve the model. This is quite difficult, since the whole continuum of productivity levels $[z_{t-T}, z_t]$ enter as state variables. But intuitively, R&D should remain procyclical as long as shocks are persistent. That is, since $\Pr(z_{t+T} = z_t \mid z_t) > \Pr(z_{t+T} \neq z_t \mid z_t)$, a potential entrant who is constrained to implement his invention only after T units of time should still expect profits at the time of implementation to be higher starting in a boom than in a recession. Diffusion lags mitigate the benefits to procyclical R&D, but R&D should still be procyclical for sufficiently large fixed costs.

Another feature the model abstracts from is the possibility that firms strategically delay the time they implement their innovation, a point raised by Shleifer (1986). This is particularly relevant when aggregate productivity varies over time, since firms can potentially undertake innovation in recessions when the cost of R&D is low but wait until booms to implement them.[15] While this intuition is suggestive, allowing for strategic delay need not eliminate the inefficient timing of innovation. This is best illustrated using the version of the model in Section 2. The next Proposition provides conditions under which the equilibrium in Proposition 4 survives even when firms can strategically delay implementing their innovations:

Proposition 5: Suppose $\lambda < e^{1-\alpha}$ and $\dfrac{\mu}{\rho+\mu} < \dfrac{Z_0}{Z_1}$. Then there exists a $F' > F^*$ as defined in Lemma 2 such that the (R_0, R_1) identified in Proposition 4 remains an equilibrium for all $F \in (F^*, F')$ even if agents can delay implementation.

Intuitively, as long as the discount rate ρ is large and regime switches are infrequent so μ is small, it will not pay to delay innovation until a boom given the long expected wait until it arrives. The notion that firms do not strategically delay innovation does have some empirical support. As already noted in an earlier footnote, Griliches (1990) notes that firms tend to take out patents – and thus reveal their new ideas – early on in the research process, long before they put their new ideas to use. Moreover, strategic delay would imply a mismatch between R&D activity and patenting over the cycle, whereas Griliches reports that R&D and patents remain highly synchronized over the business cycle.

If the assumptions of Proposition 5 are not satisfied, firms might very well prefer to delay

[15] Indeed, this is what Francois and Lloyd-Ellis (2003) find when they endogenize R&D in Shleifer's model. However, in their model the recession emerges endogenously as a result of R&D activity rather than as the result of an exogenous change in productivity.

implementation until aggregate productivity is high, and the equilibrium with procyclical innovation may not survive. In this case, something else would be required to account for the procyclical pattern in R&D we observe in the data. However, if in equilibrium some of the new ideas discovered in recessions are only implemented in booms, the inefficiency described here would simply carry over from innovation to implementation: a benevolent planner would choose to concentrate both innovation and implementation in recessions, but in the decentralized economy implementation would be procyclical even if innovation were not. The inefficient implementation of new ideas – as opposed to the inefficient allocation of resources in coming up with new ideas – is related to results in Caballero and Hammour (1996), who show in a different setting that the process of adopting new technologies can be distorted in the presence of frictions. Unfortunately, strategic delay in the presence of aggregate fluctuations is difficult to analyze. Although Francois and Lloyd-Ellis (2003) can fully characterize strategic delay in the same model as this one but where aggregate productivity z_t is fixed over time, it turns out that their approach cannot be easily extended to the case of time-varying productivity.

5. Conclusion

In recent years, a growing number of economists has argued that recessions encourage agents to invest in making their technologies more productive. But one of the important channels for productivity growth, R&D effort, appears to be procyclical. This paper provides an explanation for why R&D is procyclical even though, as the neo-Schumpeterian view argues, it is efficient to concentrate it in recessions. This captures the spirit of recent work by Caballero and Hammour (2004), who also question whether recessions promote the reallocation of resources to more productive uses. Focusing on job reallocation, they conclude that "on the contrary, cumulative restructuring is *lower* following recessions... yet – contrary to the cost-of-liquidations view – this stifling of reallocation is *costly*." This paper considers R&D rather than job reallocation, but it similarly argues that recessions lead to lower R&D activity, not higher, and that this stifling of R&D is costly. Recessions thus fail to act as an incubator for productivity growth, even when in principle they ought to play that role.[16]

At the same time, as already noted in the Introduction, some growth-enhancing activities are countercyclical. One prominent example is schooling. Betts and McFarland (1995) and Dellas and Sakellaris (2003) document countercyclical college enrollment for the U.S., especially for

[16] On a related note, Barlevy (2002) argues that search frictions may prevent recessions from reallocating resources to their most appropriate uses, even though recessions should ameliorate productive inefficiency by cleansing out less productive uses of resources.

part-time students, while Sepulveda (2002) finds participation in training courses, both on and off the job, are countercyclical. In the opposite direction, King and Sweetman (2002) argue that the number of workers who quit their job to return to school is procyclical, although full-time commitment to school is a special form of human capital accumulation that may be sensitive to the difficulty of finding jobs in recessions. The fact that schooling is countercyclical but R&D is not is consistent with the notion that schooling is not associated with dynamic externalities that would cause agents to behave in a short-sighted manner.

The quantitative analysis in this paper suggests the inefficient timing of R&D leads to a small welfare cost, although the implied cost of business cycles is larger than in Lucas (1987). Interestingly, the optimal policy response does not involve stabilization. In fact, since cyclical fluctuations allow the economy to grow at a lower overall cost, policymakers might very well welcome fluctuations (provided agents are not too risk averse), at least if they can induce agents to substitute intertemporally and undertake R&D in recessions. This could presumably be achieved through subsidies to R&D in recessions. By contrast, given evidence that relatively unconstrained firms continue to concentrate their R&D in booms, a policy of easing credit conditions in recessions by itself may not suffice to induce countercyclical R&D.

While this paper focuses primarily on business cycles, it also contributes to the literature on long-run growth. Whereas most of this literature considers steady-state growth, this paper explores non-steady-state dynamics in traditional growth models. It reveals that spillovers inherent to the R&D process can lead not only to too much or too little steady-state growth, as previous work has already documented, but also to an inefficient response to shocks around the steady state. Thus, whereas previous work suggests that there is a special case in which various forces cancel out so that steady-state equilibrium growth is optimal, the results here suggest that even in this special case, the response of R&D to shocks around its steady state is necessarily suboptimal. The ability of the decentralized market to achieve an efficient level of growth would therefore appear to be even more fragile than suggested in previous work.

Appendix

Proof of Propositions 1 and 3: For given values of $\{R_i\}_{i=0,1}$, the system given by (2.10) reduces to ordinary linear differential equations in $V(Z_i, M)$. Standard theorems ensure this system has a unique solution. Hence, starting with values for R_i, we can use the method of undetermined coefficients to find the unique value functions $V(Z_i, M)$ associated with a given pair (R_0, R_1). I conjecture that the value function $V(\cdot,\cdot)$ takes the form

$$V(Z_i, M) = v_i \lambda^{M(1-\alpha)}$$

Differentiating this function with respect to M yields

$$\frac{\partial V}{\partial M} = (1-\alpha) v_i \lambda^{M(1-\alpha)} \ln \lambda$$

which simplifies the differential equations above to a system of independent linear equations in the coefficients v_i:

$$\rho v_i = Z_i (L - R_i)^{1-\alpha} - F + \mu (v_{1-i} - v_i) + (1-\alpha) v_i \phi R_i \ln \lambda$$

This yields a unique solution (v_0, v_1) as functions of (R_0, R_1).

Since the RHS of (2.10) is strictly concave in R_i, the first order condition is both necessary and sufficient to characterize the optimal R_i. The first order condition is given by

$$-(1-\alpha) Z_i \lambda^{M(1-\alpha)} (L - R_i)^{-\alpha} + \frac{\partial V}{\partial M}\phi \leq 0 \tag{5.1}$$

with equality if $R_i > 0$. Substituting the expression for $V(\cdot,\cdot)$, we obtain

$$R_i = \begin{cases} L - \left(\dfrac{Z_i}{v_i \phi \ln \lambda}\right)^{\frac{1}{\alpha}} & \text{if } v_i > \dfrac{Z_i}{\phi L^{\alpha} \ln \lambda} \\ 0 & \text{else} \end{cases} \tag{5.2}$$

If we substitute this expression into the asset equation (2.10), we obtain a pair of equations with v_{1-i} as a function of v_i that hold at the optimal R_i:

$$v_{1-i} = g_{1-i}(v_i) = \begin{cases} \dfrac{(\rho + \mu - (1-\alpha)\phi L \ln \lambda)}{\mu} v_i - \dfrac{\alpha}{\mu} Z_i^{\frac{1}{\alpha}} (v_i \phi \ln \lambda)^{1-\frac{1}{\alpha}} + \dfrac{F}{\mu} & \text{if } v_i > \dfrac{Z_i}{\phi L^{\alpha} \ln \lambda} \\ \dfrac{\rho + \mu}{\mu} v_i - \dfrac{Z_i L^{1-\alpha}}{\mu} + \dfrac{F}{\mu} & \text{else} \end{cases}$$

The optimal program corresponds to any pair (v_0^*, v_1^*) which solves the equations

$$\begin{aligned} v_1^* &= g_1(v_0^*) \\ v_0^* &= g_0(v_1^*) \end{aligned}$$

The function $g_{1-i}(\cdot)$ is continuous and differentiable, since the left and right hand derivatives at $v_i = \dfrac{Z_i}{\phi L^{\alpha} \ln \lambda}$ are both equal to $\dfrac{\rho + \mu}{\mu}$. Since $\rho > (1-\alpha)\phi L \ln \lambda$, it follows that $\dfrac{\partial g_{1-i}(v_i)}{\partial v_i} > 1$ for all v_i. The functions $g_{1-i}(\cdot)$ are illustrated in Figure A1, suggesting that there is a unique solution (v_0^*, v_1^*). To establish this formally, I use the fact $\dfrac{dg_{1-i}}{dv_i} > 1 > 0$ for all v_i implies $g_{1-i}(\cdot)$ is invertible. An equilibrium therefore involves a value v_0^* such that $g_1(v_0^*) - g_0^{-1}(v_0^*) = 0$. Differentiating this condition with respect to v_0^* yields

$$\frac{d}{dx}\left[g_1(x) - g_0^{-1}(x)\right] = \frac{dg_1}{dx} - \left(\frac{dg_0}{dx}\right)^{-1} > 0$$

This monotonicity insures there is at most one value of v_0^*. To establish existence, note that $g_1(0) < 0$ while $g_0^{-1}(0) > 0$. Hence, $g_1(0) - g_0^{-1}(0) < 0$, and is finite. The fact that $\lim_{x\to\infty} \frac{dg_1}{dx} > 1 > \lim_{x\to\infty} \left(\frac{dg_0}{dx}\right)^{-1}$ implies $\frac{\partial}{\partial x}\left[g_1(x) - g_0^{-1}(x)\right]$ is strictly bounded away from 0, and so $g_1(x) - g_0^{-1}(x) \to \infty$ as $x \to \infty$. The existence of v_0^* follows from continuity. This implies there is a unique social solution to the social planner's problem.

Next, suppose that the optimal path dictates $R_i > 0$ for both i. I need to show $R_0 > R_1$. The proof proceeds in two steps. First, I argue that $v_1^* > v_0^*$. Since $R_i > 0$, the asset equations imply

$$\begin{aligned} v_{1-i}^* &= \frac{(\rho + \mu - (1-\alpha)\phi L \ln\lambda)}{\mu} v_i^* - \frac{\alpha}{\mu} Z_i^{\frac{1}{\alpha}} (v_i^* \phi \ln\lambda)^{1-\frac{1}{\alpha}} + \frac{F}{\mu} \\ &\equiv a v_i^* - b Z_i^{\frac{1}{\alpha}} (v_i^*)^{1-\frac{1}{\alpha}} + \frac{F}{\mu} \end{aligned}$$

Consider the fixed point $\widehat{v}_i$ which solves

$$\widehat{v}_i = a\widehat{v}_i - bZ_i^{\frac{1}{\alpha}} (\widehat{v}_i)^{1-\frac{1}{\alpha}} + \frac{F}{\mu}$$

It is easy to show $\widehat{v}_i$ exists and is unique. Implicit differentiation implies

$$\frac{d\widehat{v}_i}{dZ_i} = \frac{\frac{b}{\alpha}\left(\frac{\widehat{v}_i}{Z_i}\right)^{1-\frac{1}{\alpha}}}{(\alpha - 1) + \frac{1-\alpha}{\alpha} b \left(\frac{Z_i}{\widehat{v}_i}\right)^{\frac{1}{\alpha}}} > 0$$

so that $Z_0 < Z_1 \Rightarrow \widehat{v}_0 < \widehat{v}_1$. Since $\frac{dg_i^{-1}}{dx} < 1$, we know that for any $x < \widehat{v}_1$, it follows that $x - g_0^{-1}(x) < 0$. Hence,

$$\begin{aligned} g_1(\widehat{v}_0) - g_0^{-1}(\widehat{v}_0) &= \widehat{v}_0 - g_0^{-1}(\widehat{v}_0) \\ &< 0 \end{aligned}$$

where the inequality uses the fact that $\widehat{v}_0 > \widehat{v}_1$. Since $g_1(v_0^*) - g_0^{-1}(v_0^*) = 0$ and $g_1(x) - g_0^{-1}(x)$ is increasing in x, it follows that $v_0^* > \widehat{v}_0$. But since $\frac{dg_i^{-1}}{dx} > 1$, the fact that $g_1(\widehat{v}_0) = \widehat{v}_0$ implies $g_1(x) > x$ for any $x > \widehat{v}_0$. Hence, $g_1(v_0^*) > v_0^*$. But since $v_1^* = g_1(v_0^*)$, it follows that $v_1^* > v_0^*$.

Next, I use the fact that $v_1^* > v_0^*$ to argue $\frac{v_1}{Z_1} < \frac{v_0}{Z_0}$, which is sufficient to establish $R_1 < R_0$ from the first-order condition above. Combining the equations $v_{1-i}^* = g_{1-i}(v_i^*)$ for both values yields the equation

$$a v_0^* - bZ_0^{\frac{1}{\alpha}} (v_0^*)^{1-\frac{1}{\alpha}} - v_1^* = a v_1^* - bZ_1^{\frac{1}{\alpha}} (v_1^*)^{1-\frac{1}{\alpha}} - v_0^*$$

which can be rearranged to yield

$$\frac{v_0^*}{v_1^*} = \frac{(a+1) - b\left(\frac{Z_1}{v_1^*}\right)^{\frac{1}{\alpha}}}{(a+1) - b\left(\frac{Z_0}{v_0^*}\right)^{\frac{1}{\alpha}}}$$

so that

$$v_1^* > v_0^* \Leftrightarrow \frac{v_1^*}{v_0^*} < \frac{Z_1}{Z_0}$$

But given the expression for R_i in (5.2), this implies $R_0 > R_1$.

Finally, suppose $R_i = 0$ for some i. The proposition follows if we can rule out the case where $R_0 = 0$ and $R_1 > 0$ for sufficiently small F. Once again, it will be enough to show that $\frac{v_1^*}{v_0^*} < \frac{Z_1}{Z_0}$. Let $v_i^*(Z_0, Z_1)$ denote the values of v_i^* given Z_0 and Z_1. It will be enough to prove that

$$\frac{\partial}{\partial Z_1}\left[\frac{Z_0 v_1^*(Z_0, Z_1)}{Z_1 v_0^*(Z_0, Z_1)}\right] < 0 \tag{5.3}$$

This is because by integrating (5.3) with respect to Z_1, we obtain

$$\begin{aligned}\frac{Z_0 v_1^*(Z_0, Z_1)}{Z_1 v_0^*(Z_0, Z_1)} &= \left[\frac{Z_0 v_1^*(Z_0, Z_0)}{Z_0 v_0^*(Z_0, Z_0)}\right] + \int_{Z_0}^{Z_1} \frac{\partial}{\partial Z_1}\left[\frac{Z_0 v_1^*(Z_0, Z_1)}{Z_1 v_0^*(Z_0, Z_1)}\right] dZ_1 \\ &< \left[\frac{Z_0 v_1^*(Z_0, Z_0)}{Z_0 v_0^*(Z_0, Z_0)}\right] = 1\end{aligned}$$

Note that (5.3) holds if and only if

$$\frac{\partial}{\partial Z_1}\ln\left(\frac{Z_0 v_1^*(Z_0, Z_1)}{Z_1 v_0^*(Z_0, Z_1)}\right) < 0$$

or alternatively if

$$\frac{\partial v_1^*/\partial Z_1}{v_1^*/Z_1} < 1 + \frac{\partial v_0^*/\partial Z_1}{v_0^*/Z_1}$$

Differentiating the asset equations with respect to Z_1 yields

$$\begin{aligned}\frac{\partial v_1}{\partial Z_1} &= \begin{cases} \dfrac{(L - R_1)\phi\ln\lambda}{\phi(L - R_1)\ln\lambda + \alpha\phi R_1\ln\lambda}\dfrac{v_1}{Z_1} & \text{if } R_1 > 0 \\[2ex] \dfrac{(\rho+\mu) Z_0 L^{1-\alpha}}{(\rho+\mu) Z_1 L^{1-\alpha} + \mu\left(Z_0 L^{1-\alpha} + A\right) - (\rho + 2\mu) F}\dfrac{v_1}{Z_1} & \text{if } R_1 = 0 \end{cases} \\ \frac{\partial v_0}{\partial Z_1} &= \frac{\mu}{\rho + \mu - (1-\alpha)\phi R_{1-i}\ln\lambda}\frac{\partial v_1}{\partial Z_1}\end{aligned}$$

where A is defined by

$$Z_0 L^{1-\alpha} + A = \max_{R_0}\left\{Z_0 (L - R_0)^{1-\alpha} + (1-\alpha) v_0 \phi R_0 \ln\lambda\right\}$$

so that $A \geq 0$. Provided $F < \frac{\mu}{\rho + 2\mu} Z_0 L^{1-\alpha}$, $\frac{\partial v_1}{\partial Z_1} < \frac{v_1}{Z_1}$, and $\frac{\partial v_0}{\partial Z_1} > 0$. This insures $\frac{\partial v_1^*/\partial Z_1}{v_1/Z_1} < 1 < 1 + \frac{\partial v_0^*/\partial Z_1}{v_0^*/Z_1}$, completing the proof. ■

Lemma 1: Let $h(\xi) = \frac{\mu}{\omega(L)}\left[\frac{Z_1}{Z_0}\xi^{\alpha} - \frac{Z_0}{Z_1}\xi^{1-\alpha}\right]$, where $\omega(L) = \rho + \mu + (1 - (1-\alpha)\ln\lambda)\phi L$. There exists a unique $\xi^* > 0$ such that $1 - \xi^* \gtreqless h(\xi^*)$ if $\xi \lesseqgtr \xi^*$. This unique solution ξ^* lies in the interval $(0, 1)$.

Proof of Lemma 1: First, I claim there exists a $\xi^* \in (0, 1)$ for which $1 - \xi = h(\xi)$. This is straightforward: if $\xi = 0$, we have

$$1 - \xi = 1 > 0 = h(\xi)$$

while if $\xi = 1$, we have

$$1 - \xi = 0 < \frac{\mu}{\omega(L)}\left[\frac{Z_1}{Z_0} - \frac{Z_0}{Z_1}\right] = h(\xi)$$

where the inequality relies on the fact that $\omega(L) > 0$ given that $\rho > (1-\alpha)\phi L\ln\lambda$. The claim follows from continuity.

To prove ξ^* is unique, I proceed in two steps. Differentiating $h(\cdot)$ yields

$$h'(\xi) = \frac{\mu}{\omega(L)}\left[\alpha\frac{Z_1}{Z_0}\xi^{\alpha-1} - (1-\alpha)\frac{Z_0}{Z_1}\xi^{-\alpha}\right]$$

For $\xi \geq 1$, we have

$$\begin{aligned}\alpha\frac{Z_1}{Z_0}\xi^{\alpha-1} - (1-\alpha)\frac{Z_0}{Z_1}\xi^{-\alpha} &> -(1-\alpha)\frac{Z_0}{Z_1}\xi^{-\alpha} \\ &> -1\end{aligned}$$

Since at $\xi = 1$, $h(\xi) > 1-\xi$, a necessary condition for there to exist a $\xi^* > 1$ such that $1-\xi^* = h(\xi^*)$ is that there exists a $\xi > 1$ such that $h'(\xi) < -1$. Thus, there exists no $\xi > 1$ for which $1-\xi = h(\xi)$.

Next, I need to show there is a unique $\xi^* \in (0,1)$ for which $1-\xi^* = h(\xi^*)$. Consider first the case where $\alpha > \frac{1}{2}$. Differentiating $h(\cdot)$ establishes that $h'(\xi) \geq 0$ if and only if

$$\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2 > \xi^{1-2\alpha}$$

For $\alpha > \frac{1}{2}$, $1-2\alpha < 0$, and so $h'(\xi)$ is negative if $0 < \xi < \left[\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2\right]^{\frac{1}{1-2\alpha}}$ and positive if $\xi > \left[\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2\right]^{\frac{1}{1-2\alpha}}$. Since $h(0) = 0$, it follows that $h(\xi) < 0$ for $0 < \xi < \left[\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2\right]^{\frac{1}{1-2\alpha}}$. Hence, $1-\xi > 0 > h(\xi)$ for $\xi < \left[\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2\right]^{\frac{1}{1-2\alpha}} < 1$. Since $h'(\xi)$ is strictly positive for $\left[\frac{\alpha}{1-\alpha}\left(\frac{Z_1}{Z_0}\right)^2\right]^{\frac{1}{1-2\alpha}}$, it follows that ξ^* is unique and $1-\xi^* \gtreqless h(\xi^*)$ if $\xi \lesseqgtr \xi^*$

If $\alpha = \frac{1}{2}$, $h(\xi)$ simplifies to $\frac{\mu}{\omega(L)}\left[\frac{Z_1 - Z_0}{Z_1 Z_0}\right]\xi^{\frac{1}{2}}$ which is monotonically increasing in ξ while $1-\xi$ is monotonically decreasing. This again insures ξ^* is unique and $1-\xi^* \gtreqless h(\xi^*)$ if $\xi \lesseqgtr \xi^*$

Finally, if $\alpha < \frac{1}{2}$, it is enough to prove that $h'(\xi) > -1$ for all $\xi \in (0,1]$. Differentiating $h(\cdot)$ twice yields

$$h''(\xi) = \frac{\alpha(1-\alpha)\mu}{\omega(L)}\left[\frac{Z_0}{Z_1}\xi^{-\alpha-1} - \frac{Z_1}{Z_0}\xi^{\alpha-2}\right]$$

so that

$$h''(\xi) \gtreqless 0 \Leftrightarrow \xi \gtreqless \left(\frac{Z_1}{Z_0}\right)^{\frac{2}{1-2\alpha}}$$

Thus, the derivative attains a minimum at $\xi = \left(\frac{Z_1}{Z_0}\right)^{\frac{2}{1-2\alpha}}$ which for $\alpha < \frac{1}{2}$ is strictly greater than 1. But it was previously argued that $h'(\xi) > -1$ for all $\xi \geq 1$. Hence, $h'(\xi) > -1$ for all $\xi \in (0,\infty)$ if $\alpha < \frac{1}{2}$.

Lastly, since at $\xi = 0$, $1-\xi = 1 > 0 = h(\xi)$, continuity implies $1-\xi > h(\xi)$ for all $\xi < \xi^*$. Likewise, at $\xi = 1$, $1-\xi = 0 < h(1)$, so by continuity it follows that $1-\xi < h(\xi)$ for $\xi > \xi^*$. This establishes the lemma. ■

Proof of Proposition 2: Since $R_i \geq 0$, the case where $R_1 = 0$ trivially satisfies the claim. If $R_0 = 0$, we need to verify that $R_1 = 0$. To show this, suppose not, i.e. suppose $R_1 > 0 = R_0$. Then it follows that $\phi v_1 = 1 \geq \phi v_0$. Substituting in for v_i, we get

$$\left\{ \begin{array}{l} \omega(R_0)(L - R_1) + \\ \quad \mu \frac{Z_0}{Z_1}(L - R_0)^{1-\alpha}(L - R_1)^{\alpha} \end{array} \right\} \geq \left\{ \begin{array}{l} \omega(R_1)(L - R_0) + \\ \quad \mu \frac{Z_1}{Z_0}(L - R_1)^{1-\alpha}(L - R_0)^{\alpha} \end{array} \right\}$$

Since $v_1(R_1, R_0) = 0$ if $R_1 = L$, then $R_1 < L$ in any such equilibrium. This allows us to define ξ such that

$$R_0 = \xi R_1 + (1 - \xi) L$$

Note that since $R_1 < L$, by construction, $\xi \geq 0$, and $R_0 > R_1$ implies $\xi \in [0, 1)$ while $R_0 < R_1$ implies $\xi > 1$. After substituting in for $\omega(R)$ and rewriting R_0 in terms of R_1, we can rewrite the inequality $v_1 \geq v_0$ in terms of ξ:

$$1 - \xi \geq \frac{\mu}{\omega(L)} \left[\frac{Z_1}{Z_0}\xi^{\alpha} - \frac{Z_0}{Z_1}\xi^{1-\alpha} \right]$$

Applying lemma 1, it follows that $\xi < \xi^* < 1$, which implies $R_0 > R_1$, a contradiction. Thus, $R_0 = 0$ implies $R_1 = 0$.

Finally, if R_1 and R_0 are both positive, it must be true that $v_1 = v_0 = \frac{1}{\phi}$, which in turn implies that $1 - \xi = h(\xi)$. But from the lemma, the unique ξ^* which satisfies this equation is less than 1, which implies $R_0 \geq R_1$.

Next, I show that there exists a unique symmetric Markov-perfect equilibrium when $\lambda < e^{\frac{1}{1-\alpha}}$. This condition implies that $(1 - \alpha) \ln \lambda < 1$, which implies

$$\omega'(R) = (1 - (1 - \alpha) \ln \lambda) \phi > 0$$

Using (2.16), the fact that $\omega'(R) > 0$ can be shown to imply that $\frac{\partial v_i}{\partial R_i} < 0$ and $\frac{\partial v_i}{\partial R_{1-i}} < 0$. Recall from the proof of Proposition 2 that in any Markov-perfect equilibrium in which $\phi v_i(R_i, R_{1-i}) = 1$ for both i, the levels of innovation for the two levels of productivity are related by $R_0 = \xi^* R_1 + (1 - \xi^*) L$, where ξ^* is a constant. Hence, there can be at most one equilibrium in which $\phi v_0 = \phi v_1 = 1$. For suppose there were two such equilibria, $(R_0, R_1) \neq (R_0', R_1')$ where wlog $R_0' > R_0$. Since ξ^* is constant, it follows that $R_1' > R_1$, but since v_i is decreasing in both R_i and R_{1-i} it is impossible that $v_i(R_i, R_{1-i}) = v_i(R_i', R_{1-i}') = \frac{1}{\phi}$.

If $(R_0, R_1) = (0, 0)$ is an equilibrium, given that $\frac{\partial v_i}{\partial R_i} < 0$, it follows that for any $(R_0, R_1) \neq 0$ there always exists some $i \in \{0, 1\}$ such that $\phi v_i < 1$ but $R_i > 0$, which is inconsistent with equilibrium. In this case, $(0, 0)$ would be the unique equilibrium. Without loss of generality, then, I henceforth assume that if an equilibrium exists, it is not equal to $(0, 0)$.

I begin by arguing that if there exists an equilibrium (R_0^*, R_1^*) where $\phi v_i(R_i^*, R_{1-i}^*) = 1$ for both i, there exists no other equilibria in which $R_i = 0$ and $\phi v_i(0, R_{1-i}) < 1$ for some $i \in \{0, 1\}$. For each i, define the contour sets

$$\Omega_i = \{(R_i, R_{1-i}) \mid \phi v_i(R_i, R_{1-i}) = 1\}$$

for all values of $(R_0, R_1) \geq (0, 0)$. These sets are illustrated in Figure A2. Using the implicit function theorem and the fact that $\frac{\partial v_i}{\partial R_i}$ and $\frac{\partial v_i}{\partial R_{1-i}}$ are both strictly negative, we can establish that the graphs of Ω_i form connected, downward sloping curves in (R_0, R_1) space. If there exists an equilibrium $(R_0^*, R_1^*) \neq (0, 0)$ such that $\phi v_i(R_i^*, R_{1-i}^*) = 1$ for both i, the sets Ω_i must both be nonempty. Since $\frac{\partial v_i}{\partial R_{1-i}} < 0$ and $\phi v_i(R_i^*, R_{1-i}^*) = 1$, then $\phi v_i(R_i^*, 0) > 1$. Since $\phi v_i(L, 0) = 0$, there exists an $R_i' \geq 0$ such that $(R_i', 0) \in \Omega_i$ by continuity. Hence, The graph of Ω_i intersects the R_i axis.

Next, define $R_{1-i}'' > 0$ as the value of R_{1-i} such that $\phi v_i(0, R_{1-i}'') = 1$. If no such value exists, I adopt the convention that $R_{1-i}'' = \infty$. I now argue that $R_1'' > R_1'$. The statement follows trivially if $R_1'' = \infty$. If $R_1'' < \infty$, I argue that $\phi v_1(R_1'', 0) > 1$. For suppose not, i.e. suppose $\phi v_1(R_1'', 0) \leq 1$. Since $\phi v_0(0, R_1'') = 1$, it follows that either $(0, R_1'')$ constitutes an equilibrium, or there exists some $R_1''' \in (R_1'', L)$ such that $\phi v_1(R_1''', 0) = 1$, from which it follows that $(0, R_1''')$ is an equilibrium. Since $R_0 \geq R_1$ in any Markov-perfect equilibrium, it follows that $R_1'' = R_1''' = 0$. Since $\phi v_0(R_0^*, R_1^*) = 1$ and $(R_0^*, R_1^*) \neq (0, 0)$ by assumption, $\dfrac{\partial v_i}{\partial R_i} < 0$ implies $\phi v_0(0, 0) > 1$, a contradiction. Hence, $\phi v_1(R_1'', 0) > 1 = \phi v_1(R_1', 0)$. Since v_i is decreasing in R_i, it follows that $R_1'' > R_1'$ as claimed.

The fact that $R_1'' > R_1'$ can be used to establish that $R_0'' > R_0'$ as well. First, though, I argue that at the equilibrium (R_0^*, R_1^*),

$$\left.\frac{dR_1}{dR_0}\right|_{\phi v_1 = 1} > \left.\frac{dR_1}{dR_0}\right|_{\phi v_0 = 1}$$

To see this, consider a neighborhood around (R_0^*, R_1^*). Recall that for any admissible (R_0, R_1) where R_0 is defined as $\xi R_1 + (1 - \xi) L$ for some $\xi \geq 0$, the proof of Proposition 2 above implies that $v_1(R_1, R_0) > v_0(R_0, R_1)$ if and only if $1 - \xi > h(\xi)$, which from Lemma 1 holds if and only if $\xi < \xi^*$. Since for any $\varepsilon > 0$, $R_0^* + \varepsilon = \xi R_1^* + (1 - \xi) L$ for some $\xi < \xi^*$, it follows that

$$v_1(R_1^*, R_0^* + \varepsilon) > v_0(R_0^* + \varepsilon, R_1^*)$$

Subtracting $v_0(R_0^*, R_1^*) = v_1(R_1^*, R_0^*) = \dfrac{1}{\phi}$ from both sides, dividing by ε, and taking the limit as $\varepsilon \to 0$ implies

$$\frac{\partial v_1(R_0^*, R_1^*)}{\partial R_0} \geq \frac{\partial v_0(R_1^*, R_0^*)}{\partial R_0}$$

We can further establish this inequality is strict. This is because if $\dfrac{\partial v_0(R_1^*, R_0^*)}{\partial R_0} = \dfrac{\partial v_1(R_0^*, R_1^*)}{\partial R_0}$, the derivative of $1 - \xi - h(\xi)$ with respect to ξ would be equal to 0 at $\xi = \xi^*$, which is contradicted by the proof of Lemma 1 above. Similarly, for any $\varepsilon > 0$, Lemma 1 implies

$$v_0(R_0^*, R_1^* + \varepsilon) > v_1(R_1^* + \varepsilon, R_0^*)$$

and by an analogous argument,

$$\frac{\partial v_0(R_0^*, R_1^*)}{\partial R_1} \geq \frac{\partial v_1(R_1^*, R_0^*)}{\partial R_1}$$

Taking into account the fact that $\partial v_i / \partial R_i$ and $\partial v_i / \partial R_{1-i}$ are both negative, it follows that

$$\frac{\partial v_0 / \partial R_0}{\partial v_0 / \partial R_1} > \frac{\partial v_1 / \partial R_0}{\partial v_1 / \partial R_1}$$

which implies

$$\left.\frac{dR_1}{dR_0}\right|_{\phi v_1 = 1} = -\frac{\partial v_1 / \partial R_0}{\partial v_1 / \partial R_1} > -\frac{\partial v_0 / \partial R_0}{\partial v_0 / \partial R_1} = \left.\frac{dR_1}{dR_0}\right|_{\phi v_0 = 1}$$

Since there can be only one point at which $\phi v_i(R_i, R_{1-i}) = 1$, the two contours sets Ω_i intersect only at (R_0^*, R_1^*), and by continuity it follows that $R_0'' > R_0'$.

With these observations, I can finally establish that there exists no other equilibrium $\left(\widehat{R}_i, \widehat{R}_{1-i}\right)$ in which $\widehat{R}_i = 0$ for some $i \in \{0, 1\}$ and $\phi v_i\left(0, \widehat{R}_{1-i}\right) < 1$. This is because if such an equilibrium existed, by definition it must be true that $\phi v_i\left(0, \widehat{R}_{1-i}\right) \leq 1$. Since $\phi v_i\left(0, R''_{1-i}\right) = 1$ by definition, monotonicity implies $\widehat{R}_{1-i} \geq R''_{1-i} > R'_{1-i}$. But since $R_{1-i} > R'_{1-i}$, it follows that $\phi v_{1-i}\left(R_{1-i}, 0\right) < 1$. For this to be an equilibrium, $\widehat{R}_{1-i} = 0$. But since there exists an $\left(R_0^*, R_1^*\right) \neq (0,0)$ such that $\phi v_i\left(R_i^*, R_{1-i}^*\right) = 1$, it follows that $\phi v_i(0,0) > 1$, a contradiction.

Finally, suppose there exists no pair $\left(R_0^*, R_1^*\right)$ such that $\phi v_0\left(R_0^*, R_1^*\right) = \phi v_1\left(R_0^*, R_1^*\right) = 1$. We need to establish that there still exists a unique Markov-perfect equilibrium. Suppose first that $\phi v_0(0,0) \leq 1$. Then for any $\left(R_0, R_1\right) \geq (0,0)$ where $\left(R_0, R_1\right) \neq (0,0)$, it must be the case that $\phi v_0\left(R_0, R_1\right) < 1$. This implies $R_0 = 0$ in any equilibrium, and since $R_0 \geq R_1$ in any equilibrium according to Proposition 2, $R_0 = R_1 = 0$ must be the unique equilibrium. If we rewrite R_0 as $\xi R_1 + (1-\xi) L$, then $\xi = 1 > \xi^*$. But recall from Lemma 1 that this implies $v_1(0,0) < v_0(0,0)$. Since $\phi v_0(0,0) \leq 1$, it follows that $(0,0)$ is in fact an equilibrium.

This leaves the case where (i) there exists no pair $\left(R_0^*, R_1^*\right)$ such that $\phi v_0\left(R_0^*, R_1^*\right) = \phi v_1\left(R_0^*, R_1^*\right) = 1$ and (ii) $\phi v_0(0,0) > 1$. By continuity, there exists an $R'_0 < L$ such that $\phi v_0\left(R'_0, 0\right) = 1$. Then I claim $\left(R_0, R_1\right) = \left(R'_0, 0\right)$ is the unique Markov perfect equilibrium. Consider again two cases. First, suppose that $\phi v_1(0,0) \leq 1$. In that case, monotonicity implies $\phi v_1\left(0, R'_0\right) < 1$ given that $R'_0 > 0$, so that $\phi v_1\left(0, R'_0\right) \leq 1$ and $\left(R'_0, 0\right)$ is indeed an equilibrium. Moreover, since $\phi v_1(0,0) \leq 1$, then $R_1 = 0$ in any equilibrium, and it follows that $\left(R'_0, 0\right)$ is the unique equilibrium. Lastly, suppose $\phi v_1(0,0) > 1$. Once again, define R''_0 such that $\phi v_0\left(0, R''_0\right) = 1$, with the convention of setting $R''_0 = \infty$ if no such value exists. We need to show that $R'_0 > R''_0$, which insures $\phi v_1\left(0, R'_0\right) \leq 1$. Suppose not, i.e. suppose $R''_0 \geq R'_0$. But using the same argument as before, we know that $R''_1 > R'_1$. If $R''_0 \geq R'_0$, then by continuity Ω_0 and Ω_1 must intersect, which contradicts the supposition that there exists no solution $\left(R_0^*, R_1^*\right)$. Hence, $\phi v_1\left(0, R'_0\right) \leq 1$, so that $\left(R'_0, 0\right)$ is an equilibrium, and since $R_1 = 0$ in any equilibrium, which insures the equilibrium above is unique.$\blacksquare$

Proof of Lemma 2 (in text): Consider the equation $v_0\left(R^*, R^*\right) = v_1\left(R^*, R^*\right)$. It implies

$$\frac{\omega\left(R^*\right) - \mu \dfrac{Z_1}{Z_0}}{(1-\alpha) Z_1 \left(L - R^*\right)^{1-\alpha} - F} - \frac{\omega\left(R^*\right) - \mu \dfrac{Z_0}{Z_1}}{(1-\alpha) Z_0 \left(L - R^*\right)^{1-\alpha} - F} = \frac{\mu}{\lambda}\left[\frac{Z_0}{Z_1} - \frac{Z_1}{Z_0}\right]$$

Differentiate left hand side yields

To establish the existence and uniqueness of R^*, define $y_i = L - R_i$. Using the assumption that $R_i < L$ implies $y_i \neq 0$, we can rearrange the condition $v_0(R,R) = v_1(R,R)$ by dividing both sides by y^α and expanding out $\omega(L-y)$ to obtain

$$\mu\left(Z_1 + Z_0\right) \frac{y^{1-\alpha}}{F} + \frac{\lambda}{(\lambda-1)(1-\alpha)}\left(1 - (1-\alpha)\ln\lambda\right)\phi y = \frac{\lambda\left(\omega(L) + \mu\right)}{(\lambda-1)(1-\alpha)} \tag{5.4}$$

The LHS of this equation is monotonically increasing in y, and ranges from 0 to ∞ as y ranges from 0 to ∞. Since the RHS above is strictly positive, there exists a unique value y^* for which the equation is satisfied. This translates into a unique value $R^* = L - y^*$.

Note that y^* is monotonically increasing in F. Taking limits, $y^* \to 0$ as $F \to 0$, while $y^* \to L + \dfrac{\rho + 2\mu}{(1-(1-\alpha)\ln\lambda)\phi}$ as $F \to \infty$, at which point $\omega\left(L - y^*\right) = -\mu$. If we evaluate $v_i\left(R^*, R^*\right)$ and substitute in from (5.4), we obtain

$$
\begin{aligned}
v_i &= (\lambda-1)\frac{\left(\omega\left(L-y\right)+\mu\frac{Z_{1-i}}{Z_i}\right)y^*-\frac{\left(\omega\left(L-y\right)+\mu\right)\lambda F\left(y^*\right)^\alpha}{\left(\lambda-1\right)\left(1-\alpha\right)Z_i}}{\omega^2\left(L-y_F^*\right)-\mu^2} \\
&= (\lambda-1)\frac{\left(\omega\left(L-y^*\right)+\mu\frac{Z_{1-i}}{Z_i}\right)y^*-\frac{\mu\left(Z_1+Z_0\right)y^*}{Z_i}}{\omega^2\left(L-y_F^*\right)-\mu^2} \\
&= \frac{\left(\lambda-1\right)y^*}{\omega\left(L-y^*\right)+\mu}
\end{aligned}
$$

Hence, $v_i\left(R^*,R^*\right)$ is monotonically increasing in y^*, which in turn is monotonically increasing in F. As noted above, for different values of F, $y^*\in\left[0,L+\frac{\rho+2\mu}{\left(1-\left(1-\alpha\right)\ln\lambda\right)\phi}\right)$, which implies v_i ranges between 0 and ∞. The existence of F^* follows from continuity. ■

Proof of Proposition 4: Consider a fixed value $F>F^*$, where F^* is defined in Lemma 2. As in the proof of Lemma 2, it will be useful to work with $y_i=L-R_i$. Consider the set

$$
S=\left\{\left(y_0,y_1\right)\ \mid v_0\left(L-y_0,L-y_1\right)=v_1\left(L-y_1,L-y_0\right)\right\}
$$

The proposition follows if I can show that within this set there exists an element $\left(y_0,y_1\right)\in S$ such that $y_0>y_1$ and $\phi v_i\left(L-y_i,L-y_{1-i}\right)=1$ for $i\in\{0,1\}$.

Consider first the case where $y_1=0$. For this value, we have

$$
\begin{aligned}
v_0 &= \frac{\left(\lambda-1\right)\omega\left(L\right)y_0-\frac{\omega\left(L\right)+\mu\lambda F}{\left(1-\alpha\right)Z_0}y_0^\alpha}{\omega\left(L-y_0\right)\omega\left(L\right)-\mu^2} \\
v_1 &= 0
\end{aligned}
$$

Hence, there are exactly two values of y_0 for which $v_0\left(L-y_0,L\right)=v_1\left(L,L-y_0\right)$, namely $y_0=0$ and

$$
y_0=\widetilde{y}_0\equiv\left[\frac{\left(\omega\left(L\right)+\mu\right)\lambda F}{\omega\left(L\right)\left(\lambda-1\right)\left(1-\alpha\right)Z_0}\right]^{\frac{1}{1-\alpha}}
$$

By the implicit function theorem, there exist continuous functions $y_0\left(\cdot\right)$ defined in a neighborhood of $y_1=0$ such that $y_0\left(y_1\right)\to 0$ and $y_0\left(y_1\right)\to\widetilde{y}_0$ as $y_1\to 0$ which satisfy $v_0\left(L-y_0\left(y_1\right),L-y_1\right)=v_0\left(L-y_1,L-y_0\left(y_1\right)\right)$.

For $y_1\neq 0$, we can rewrite the equation $v_0=v_1$ in terms of y_1 and $\xi=\frac{y_0}{y_1}$. The condition that $v_1=v_0$ can be rewritten as

$$
\frac{\lambda F y_1^{\alpha-1}}{Z_1\left(\lambda-1\right)\left(1-\alpha\right)}\left(A_0-A_1\xi-A_2\xi^\alpha\right)-1+\xi+h\left(\xi\right)=0 \tag{5.5}
$$

where

$$
\begin{aligned}
A_0 &= \frac{\omega\left(L\right)+\mu}{\omega\left(L\right)} \\
A_1 &= \frac{\left(1-\left(1-\alpha\right)\ln\lambda\right)\phi y_1}{\omega\left(L\right)} \\
A_2 &= \frac{\omega\left(L-y_1\right)+\mu}{\omega\left(L\right)}\frac{Z_1}{Z_0}
\end{aligned}
$$

and as in Lemma 1,

$$h(\xi) = \frac{\mu}{\omega(L)}\left[\frac{Z_1}{Z_0}\xi^{\alpha} - \frac{Z_0}{Z_1}\xi^{1-\alpha}\right]$$

For notational convenience, I will rewrite (5.5) more compactly as

$$Q(\xi; y_1) = 0$$

The implicit functions $y_0(y_1)$ described above which limit to 0 and $\widetilde{y}_0$ establish the existence of functions $\xi(y_1)$ defined locally near $y_1 = 0$ that limit to $\xi = \left(\frac{Z_0}{Z_1}\right)^{\frac{1}{\alpha}}$ and $\xi = \infty$, respectively, as $y_1 \to 0$.

Define y^* as in Lemma 2. Differentiate $Q(\xi; y_1)$ with respect to ξ twice to obtain

$$\frac{\partial^2 Q}{\partial \xi^2} = h''(\xi) + \alpha(1-\alpha)\frac{\lambda F \xi^{\alpha-2}}{Z_1(\lambda-1)(1-\alpha)}A_2 y_1^{\alpha-1}$$

Substituting in for $h''(\xi)$, we obtain

$$\frac{\partial^2 Q}{\partial \xi^2} = \alpha(1-\alpha)\left(\frac{\mu}{\omega(L)}\left[\frac{Z_0}{Z_1}\xi^{-\alpha-1} - \frac{Z_1}{Z_0}\xi^{\alpha-2}\right] + \frac{\lambda F \xi^{\alpha-2}}{Z_1(\lambda-1)(1-\alpha)}A_2 y_1^{\alpha-1}\right)$$

Now, note that

$$\begin{aligned} A_2 y_1^{\alpha-1} &= \frac{\omega(L-y_1)+\mu}{\omega(L)}\frac{Z_1}{Z_0}y_1^{\alpha-1} \\ &= \left[\frac{\omega(L)+\mu}{\omega(L)}y_1^{\alpha-1} - \frac{(1-(1-\alpha)\ln\lambda)\phi y_1^{\alpha}}{\omega(L)}\right]\frac{Z_1}{Z_0} \end{aligned}$$

is decreasing in y_1. Hence, if we can show that $\frac{\partial^2 Q}{\partial \xi^2} > 0$ for some y_1', it follows that $\frac{\partial^2 Q}{\partial \xi^2} > 0$ for all $y_1 \leq y_1'$. For $y_1 = y^*$, we know from the proof of Lemma 2 that y^* satisfies

$$\frac{\lambda F (y^*)^{\alpha-1}}{Z_1(\lambda-1)(1-\alpha)} = \frac{\mu}{\omega(L-y^*)+\mu}\left(\frac{Z_1+Z_0}{Z_1}\right)$$

Substituting this into the expression for $\frac{\partial^2 Q}{\partial \xi^2}$ yields

$$\frac{\partial^2 Q}{\partial \xi^2} = \alpha(1-\alpha)\left(\frac{\mu}{\omega(L)}\frac{Z_0}{Z_1}\xi^{-\alpha-1} + \frac{\mu}{\omega(L)}\xi^{\alpha-2}\right) > 0$$

Hence, for all $y_1 \leq y^*$, $Q(\xi; y_1)$ is convex in ξ. This will prove important in what follows.

Before I proceed, I introduce the notation $(y_0, y_1) \rightsquigarrow (y_0', y_1')$ to denote the case in which there exists a continuous mapping $y_1(\tau) > 0$ and a continuous mapping $y_0(\tau)$ defined for $\tau \in (0,1)$ such that

1. $\lim_{\tau\to 0} y_1(\tau) = y_1$ and $\lim_{\tau\to 0} y_0(\tau) = y_0$
2. $\lim_{\tau\to 1} y_1(\tau) = y_1'$ and $\lim_{\tau\to 1} y_0(\tau) = y_0'$
3. For all $\tau \in (0,1)$, $(y_0(\tau), y_1(\tau)) \in S$, i.e. $Q(\xi(\tau); y_1(\tau)) = 0$ for $\xi(\tau) = \frac{y_0(\tau)}{y_1(\tau)}$

The notation $\lim_{\tau\to 1} y(\tau) = \infty$ denotes, as usual, that for every $N > 0$, there exists a τ_N such that $y(\tau) > N$ for all $\tau > \tau_N$. Thus, we can describe a path in which $y_i' = \infty$ for some i. Note that if we can establish that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$, the statement of the proposition follows from a simple continuity argument: since

$\phi v_0 (L - y^*, L - y^*) > 1$ for $F > F^*$ but $\phi v_0 (\widetilde{y}_0, 0) = 0$, there exists some τ for which $(y_0(\tau), y_1(\tau)) \in S$ and where $\phi v_i (L - y_i(\tau), L - y_{1-i}(\tau)) = 1$. Since $v_0(L - y, L - y) = v_1(L - y, L - y)$ if and only if $y = y^*$, and since $\widetilde{y}_0 > 0$, it follows that $y_0(\tau) > y_1(\tau)$ by continuity.

I now break down my analysis into different cases, depending on the sign of $\frac{\partial Q}{\partial \xi}$ evaluated at $\xi = 1$ and $y_1 = y^*$. For convenience, a graphical view of the set S corresponding to each of the three cases (for particular parameter values) is provided in Figure A3.

Case I: $\frac{\partial Q(1; y^*)}{\partial \xi} > 0$

I claim this is enough to establish that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$, which from above is enough to establish the proposition. I first argue that there exists a $y_1' \in [0, y^*)$ such that $(y^*, y^*) \rightsquigarrow (y_0', y_1')$ where $\frac{y_0'}{y_1'} = \infty$. For suppose not. Since $\frac{\partial Q(1; y_1)}{\partial y_1} > 0$ for all y_1, it follows that $Q(1; y_1) < 0$ for all $y_1 < y^*$. Furthermore, since $Q(\xi; y^*)$ is convex in ξ, it also follows that $Q(\xi; y^*) > 0$ for all $\xi > 1$. By continuity, then, the assumption that $(y^*, y^*) \not\rightsquigarrow (\infty, y_1)$ for all $y_1 \in (0, y^*)$ implies that for each $y_1 \in (0, y^*)$ there must exist some $\xi(y_1) > 1$ such that $Q(\xi(y_1); y_1) > 0$. Applying the intermediate value theorem, we can deduce that for every $y_1 \in (0, y^*)$ there exists a $\xi^*(y_1) > 1$ such that $Q(\xi^*(y_1); y_1) = 0$. Since Q is continuous and convex in ξ for all $y_1 \leq y^*$, the root $\xi^*(y_1)$ is the unique value of $\xi > 1$ such that $Q(\xi^*(y_1); y_1) = 1$, is continuous in y_1, and $\lim_{y_1 \uparrow y_F^*} \xi^*(y_1) = 1$. Hence, $(y^*, y^*) \rightsquigarrow (\xi^*(y_1), y_1)$. Taking the limit as $y_1 \downarrow 0$, it follows that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$, since the unique root greater than one for which $\lim_{y_1 \to 0} Q(\xi; y_1) = 0$ limits to ∞. But then $(y^*, y^*) \rightsquigarrow (y_0', y_1')$ such that $\frac{y_0'}{y_1'} = \infty$, which is a contradiction. Since $\lim_{\xi \to \infty} \frac{\partial Q}{\partial y_1} < 0$ for all $y_1 > 0$, there can exist at most one y_1 for which $\lim_{\xi \to \infty} Q(\xi; y_1) = 0$. Since $\lim_{\xi \to \infty} Q(\xi; 0) = 0$, it follows that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$.

Case II: $\frac{\partial Q(1; y^*)}{\partial \xi} = 0$.

Since $Q(1; y^*)$ is strictly convex, it follows that $Q(\xi; y^*) > 0$ for all $\xi \neq 1$. The fact that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$ then follows from the same argument as in Case I.

Case III: $\frac{\partial Q(1; y^*)}{\partial \xi} < 0$

In this case, the arguments above can no longer be used to establish that $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$. However, I argue that if $(y^*, y^*) \not\rightsquigarrow (\widetilde{y}_0, 0)$, then there exists some $y_1 > 0$ such that $(y^*, y^*) \rightsquigarrow (\infty, y_1)$ where $\lim_{\tau \to 1} \frac{y_0(\tau)}{y_1(\tau)} > 1$ along any such connecting path. As I argue below, this condition is also sufficient to establish the proposition.

Again, the proof is by contradiction. Suppose the claim is false, i.e. suppose $(y^*, y^*) \not\rightsquigarrow (\widetilde{y}_0, 0)$ and $(y^*, y^*) \not\rightsquigarrow (\infty, y_1)$ for all $y_1 > 0$, including $y_1 = \infty$. If we differentiate ξ with respect to y_1 along the curve $Q(\xi; y_1) = 0$, we obtain

$$\left.\frac{d\xi}{dy_1}\right|_{(\xi; y_1) = (1; y^*)} = -\frac{\partial Q / \partial y_1}{\partial Q / \partial \xi} > 0$$

where the last inequality follows from the fact that $\frac{\partial Q(1; y_1)}{\partial y_1} > 0$ for all y_1. Hence, if $(y^*, y^*) \rightsquigarrow (y_0, y_1)$ for some $y_0 > y^*$, it follows from continuity and the uniqueness of y^* that $y_0 > y_1$. Next, since $(y^*, y^*) \not\rightsquigarrow (\infty, y_1)$

for all y_1 (including $y_1 = \infty$) by assumption, it follows that

$$\overline{y} = \sup \{ y_0 \mid (y^*, y^*) \rightsquigarrow (y_0, y_1) \text{ for some } y_1 \}$$

is finite. It follows that for any $y_0 > y^*$, it must be the case that $(y^*, y^*) \not\rightsquigarrow (y_0, y_1)$ if $y_1 \geq \overline{y}$. Thus, any continuous path that originates at (y^*, y^*) for which $y_0(\tau) > y^*$ is bounded in its y_1 term from above by $\overline{y}$. But the fact that $\lim_{y_1 \to 0} Q(\xi; y_1) < 0$ for all finite $\xi > 0$, together with continuity and the uniqueness of y^*, implies that this occurs only if $(y^*, y^*) \rightsquigarrow (y_0, y_1)$ for some $y_0 > y^*$ and some y_1 such that $\lim_{\tau \to 1} \frac{y_0(\tau)}{y_1(\tau)} = \infty$. Since $y_0(\tau) \leq \overline{y}$ for all τ, this requires that $\lim_{\tau \to 1} y_1(\tau) = 0$. But this contradicts the fact that $(y^*, y^*) \not\rightsquigarrow (\widetilde{y}_0, 0)$. It follows that either $(y^*, y^*) \rightsquigarrow (\widetilde{y}_0, 0)$ or $(y^*, y^*) \rightsquigarrow (\infty, y_1)$ where $\lim_{\tau \to 1} \frac{y_0(\tau)}{y_1(\tau)} > 1$ along this path.

The final step is to prove that the fact that $(y^*, y^*) \rightsquigarrow (\infty, y_1)$ where $\lim_{\tau \to 1} \frac{y_0(\tau)}{y_1(\tau)} > 1$ implies there exists a solution (y_0, y_1) with $y_0 > y_1$ such that $\phi v_0(L - y_0, L - y_1) = \phi v_1(L - y_1, L - y_0) = 1$. Consider

$$\lim_{y_i \to \infty} v_i(L - y_i, L - y_{1-i}) = \lim_{y_i \to \infty} \frac{(\lambda - 1)\left[\omega(L - y_{1-i}) y_i + \left(\mu \frac{Z_{1-i}}{Z_i} y_{1-i}^{1-\alpha} - \frac{(\omega(L - y_{1-i}) + \mu)\lambda F}{(1-\alpha)(\lambda - 1) Z_i}\right) y_i^{\alpha}\right]}{\omega(L - y_i)\,\omega(L - y_{1-i}) - \mu^2}$$

As $y_i \to \infty$, the numerator converges to $\pm\infty$, depending on the sign of $\omega(L - y_{1-i})$, and since $\omega(L - y_i) \to -\infty$, the denominator converges to $\pm\infty$, again depending on the sign of $\omega(L - y_{1-i})$. Applying L'Hopital's rule, we obtain

$$\lim_{y_i \to \infty} v_i(L - y_i, L - y_{1-i}) = -\frac{(\lambda - 1)\,\omega(L - y_{1-i})}{\omega'(\cdot)\,\omega(L - y_{1-i})} < 0$$

Hence, since $v_0(L - y_0, L - y_1) < 0$ as $y_0 \to \infty$, it follows by continuity that there exists a pair (y_0, y_1) such that $\phi v_i(L - y_i, L - y_{1-i}) = 1$. Again, since $v_0(L - y, L - y) = v_1(L - y, L - y)$ if and only if $y = y^*$ and $\lim_{\tau \to 1} \frac{y_0(\tau)}{y_1(\tau)} > 1$, it follows that $y_0(\tau) > y_1(\tau)$ by continuity.

Remark: all cases for $\frac{\partial Q(1; y^*)}{\partial \xi}$ are possible, depending on parameter values. In cases II and III, there will be multiple solutions to the problem $\phi v_i(R_i, R_{1-i}) = 1$, i.e. in addition to the solution identified above, there also exists a second solution with $R_0 > R_1$. However, the existence of multiple solutions does not necessarily imply multiple equilibria, since these solutions may involve negative values of R_i. ■

Proof of Proposition 5: Given that firms maximize expected profits, a firm that has successfully innovated will choose the time to implement by solving

$$\max_s E_t \left[e^{-\rho s} \frac{1/P_{t+s}}{1/P_t} v_{t+s} \right]$$

Suppose the solution in Proposition 4 is an equilibrium. The Proposition follows if we can show that $s = 0$. Clearly, if $z_{t+s} = z_t$, there is no benefit from delay, since $v_{t+s} = v_t$ and so in the best case scenario the firm becomes the leader and earns v_t discounted at a positive rate. Thus, given other agents are implementing immediately and innovating in accordance with Proposition 4, a firm will only delay implementation until a change in the level of productivity. If the current level of productivity is equal to Z_1, then given $v_0 = v_1 = \frac{1}{\phi}$ in equilibrium, waiting until a regime change yields at most

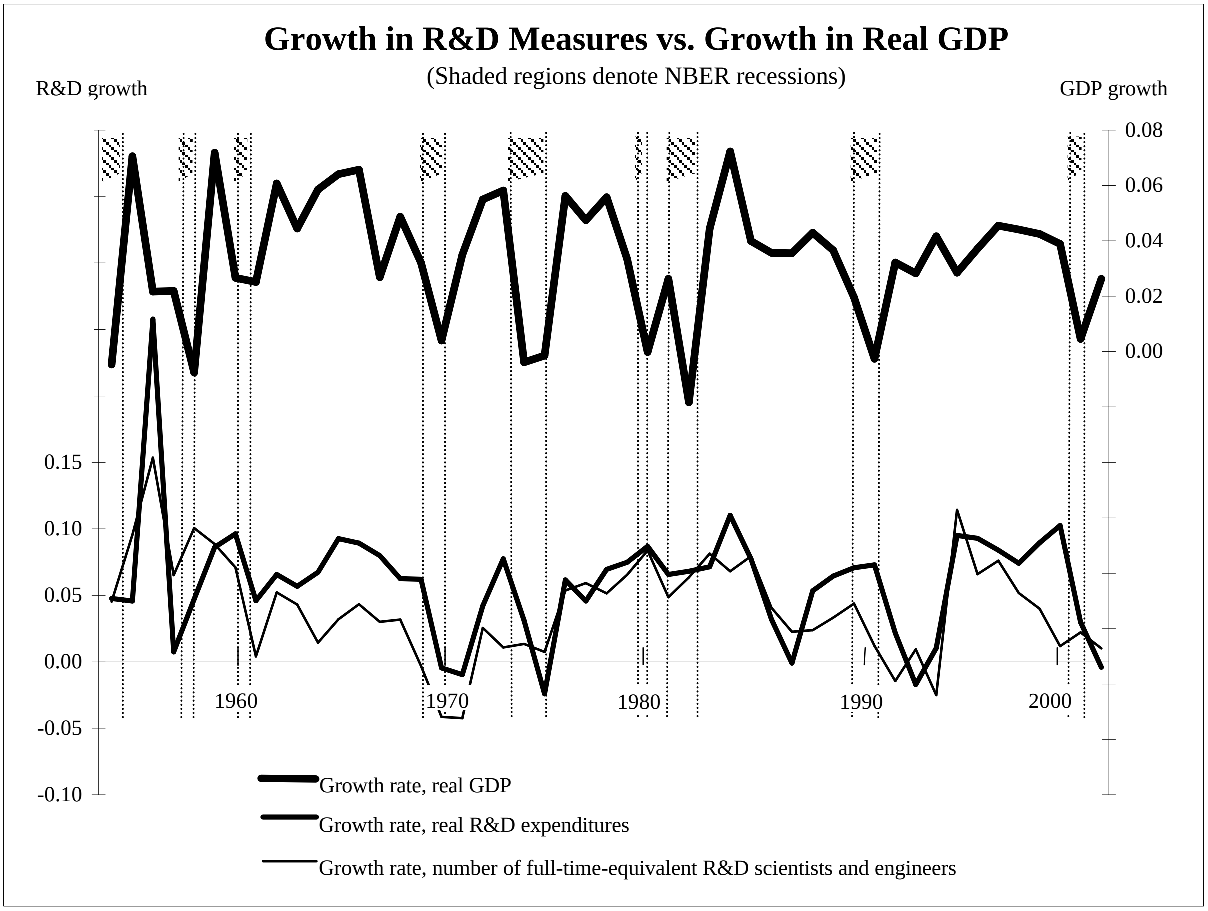
$$\frac{\mu}{\rho + \mu} \frac{P_1}{P_0} v = \frac{\mu}{\rho + \mu} \frac{Z_0}{Z_1} \left(\frac{L - R_1}{L - R_0} \right)^{\alpha} v$$

assuming the firm is the leading producer when it implements. Since $R_1 > R_0$, it follows that this is less than v. Thus, there is no reason to delay an innovation uncovered when productivity is high. Conversely, there will be no reason to delay an innovation that is discovered when productivity is low if

$$\frac{\mu}{\rho + \mu} \frac{Z_1}{Z_0} \left(\frac{L - R_0}{L - R_1} \right)^{\alpha} < 1$$

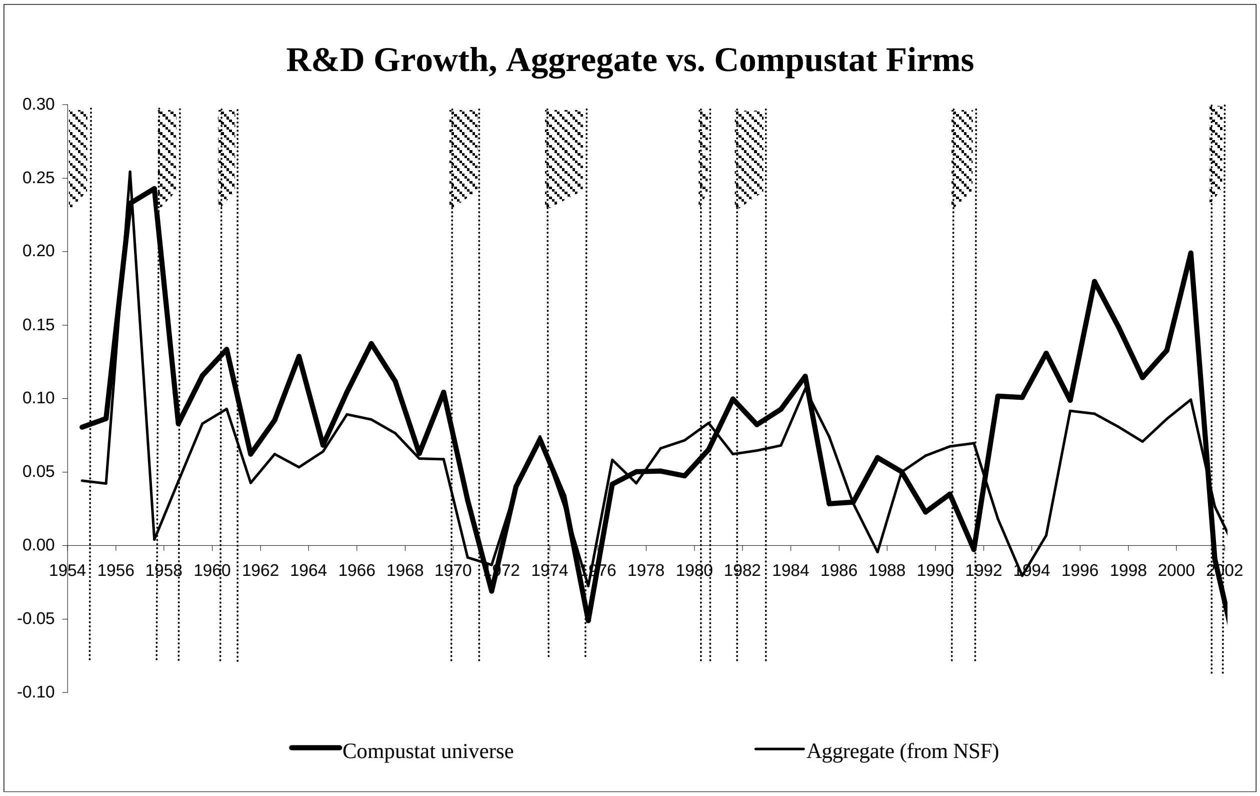
By assumption, $\frac{\mu}{\rho + \mu} \frac{Z_1}{Z_0} < 1$. Moreover, by continuity, the solution (R_0, R_1) identified in Proposition 4 limits to (R^*, R^*) as $F \to F^*$. Thus, there will be no benefit from delay even though $R_1 > R_0$ for F close to F^*.

Figure 1



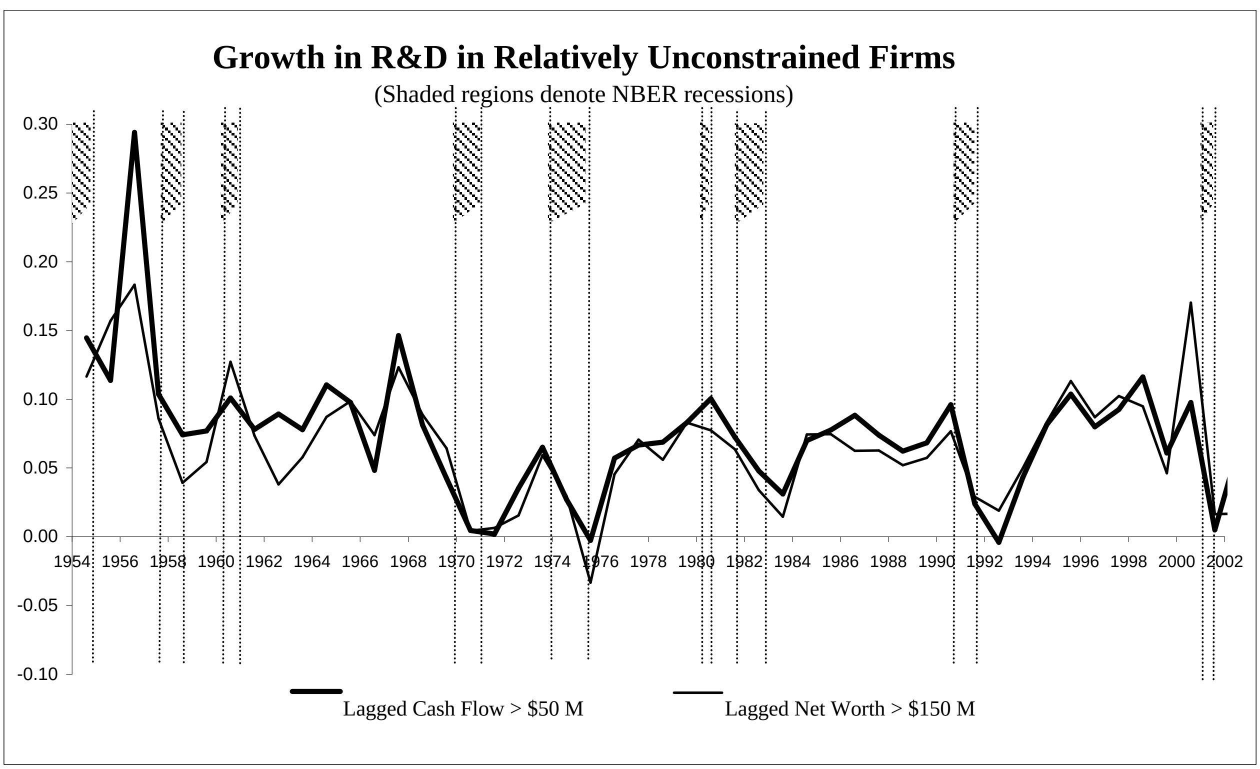
Source: National Science Foundation

Figure 2



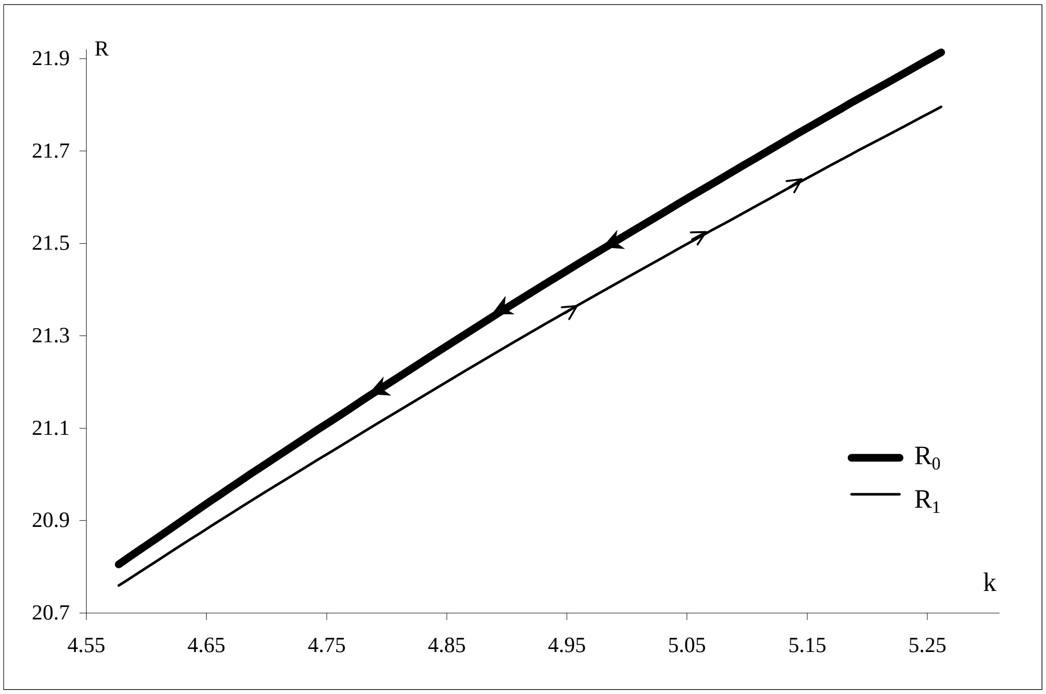
Source: National Science Foundation and S&P's Compustat database

Figure 3

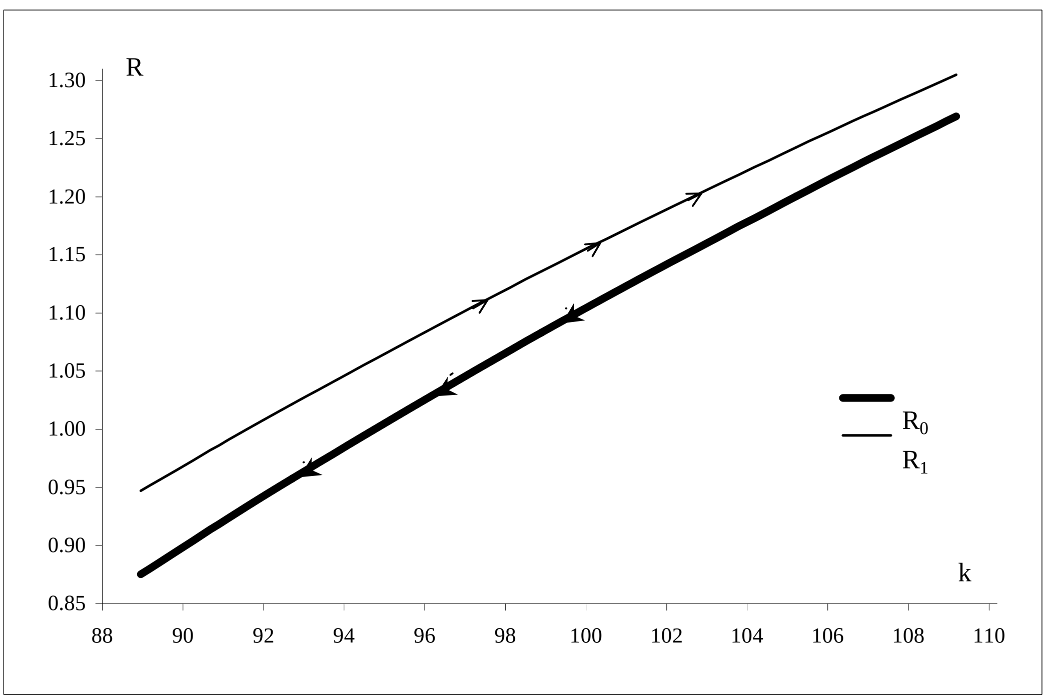


Source: S&P Compustat database

Figure 4



Optimal R&D as a function of capital ratio k



Equilibrium R&D as a function of capital ratio k

Note: arrows indicate direction of capital accumulation for each respective level of productivity

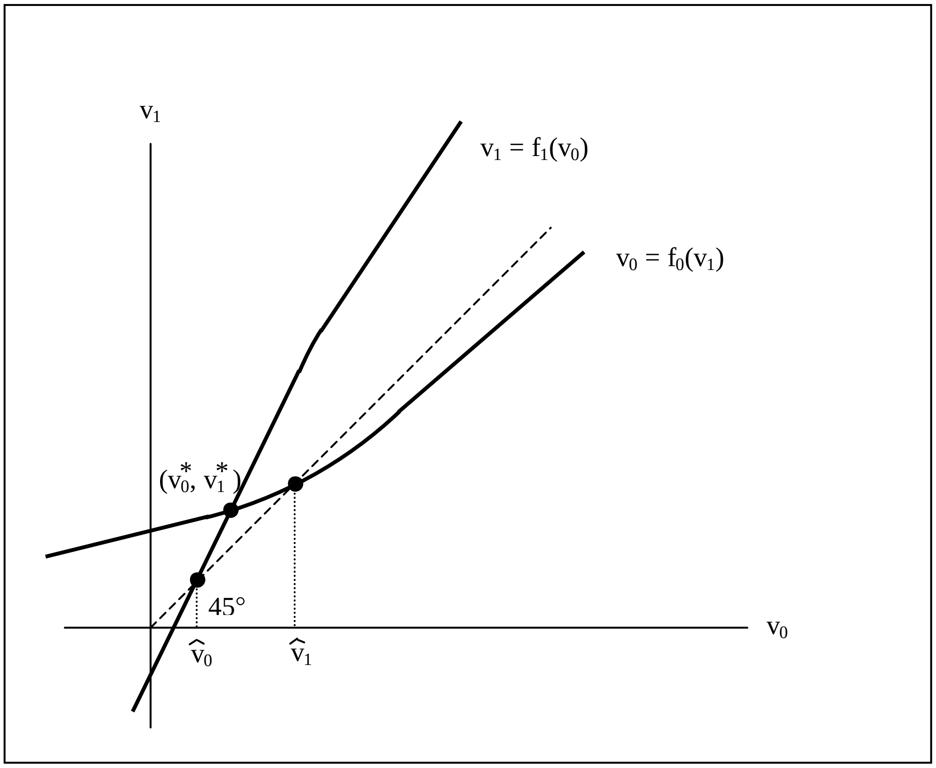


Figure A1: Uniqueness of Social Optimum

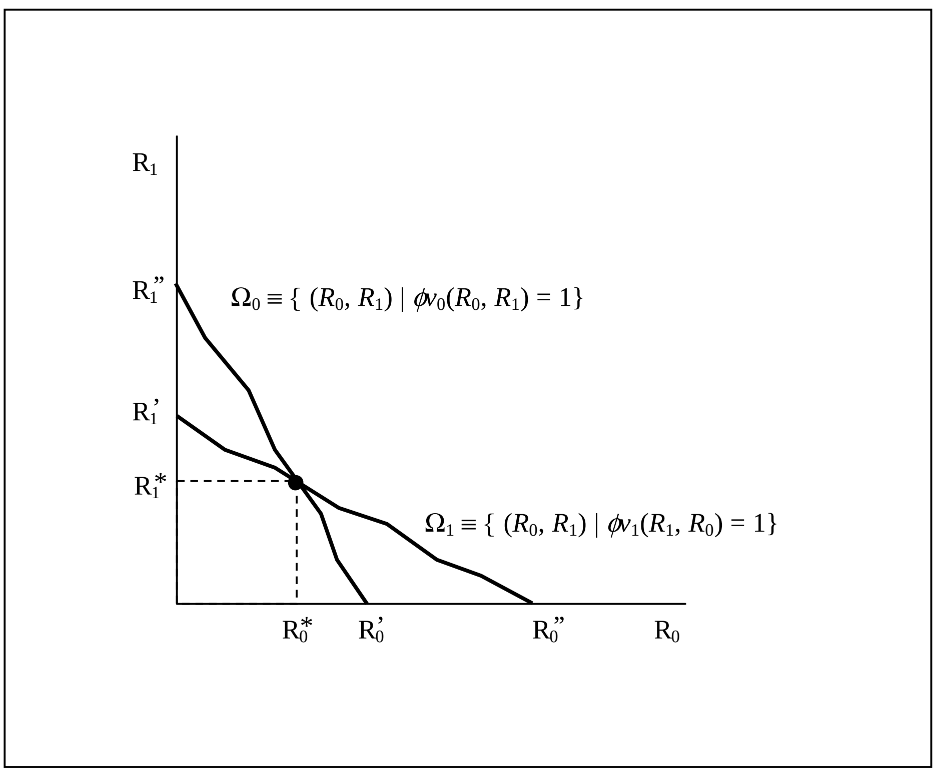
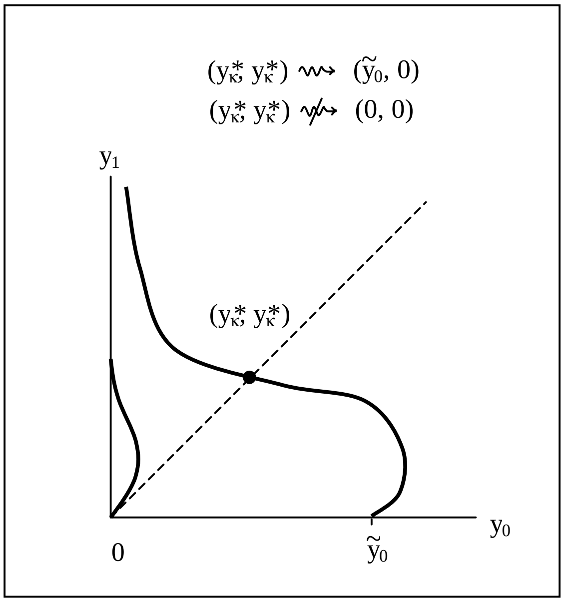
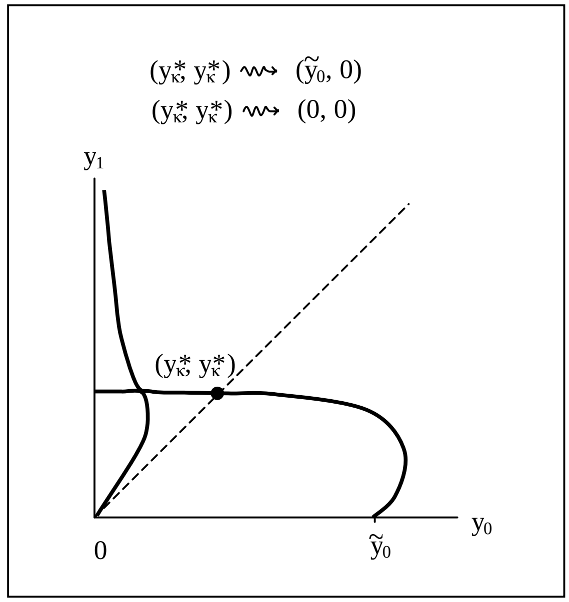


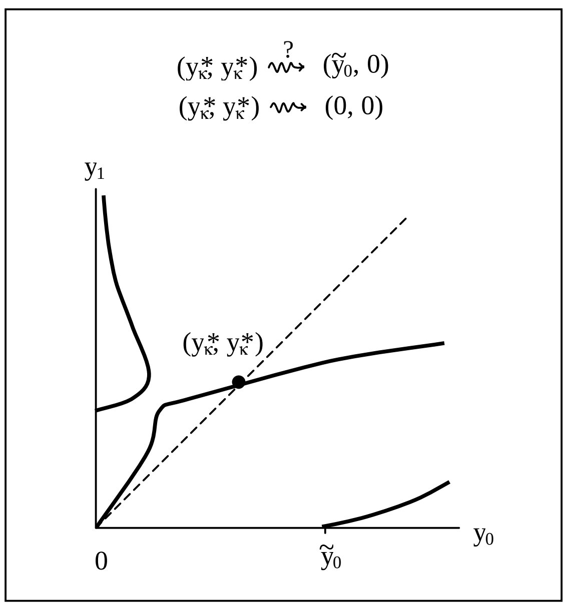
Figure A2: Uniqueness of Markov-Perfect Equilibrium



(A)



(B)



(C)

Figure A3: The Evolution of the set S in Proposition 4

Panel (A) corresponds to Case I in proof of Proposition 4
Panel (B) corresponds to Case II in proof of Proposition 4
Panel (C) corresponds to Case III in proof of Proposition 4

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