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Abstract: This paper introduces a general method for computing aggregate fluctuations in economies with private information. Instead of the cross-sectional distribution of agents across individual states, the method uses as a state variable a vector of spline coefficients describing a long history of past individual decision rules. The model is then linearized with respect to that vector. Applying the computational method to a Mirrlees RBC economy with known analytical solution recovers the solution perfectly well. This test provides significant confidence on the accuracy of the method.

Keywords: Computational methods, heterogeneous agents, business cycles, private information.

JEL codes: C63, D82, E32.

1 Introduction

This paper introduces a general method for computing aggregate fluctuations of economies with private information. Economies with private information are difficult to solve because promised

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values are contingent on the realization of the aggregate shocks. This makes one of the endogenous state variables, the distribution of agents across promised values, not only infinite dimensional but state-contingent. The computational method described in this paper can handle this case without difficulty. In addition, the method displays three very attractive features: 1) it keeps track of the full distribution of agents across individual states, 2) it can handle irregular shapes for this distribution, and 3) it incorporates the distribution's exact law of motion. However, the method is extremely slow compared to other alternatives, discouraging its use as all-purpose computational method. Where the method excels is with private information economies since not only none of them have been used to compute such an economy, but it is unclear how they could handle their state-contingent distributions.

My basic strategy for the computational method is to parametrize individual decision rules as spline approximations and to keep long histories of the spline coefficients as state variables.[2] Starting from the deterministic steady-state distribution, I use the history of decision rules implied by the spline coefficients to obtain the current distribution of agents across individual states. I do this by performing Monte Carlo simulations on a large panel of agents. All individual first-order conditions and aggregate feasibility constraints are then linearized with respect to the history of spline coefficients. The resulting linear model is then solved using standard methods. I show that a simple transformation can be applied to this solution in order to handle the case of contingent endogenous state variables.

In order to facilitate the comprehension of the computational method, and for sake of concreteness, I introduce it in the context of a mechanism design problem for a Mirrlees economy with aggregate shocks.[3] The economy, which belongs to the class considered in Veracierto (2021), is populated by agents that value consumption and leisure using logarithmic utility functions and which are subject to idiosyncratic shocks to their value of leisure. These shocks take only two possible values, are i.i.d. over time and across individuals, and are private information. Output, which can be consumed or invested, is produced using capital and labor as inputs to a Cobb-Douglas production function subject to aggregate productivity shocks. The aggregate shocks follow a standard AR(1) process.

[2] In practice, I use the monotonicity preserving cubic splines described in Steffen (1990).

[3] The Appendix describes the computational method in general terms.

A social planner designs dynamic contracts for the agents in this Mirrlees real business cycle (RBC) economy. Following the literature, a dynamic contract is given a standard recursive formulation where a promised value to the agent describes its state. Given the current state, the contract specifies current consumption, current hours worked, and next-period state-contingent promised values as a function of the value of leisure reported by the agent. Since the model has a large number of agents and the shocks to the value of leisure are idiosyncratic, the social planner needs to keep track of the whole distribution of promised values across individuals as a state variable. Given this distribution, the aggregate stock of capital, and the aggregate productivity level, the social planner seeks to maximize the present discounted utility of agents subject to incentive compatibility, promise keeping, and aggregate resource feasibility constraints.

Solving this mechanism design problem not only illustrates the computational method by applying it to a workhorse private information economy in the literature, but provides a strong test for it. The reason is that in Veracierto (2021) I provide a sharp analytical characterization of the solution to this mechanism design problem. In particular, I characterize the cyclical behavior of the consumption and leisure allocation rules across promised values, as well as the optimal amount of cross-sectional inequality in consumption and leisure over the business cycle. I also show that all macroeconomic variables in the private information economy are exactly the same as under full information. This provides an important quantitative test of the computational model, since we can compare the computed macroeconomic variables of the Mirrlees economy with those from the representative agent economy. I find that the computational method passes these tests extremely well: It recovers all the results, both qualitative and quantitative, almost exactly. Since nothing in the computational method exploits the functional forms or structure of the Mirrlees RBC economy considered, this provides significant evidence about its accuracy. This finding indicates that the method should prove useful in a variety of other private information economies.

The paper is closely related to a vast literature on computational methods, but it has salient differences.[4] The seminal papers by Krusell et al. (1998) and Den Haan (1996) summarize the cross-sectional distribution with a small set of moments. In contrast, the method in this paper keeps track of the whole distribution. Den Haan (1997) and Algan et al. (2008) also keep track of the whole distribution but parametrize the distribution with a flexible exponential polynomial

[4]See Algan et al. (2014) for a survey of computational methods.

form, allowing them to solve the model using quadrature and projection techniques. For many applications this may be an accurate and convenient approach, but for economies with odd-shaped distributions, it may not be. The method in this paper is able to handle odd shapes for the cross-sectional distribution as long as it is generated by smooth individual decision rules.

In addition to projection methods, the literature has explored perturbation methods, which are essentially local approximation methods around a deterministic steady state. Early versions include Campbell (1998), Dotsey et al. (1999), and Veracierto (2002) – the last two in the context of (S,s) economies.[5] Perhaps the most widely known perturbation method is Reiter (2009), which is closely related to Campbell (1998).[6] Instead of parametrizing the cross-sectional distribution as a polynomial, Reiter (2009) keeps a finite histogram of the distribution as a state variable. While the perturbation method allows him to greatly reduce the coarseness of the histogram, a limitation of Reiter's method is that the law of motion for the distribution needs to be approximated, and this can be a highly non-linear mapping.[7] Instead, my method here embodies the exact law of motion for the distribution. Winberry (2018) introduces a very interesting perturbation method which, similarly to Algan et al. (2008), parametrizes the distribution with a flexible exponential polynomial form. The perturbation method allows him to carry a polynomial of large order as a

[5]The method in this paper actually generalizes the approach in Veracierto (2002) to economies with general decision rules and cross-sectional distributions of agents with infinite support.

[6]The recent method in Ahn et al. (2018) is an adaptation of Reiter's method to continuous time. Other perturbation methods in the literature include Preston and Roca (2007) and Mertens and Judd (2018), both of which perturb a deterministic steady state with no aggregate or idiosyncratic shocks. In contrast, the method in this paper perturbs a deterministic steady state with no aggregate shocks but positive idiosyncratic uncertainty.

[7]For instance, consider the Krusell et al. (1998) model. In this model there is generally a mass of agents with the lowest idiosyncratic income level and zero assets (these agents are at the borrowing constraint). Now consider the steady state assets level chosen by these agents when they transit to a higher idiosyncratic income level. Suppose that this assets level falls within the first range of the histogram. Whenever there is a positive aggregate productivity shock, this choice of assets will generally increase. If the aggregate shock happens to be small enough that the modified assets level still falls within the first range of the histogram, there will be no effects at all on the histogram. However, if the shock is large enough that the modified assets level falls within the second range of the histogram, there will be a discrete reduction in the size of the first bar of the histogram and a discrete increase in the size of the second bar. Thus, the histogram bars change quite non-linearly with respect to the assets level chosen. This non-linearity problem can only be exacerbated when reducing the coarseness of the histogram.

state variable (or, equivalently, a large number of moments), which greatly improves the description of the distribution. However, his method also relies on an approximation for the law of motion of the cross-sectional distribution.

Another powerful method has been introduced by Boppart et al. (2018) and improved by Auclert et al. (2021). This method requires computing transitionary dynamics after an unexpected aggregate shock, starting from a given deterministic steady state. In most contexts this can be readily done. However, the approach is ill suited for economies with private information. The reason is that since the distribution of agents across individual states is state-contingent, when the shock hits the economy the distribution shifts endogenously. As a consequence, there is no fixed starting point from which to start the deterministic transitionary dynamics. Ignoring the endogenous shift in the distribution on impact (assuming that it is fixed at its deterministic steady state position) would give the wrong results.

In order to see this in a simpler case, consider a full information RBC model with perpetual youth and identical agents, in which the social planner must design recursive contracts with a promised value as their state. Depending on the realization of the aggregate productivity shock and the stock of capital the planner must decide current consumption, current hours worked and next-period promised values (contingent on the realization of next-period aggregate productivity). The planner is committed to delivering the promised value of the recursive contract to the old agents (those born in a previous period) but can choose any value for the young agents (those born in the current period). Since agents are identical, we know that the optimal allocation must be the same for all agents of all generations. However, consider computing the transitionary dynamics after an unexpected positive aggregate shock starting from the deterministic steady state (as in Auclert et al. (2021)), assuming that the promised value to the initial old agents is fixed at the deterministic steady-state level. Since the planner is committed to delivering that steady-state promised value to the initial old agents but is free to choose any allocation for the initial young, this second group will receive all the benefits from the higher productivity while the first group will be left behind. This asymmetric treatment of identical agents in different generations is suboptimal from an ex-ante perspective. The source of the inefficiency is failing to recognize that the promised value of the initial old should have been contingent on the realization of the aggregate productivity shock and that this shock cannot have been completely unexpected.

In fact, not only Boppart et al. (2018) and Auclert et al. (2021) have difficulties handling state-contingent distributions: none of the other papers cited above have addressed this case.[8] On the contrary, the contribution of this paper is to provide an extension of the perturbation approach to handle state-contingent distributions. This is crucial for analyzing economies with private information and aggregate shocks because optimal next-period promised values are contingent on the realization of next-period aggregate shocks, making the distribution of agents across promised values state-dependent.

The paper is organized as follows. Section 2 describes the Mirrlees RBC economy with known solution. Section 3 introduces the computational method in the context of the Mirrlees RBC economy. Section 4 tests the method's accuracy. Section 5 discusses its applicability. Finally, Section 6 concludes the paper. Appendix 8 presents the computational method in general form while all proofs are provided in an accompanying Technical Appendix.

2 A Mirrlees RBC economy with known solution

The economy is populated by a unit measure of agents subject to stochastic lifetimes. Whenever an agent dies they are immediately replaced by a newborn, leaving the aggregate population level constant over time.[9] The preferences of an individual born at date T are given by

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[\ln(c_t) + \alpha_t \ln(1-h_t) \right] \right\}, \tag{2.1}$$

where σ is the survival probability, $0 < \beta < 1$ is the discount factor, and $\alpha_t \in \{\alpha_L, \alpha_H\}$ is the idiosyncratic value of leisure (where $\alpha_L < \alpha_H$). Realizations of α_t are assumed to be i.i.d. both across individuals and across time. The probability that $\alpha_t = \alpha_s$ is given by ψ_s. A key assumption is that α_t is private information of the individual.

Output, which can be consumed or invested, is produced with the following production function:

$$Y_t = e^{z_t} K_{t-1}^{\gamma} H_t^{1-\gamma}, \tag{2.2}$$

[8] This does not mean that none of these other methods could handle state-contingent distributions. In particular, the method in Krusell et al. (1998) may be flexible enough to do so. However, its implementation details and computational costs are unclear at this point

[9] As in Phelan (1994), the stochastic lifetime guarantees that there will be a stationary distribution of agents across individual states.

where $0 < \gamma < 1$, Y_t is output, z_t is aggregate productivity, K_{t-1} is capital, and H_t is hours worked. The aggregate productivity level z_t follows a standard AR(1) process given by:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}, \tag{2.3}$$

where $0 < \rho < 1$, and ε_{t+1} is normally distributed with mean zero and standard deviation σ_ε.

Capital is accumulated using a standard linear technology given by

$$K_t = (1 - \delta) K_{t-1} + I_t, \tag{2.4}$$

where I_t is gross investment and $0 < \delta < 1$.

2.1 The mechanism design problem

In what follows, I will describe the mechanism design problem for this economy. To do this, it will be convenient to distinguish between two types of agents: young and old. A young agent is one that has been born at the beginning of the current period. An old agent is one that has been born in some previous period. The social planner must decide recursive plans for both types of agents. The state of a recursive plan is the value (i.e., discounted expected utility) that the agent is entitled to at the beginning of the period. Given this promised value, the recursive plan specifies the current utility of consumption, the current utility of leisure, and next-period promised values as functions of the value of leisure currently reported by the agent. The social planner is fully committed to the recursive plans they choose and agents have no outside opportunities available.

A key difference between the young and the old is in terms of promised values. Since during the previous period the social planner has already decided on some recursive plan for a currently old agent, the planner is restricted to delivering the corresponding promised value during the current period. In contrast, the social planner is free to deliver any value to a currently young agent since this is the first period they are alive. Reflecting this difference, I will specify the individual state of an old agent to be their promised value v and their current value of leisure s (henceforth, I will refer to the value of leisure α_s by its subindex $s \in \{L, H\}$). At date t, their current utility of consumption, utility of leisure, and next-period promised value are denoted by $u_{ost}(v)$, $n_{ost}(v)$ and $w_{os,t+1}(v)$, respectively, where $w_{os,t+1}(v)$ is a random variable contingent on the realization of z_{t+1}. In turn, the individual state of a young agent is solely given by their current value of leisure s. At date t, the agent's current utility of consumption, utility of leisure, and next-period promised value are denoted by u_{yst}, n_{yst} and $w_{ys,t+1}$ respectively, where $w_{ys,t+1}$ is also contingent on the realization of z_{t+1}.

The social planner seeks to maximize the weighted sum of the welfare levels of the current and future generations of young agents, subject to individual incentive compatibility and promise keeping constraints, as well as aggregate feasibility constraints.[10] Veracierto (2021) describes this economy-wide planning problem in detail. However, for computational purposes, it will be convenient to decompose that planning problem into a sequence of sub-planning problems and focus on these problems instead. In each period t, there are two sub-planning problems: one sub-planning problem concerned with providing insurance and incentives to individuals, and another sub-planning problem concerned with making production and investment decisions. In these sub-planning problems, the joint stochastic process for the shadow price of labor (in terms of the consumption good), q_t, and the shadow price of the consumption good (in utiles), λ_t, are taken as given. The solutions to these sub-planning problems correspond to that of the economy-wide planning problem if certain side conditions are satisfied.

The sub-planning problems for individuals differ depending on whether the individual is young or old. However, both planning problems face a similar trade-off: since agents are risk averse, the social planners would like to insure them as much as possible. However, full insurance cannot be provided because the agents would always report to have a high value of leisure (in order to work less). As a consequence, when an agent reports that they have a high value of leisure the social planner allows them to work less but punishes them by providing a lower consumption level and promising to treat them worse in the future. The reason why it is efficient to lower their future promised values is that it allows the planner to smooth the punishment over time (and agents value smoother allocations).

For every date t, the sub-planning problem for old individuals is as follows:

$$P_{ot}\left(v\right) = \max_{\{u_{ost}, n_{ost}, w_{os,t+1}\}_s} \sum_s \psi_s \left\{ q_t h(n_{ost}) - c\left(u_{ost}\right) + \theta\sigma E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{o,t+1}\left(w_{os,t+1}\right) \right] \right\} \tag{2.5}$$

subject to

$$u_{oLt} + \alpha_L n_{oLt} + \beta\sigma E_t \left[w_{oL,t+1}\right] \geq u_{oHt} + \alpha_L n_{oHt} + \beta\sigma E_t \left[w_{oH,t+1}\right], \tag{2.6}$$

[10]The welfare levels of the current old agents are predetermined by their promised values at the beginning of the period.

$$v = \sum_s \left\{ u_{ost} + \alpha_s n_{ost} + \beta\sigma E_t \left[w_{os,t+1} \right] \right\} \psi_s, \tag{2.7}$$

where $h(n)$ are the hours worked implied by the utility of leisure n (i.e. $h(n) = 1 - e^n$), and $c(u)$ is the consumption level implied by the utility of consumption u (i.e. $c(u) = e^u$). Observe that the current "social profits" in equation (2.5) are given by the social value of the hours worked by the old agent, net of the consumption goods that are transferred to them. Also observe that the sub-planner discounts the future social profits of the old individual using the social discount factor θ, the survival probability σ, and the stochastic social discount factor λ_{t+1}/λ_t. The social discount rate θ is the Pareto weight of the next-period generation of young agents relative to the Pareto weight of the current generation of young agents.[11] Equation (2.6) is the binding incentive compatibility constraint. It states that the expected value to the individual of truthfully reporting the low value of leisure L must be at least as large as the expected value to the individual of misreporting the high value of leisure H.[12] Equation (2.7) is the promise-keeping constraint. It states that the social sub-planner must deliver the expected value v that was promised at the beginning of the period.[13] Hereon, the solution to this sub-planning problem will be denoted as $\left\{ u_{ost}(v), n_{ost}(v), w_{os,t+1}(v) \right\}_s$, indicating its dependence on the state variable v. This solution must satisfy the following first-order conditions:

$$0 = -e^{u_{oLt}(v)} \psi_L + \xi_{ot}(v) + \eta_t(v) \psi_L, \tag{2.8}$$

$$0 = -e^{u_{oHt}(v)} \psi_H - \xi_{ot}(v) + \eta_t(v) \psi_H, \tag{2.9}$$

$$0 = -q_t e^{n_{oLt}(v)} \psi_L + \alpha_L \xi_{ot}(v) + \eta_t(v) \alpha_L \psi_L, \tag{2.10}$$

$$0 = -q_t e^{n_{oHt}(v)} \psi_H - \alpha_L \xi_{ot}(v) + \eta_t(v) \alpha_H \psi_H, \tag{2.11}$$

$$0 = \lambda_t \beta\sigma \xi_{ot}(v) + \lambda_t \eta_t(v) \beta\sigma\psi_L - \theta\lambda_{t+1}\sigma\psi_L \eta_{t+1}\left[w_{oL,t+1}(v) \right], \tag{2.12}$$

$$0 = -\lambda_t \beta\sigma \xi_{ot}(v) + \lambda_t \eta_t(v) \beta\sigma\psi_H - \theta\lambda_{t+1}\sigma\psi_H \eta_{t+1}\left[w_{oH,t+1}(v) \right], \tag{2.13}$$

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[11] I assume that $\beta\sigma < \theta < 1$.

[12] Implicitly, the planning problem (2.5)-(2.7) only considers one-time deviations to the incentive compatibility constraint (2.6). There is no loss of generality in doing this because it can be shown that if an agent has lied in the past it always dominates to tell the truth in the future (see Fernandes and Phelan (2000)). In fact, this is why the recursive representation in Fernandes and Phelan (2000) is even possible.

[13] The expectation operator $E_t[.]$ in equations (2.5)-(2.7) conditions on the aggregate productivity value z_t.

where $\lambda_t \xi_{ot}(v)$ and $\lambda_t \eta_t(v)$ are the Lagrange multipliers of equations (2.6) and (2.7), respectively.

For every date t, the sub-planning problem for young individuals is as follows:

$$\begin{aligned} P_{yt} \quad = \quad & \max_{\{u_{yst}, n_{yst}, w_{ys,t+1}\}_s} \sum_s \psi_s \left\{ \frac{u_{yst} + \alpha_s n_{yst} + \beta\sigma E_t \left[w_{ys,t+1}\right]}{\lambda_t} + q_t h(n_{yst}) - c\left(u_{yst}\right) \right. \\ & \left. + \theta\sigma E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{o,t+1}\left(w_{ys,t+1}\right) \right] \right\} \end{aligned} \tag{2.14}$$

subject to

$$u_{yLt} + \alpha_L n_{yLt} + \beta\sigma E_t \left[w_{yL,t+1}\right] \geq u_{yHt} + \alpha_L n_{yHt} + \beta\sigma E_t \left[w_{yH,t+1}\right]. \tag{2.15}$$

Observe that in this case the social surplus is given by the expected lifetime utility level of the young agent (in current consumption units), plus the expected social value of the hours worked by the agent, net of the expected consumption goods transferred to them. Since, conditional on surviving the young agent becomes old after one period, the function used to evaluate next-period continuation values is $P_{o,t+1}$. The reason why the lifetime utility level of the young agent enters equation (2.14) but the promised value of an old agent does not enter equation (2.5) is that the lifetime utility level of a current young agent directly enters into the objective function of the economy-wide social planner while the promised value to an old agent is a bygone (which the economy-wide social planner is committed to deliver). The first-order conditions are the following:

$$0 = \psi_L - \lambda_t e^{u_{yLt}} \psi_L + \lambda_t \xi_{yt}, \tag{2.16}$$

$$0 = \psi_H - \lambda_t e^{u_{yHt}} \psi_H - \lambda_t \xi_{yt}, \tag{2.17}$$

$$0 = \alpha_L \psi_L - \lambda_t q_t e^{n_{yLt}} \psi_L + \alpha_L \lambda_t \xi_{yt}, \tag{2.18}$$

$$0 = \alpha_H \psi_H - \lambda_t q_t e^{n_{yHt}} \psi_H - \alpha_L \lambda_t \xi_{yt}, \tag{2.19}$$

$$0 = \beta\sigma\psi_L + \lambda_t \beta\sigma\xi_{yt} - \theta\lambda_{t+1}\sigma\psi_L \eta_{t+1}\left(w_{yL,t+1}\right), \tag{2.20}$$

$$0 = \beta\sigma\psi_H - \lambda_t \beta\sigma\xi_{yt} - \theta\lambda_{t+1}\sigma\psi_H \eta_{t+1}\left(w_{yH,t+1}\right), \tag{2.21}$$

where $\lambda_t \xi_{yt}$ is the Lagrange multiplier of equation (2.15).

For every date t, the sub-planning problem for production decisions is

$$P_{pt}\left(K_{t-1}\right) = \max_{\{H_t, I_t\}} \left\{ e^{z_t} K_{t-1}^{\gamma} H_t^{1-\gamma} - q_t H_t - I_t + \theta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{p,t+1}\left(\left(1-\delta\right) K_{t-1} + I_t\right) \right] \right\}. \tag{2.22}$$

Observe that the social surplus generated by this planning problem is given by output net of the value of the labor input and the value of investment. Its first order conditions are standard given the neoclassical growth structure:

$$0 = q_t - e^{z_t} K_{t-1}^{\gamma} (1-\gamma) H_t^{-\gamma}, \tag{2.23}$$

$$0 = -\lambda_t + \theta E_t \left\{ \lambda_{t+1} \left[e^{z_{t+1}} \gamma K_t^{\gamma-1} H_{t+1}^{1-\gamma} + 1 - \delta \right] \right\}, \tag{2.24}$$

The economy-wide distribution of old agents across promised values v at the beginning of period t is given by a measure μ_t, while the number of young agents is constant over time and given by $1-\sigma$.[14] Given the stochastic sequence of decision rules $[u_{ost}, n_{ost}, w_{os,t+1}, u_{yst}, n_{yst}, w_{ys,t+1}]_s$ that solve the corresponding sub-planning problems for individuals, the law of motion for μ_t is given as follows:

$$\mu_{t+1}(B) = \sigma \sum_s \int_{\{v:\ w_{os,t+1}(v) \in B\}} \psi_s d\mu_t + (1-\sigma)\sigma \sum_{s:\ w_{ys,t+1} \in B} \psi_s, \tag{2.25}$$

for every Borel set B. Equation (2.25) states that the number of old agents that have a promised value in the Borel set B at the beginning of the following period is given by the sum of two terms. The first term sums all currently old agents that receive a next-period promised value in the set B and do not die. The second term does the same for all currently young agents. Observe that, since the promised values $w_{os,t+1}$ and $w_{ys,t+1}$ are contingent on the realization of the aggregate productivity shock z_{t+1} at the beginning of period $t+1$, the distribution μ_{t+1} is also state contingent.[15]

The economy-wide stock of capital at the beginning of period t is equal to K_{t-1}. Given the stochastic-process for I_t that solves the sub-planning production problems, K_t follows a stochastic process given by

$$K_t = (1-\delta) K_{t-1} + I_t. \tag{2.26}$$

[14] Since agents are not fully insured, the social planners generate heterogeneity across agents. In fact, the amount of inequality increases over time within a same cohort of agents. The reason why there is a well defined cross-sectional distribution of agents across cohorts is that agents die stochastically and are replaced by newborns, leading to a strong "reversion to the mean".

[15] The state-contingency of this economy-wide distribution is what makes solving economies with private information so different from solving standard problems.

The side conditions that the stochastic shadow prices $\{q_t, \lambda_t\}_{t=1}^{\infty}$ need to satisfy are that

$$(1-\sigma)\sum_s c(u_{yst})\psi_s + \int \sum_s c(u_{ost}(v))\psi_s d\mu_t + I_t = e^{z_t} K_{t-1}^{\gamma} H_t^{1-\gamma}, \tag{2.27}$$

and

$$H_t = (1-\sigma)\sum_s h(n_{yst})\psi_s + \int \sum_s h(n_{ost}(v))\psi_s d\mu_t, \tag{2.28}$$

almost surely at every period t. Equation (2.27) describes the aggregate feasibility constraint for the consumption good. It states that the total consumption of young and old agents, plus aggregate investment cannot exceed aggregate output. Equation (2.28) is the aggregate labor feasibility constraint. It states that the input of hours into the production function cannot exceed the total hours worked by young and old agents.

3 Computations

This section introduces the computational method by applying it to the Mirrlees economy described in the previous section.[16] The purpose is to familiarize the reader with the computational method using a concrete example (the various elements of the method will reappear much more abstractly in the general specification provided in Appendix 8). Its main features are the following. Instead of keeping track of the distribution of promised values μ as a state variable, what the computational method keeps track of is a long history of individual decision rules w_{os} and w_{ys}. Since the individual decision rules w_{os} are parametrized as spline approximations, the computational method only needs to keep track of a long but finite history of spline coefficients. The current distribution of promised values is then recovered by simulating the evolution of a large number of agents (and their descendants) over time using the history of individual decision rules kept as state variables.[17] The next period distribution of promised values is then obtained by simply updating by one period the history of individual decision rules using the decision rules chosen during the

[16] Technical Appendix 10 provides further implementation details.

[17] Because of the stochastic lifetimes, the truncation introduced by the finite history of decision rules generates arbitrarily small approximation errors as the length of the history becomes large. In fact, when this length becomes large the distribution used for drawing initial promised values for the simulations becomes irrelevant (although, in practice, I use the invariant distribution of the deterministic steady state).

current period. All first order conditions and aggregate feasibility constraints are then linearized with respect to the spline coefficients describing current and past individual decision rules.[18] This delivers a linear rational expectations model which, despite of its high dimensionality, can be solved for using standard methods.

3.1 Computing the deterministic steady state

While computing the deterministic steady state of the model is completely standard, this section describes the algorithm in detail since this will introduce objects and notation that will be needed later on.

Observe that the shadow value of labor q is known from the steady state versions of equations (2.23) and (2.24). In particular it is given by

$$q = (1-\gamma)\left\{\frac{1}{\gamma}\left[\frac{1}{\theta}-1+\delta\right]\right\}^{\frac{\gamma}{1-\gamma}}.$$

Given this value of q, the steady state decision rules for old agents can then be solved for. To this end, I find it convenient to use cubic spline approximations and iterate with the steady state versions of equations (2.8)-(2.13).[19] In order to do this, I first restrict the promised values to lie on a closed interval $[v_{\min}, v_{\max}]$ and define an equidistant vector of grid points $(v_j)_{j=1}^J$, with $v_1 = v_{\min}$ and $v_J = v_{\max}$.[20] Given the function η from the previous iteration, which is used to value next period promised values in the steady state versions of equations (2.12) and (2.13), the values of $[u_{os}(v_j), n_{os}(v_j), w_{os}(v_j), \xi_o(v_j), \eta(v_j)]_{j=1}^J$ that satisfy the steady state versions of equations (2.8)-(2.13) are then solved for at the grid points $(v_j)_{j=1}^J$. Once these values are found, the functions are extended to the full domain $[v_{\min}, v_{\max}]$ using cubic splines. The iterations continue until the values for $[u_{os}(v_j), n_{os}(v_j), w_{os}(v_j), \xi_o(v_j), \eta(v_j)]_{j=1}^J$ converge. Observe, that this solution does not depend on any other endogenous values, so it forms part of the steady state.

[18] This is the computationally most intensive part of the method. The reason is that we need to take numerical derivatives with respect to each spline coefficient in the history, and each of these calculations requires simulating the evolution of a large panel of agents over the entire history of individual decision rules kept as state variables.

[19] Observe that the shadow value of consumption λ does not appear in the steady state version of these equations,

[20] When restricting promised values to lie in the interval $[v_{\min}, v_{\max}]$, the first order conditions (2.12)-(2.13) and (2.20)-(2.21) change by incorporating inequalities that check for corner solutions.

Given the steady state solution for η, the steady state decisions for young agents can be solved for next. This is straightforward: conditional on a value for λ, the steady state versions of equations (2.16)-(2.21) can be solved for the finite numbers of unknowns $(u_{ys}, n_{ys}, w_{ys}, \xi_y)$ in one step (no iterations are needed here). Later on I will have to provide the side condition that λ must satisfy for this to form part of the steady state.

The steady state version of equation (2.25) describes the recursion that the invariant μ has to satisfy. This equation corresponds to the case of a continuum of agents. However, I find it convenient to work with a large, but finite number of agents, and perform the recursion for this case. In particular, consider a large but finite number of agents I and endow them with promised values in the interval $[v_{\min}, v_{\max}]$. Using the functions w_{os} and the values w_{ys} already obtained, simulate the evolution of the promised values of these I agents and their descendants for a large number of periods T. To be precise, if agent i was promised a value v at the beginning of the current period (conditional on being alive), then his promised value (or his descendant's, in case the agent dies) at the beginning of the following period will be given by:

$$v' = \begin{cases} w_{os}(v), \text{ with probability } \sigma\psi_s, \\ w_{ys}, \text{ with probability } (1-\sigma)\psi_s, \end{cases} \tag{3.1}$$

Simulating the I agents for T periods using equation (3.1) we obtain a realized distribution $(\bar{v}_i)_{i=1}^I$ of promised values (conditional on being alive) across the I agents. Observe that the last iteration of equation (3.1) also gives the corresponding realized values of leisure $(\bar{\alpha}_i)_{i=1}^I$ across the I agents. The joint realized distribution of promised values and values of leisure $(\bar{v}_i, \bar{\alpha}_i)_{i=1}^I$ can then be used to compute statistics under the invariant distribution. In particular, aggregate consumption can be obtained as

$$C = \sigma \frac{1}{I} \sum_{i=1}^{I} c\left(u_{o,\bar{s}_i}(\bar{v}_i)\right) + (1-\sigma) \sum_s c(u_{ys})\psi_s. \tag{3.2}$$

To understand this expression, suppose that we are at the beginning of period $T+1$. The joint realized distribution $(\bar{v}_i, \bar{\alpha}_i)_{i=1}^I$ now corresponds to agents that were alive in the previous period, and thus a fraction σ of them will have survived and a fraction $(1-\sigma)$ of them will have died. The first term in equation (3.2) corresponds to those who have survived. It averages the consumption of these agents and multiplies the result by the probability of surviving σ. The second term corresponds to those who have died and thus have been replaced by young agents. It averages the consumption of young agents and multiplies the result by the probability of dying $(1-\sigma)$.

Aggregate hours worked can be similarly computed as

$$H = \sigma \frac{\sum_{i=1}^{I} h\left(n_{o,\bar{s}_i}\left(\bar{v}_i\right)\right)}{I} + (1-\sigma)\sum_{s} h\left(n_{ys}\right)\psi_s. \tag{3.3}$$

Observe that by a law of large numbers equations (3.2) and (3.3) will become arbitrarily good approximations to the steady state versions of equations (2.27) and (2.28) as I and T tend to infinity.

Given aggregate hours worked, aggregate capital can be then obtained from the fact that the social planner equates the marginal productivity of capital to its shadow price. In particular, from the steady state version of equation (2.24) we have that aggregate capital is given by

$$K = \left(\frac{\gamma}{\frac{1}{\theta} - 1 + \delta}\right)^{\frac{1}{1-\gamma}} H. \tag{3.4}$$

The last equation that needs to be satisfied is the feasibility condition for consumption,

$$C + \delta K = K^{\gamma} H^{1-\gamma}. \tag{3.5}$$

This is the side condition mentioned above for the shadow value of consumption λ. The shadow value of consumption determines the consumption, hours worked and promised values of young agents, and therefore each of the variables in equation (3.5). Therefore, λ must be changed until equation (3.5) holds.

3.2 Computing business cycle fluctuations

As has already been mentioned, computing business cycle fluctuations requires linearizing the first order conditions and aggregate feasibility constraints with respect to a convenient set of variables. Linearizing equations (2.6)-(2.13), (2.15)-(2.21) and (2.23)-(2.28) present different types of issues. As a consequence, I classify them into different categories.

The first category is constituted by equations that only involve scalar variables. Equations (2.15)-(2.19), (2.23)-(2.24) and (2.26) fall into this category. For example, consider equation (2.17). This equation is a function of $\{\lambda_t, u_{yHt}, \xi_{yt}\}$, which are all scalars. Linearizing this equation around the deterministic steady state values $\left\{\bar{\lambda}, \bar{u}_{yHt}, \bar{\xi}_{yt}\right\}$ poses no difficulty.[21]

[21] Although in this case derivatives can be taken analytically, throughout the section derivatives are assumed to be numerically obtained.

The second category is constituted by a continuum of equations that only involve scalar variables. Equations (2.6)-(2.11) fall into this category. Consider, for example, equation (2.9). This equation depends on $\{u_{oHt}(v), \xi_{ot}(v), \eta_t(v)\}$ which are all scalars. The problem is that there is one of these equations for every value of v in the interval $[v_{\min}, v_{\max}]$. In this case the "curse of dimensionality" is solved by considering this equation only at the grid points $(v_j)_{j=1}^J$ that were used in the computation of the deterministic steady state. It is now straightforward to linearize each of these J equations with respect to $\{u_{oHt}(v_j), \xi_{ot}(v_j), \eta_t(v_j)\}$ at their deterministic steady state values $\left\{\bar{u}_{oH}(v_j), \bar{\xi}_o(v_j), \bar{\eta}(v_j)\right\}$. Extending $\{u_{oHt}(v), \xi_{ot}(v), \eta_t(v)\}$ to the full domain $[v_{\min}, v_{\max}]$ using cubic splines will make equation (2.9) hold only approximately outside of the grid points $(v_j)_{j=1}^J$. The quality of this approximation will depend on how many grid points J we work with.

The third category is constituted by equations that involve both scalars and functions. Equations (2.20) and (2.21) fall in this category. For example, consider equation (2.21). This equation depends on λ_t, ξ_{yt}, λ_{t+1}, $w_{yH,t+1}$ and on the function η_{t+1}, which is a high dimensional object. In this case the "curse of dimensionality" is broken by considering that η_{t+1} is a spline approximation and, therefore, is completely determined by the finite set of values $\{\eta_{t+1}(v_j)\}_{j=1}^J$, i.e. the value of the function at the grid points. The equation can then be linearized with respect to $\left[\lambda_t, \xi_{yt}, \lambda_{t+1}, w_{yH,t+1}, \{\eta_{t+1}(v_j)\}_{j=1}^J\right]$ at the steady state values $\left[\bar{\lambda}, \bar{\xi}_y, \bar{\lambda}, \bar{w}_{yH}, \{\bar{\eta}(v_j)\}_{j=1}^J\right]$.

The fourth category is a combination of the previous two: it is constituted by a continuum of equations that involve both scalars and functions. Equations (2.12) and (2.13) fall in this category. For example, consider equation (2.13). Similarly to the third category, this equation depends on the scalars λ_t, $\xi_{ot}(v)$, λ_{t+1}, $w_{oH,t+1}(v)$ and on the function η_{t+1}. Similarly to the second category there is one of these equations for every value of v in the interval $[v_{\min}, v_{\max}]$. Given these similarities we can use the same strategy. In particular, we can consider this equation only at the grid points $(v_j)_{j=1}^J$ and linearize each of these J equations with respect to $\left[\lambda_t, \xi_{ot}(v_j), \lambda_{t+1}, w_{oH,t+1}(v_j), \{\eta_{t+1}(v_k)\}_{k=1}^J\right]$ at the deterministic steady state values $\left[\bar{\lambda}, \bar{\xi}_o(v_j), \bar{\lambda}, \bar{w}_{oH}(v_j), \{\bar{\eta}(v_k)\}_{k=1}^J\right]$.

The fifth category is much more complicated. It is constituted by equations that involve scalars and integrals of variables with respect to the distribution μ_t. Equations (2.27) and (2.28) fall in this category. For example, consider equation (2.27). This equation depends on the real numbers $u_{yL,t}$, $u_{yH,t}$, z_t, K_t, K_{t-1}, and H_t, and on the integrals $\int c(u_{ost}(v))\, d\mu_t$. To make progress it will be important to represent these integrals with a convenient finite set of variables. In order to do this, I will follow a strategy that is closely related to the one that was used in Section 3.1 for computing statistics under the invariant distribution. In particular, consider the same large but finite number of agents I that was used in that section and endow them with the same realized distribution of promised values $(\bar{v}_i)_{i=1}^I$ that was obtained when computing the steady state. Now, assume that these agents populated the economy M time periods ago and consider the history

$$\{w_{oL,t-m}, w_{oH,t-m}, w_{yL,t-m}, w_{yH,t-m}\}_{m=0}^M ,$$

which describes the allocation rules for next-period promised values that were chosen during the last M periods (where t is considered to be the current period). Observe that since $w_{oL,t-m}$ and $w_{oH,t-m}$ are spline approximations, this history can be represented by the following finite list of values:

$$\left\{ [w_{oL,t-m}(v_j)]_{j=1}^J , [w_{oH,t-m}(v_j)]_{j=1}^J , w_{yL,t-m}, w_{yH,t-m} \right\}_{m=0}^M . \tag{3.6}$$

Using the history of allocation rules for next-period promised values, we can simulate the evolution of promised values for the I agents and their descendants during the last M time periods to update the distribution of promised values from the initial $(\bar{v}_i)_{i=1}^I$ to a current distribution $(v_{i,t})_{i=1}^I$.

In particular, we can initialize the distribution of promised values at the beginning of period $t-M-1$ as follows:

$$v_{i,t-M-1} = \bar{v}_i,$$

for $i = 1, ..., I$. Given a distribution of promised values at the beginning of period $t-m-1$, the distribution of promised values at period $t-m$ is then obtained from the following equation:

$$v_{i,t-m} = \begin{cases} w_{os,t-m}(v_{i,t-m-1}), \text{ with probability } \sigma\psi_s, \\ w_{ys,t-m}, \text{ with probability } (1-\sigma)\psi_s, \end{cases} \tag{3.7}$$

for $i = 1, ..., I$. Proceeding recursively for $m = M, M-1, ..., 0$, we obtain a realized distribution of promised values $(v_{i,t})_{i=1}^I$ at the beginning of period t.

Observe that the last iteration of equation (3.7) also gives the corresponding realized values of leisure $(\alpha_{it})_{i=1}^I$ across the I agents. The joint realized distribution of promised values and values of leisure $(v_{it}, \alpha_{it})_{i=1}^I$ can then be used to compute statistics under the distribution μ_t. In particular, equation (2.27) can be re-written as:

$$0 = (1-\sigma)\left[e^{u_{yL,t}}\psi_L + e^{u_{yH,t}}\psi_H\right] + \sigma\frac{1}{I}\sum_{i=1}^I e^{u_{os_{it}}(v_{it})} + K_t - (1-\delta)K_{t-1} - e^{z_t}K_{t-1}^\gamma H_t^{1-\gamma}. \tag{3.8}$$

Since $u_{oL,t}$ and $u_{oH,t}$ are splines approximations, they can be summarized by their values at the grid points $(v_j)_{j=1}^J$. Therefore, equation (3.8) can be linearized with respect to

$$\begin{aligned}&z_t, K_t, K_{t-1}, H_t, u_{yL,t}, u_{yH,t}, \left[u_{oL,t}(v_j)\right]_{j=1}^J, \left[u_{oH,t}(v_j)\right]_{j=1}^J, \\ &\left\{\left[w_{oL,t-m}(v_j)\right]_{j=1}^J, \left[w_{oH,t-m}(v_j)\right]_{j=1}^J, w_{yL,t-m}, w_{yH,t-m}\right\}_{m=0}^M\end{aligned} \tag{3.9}$$

at their deterministic steady state values.

Observe that equation (3.9) provides a large but finite list of variables. In particular, there are $(M+1)(2J+2)$ variables in the second line of equation (3.9). Taking numerical derivatives with respect to each of these variables requires simulating I agents over M periods. As a consequence, linearizing equation (3.8) requires performing a massive number of Monte Carlo simulations. While this seems a daunting task it is easily parallelizable. Thus, using massively parallel computer systems can play an important role in reducing computing times and keeping the task manageable.[22]

The last category of equations has only one element: equation (2.25), which describes the law of motion for the distribution μ_t. While daunting at first sight, this equation is greatly simplified by our approach of representing the distribution μ_t using the history of values given by equation (3.6). In fact, updating the distribution μ_t is merely reduced to updating this history. In particular, the date-$(t+1)$ history can be obtained from the date-t history and the current values of $\left[w_{oL,t+1}(v_j)\right]_{j=1}^J$, $\left[w_{oH,t+1}(v_j)\right]_{j=1}^J$, $w_{yL,t+1}$ and $w_{yH,t+1}$ using the following equations:

$$\left[w_{os,(t+1)-m}(v_j)\right]_{j=1}^J = \left[w_{os,t-(m-1)}(v_j)\right]_{j=1}^J \tag{3.10}$$

$$w_{ys,(t+1)-m} = w_{ys,t-(m-1)} \tag{3.11}$$

for $s = L, H$ and $m = 1, ..., M$. Observe that the law of motion described by equations (3.10) and (3.11) is already linear, so no further linearization is needed. Also observe that the variables that are M periods old in the date-t history are dropped from the date-$(t+1)$ history. Thus, the law of motion described by equations (3.10)-(3.11) introduces a truncation. However, the consequences of this truncation are expected to be negligible. The reason is that the truncation only affects the agents that had survived for M consecutive periods, and given a sufficiently small survival probability σ and/or a sufficiently large M there will be very few of these agents. Aside from this

[22] In practice, I heavily rely on GPU computing for performing the Monte Carlo simulations.

negligible truncation there are no further approximations errors in the representation of the law of motion given by equation (2.25). As has been already stated, this is an important benefit of using the computational method described here.

3.3 Solving the linearized model

Once all equations have been linearized, we are left with a stochastic linear rational expectations model with a non-standard feature – namely, that some of the decision variables during the current period and some of the endogenous states during the next period are contingent on the realization of the aggregate shocks during the next period. Fortunately, this difficulty can be handled easily. The reason is that the stochastic state-contingent solution that we seek can be easily constructed from the solution to the deterministic version of the model, and this version has a standard structure that can be solved using well known methods. In what follows, I describe the linear stochastic model in detail and show how to perform this transformation.

Define the following vectors of variables,

$$x_t^1 = \left\{ \triangle w_{yL,t-n}, \triangle w_{yH,t-n}, \left[\triangle w_{oL,t-n}\left(\bar{v}_j\right)\right]_{j=J_1}^{J_2}, \left[\triangle w_{oH,t-n}\left(\bar{v}_j\right)\right]_{j=J_3}^{J_4} \right\}_{n=0}^{N} \tag{3.12}$$

$$x_{t-1}^2 = \{\triangle \ln K_{t-1}\} \tag{3.13}$$

$$y_{t+1}^1 = \left\{ \triangle w_{yL,t+1}, \triangle w_{yH,t+1}, \left[\triangle w_{oL,t+1}\left(\bar{v}_j\right)\right]_{j=J_1}^{J_2}, \left[\triangle w_{oH,t+1}\left(\bar{v}_j\right)\right]_{j=J_3}^{J_4} \right\} \tag{3.14}$$

$$\begin{aligned} y_t^2 = \Big\{ & \triangle u_{yL,t}, \triangle u_{yH,t} \triangle n_{yL,t}, \triangle n_{yH,t} \triangle \ln \xi_{yt}, \triangle \ln \lambda_t, \triangle \ln q_t, \left[\triangle \ln \eta_t\left(\bar{v}_j\right)\right]_{j=1}^{J}, \\ & \left[\triangle u_{oL,t}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \left[\triangle u_{oH,t}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \left[\triangle n_{oL,t}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \left[\triangle n_{oH,t}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \\ & \left[\triangle \ln \xi_{ot}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \triangle \ln H_t, \triangle I_t \Big\} \end{aligned} \tag{3.15}$$

where $\triangle$ represents the deviation of a variable from its deterministic steady state value, J_1 (J_2) is the lowest (largest) grid point j for which the steady-state value $w_{oL}\left(\bar{v}_j\right)$ is interior, and J_3 and J_4 are similarly defined for w_{oH}.[23] Observe that x_t^1 lists all z_t-contingent state variables, x_{t-1}^2 lists all non-contingent state variables, y_{t+1}^1 lists all z_{t+1}-contingent decision variables, and y_t^2 lists all non-contingent jump and decision variables.

[23] If for some grid point j, $w_{oL}\left(\bar{v}_j\right)$ or $w_{oH}\left(\bar{v}_j\right)$ are corner solutions at the deterministic steady, their values are fixed along the stochastic solution.

Once the model is linearized as in Section 3.2 it can be written as follows:

$$
\begin{aligned}
0 &= B_{11}x_t^1 + B_{12}x_{t-1}^2 + C_{12}y_t^2 + D_1 z_t, && (3.16)\\
0 &= A_{21}x_{t+1}^1 + B_{21}x_t^1 + B_{22}x_{t-1}^2 + C_{21}y_{t+1}^1, && (3.17)\\
0 &= A_{32}x_t^2 + B_{31}x_t^1 + B_{32}x_{t-1}^2 + C_{32}y_t^2, && (3.18)\\
0 &= H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}y_{t+1}^2 + K_{41}y_{t+1}^1 + K_{42}y_t^2 + M_4 z_t, && (3.19)\\
0 &= E_t\left\{F_{52}x_{t+1}^2 + G_{52}x_t^2 + H_{51}x_t^1 + H_{52}x_{t-1}^2 + J_{52}y_{t+1}^2 + K_{51}y_{t+1}^1\right. ,\\
&\quad \left. + K_{52}y_t^2 + L_5 z_{t+1} + M_5 z_t\right\} && (3.20)\\
z_{t+1} &= N z_t + \varepsilon_{t+1}, && (3.21)
\end{aligned}
$$

where (3.16) represents the aggregate feasibility constraints (equations 2.27-2.28), (3.17) is the law of motion for x_t^1 (equations 3.10-3.11), (3.18) is the law of motion for x_{t-1}^2 (equation 2.26), (3.19) is the first-order conditions for y_{t+1}^1 (equations 2.12-2.13 and 2.20-2.21) which must hold almost surely, and (3.20) represents the IC and PK constraints (equations 2.6-2.7 and 2.15) and the first-order conditions for y_t^2 (equations 2.8-2.11, 2.16-2.19, and 2.23-2.24), which must hold in expectation. Equation (3.21) is simply the stochastic equation for z_{t+1}(equation 2.3).[24]

I seek a recursive solution to equations (3.16)-(3.21) of the following form:

$$
\begin{aligned}
x_{t+1}^1 &= \Omega_{11}x_t^1 + \Omega_{12}x_{t-1}^2 + \Psi_1 z_t + \Theta_1 z_{t+1}, && (3.22)\\
x_t^2 &= \Omega_{21}x_t^1 + \Omega_{22}x_{t-1}^2 + \Psi_2 z_t, && (3.23)\\
y_{t+1}^1 &= \Phi_{11}x_t^1 + \Phi_{12}x_{t-1}^2 + \Gamma_1 z_t + \Lambda_1 z_{t+1}, && (3.24)\\
y_t^2 &= \Phi_{21}x_t^1 + \Phi_{22}x_{t-1}^2 + \Gamma_2 z_t. && (3.25)
\end{aligned}
$$

My strategy will be to construct it from the recursive solution to the deterministic version of equations (3.16)-(3.21), in which ε_{t+1} is set to zero and the expectations operator is dropped. This deterministic version has identical structure as the system analyzed in Uhlig (1999) and can be

[24] Actually, in the Mirrlees economy considered here $B_{22} = B_{31} = H_{41} = H_{42} = M_4 = F_{52} = H_{51} = 0$. They are solely included in equations (3.16)-(3.21) so that the general computational method described in the Appendix can refer to this same set of equations.

solved using identical methods.[25] Its solution has the following form:

$$x_{t+1}^1 = P_{11}x_t^1 + P_{12}x_{t-1}^2 + Q_1 z_t, \tag{3.26}$$

$$x_t^2 = P_{21}x_t^1 + P_{22}x_{t-1}^2 + Q_2 z_t, \tag{3.27}$$

$$y_{t+1}^1 = R_{11}x_t^1 + R_{12}x_{t-1}^2 + S_1 z_t, \tag{3.28}$$

$$y_t^2 = R_{21}x_t^1 + R_{22}x_{t-1}^2 + S_2 z_t. \tag{3.29}$$

Proposition 1 *Let (3.26)-(3.29) be the solution to the deterministic version of equations (3.16)-(3.21). Define* $\Omega_{11} = P_{11}$, $\Omega_{12} = P_{12}$, $\Omega_{21} = P_{21}$, $\Omega_{22} = P_{22}$, $\Psi_2 = Q_2$, $\Phi_{11} = R_{11}$, $\Phi_{12} = R_{12}$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, $\Gamma_2 = S_2$, *and*

$$\Theta_1 = \Upsilon A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2, \tag{3.30}$$

$$\Psi_1 = \Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right], \tag{3.31}$$

$$\Lambda_1 = -K_{41}^{-1}J_{42}R_{21}\Theta_1 - K_{41}^{-1}J_{42}S_2, \tag{3.32}$$

$$\Gamma_1 = -K_{41}^{-1}J_{42}R_{21}\Psi_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4, \tag{3.33}$$

where

$$\Upsilon = \left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]^{-1} \tag{3.34}$$

Then, (3.22)-(3.25) solves the stochastic system (3.16)-(3.21).

Proof. *The solution is verified using algebraic manipulations and the law of iterated expectations.*[26] ∎

4 Testing the computational method

The Mirrlees RBC model considered so far provides an ideal test case for the accuracy of the computational method. The reason is that in Veracierto (2021) I establish key features of the stationary solution analytically. In particular, I demonstrate that under the logarithmic preferences assumed here that the following properties hold:

[25] In fact, I use the same notation as Uhlig (1999), page 38, to facilitate comparisons. The only difference is that the variables here written as x_t^1 and y_{t+1}^1 are there written as x_{t-1}^1 and y_t^1. However, in a deterministic context this difference is immaterial (it can be considered a simple notational issue).

[26] See Technical Appendix 9 for a complete proof.

Property 1: u_{yst}, n_{yst}, and $w_{ys,t+1}$ fluctuate over the business cycle by amounts that are independent of the reported type s,

Property 2: each of the allocation rules $u_{ost}(v)$, $n_{ost}(v)$, and $w_{os,t+1}(v)$ are strictly increasing linear functions that are parallel across reported types,

Property 3: $u_{ost}(v)$, $n_{ost}(v)$, and $w_{os,t+1}(v)$ shift over the business cycle while keeping their slopes constant, and the shifts are independent of the reported type s,

Property 4: the cross-sectional distributions of promised values v, of log-consumption u_{ost}, and of log-leisure n_{ost}, shift horizontally over the business cycle while maintaining their shapes,

Property 5: aggregate consumption C_t, aggregate hours worked H_t, and aggregate capital K_t are exactly the same as in the stationary solution to the following representative agent planning problem:

$$V(z_t, K_{t-1}) = \max\left\{u(C_t) + \bar{\alpha} n(1 - H_t) + \theta E_t\left[V(z_{t+1}, K_t)\right]\right\} \tag{4.1}$$

subject to

$$C_t + K_t - (1-\delta) K_{t-1} \leq e^{z_t} K_{t-1}^{\gamma} H_t^{1-\gamma}, \tag{4.2}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}, \tag{4.3}$$

where $\bar{\alpha} = \alpha_L \psi_L + \alpha_H \psi_H$.

While Properties 1-4 appear to be qualitative in nature, Property 5 provides a strong quantitative test of the computational method.[27] The reason is that the optimal allocation of the representative agent economy can be computed using standard methods and compared with the aggregate variables of the Mirrlees economy computed with the method described in this paper. In fact, this also provides a quantitative test of Property 3, since the vertical parallel shifts in Property 3 must be exactly those needed to deliver Property 5. The essential reason why Property 5 holds is that the optimal allocations in the Mirrlees economy must satisfy Inverse Euler conditions, and under logarithmic preferences these Inverse Euler conditions become linear with respect to individual consumption and leisure. Thus when these Inverse Euler conditions are integrated in the cross section they deliver the (direct) Euler conditions of the representative agent economy.[28]

[27] Property 5 generally does not hold in economies with private information. It would not even hold in the Mirlees economy considered here if preferences were different from logarithmic (see Veracierto (2021)).

[28] I refer the reader to Veracierto (2021) for the details.

In order to show that the computational method recovers the above properties exactly I must first parametrize the model. Following the RBC literature, I select a labor share $1-\gamma$ of 0.64, a depreciation rate δ of 0.10, a private discount factor β of 0.96, a persistence of aggregate productivity ρ of 0.95, and a variance of the innovations to aggregate productivity σ_{ε}^{2} equal to 4×0.007^{2}, all corresponding to a time period of one year. The social discount factor θ is chosen to be the same as the private discount factor β. The values of leisure α_L and α_H are chosen to satisfy two criteria: That aggregate hours worked H equal 0.31 (a standard target in the RBC literature) and that the hours worked by old agents with the high value of leisure and the highest possible promised value $n_{oH}(v_{\max})$ be a small but positive number. The rationale for this second criterion is that I want to maximize the relevance of the information frictions while keeping an internal solution for hours worked. The resulting values for α_L and α_H are 1.643 and 2.177, respectively. I treat both values of the idiosyncratic shock symmetrically and chose $\psi_L=\psi_H=0.50$. In terms of the life-cycle structure, I choose $\sigma=0.975$ to generate an expected lifespan of 40 years.

While the above parameters are structural, there are a number of computational parameters to be determined. The number of grid points in the spline approximations J, the total number of agents simulated I, the length of the simulations for computing the invariant distribution T, and the length of the histories kept as state variables when computing the business cycles N are all chosen to be as large as possible, while keeping the computational task manageable and results being robust to non-trivial changes in their values. Their chosen values are 20, 2^{23}, 1000, and 273, respectively.[29] It turns out that under these computational parameters, the linearized system (3.16)-(3.21) has about $12,000$ variables (a large system indeed).

Finally, the lower and upper bounds for the range of possible promised values $v_{\min}$ and $v_{\max}$ were chosen so that the fraction of agents in the intervals $[v_1, v_2]$ and $[v_{J-1}, v_J]$ are each less than 0.01%. Thus, truncating the range of possible values at $v_{\min}$ and $v_{\max}$ should not play an important role in the results. The chosen values for $v_{\min}$ and $v_{\max}$ are -35.0 and -16.3, respectively.

Before turning to the business cycle results, I illustrate different features of the model at its deterministic steady state. Figure 1 shows the invariant distribution of promised values across the

[29] Given the value selected for the survival probability σ, less than 0.1% of individuals survive more than N periods. Thus, the truncation imposed by keeping track of a finite history of decision rules introduces a very small approximation error.

$J-1$ intervals $[v_j, v_{j+1}]_{j=1}^{J-1}$, defined by the grid points of the spline approximations. We see that the invariant distribution puts very little mass at extreme values. In consequence, in what follows I will report allocation rules only between the 6th and 16th grid points. The reason is not only that there are too few agents at the tails of the distribution for them to matter, but also that being close to the artificial bounds $v_{\min}$ and $v_{\max}$ greatly distorts the shape of the allocation rules.

Figure 2.A reports utilities of consumption for old agents $u_{oL}(v)$ and $u_{oH}(v)$ across promised values v, at the deterministic steady state. We see that both u_{oL} and u_{oH} are strictly increasing in the promised value v, are linear, and are parallel to each other. Figure 2.B shows the same for the utilities of leisure $n_{oL}(v)$ and $n_{oH}(v)$, and Figure 2.C for the next-period promised values $w_{oL}(v)$ and $w_{oH}(v)$. Since these figures could be tricking the naked eye, Figure 2.D, depicts the vertical differences across reported types $u_{oH}(v)-u_{oL}(v)$, $n_{oH}(v)-n_{oL}(v)$ and $w_{oH}(v)-w_{oL}(v)$. We see that the different pairs of functions are indeed parallel to each other. Thus, Figure 2 verifies that Property 2 holds at the deterministic steady state.

The discussion of business cycle dynamics that follows is centered around the analysis of the impulse responses of different variables to a one standard deviation increase in aggregate productivity. Figure 3.A shows the impulse responses of the utility of consumption of young agents u_{yL} and u_{yH}. We see that the two impulse responses overlap perfectly, thus satisfying Property 1. Figure 3.B shows the impulse response of the utility of consumption of old agents with a low value of leisure $u_{oL}(v)$, at each of the eleven grid points $(v_j)_{j=6}^{16}$. While the figure shows eleven impulse responses, only one of them is actually seen because they overlap perfectly. This means that, in response to the aggregate productivity shock, the linear function u_{oL} depicted in Figure 2.A shifts vertically over time while keeping its slope constant. Figure 3.C, which does the same for u_{oH}, is identical to Figure 3.B. Thus, not only u_{oH} shifts over time keeping its slope constant, but its increments are the same as those of u_{oL}. We have thus verified that u_{oLt} and u_{oHt} satisfy Property 3. Figure 4 is analogous to Figure 3, except that it depicts the behavior of the utilities of leisure n_{yst} and $[n_{ost}(v_j)]_{j=6}^{16}$. Figure 5 is also analogous to Figure 3 but depicts the behavior of the promised values $w_{ys,t+1}$ and $[w_{os,t+1}(v_j)]_{j=6}^{16}$. A quick inspection verifies that Figures 4 and 5 have the same characteristics as Figure 3. Thus, Properties 1 and 3 are fully satisfied.

Figure 6 shows the impulse responses of the cross sectional standard deviations of promised values, log-consumption and log-leisure. We see that in response to a positive aggregate productivity shock, all these standard deviations remain flat. Thus, Property 4 is satisfied.

Finally, Figure 7.A shows the impulse responses of aggregate output Y_t, aggregate consumption C_t, aggregate investment I_t, aggregate hours worked H_t and aggregate capital K_{t-1} for the Mirrlees RBC economy. Figure 7.B reports the impulse responses for the same variables but for the representative agent economy planning problem (4.1)-(4.2). We see that Figures 7.A and 7.B are the same. Figure 7.C verifies this by reporting the differences between the Mirrlees economy and the representative agent economy, for each of the macro variables considered. Thus, Property 5 is perfectly satisfied.

We have thus verified that the computational method, when applied to the Mirrlees RBC economy with logarithmic preferences, reproduces all the properties found in Veracierto (2021) as well as a crucial quantitative test: That all aggregate variables in the economy with private information perfectly reproduce the impulse responses of the same variables in the representative agent economy. Since nothing in the computational method exploits the functional forms or structure of the economy considered, this provides significant evidence about its accuracy. This finding indicates that the computational method introduced in this paper should prove useful in a variety of other private information settings.

5 Applicability

The computational method just described is applicable to a wide variety of economies with private information, but these models must satisfy certain conditions. A key feature of the computational method is that it uses a finite history of past decision rules to describe the current cross-sectional distributions of agents across individual states. For this strategy to work, the model considered should incorporate a significant life-cycle structure for the non-representative types of agents. In particular, their expected (or deterministic) lifetimes should be sufficiently short relative to the model time period. Otherwise, one may have to use prohibitively long histories of past decision rules in order to characterize the cross-sectional amount of heterogeneity accurately.

Another feature of the computational method is that it uses spline approximations to describe decision rules. This approach can accommodate a large class of decision rules but could become quite costly in certain cases. If the decision rules have ranges with sharp non-linearities, describing them accurately may require adding many grid points at those ranges. This could increase the computational costs significantly, since introducing more spline coefficients increases the number of aggregate state variables in the system (the computational method must keep track of the history of the additional coefficients). Another reason for the added complexity is that calculating numerical derivatives accurately at narrowly separated grid points requires having a good definition of the invariant distribution over the subranges that they define.[30] Since the invariant distribution is obtained by performing Monte Carlo simulations, this may require working with a huge panel of agents. For these reasons, it is important to inspect the invariant distribution and decision rules at the deterministic steady state of the model and evaluate if the invariant distribution puts enough mass on ranges of non-linearities to justify the added complexity.

The spline decision rules will also only approximately describe the critical values at which a constraint becomes binding. Moreover, the computational method assumes that if a constraint binds (does not bind) at a given grid point in the deterministic steady state, that it will always bind (not bind) in the stochastic solution. While this assumption is likely to hold at most grid points, it may not hold at grid points that are sufficiently close to true critical values. In many cases these approximation errors will have unimportant consequences for the aggregate dynamics of the model. For example, if the invariant distribution puts little mass around the computed critical values, it will be largely irrelevant what happens in those ranges. Even if the invariant distribution puts significant mass around those critical values, the consequences of missing the associated constraints by small amounts are likely to be unimportant if the decision rules are sufficiently smooth. Problems may arise when the invariant distribution puts considerable mass close to the critical values and the decision rules are sharply non-linear around them. In these cases, the computational method may fail to capture the aggregate dynamics of the model correctly.

While this paper has applied the computational method to a RBC Mirrlees economy with logarithmic preferences, in Veracierto (2021) I have used it to compute the solution under more general CRRA preferences and for preferences with a constant Frisch elasticity of labor supply. In Veracierto (2022) I also use it to study the optimal provision of unemployment insurance over the business cycle in an economy with moral hazard. In particular, in that paper I consider a RBC structure in which all production is performed in a central island, agents get separated exogenously from that island, and in order to get back they need to search. An unemployed

[30] Recall the linearization of equation (2.27).

agent's probability of arriving to the production island depends on their own search intensity, which (similarly to Hopenhayn and Nicolini (1997)) is private information. For that framework I use the computational method described here to solve the economy-wide mechanism design problem. While I admit that two applications is not a long list, it is certainly a start. I hope that the reader will appreciate the applicability of the method and be motivated to use it in a variety of other settings.

6 Conclusions

In this paper I introduced a method for computing aggregate fluctuations of economies with private information. Its basic strategy is to parametrize individual decision rules as spline approximations and to keep long histories of the spline coefficients as state variables. The resulting representation of the model is then linearized at the deterministic steady-state. The computational method has three attractive features: 1) it keeps track of the full distribution of agents across individual states, 2) it can handle irregular shapes for this distribution, and 3) it incorporates the distribution's exact law of motion. However, the main advantage of the computational method is that it is able to handle cases in which the cross-sectional distribution of agents is state-contingent, which is a prevalent feature of economies with private information.

The computational method was illustrated using a Mirrlees RBC economy with known analytical solution. Contrasting the numerical solution to the theoretical solution allowed me to test the accuracy of the computational method. The method passed the test with flying colors: it reproduced all the qualitative and quantitative properties of the solution perfectly well. This finding suggests that the computational method should prove useful in a variety of other private information applications.

Having said this, I would like to conclude the paper with two caveats. The first one is that, since linearizing the model with respect to each of the elements in the history of spline coefficients requires performing a massive number of Monte Carlo simulations, the method turns out to be very slow.[31] This should not be a problem when calibrating the deterministic steady-state of a model,

[31]Linearizing the Mirlees RBC model under the computational parameters described in Section 4 takes about four hours in a system equipped with four NVIDIA Tesla V100 GPUs.

since the computational method needs to be applied only once (after all parameter values have been determined). However, it makes it impractical for estimating a model using formal econometric methods. The second caveat is that the computational complexity grows exponentially with the number of endogenous individual state variables. The reason is that as the spline approximations are defined over state spaces of increasing dimensionality, the number of spline coefficients in the system grow accordingly. As a result, models with two or more endogenous individual state variables could only be handled if the decision rules are sufficiently smooth to be described with a relatively small number of grid points.

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8 Appendix: The general computational method

This Appendix describes the general method for computing stationary equilibria of economies with private information and aggregate shocks. While the main features of the method have already been discussed in the main text, here it is described in general terms to make it applicable to a wide variety of settings.

The basic framework is as follows.[32] The economy is populated by individual decision makers that solve maximization problems of the following form at every time period t:

$$v_{ht}(a,x_1,x_2) =$$

$$\max_{\left[u_{h1,t+1}(s,a')\right]_{s,a'},\left[u_{h2t}(s,a')\right]_{s,a'}} \left\{ E_t\left[\sum_s R_h(s,a,x_1,x_2,\left[u_{h1,t+1}(s,a')\right]_{a'},\left[u_{h2t}(s,a')\right]_{a'},z_t,p_t,p_{t+1})\psi_s\right] + \right.$$

$$\left. E_t\left[\sum_s\sum_{a'}\beta_h(a,a',z_t,p_t,p_{t+1})v_{h,t+1}\left(a',x_1'(s,a'),x_2'(s,a')\right)\pi_h\left[a,a',u_{h1,t+1}(s,a'),u_{2ht}(s,a')\right]\psi_s\right]\right\} \tag{8.1}$$

subject to

$$x_1'(s,a') = G_{h1}\left(a,x_1,x_2,s,a',u_{h1,t+1}(s,a')\right), \tag{8.2}$$

$$x_2'(s,a') = G_{h2}\left(a,x_1,x_2,s,a',u_{h2t}(s,a')\right), \tag{8.3}$$

$$0 \leq E_t\left[C_h\left(a,x_1,x_2,\left[u_{h1,t+1}(s,a')\right]_{s,a'},\left[u_{h2t}(s,a')\right]_{s,a'},z_t,p_t,p_{t+1}\right)\right], \tag{8.4}$$

where h is the permanent type of the individual (e.g., being a household or a firm), a is a vector of individual states that take a finite number of values (e.g., persistent idiosyncratic shocks), z_t is a vector of aggregate shocks, x_1 is a vector of individual state variables whose values are contingent on the realizations of a and z_t, x_2 is a vector of individual state variables whose values are contingent on the realization of a but independent of z_t, s is a vector of i.i.d. idiosyncratic shocks with distribution ψ, $u_{h1,t+1}(s,a')$ is a vector of (s,a',z_{t+1})-contingent decision variables, $u_{h2t}(s,a')$ is a vector of (s,a')-contingent decision variables, p_t is a vector of equilibrium prices (whose stochastic process is taken as given by the individual), G_{h1} and G_{h2} define the laws of motion for x_1 and x_2, respectively, C_h is a vector valued function defining constraints on $\left[u_{h1,t+1}(s,a')\right]_{s,a'}$, and $\left[u_{h2t}(s,a')\right]_{s,a'}$, β_h is a function that describes the discounting of future payoffs (allowing for idiosyncratic and/or aggregate preference shocks, as well as discounting using market prices), and π_h describes the transition probabilities for a (potentially affected by the individual's decisions).[33]

[32]Hereon I use the convention that a variable is dated t if its value becomes known when the date-t aggregate shocks realize. If the dating of a variable x is clear from the context, I avoid dating it explicitly and its next period value will be denoted by x'. In particular, I avoid dating the arguments of individual value functions and decision rules.

[33]The [] notation is used for a list of variables. For example, $\left[u_{h1,t+1}(s,a')\right]_{s,a'}$lists the values of $u_{h1,t+1}(s,a')$

While a and s take a finite number of values, all other variables take real values.[34] The solution to this sequence of maximization problems is a stochastic process for v_{ht}, $[u_{h1,t+1}(s,a')]_{s,a'}$, and $[u_{h2t}(s,a')]_{s,a'}$, which are all functions over (a, x_1, x_2).[35] The permanent type h implicitly defines the space in which (a, x_1, x_2) lie.[36]. There is a finite number of different permanent types h in the economy.[37]

The distribution of h-type agents across individual states (a, x_1, x_2) at the beginning of period t is described by a measure μ_{ht}. The law of motion for μ_{ht} is given by the following equation:

$$\begin{aligned} & \mu_{h,t+1}\left(\{a'\} \times \mathcal{X}_1 \times \mathcal{X}_2\right) \\ = \;& \phi_h\left(\{a'\} \times \mathcal{X}_1 \times \mathcal{X}_2\right) + \sum_s \left(\int_{\mathcal{B}} \pi_h \left[a, a', u_{h1,t+1}\left(a, x_1, x_2, s, a'\right), u_{h2t}\left(a, x_1, x_2, s, a'\right)\right] d\mu_{ht} \right) \psi_s, \end{aligned} \tag{8.5}$$

for every a' and Borel sets $\mathcal{X}_1$ and $\mathcal{X}_2$, where

$$\begin{aligned} \mathcal{B} \;=\; & \{(a, x_1, x_2) : G_{h1}\left(a, x_1, x_2, s, a', u_{h1,t+1}\left(a, x_1, x_2, s, a'\right)\right) \in \mathcal{X}_1 \\ & \text{and } G_{h2}\left(a, x_1, x_2, s, a', u_{h2t}\left(a, x_1, x_2, s, a'\right)\right) \in \mathcal{X}_2\}. \end{aligned} \tag{8.6}$$

The measure ϕ_h describes an exogenous endowment of new agents (e.g., to accommodate exogenous entry of firms in a firm dynamics context or newborns in a households life cycle context), while

across all possible values of s and a', while $[u_{h1,t+1}(s,a')]_{a'}$ lists the values of $u_{h1,t+1}(s,a')$ across all possible values of a' fixing the value of s.

[34]The reason I introduce the i.i.d. shocks s explicitly instead of subsuming them in the vector a is because of the restrictions across realizations of s that equation (8.4) allows for. These cross-restrictions play a crucial role in certain economies with private information (e.g. representing incentive compatibility constraints).

[35]While the dependence of $[u_{h1,t+1}(s,a')]_{s,a'}$ or $[u_{h2t}(s,a')]_{s,a'}$ on a' is not critical, the dependence of $[u_{h1,t+1}(s,a')]_{s,a'}$ on z_{t+1} is what distinguishes it from $[u_{h2t}(s,a')]_{s,a'}$. Any decision variable that is not contingent on z_{t+1} is assumed to be included in $[u_{h2t}(s,a')]_{s,a'}$. The same assumptions apply to x_1 and x_2. The presence of individual state and decision variables that depend on the realization of the aggregate shocks plays a crucial role in economies with private information, since promised values are contingent on the realization of the aggregate shocks.

[36]I avoid introducing a subscript h for these variables in order to simplify notation. However, the context will always make clear the permanent type h that they correspond to.

[37]Observe that, similarly to Section 2.1, s refers to the index of an i.i.d. idiosyncratic shock. Thus, similarly to equation 2.5, the sum across s in equation 8.1 is introduced to take expectations across all possible realizations of s, each of which takes place with probability ψ_s.

the second term describes the endogenous evolution of the distribution. Observe that since $u_{h1,t+1}$ is contingent on the realization of z_{t+1}, the same is generally true for $\mu_{h,t+1}$. I assume that μ_{h1}, ϕ_h, and π_h are such that the total number of h-type agents μ_{ht} is constant over time and equal to Γ_h, independent of the stochastic process $\{u_{h1,t+1}, u_{h2t}\}_{t=1}^{\infty}$.

In what follows, it will be useful to differentiate the h-type of agents that are infinitely lived and for which the maximization problem (8.1)-(8.4) is independent of a and s. Henceforth, all variables corresponding to such "representative" types of agents will be denoted with a subscript r, while the h subscript will be reserved for heterogeneous types. An important characteristic of representative types of agents is that the measure μ_{rt} describing their distribution across individual states will have mass at a single point $(x_{r1t}, x_{r2,t-1})$. Therefore, it will be convenient to replace μ_{rt} with that single point and replace the law of motion (8.5)-(8.6) with

$$x_{r1,t+1} = G_{r1}\left(x_{r1,t}, x_{r2,t-1}, u_{r1,t+1}\left(x_{r1,t}, x_{r2,t-1}\right)\right), \tag{8.7}$$

$$x_{r2t} = G_{r2}\left(x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1,t}, x_{r2,t-1}\right)\right). \tag{8.8}$$

The stochastic process for p_t, which is taken as given in the maximization problems (8.1)-(8.4), is an equilibrium process if for every t,

$$Q\left(z_t, \left[\sum_s \left(\int M_h(a, x_1, x_2, \left[u_{h2t}\left(a, x_1, x_2, s, a'\right)\right]_{a'}) d\mu_{ht}\right)\psi_s\right]_h, \left[x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1t}, x_{r2,t-1}\right)\right]_r\right) = 0, \tag{8.9}$$

almost surely, where Q is a vector valued function (of the same dimensionality as p_t) describing aggregate feasibility and/or market clearing conditions, M_h is a vector valued function that determines which moments of μ_{ht} are arguments of Q, and $(x_{r1t}, x_{r2,t-1}, u_{r2t})$ are the states and decision functions of the r-type of representative agents. Observe that the z_{t+1}-contingent decision variables $\left[u_{h1,t+1}\left(s, a'\right)\right]_{s,a'}$ and $u_{r1,t+1}$ do not enter Q.

The vector of aggregate shocks z_t follows an AR(1) process $z_{t+1} = N z_t + \varepsilon_{t+1}$, where $E_t\left[\varepsilon_{t+1}\right] = 0$.

8.1 Computing the deterministic steady state

In order to compute a steady state, I start by making z_t identical to zero and fixing the price vector at some value p. For each r-type of representative agent, the vector of time invariant state and decision variables $(x_{1r}, x_{2r}, u_{1r}, u_{2r})$ can then be directly obtained from the first-order conditions of the corresponding maximization problem.

I solve the maximization problems given by equations (8.1)-(8.4) using spline approximations and value function iterations.[38] To start, I restrict each component of the vector of endogenous individual state variables (x_1, x_2) for each h-type agent to lie in a closed interval and define a set of grid points in it that includes the extremes.[39] The Cartesian product of all these sets of grid points defines a finite set of grid points for (x_1, x_2), which is described by a vector $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$. Given the value function v_h from the previous iteration, which is used to evaluate (x_1', x_2') (possibly outside the grid points), the maximization problem in equations (8.1)-(8.4) is solved for only at the grid points $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$. Once, the vectors of new values $\bar{v}_h = [v_h(a, \bar{x}_{1j}, \bar{x}_{2j})]_{a,j}$, $\bar{u}_{h1} = [u_{h1}(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a')]_{a,j,s,a'}$, and $\bar{u}_{h2} = [u_{h2}(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a')]_{a,j,s,a'}$ are obtained, I extend their values to the full domain of (x_1, x_2) using splines. These value function iterations continue until $\bar{v}_h$ converges. Observe that the solution obtained depends on the price vector p, which has been fixed.

For heterogenous agents, the steady state version of equations (8.5)-(8.6) describes the recursion that the time invariant distribution μ_h has to satisfy. This equation corresponds to the case of a continuum of agents. However, I perform the recursion in the case of a large but finite number of agents. In particular, consider a large but finite number I_h of h-type agents and endow them with some individual states (a, x_1, x_2). Using the functions u_{h1} and u_{h2} already obtained, I simulate the evolution of the individual states of these I_h agents for a large number of periods T. To be precise, if an h-type agent i has the individual state (a, x_1, x_2) at the beginning of the current period, then the individual state (a', x_1', x_2') at the beginning of the following period is randomly

[38]For representative agents with state contingent state variables x_{1r}, it will be important to follow the procedure described in this paragraph as well since the steady state objects described here will be needed later on.

[39]When restricting each of these variables to lie in a closed interval, one should modify the steady state maximization problem (8.1)-(8.4) to incorporate the corresponding constraints on x_1' and x_2'. The use of splines is what requires each component of (x_1, x_2) to lie in a closed interval.

determined as follows:

$$\begin{array}{lll} \text{(i)} & \text{with probability } \pi_h\left[a,a',u_{h1}\left(a,x_1,x_2,s,a'\right),u_{h2}\left(a,x_1,x_2,s,a'\right)\right]\psi_s, \text{ it is given by} & (8.10) \\ & \left[a',G_{h1}\left(a,x_1,x_2,s,a',u_{h1}\left(a,x_1,x_2,s,a'\right)\right),G_{h2}\left(a,x_1,x_2,s,a',u_{h2}\left(a,x_1,x_2,s,a'\right)\right)\right], & \\ \text{(ii)} & \text{with probability } 1-\sum_{s,a'}\pi\left[a,a',u_{h1}\left(a,x_1,x_2,s,a'\right),u_{h2}\left(a,x_1,x_2,s,a'\right)\right]\psi_s \text{ it is determined by } \phi_h. & \end{array}$$

Observe that the transition in (ii) takes place when the individual dies and is replaced by a newborn whose initial state is unrelated to the state of the predecessor.

Simulating the I_h agents and their descendants for T periods using the law of motion in (8.10), I obtain a realized distribution $(a^i, x_1^i, x_2^i)_{i=1}^{I_h}$ of individual states across the I_h agents. Doing this for every h-type, the aggregate feasibility conditions can then be computed as

$$Q\left(0,\left[\sum_s\left(\Gamma_h\frac{1}{I_h}\sum_{i=1}^{I_h}M_h(a^i,x_1^i,x_2^i,\left[u_{h2}\left(a^i,x_1^i,x_2^i,s,a'\right)\right]_{a'})\right)\psi_s\right]_h,\left[x_{r1},x_{r2},u_{r2}\right]_r\right)=0. \quad (8.11)$$

Observe that by the law of large numbers, equation (8.11) will become an arbitrarily good approximation of equation (8.9) as all I_h and T tend to infinity.

If equation (8.11) is not satisfied, the price vector p must be changed until it is. This represents a standard root finding problem.

8.2 Computing the stationary stochastic solution

Computing the stationary stochastic solution requires linearizing the first-order conditions to the maximization problems given by equations (8.1)-(8.4), the laws of motion (8.5)-(8.6), the laws of motion (8.7)-(8.8), and the aggregate feasibility conditions given by equation (8.9) with respect to a convenient set of variables.

In order to illustrate some of the issues involved in the linearization of the first-order conditions, I will use equation (8.1) as an example since it represents the most complex type.[40] The first issue

[40]Equation (8.1) enters the set of first order conditions if the transition probabilities π_h depend on $u_{h1,t+1}$ or u_{h2t}. In this case, the level of v_{ht} enters the first order conditions and the definitional equation (8.1) must be included. If π_h does not depend on $u_{h1,t+1}$ or u_{h2t}, only the derivatives of v_{ht} enter the first order conditions. However, the issues discussed here in the context of equation (8.1) apply to other first-order conditions, including the definitional equation for the derivatives of v_{ht}. For reasons I will explain in Section 8.3, it is important to write first order conditions using the derivatives of the value function and not as second order stochastic difference equations.

is the existence of a continuum of equations (8.1), since (x_1, x_2) take a continuum of values. I solve this "curse of dimensionality" by considering the equation only at the grid points $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$ that were used in the computation of the deterministic steady state. Another issue is that each of this finite number of equations depends on the infinite dimensional object $v_{h,t+1}$, since it is a function of (x_1', x_2'), and I need to evaluate these variables outside the grid points. In this case, I solve the "curse of dimensionality" by considering that $v_{h,t+1}$ is a spline approximation and, therefore, is completely determined by the vector $\bar{v}_{h,t+1} = [v_{h,t+1}(a, \bar{x}_{1j}, \bar{x}_{2j})]_{a,j}$, i.e., by the value of the function at the grid points. Consequently, after substituting equations (8.2)-(8.3) into equation (8.1) and linearizing at the corresponding steady state values, I am left with the following finite set of equations:

$$0 = E_t \left\{ \mathcal{L}_h^v \left(\bar{v}_{h,t}, \bar{u}_{h1,t+1}, \bar{u}_{h2t}, z_t, p_t, p_{t+1}, \bar{v}_{h,t+1} \right) \right\}, \tag{8.12}$$

where $\bar{u}_{h1,t+1} = [u_{h1,t+1}(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a')]_{a,j,s,a'}$, $\bar{u}_{h2t} = [u_{h2t}(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a')]_{a,j,s,a'}$ and $\mathcal{L}_h^v$ is a vector valued linear function with the same dimensionality as $\bar{v}_{ht}$.

Particular attention should be given to the first-order conditions corresponding to grid points $(a, \bar{x}_{1j}, \bar{x}_{2j})$ for which the deterministic steady state choice of some component of $x_1'(s, a')$ or $x_2'(s, a')$ hits one of the extremes imposed by the use of spline approximations. At these grid points, the maximization problem (8.1)-(8.4) should be modified by imposing the constraint that the corresponding component of equation (8.2) or (8.3) must evaluate to the corresponding extreme. The first-order conditions used at these grid points should be those of the modified problem. A consequence of this is that if the optimal choice of some component of $x_1'(s, a')$ or $x_2'(s, a')$ hits an extreme in the steady state solution, it will always hit it in the stochastic solution. This will certainly distort the stochastic decision rules close to the extremes, so in practice one should choose these extremes far enough that the invariant distribution μ_h puts little mass close to them (minimizing the relevance of these distortions).

Linearizing the aggregate feasibility conditions described by equation (8.9) presents more complicated issues because of their dependence on the integrals $\left[\int M_h d\mu_{ht}\right]_h$. In order to do this linearization I do the following. For each heterogenous type of agent h I consider the same large but finite number of agents I_h used in Section 3.1 and endow them with the same realized distribution of individual states $(a^i, x_1^i, x_2^i)_{i=1}^{I_h}$ that was obtained when computing the steady state. Now, assume that these agents populated the economy N time periods ago and consider the history

$\{u_{h1,t+1-n}, u_{h2,t-n}\}_{n=1}^{N}$ of decision rules that were realized during the last N periods (where t is considered to be the current period). Since these decision rules are spline approximations, this history can be represented by the finite list of values $\{\bar{u}_{h1,t+1-n}, \bar{u}_{h2,t-n}\}_{n=1}^{N}$. Using this history of decision rules, I can simulate the evolution of individual states for the I_h agents and their descendants during the last N time periods to update the distribution of individual states from the initial $(a^i, x_1^i, x_2^i)_{i=1}^{I_h}$ to a current distribution $\left(a_t^i, x_{1t}^i, x_{2,t-1}^i\right)_{i=1}^{I_h}$. In particular, I can initialize the distribution of individual states at the beginning of period $t-N$ as $\left(a_{t-N}^i, x_{1,t-N}^i, x_{2,t-N-1}^i\right) = (a^i, x_1^i, x_2^i)$, for $i = 1, ..., I_h$. Given a distribution of individual states $\left[\left(a_{t-n}^i, x_{1,t-n}^i, x_{2,t-n-1}^i\right)\right]_{i=1}^{I_h}$ at period $t-n$, the individual state $\left(a_{t-n+1}^i, x_{1,t-n+1}^i, x_{2,t-n}^i\right)$ of each agent i at period $t-n+1$ is randomly determined as follows:

(i) with probability $\pi_h\left[a_{t-n}^i, a', u_{h1,t+1-n}^i(s,a'), u_{h2,t-n}^i(s,a')\right]\psi_s$, it is given by $\left(a', G_{h1,t+1-n}^i(s,a'),\right.$ $\left.G_{h2,t-n}^i(s,a')\right)$, where $\left(u_{h1,t+1-n}^i(s,a'), u_{h2,t-n}^i(s,a'), G_{h1,t+1-n}^i(s,a'), G_{h2,t-n}^i(a')\right)$ are the values of $(u_{h1,t+1-n}, u_{h2,t-n}, G_{h1}, G_{h2})$ evaluated at $\left(a_{t-n}^i, a', x_{1,t-n}^i, x_{2,t-n-1}^i, s, a'\right)$,

(ii) with probability $1 - \sum_{s,a'} \pi_h\left[a_{t-n}^i, a', u_{h1,t+1-n}^i(s,a'), u_{h2,t-n}^i(s,a')\right]\psi_s$, it is determined by ϕ_h.

Proceeding recursively for $n = N, N-1, ..., 1$, I obtain a realized distribution $\left(a_t^i, x_{1t}^i, x_{2,t-1}^i\right)_{i=1}^{I_h}$ at the beginning of period t. This distribution can be used to compute statistics under the distribution μ_{ht}. In particular, having followed the above procedure for each h-type of heterogeneous agents, I can rewrite equation (8.9) as

$$
\begin{aligned}
0 \quad = \quad & Q\left(z_t, \left[\sum_s \left(\Gamma_h \frac{1}{I_h}\sum_{i=1}^{I_h} M_h(a_t^i, x_{1t}^i, x_{2,t-1}^i, \left[u_{h2t}\left(a_t^i, x_{1t}^i, x_{2,t-1}^i, s, a'\right)\right]_{a'})\right)\psi_s\right]_h, \right. \\
& \left., \left[x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1t}, x_{r2,t-1}\right)\right]_r\right) \qquad (8.13)
\end{aligned}
$$

Since u_{h2t} and u_{r2t} are spline approximations, they can also be summarized by their values at the grid points $\bar{u}_{h2t}$ and $\bar{u}_{r2t}$.[41] As a consequence, equation (8.13) can be linearized at the deterministic

[41]For simplicity, I assume here that all representative agents have state-contingent states x_{1r}. However, for representative agents with no state-contingent states, instead of writing equation (8.13) in terms of their decision rules u_{r2t}, it is often more convenient to write it directly in terms of the values of their type-2 decision variables at date t. Consequently, for this type of representative agents, $\bar{u}_{r2t}$ in equations (8.14) and (8.18) is not a vector of spline coefficients but a vector of values for type-2 decision variables.

steady state values to get the following finite set of equations:

$$0 = \mathcal{L}^Q \left(z_t, \left[\{\bar{u}_{h1,t+1-n}\}_{n=1}^N, \{\bar{u}_{h2,t-n}\}_{n=0}^N \right]_h, [x_{r1t}, x_{r2,t-1}, \bar{u}_{r2t}]_r \right) \tag{8.14}$$

where $\mathcal{L}^Q$ is a vector valued linear function.[42]

My approach of representing the distribution μ_{ht} with a finite history of values greatly simplifies the description of the law of motion in equations (8.5)-(8.6). In fact, updating the distribution μ_{ht} is merely reduced to updating those histories. In particular, the date-$(t+1)$ histories can be obtained from the date-t histories and the current values $\bar{u}_{h1,t+1}$ and $\bar{u}_{h2t}$ using the following equations:

$$\bar{u}_{h1,(t+1)-n} = \bar{u}_{h1,t-(n-1)} \tag{8.15}$$

$$\bar{u}_{h2,(t+1)-n} = \bar{u}_{h2,t-(n-1)}, \tag{8.16}$$

for $n = 1, ..., N$. Observe that the law of motion described by equations (8.15)-(8.16) is already linear, so no further linearization is needed. Also observe that the variables that are N periods old in the date-t history are dropped from the date-$(t+1)$ history. Thus, the law of motion described by these equations introduces a truncation. However, introducing a life cycle structure to the h-type of heterogenous agents will make the consequences of this truncation negligible. The reason is that the truncation only affects agents surviving for N consecutive periods and, given sufficiently small survival probabilities and/or a sufficiently large N, there will be very few of these agents. Apart from this negligible truncation, there are no further approximations errors in the representation of the law of motion given by equations (8.5)-(8.6) – a benefit of using the computational method described in this paper.

Since all $u_{r1t,t+1}$ and u_{r2t} are also spline approximations they are summarized by their values

[42] Taking numerical derivatives of equation (8.13) with respect to each spline coefficient in the list $\left[\{\bar{u}_{h1,t+1-n}\}_{n=1}^N, \{\bar{u}_{h2,t-n}\}_{n=0}^N \right]_h$ requires simulating I_h agents over N periods. Thus, obtaining the linear function $\mathcal{L}^Q$ requires performing a large number of Monte Carlo simulations. Moreover, minimizing sampling errors requires a large value for I_h (in practice I work with panels of about 10 million individuals). While this seems a daunting task, it is easily parallelizable. Thus, using massively parallel computer systems (such as GPU accelerators) can play an important role in reducing computing times and keeping the task manageable.

at the grid points $\bar{u}_{r1t,t+1}$ and $\bar{u}_{r2t}$. The laws of motion (8.7)-(8.8) can then be linearized to obtain

$$0 = \mathcal{L}^{G_{r1}}\left(x_{r1,t+1}, x_{r1t}, x_{r2,t-1}, \bar{u}_{r1t,t+1}\right), \tag{8.17}$$

$$0 = \mathcal{L}^{G_{r2}}\left(x_{r2t}, x_{r1t}, x_{r2,t-1}, \bar{u}_{r2t}\right), \tag{8.18}$$

where $\mathcal{L}^{G_{r1}}$ and $\mathcal{L}^{G_{r2}}$ are vector valued linear functions of the same dimensionality as $x_{r1,t+1}$ and x_{r2t}, respectively.

8.3 Solving the linearized model

Define the following vectors:

$$x_t^1 = \left[\left[\{\Delta\bar{u}_{h1,t+1-n}\}_{n=1}^N\right]_h, [\Delta x_{r1t}]_r\right], \tag{8.19}$$

$$x_{t-1}^2 = \left[\left[\{\Delta\bar{u}_{h2,t-n}\}_{n=1}^N\right]_h, [\Delta x_{r2,t-1}]_r\right], \tag{8.20}$$

$$y_{t+1}^1 = \left[[\Delta\bar{u}_{h1,t+1}]_h, [\Delta\bar{u}_{r1,t+1}]_r\right], \tag{8.21}$$

$$y_t^2 = \left[\left[\Delta\bar{v}_{ht}, \Delta\left(\frac{\partial\bar{v}_{ht}}{\partial x}\right), \Delta\bar{q}_{ht}, \Delta\bar{u}_{h2t}\right]_h, \left[\Delta\bar{v}_{rt}, \Delta\left(\frac{\partial\bar{v}_{rt}}{\partial x}\right), \Delta\bar{q}_{rt}, \Delta\bar{u}_{r2t}\right]_r, \Delta p_t\right], \tag{8.22}$$

where Δ represents deviations from steady state values. $\partial\bar{v}_{ht}/\partial x$ and $\bar{q}_{ht}$ are the derivatives of v_{ht} and the Lagrange multipliers of constraints (8.4), respectively, evaluated at the grid points of the h-type of heterogeneous agents. $\partial\bar{v}_{rt}/\partial x$ and $\bar{q}_{rt}$ are similar objects but for the r-type of representative agents.

The linearized model can then be written as equations (3.16)-(3.21), where (3.16) represents the aggregate feasibility constraints (equation 8.14), (3.17) is the law of motion for x_t^1 (equations 8.15 and 8.17), (3.18)is the law of motion for x_{t-1}^2 (equations 8.16 and 8.18), (3.19) is the first-order conditions for $u_{h1,t+1}$ and $u_{r1,t+1}$ evaluated at the grid points (which must hold almost surely), and (3.20) represents the constraints (8.4), the first-order conditions for u_{h2t} and u_{r2t}, the definitions of $\bar{v}_{ht}$ and $\bar{v}_{rt}$ (e.g., equation 8.12), and the envelope conditions for $\partial\bar{v}_{ht}/\partial x$ and $\partial\bar{v}_{rt}/\partial x$, all evaluated at the grid points (these equations must all hold in expectation).[43]

The strategy for computing the solution to equations (3.16)-(3.21) is described in Section 3.3.

[43] Actually, only the constraints in (8.4) that hold with equality are included in the system of equations. Also, only the Lagrange multipliers of these constraints are included in $\bar{q}_{ht}$ and $\bar{q}_{rt}$ in equation 8.22.

Figure 1: Invariant distribution of promised values

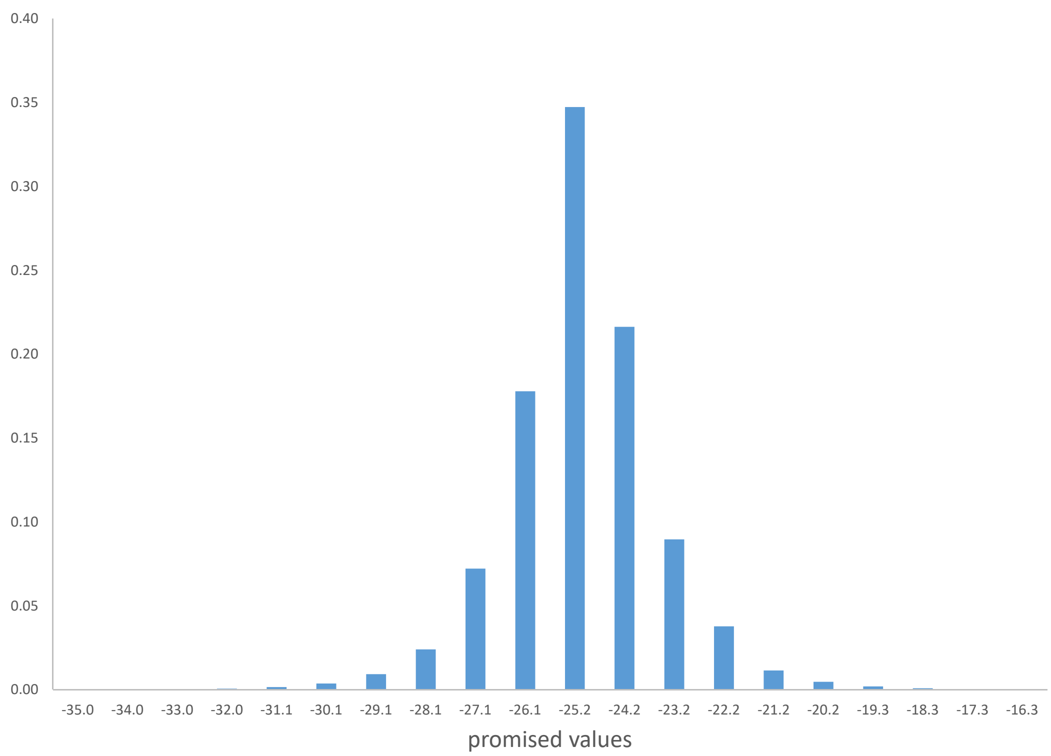


Figure 2: Steady state allocation rules

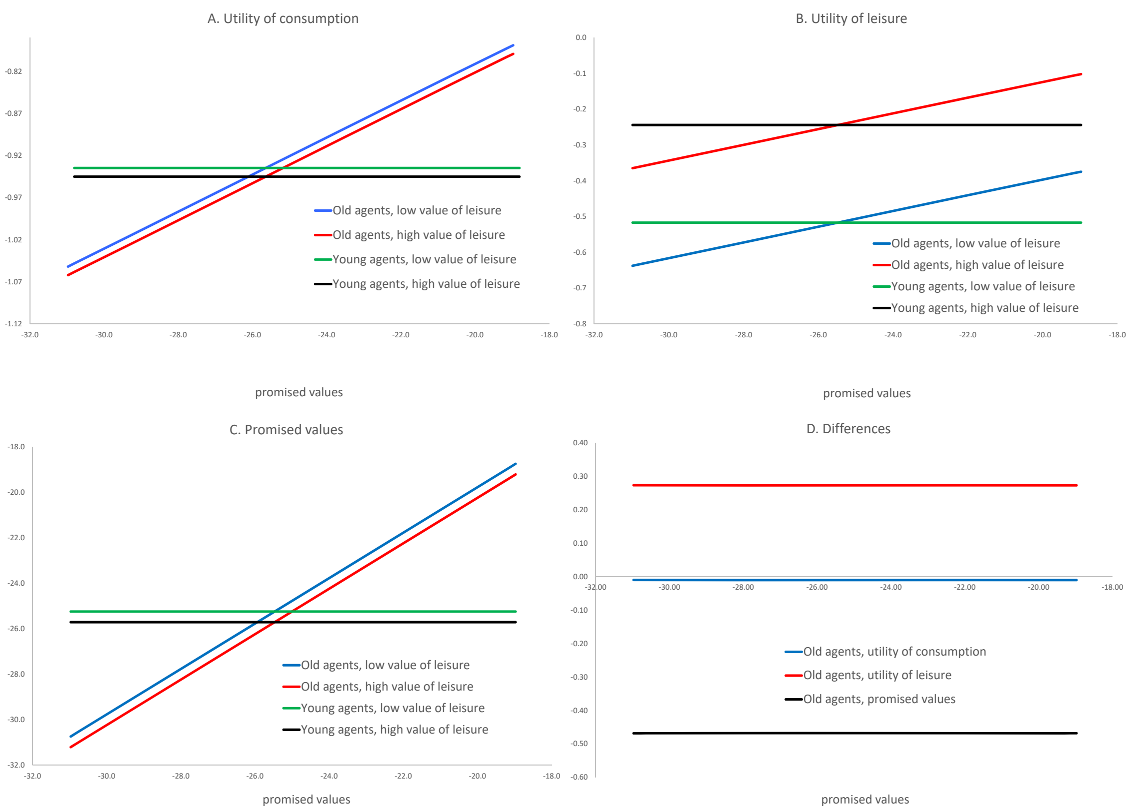


Figure 3: Impulse responses for consumption utilities

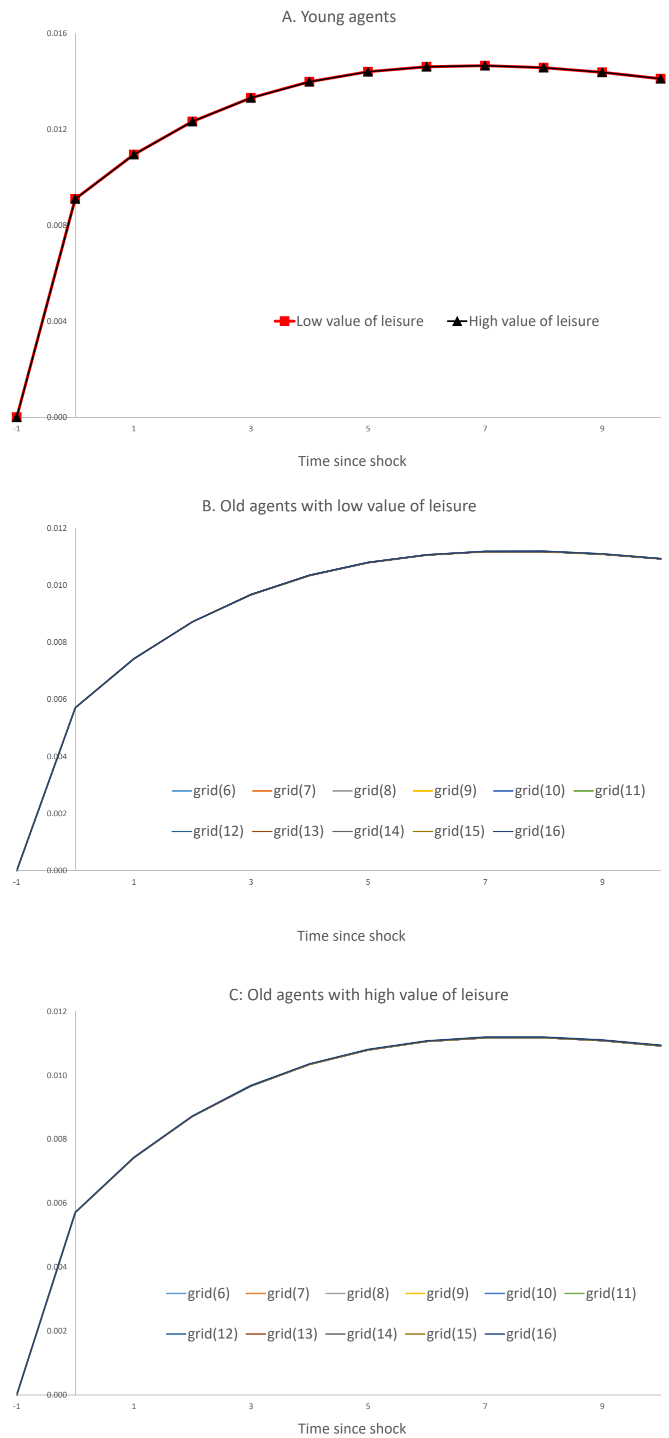


Figure 4: Impulse responses for leisure utilities

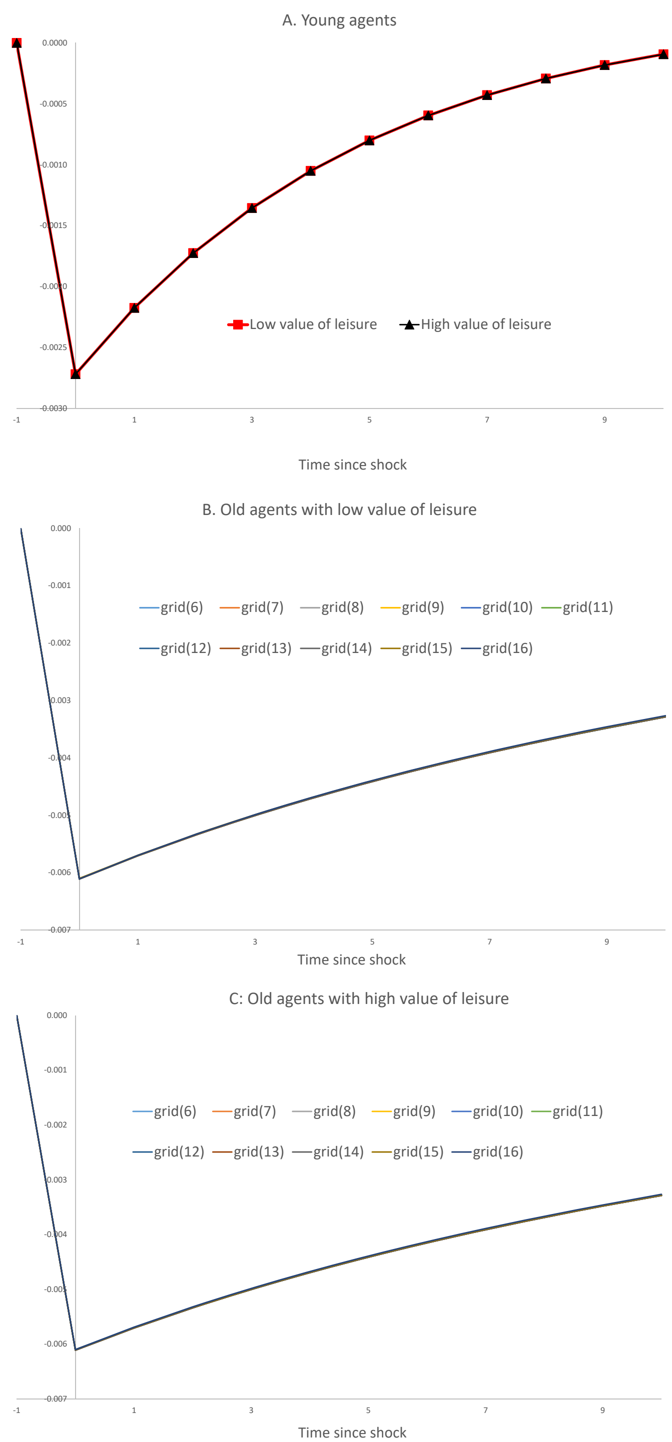


Figure 5: Impulse responses for promised values

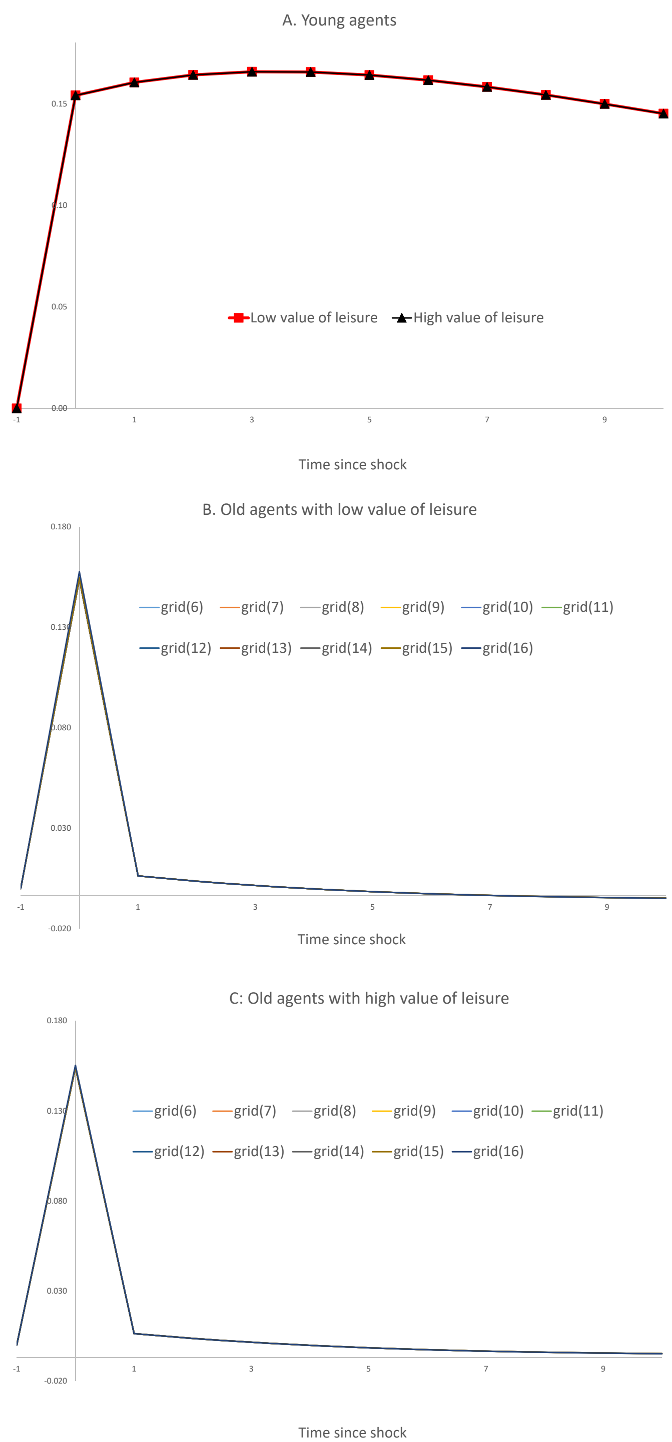


Figure 6: Cross-sectional heterogeneity

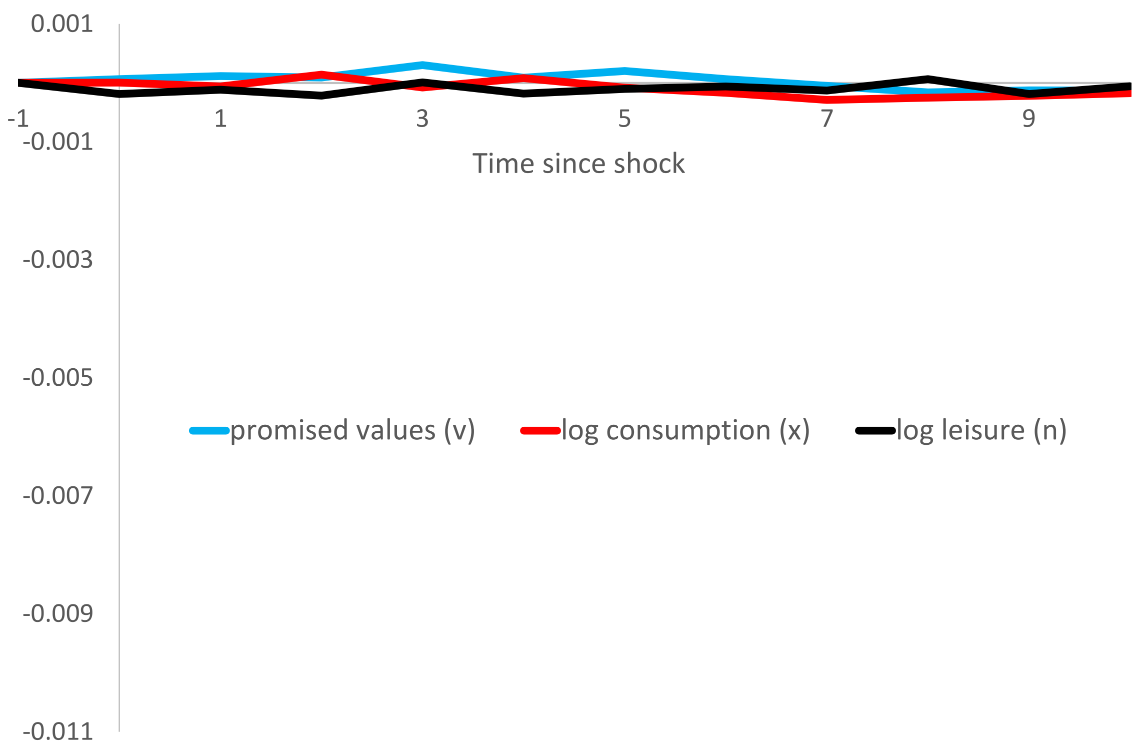
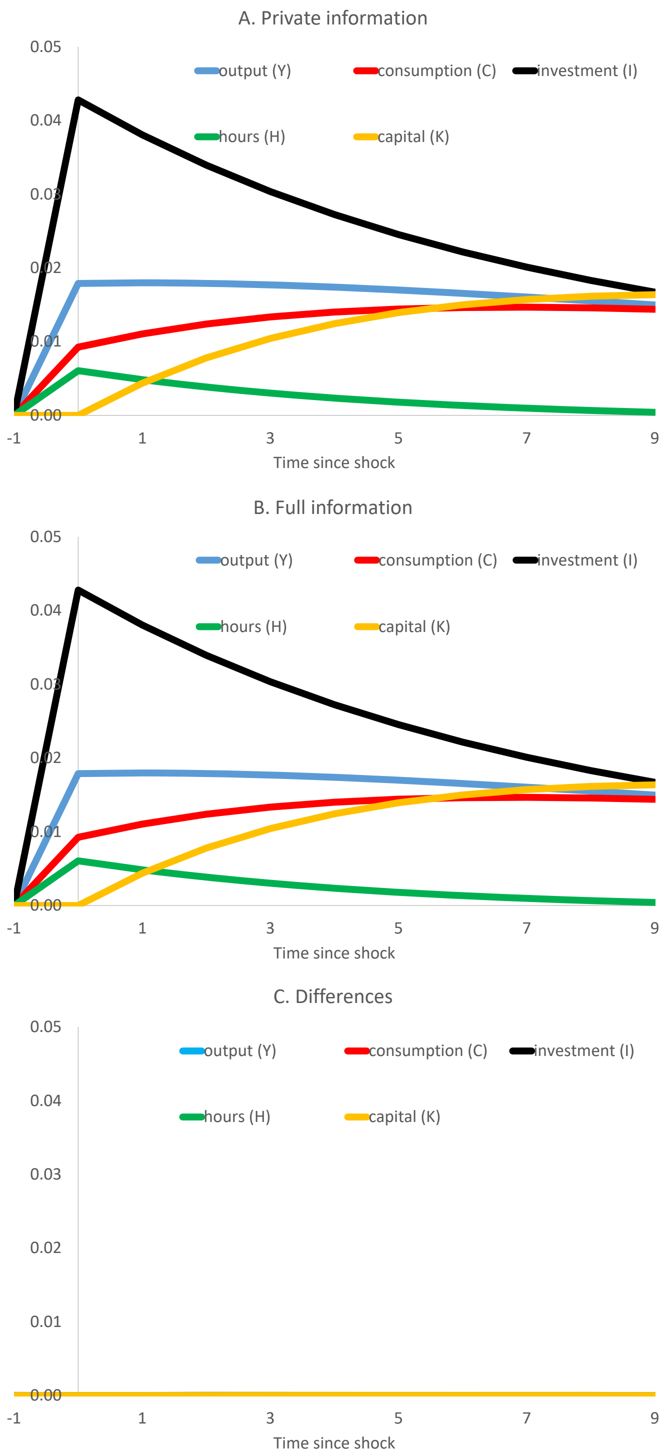


Figure 7: Macro variables



Technical Appendices for

Computing Aggregate Fluctuations of Economies with Private Information

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NOT FOR PUBLICATION

This document contains the following appendices:

Technical Appendix 9: Proof of Proposition 1.

Technical Appendix 10: Linearization of the Mirrlees RBC economy

9 Proof of Proposition 1

The deterministic version of equations (3.16)-(3.21) is given by:

$$
\begin{aligned}
0 &= B_{11}x_t^1 + B_{12}x_{t-1}^2 + C_{12}y_t^2 + D_1 z_t, && (9.1)\\
0 &= A_{21}x_{t+1}^1 + B_{21}x_t^1 + B_{22}x_{t-1}^2 + C_{21}y_{t+1}^1, && (9.2)\\
0 &= A_{32}x_t^2 + B_{31}x_t^1 + B_{32}x_{t-1}^2 + C_{32}y_t^2, && (9.3)\\
0 &= H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}y_{t+1}^2 + K_{41}y_{t+1}^1 + K_{42}y_t^2 + M_4 z_t, && (9.4)\\
0 &= F_{52}x_{t+1}^2 + G_{52}x_t^2 + H_{51}x_t^1 + H_{52}x_{t-1}^2 + J_{52}y_{t+1}^2 + K_{51}y_{t+1}^1 + K_{52}y_t^2 + L_5 z_{t+1} + M_5 z_t, && (9.5)\\
z_{t+1} &= N z_t. && (9.6)
\end{aligned}
$$

The following Lemmas is used in the proof of Proposition **1.**

Lemma 2 *: Suppose that equations (3.26)-(3.29) are the recursive solution to equations (9.1)-(9.6). Then,*

$$
\begin{aligned}
P_{11} &= -A_{21}^{-1}B_{21} - A_{21}^{-1}C_{21}R_{11} && (9.7)\\
P_{12} &= -A_{21}^{-1}B_{22} - A_{21}^{-1}C_{21}R_{12} && (9.8)\\
Q_1 &= -A_{21}^{-1}C_{21}S_1 && (9.9)\\
R_{11} &= -K_{41}^{-1}H_{41} - K_{41}^{-1}J_{42}R_{21}P_{11} - K_{41}^{-1}J_{42}R_{22}P_{21} - K_{41}^{-1}K_{42}R_{21} && (9.10)\\
R_{12} &= -K_{41}^{-1}H_{42} - K_{41}^{-1}J_{42}R_{21}P_{12} - K_{41}^{-1}J_{42}R_{22}P_{22} - K_{41}^{-1}K_{42}R_{22} && (9.11)\\
S_1 &= -K_{41}^{-1}J_{42}R_{21}Q_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}J_{42}S_2N - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4 && (9.12)
\end{aligned}
$$

Also,

$$
\begin{aligned}
0 = {} & \left[F_{52}P_{21}P_{11} + F_{52}P_{22}P_{21} + G_{52}P_{21} + H_{51} + J_{52}R_{21}P_{11} + J_{52}R_{22}P_{21} + K_{51}R_{11} + K_{52}R_{21}\right] x_t^1 \\
& + \left[F_{52}P_{21}P_{12} + F_{52}P_{22}P_{22} + G_{52}P_{22} + H_{52} + J_{52}R_{21}P_{12} + J_{52}R_{22}P_{22} + K_{51}R_{12} + K_{52}R_{22}\right] x_{t-1}^2 \\
& + \left[F_{52}P_{21}Q_1 + F_{52}P_{22}Q_2 + F_{52}Q_2N + G_{52}Q_2 + J_{52}R_{21}Q_1 + J_{52}R_{22}Q_2 \right.\\
& \left. + J_{52}S_2N + K_{51}S_1 + K_{52}S_2 + L_5N + M_5\right] z_t \qquad (9.13)
\end{aligned}
$$

Proof: From equation (9.2) we have that

$$\begin{aligned}
x^1_{t+1} &= -A_{21}^{-1}B_{21}x^1_t - A_{21}^{-1}B_{22}x^2_{t-1} - A_{21}^{-1}C_{21}y^1_{t+1} \qquad (9.14)\\
&= -A_{21}^{-1}B_{21}x^1_t - A_{21}^{-1}B_{22}x^2_{t-1} - A_{21}^{-1}C_{21}\left[R_{11}x^1_t + R_{12}x^2_{t-1} + S_1 z_t\right]\\
&= -A_{21}^{-1}B_{21}x^1_t - A_{21}^{-1}B_{22}x^2_{t-1} - A_{21}^{-1}C_{21}R_{11}x^1_t - A_{21}^{-1}C_{21}R_{12}x^2_{t-1} - A_{21}^{-1}C_{21}S_1 z_t\\
&= \left[-A_{21}^{-1}B_{21} - A_{21}^{-1}C_{21}R_{11}\right]x^1_t + \left[-A_{21}^{-1}B_{22} - A_{21}^{-1}C_{21}R_{12}\right]x^2_{t-1} + \left[-A_{21}^{-1}C_{21}S_1\right]z_t
\end{aligned}$$

where the second equality uses equation (3.28). Equating coefficients with equation (3.26) gives equations (9.7)-(9.9).

From equation (9.4) we have that

$$\begin{aligned}
y^1_{t+1} &= -K_{41}^{-1}H_{41}x^1_t - K_{41}^{-1}H_{42}x^2_{t-1} - K_{41}^{-1}J_{42}y^2_{t+1} - K_{41}^{-1}K_{42}y^2_t - K_{41}^{-1}M_4 z_t\\
&= -K_{41}^{-1}H_{41}x^1_t - K_{41}^{-1}H_{42}x^2_{t-1} - K_{41}^{-1}J_{42}\left[R_{21}x^1_{t+1} + R_{22}x^2_t + S_2 z_{t+1}\right]\\
&\quad -K_{41}^{-1}K_{42}\left[R_{21}x^1_t + R_{22}x^2_{t-1} + S_2 z_t\right] - K_{41}^{-1}M_4 z_t\\
&= -K_{41}^{-1}H_{41}x^1_t - K_{41}^{-1}H_{42}x^2_{t-1} - K_{41}^{-1}J_{42}R_{21}x^1_{t+1} - K_{41}^{-1}J_{42}R_{22}x^2_t - K_{41}^{-1}J_{42}S_2 N z_t\\
&\quad -K_{41}^{-1}K_{42}R_{21}x^1_t - K_{41}^{-1}K_{42}R_{22}x^2_{t-1} - K_{41}^{-1}K_{42}S_2 z_t - K_{41}^{-1}M_4 z_t\\
&= -K_{41}^{-1}H_{41}x^1_t - K_{41}^{-1}H_{42}x^2_{t-1} - K_{41}^{-1}J_{42}R_{21}\left[P_{11}x^1_t + P_{12}x^2_{t-1} + Q_1 z_t\right]\\
&\quad -K_{41}^{-1}J_{42}R_{22}\left[P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t\right] - K_{41}^{-1}J_{42}S_2 N z_t\\
&\quad -K_{41}^{-1}K_{42}R_{21}x^1_t - K_{41}^{-1}K_{42}R_{22}x^2_{t-1} - K_{41}^{-1}K_{42}S_2 z_t - K_{41}^{-1}M_4 z_t\\
&= -K_{41}^{-1}H_{41}x^1_t - K_{41}^{-1}H_{42}x^2_{t-1} - K_{41}^{-1}J_{42}R_{21}P_{11}x^1_t - K_{41}^{-1}J_{42}R_{21}P_{12}x^2_{t-1} - K_{41}^{-1}J_{42}R_{21}Q_1 z_t\\
&\quad -K_{41}^{-1}J_{42}R_{22}P_{21}x^1_t - K_{41}^{-1}J_{42}R_{22}P_{22}x^2_{t-1} - K_{41}^{-1}J_{42}R_{22}Q_2 z_t - K_{41}^{-1}J_{42}S_2 N z_t\\
&\quad -K_{41}^{-1}K_{42}R_{21}x^1_t - K_{41}^{-1}K_{42}R_{22}x^2_{t-1} - K_{41}^{-1}K_{42}S_2 z_t - K_{41}^{-1}M_4 z_t\\
&= \left[-K_{41}^{-1}H_{41} - K_{41}^{-1}J_{42}R_{21}P_{11} - K_{41}^{-1}J_{42}R_{22}P_{21} - K_{41}^{-1}K_{42}R_{21}\right]x^1_t\\
&\quad + \left[-K_{41}^{-1}H_{42} - K_{41}^{-1}J_{42}R_{21}P_{12} - K_{41}^{-1}J_{42}R_{22}P_{22} - K_{41}^{-1}K_{42}R_{22}\right]x^2_{t-1}\\
&\quad + \left[-K_{41}^{-1}J_{42}R_{21}Q_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}J_{42}S_2 N - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4\right]z_t
\end{aligned}$$

where the second equality uses equation (3.29), the third equality uses equation (9.6), and the fourth equality uses equations (3.26) and (3.27). Equating coefficients with equation (3.28) gives equations (9.10)-(9.12).

Finally, from equations (3.29), (3.26), (3.27), and (9.6) we have

$$\begin{aligned} y^2_{t+1} &= R_{21}x^1_{t+1} + R_{22}x^2_t + S_2 z_{t+1} \qquad (9.15) \\ &= R_{21}\left[P_{11}x^1_t + P_{12}x^2_{t-1} + Q_1 z_t\right] + R_{22}\left[P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t\right] + S_2 N z_t \\ &= R_{21}P_{11}x^1_t + R_{21}P_{12}x^2_{t-1} + R_{21}Q_1 z_t + R_{22}P_{21}x^1_t + R_{22}P_{22}x^2_{t-1} + R_{22}Q_2 z_t + S_2 N z_t \\ &= \left[R_{21}P_{11} + R_{22}P_{21}\right]x^1_t + \left[R_{21}P_{12} + R_{22}P_{22}\right]x^2_{t-1} + \left[R_{21}Q_1 + R_{22}Q_2 + S_2 N\right]z_t \end{aligned}$$

From equations (9.5), (9.15), (3.26), (3.27), (3.28) and (3.29) we then have

$$\begin{aligned} 0 &= F_{52}x^2_{t+1} + G_{52}x^2_t + H_{51}x^1_t + H_{52}x^2_{t-1} + J_{52}y^2_{t+1} + K_{51}y^1_{t+1} + K_{52}y^2_t + L_5 z_{t+1} + M_5 z_t \\ &= F_{52}x^2_{t+1} + G_{52}x^2_t + H_{51}x^1_t + H_{52}x^2_{t-1} + J_{52}\left[R_{21}x^1_{t+1} + R_{22}x^2_t + S_2 z_{t+1}\right] \\ &\quad + K_{51}y^1_{t+1} + K_{52}y^2_t + L_5 z_{t+1} + M_5 z_t \\ &= F_{52}\left[P_{21}x^1_{t+1} + P_{22}x^2_t + Q_2 z_{t+1}\right] + G_{52}\left[P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t\right] + H_{51}x^1_t + H_{52}x^2_{t-1} \\ &\quad + J_{52}R_{21}\left[P_{11}x^1_t + P_{12}x^2_{t-1} + Q_1 z_t\right] + J_{52}R_{22}\left[P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t\right] \\ &\quad + J_{52}S_2 N z_t + K_{51}y^1_{t+1} + K_{52}y^2_t + L_5 z_{t+1} + M_5 z_t \\ &= F_{52}P_{21}\left[P_{11}x^1_t + P_{12}x^2_{t-1} + Q_1 z_t\right] + F_{52}P_{22}\left[P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t\right] + F_{52}Q_2 N z_t \\ &\quad + G_{52}P_{21}x^1_t + G_{52}P_{22}x^2_{t-1} + G_{52}Q_2 z_t + H_{51}x^1_t + H_{52}x^2_{t-1} + J_{52}\left[R_{21}P_{11} + R_{22}P_{21}\right]x^1_t \\ &\quad + J_{52}\left[R_{21}P_{12} + R_{22}P_{22}\right]x^2_{t-1} + J_{52}\left[R_{21}Q_1 + R_{22}Q_2 + S_2 N\right]z_t \\ &\quad + K_{51}\left[R_{11}x^1_t + R_{12}x^2_{t-1} + S_1 z_t\right] + K_{52}\left[R_{21}x^1_t + R_{22}x^2_{t-1} + S_2 z_t\right] + L_5 z_{t+1} + M_5 z_t \\ &= F_{52}P_{21}P_{11}x^1_t + F_{52}P_{21}P_{12}x^2_{t-1} + F_{52}P_{21}Q_1 z_t + F_{52}P_{22}P_{21}x^1_t + F_{52}P_{22}P_{22}x^2_{t-1} + F_{52}P_{22}Q_2 z_t \\ &\quad + F_{52}Q_2 N z_t + G_{52}P_{21}x^1_t + G_{52}P_{22}x^2_{t-1} + G_{52}Q_2 z_t + H_{51}x^1_t + H_{52}x^2_{t-1} + J_{52}\left[R_{21}P_{11} + R_{22}P_{21}\right]x^1_t \\ &\quad + J_{52}\left[R_{21}P_{12} + R_{22}P_{22}\right]x^2_{t-1} + J_{52}\left[R_{21}Q_1 + R_{22}Q_2 + S_2 N\right]z_t \\ &\quad + K_{51}\left[R_{11}x^1_t + R_{12}x^2_{t-1} + S_1 z_t\right] + K_{52}\left[R_{21}x^1_t + R_{22}x^2_{t-1} + S_2 z_t\right] + L_5 z_{t+1} + M_5 z_t \\ &= \left[F_{52}P_{21}P_{11} + F_{52}P_{22}P_{21} + G_{52}P_{21} + H_{51} + J_{52}R_{21}P_{11} + J_{52}R_{22}P_{21} + K_{51}R_{11} + K_{52}R_{21}\right]x^1_t \\ &\quad + \left[F_{52}P_{21}P_{12} + F_{52}P_{22}P_{22} + G_{52}P_{22} + H_{52} + J_{52}R_{21}P_{12} + J_{52}R_{22}P_{22} + K_{51}R_{12} + K_{52}R_{22}\right]x^2_{t-1} \\ &\quad + \left[F_{52}P_{21}Q_1 + F_{52}P_{22}Q_2 + F_{52}Q_2 N + G_{52}Q_2 + J_{52}R_{21}Q_1 + J_{52}R_{22}Q_2\right. \\ &\quad \left. + J_{52}S_2 N + K_{51}S_1 + K_{52}S_2 + L_5 N + M_5\right]z_t \end{aligned}$$

Thus, equation (9.13) is satisfied. ■

Proof of Proposition 1: By assumption, equations (9.1) and (9.3) are satisfied by equations

(3.27) and (3.29). Since $\Omega_{21} = P_{21}$, $\Omega_{22} = P_{22}$, $\Psi_2 = Q_2$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, $\Gamma_2 = S_2$, equations (3.16) and (3.18) are then satisfied by equations (3.23) and (3.25).

Observe that equation (3.17) evaluates as follows:

$$\begin{aligned}
& A_{21}x_{t+1}^1 + B_{21}x_t^1 + B_{22}x_{t-1}^2 + C_{21}y_{t+1}^1 \qquad (9.16) \\
= \; & A_{21}\left[\Omega_{11}x_t^1 + \Omega_{12}x_{t-1}^2 + \Psi_1 z_t + \Theta_1 z_{t+1}\right] + B_{21}x_t^1 + B_{22}x_{t-1}^2 \\
& + C_{21}\left[\Phi_{11}x_t^1 + \Phi_{12}x_{t-1}^2 + \Gamma_1 z_t + \Lambda_1 z_{t+1}\right] \\
= \; & A_{21}P_{11}x_t^1 + A_{21}P_{12}x_{t-1}^2 + A_{21}\Psi_1 z_t + A_{21}\Theta_1 z_{t+1} + B_{21}x_t^1 + B_{22}x_{t-1}^2 \\
& + C_{21}R_{11}x_t^1 + C_{21}R_{12}x_{t-1}^2 + C_{21}\Gamma_1 z_t + C_{21}\Lambda_1 z_{t+1} \\
= \; & \left[A_{21}P_{11} + B_{21} + C_{21}R_{11}\right]x_t^1 + \left[A_{21}P_{12} + B_{22} + C_{21}R_{12}\right]x_{t-1}^2 \\
& + \left[A_{21}\Psi_1 + C_{21}\Gamma_1\right]z_t + \left[A_{21}\Theta_1 + C_{21}\Lambda_1\right]\left[Nz_t + \varepsilon_{t+1}\right] \\
= \; & \left[A_{21}P_{11} + B_{21} + C_{21}R_{11}\right]x_t^1 + \left[A_{21}P_{12} + B_{22} + C_{21}R_{12}\right]x_{t-1}^2 \\
& + \left[A_{21}\Psi_1 + C_{21}\Gamma_1 + A_{21}\Theta_1 N + C_{21}\Lambda_1 N\right]z_t + \left[A_{21}\Theta_1 + C_{21}\Lambda_1\right]\varepsilon_{t+1} \\
= \; & \left[-B_{21} - C_{21}R_{11} + B_{21} + C_{21}R_{11}\right]x_t^1 + \left[-B_{22} - C_{21}R_{12} + B_{22} + C_{21}R_{12}\right]x_{t-1}^2 \\
& + \left[A_{21}\Psi_1 + C_{21}\Gamma_1 + A_{21}\Theta_1 N + C_{21}\Lambda_1 N\right]z_t + \left[A_{21}\Theta_1 + C_{21}\Lambda_1\right]\varepsilon_{t+1} \\
= \; & \left[A_{21}\Psi_1 + C_{21}\Gamma_1 + A_{21}\Theta_1 N + C_{21}\Lambda_1 N\right]z_t + \left[A_{21}\Theta_1 + C_{21}\Lambda_1\right]\varepsilon_{t+1}
\end{aligned}$$

where the first equality uses equations (3.22) and (3.24), the second equality uses the fact that $\Omega_{11} = P_{11}$, Ω_{12}, $\Phi_{11} = R_{11}$, $\Phi_{12} = R_{12}$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, and $\Gamma_2 = S_2$, the third equality uses equation (3.21), and the fifth equality uses equations (9.7) and (9.8).

Observe that

$$\begin{aligned}
A_{21}\Theta_1 + C_{21}\Lambda_1 \; = \; & A_{21}\Theta_1 - C_{21}K_{41}^{-1}J_{42}R_{21}\Theta_1 - C_{21}K_{41}^{-1}J_{42}S_2 \qquad (9.17) \\
= \; & A_{21}\left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]\Theta_1 - C_{21}K_{41}^{-1}J_{42}S_2 \\
= \; & A_{21}\left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]\left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]^{-1}A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2 \\
& - C_{21}K_{41}^{-1}J_{42}S_2 \\
= \; & A_{21}A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2 - C_{21}K_{41}^{-1}J_{42}S_2 \\
= \; & C_{21}K_{41}^{-1}J_{42}S_2 - C_{21}K_{41}^{-1}J_{42}S_2 \\
= \; & 0
\end{aligned}$$

where the first equality uses equation (3.32), and the third equality uses equations (3.30) and

(3.34).

Also observe that

$$
\begin{aligned}
& A_{21}\Psi_1 + C_{21}\Gamma_1 + A_{21}\Theta_1 N + C_{21}\Lambda_1 N \qquad (9.18)\\
= \;& A_{21}\Psi_1 + C_{21}\Gamma_1 \\
= \;& A_{21}\Psi_1 - C_{21}K_{41}^{-1}J_{42}R_{21}\Psi_1 - C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 - C_{21}K_{41}^{-1}K_{42}S_2 - C_{21}K_{41}^{-1}M_4 \\
= \;& A_{21}\left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]\Psi_1 - C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 - C_{21}K_{41}^{-1}K_{42}S_2 - C_{21}K_{41}^{-1}M_4 \\
= \;& A_{21}\left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]\Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right] \\
& -C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 - C_{21}K_{41}^{-1}K_{42}S_2 - C_{21}K_{41}^{-1}M_4 \\
= \;& A_{21}\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right] \\
& -C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 - C_{21}K_{41}^{-1}K_{42}S_2 - C_{21}K_{41}^{-1}M_4 \\
= \;& C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + C_{21}K_{41}^{-1}K_{42}S_2 + C_{21}K_{41}^{-1}M_4 - C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 \\
& -C_{21}K_{41}^{-1}K_{42}S_2 - C_{21}K_{41}^{-1}M_4 \\
= \;& 0
\end{aligned}
$$

where the first equality uses equation (9.17), the second equality uses equation (3.33), the fourth equality uses equation (3.31) and the fifth equality uses equation (3.34).

From equations (9.16), (9.17) and (9.18) it follows that equation (3.17) is satisfied.

Observe that equation (3.19) evaluates as follows:

$$
\begin{aligned}
& H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}y_{t+1}^2 + K_{41}y_{t+1}^1 + K_{42}y_t^2 + M_4 z_t && (9.19)\\
= \; & H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}\left[\Phi_{21}x_{t+1}^1 + \Phi_{22}x_t^2 + \Gamma_2 z_{t+1}\right] + K_{41}\left[\Phi_{11}x_t^1 + \Phi_{12}x_{t-1}^2 + \Gamma_1 z_t + \Lambda_1 z_{t+1}\right] \\
& + K_{42}\left[\Phi_{21}x_t^1 + \Phi_{22}x_{t-1}^2 + \Gamma_2 z_t\right] + M_4 z_t \\
= \; & H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}R_{21}x_{t+1}^1 + J_{42}R_{22}x_t^2 + J_{42}S_2 z_{t+1} + K_{41}R_{11}x_t^1 + K_{41}R_{12}x_{t-1}^2 \\
& + K_{41}\Gamma_1 z_t + K_{41}\Lambda_1 z_{t+1} + K_{42}R_{21}x_t^1 + K_{42}R_{22}x_{t-1}^2 + K_{42}S_2 z_t + M_4 z_t \\
= \; & H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}R_{21}\left[\Omega_{11}x_t^1 + \Omega_{12}x_{t-1}^2 + \Psi_1 z_t + \Theta_1 z_{t+1}\right] \\
& + J_{42}R_{22}\left[\Omega_{21}x_t^1 + \Omega_{22}x_{t-1}^2 + \Psi_2 z_t\right] + J_{42}S_2 z_{t+1} + K_{41}R_{11}x_t^1 + K_{41}R_{12}x_{t-1}^2 + K_{41}\Gamma_1 z_t \\
& + K_{41}\Lambda_1 z_{t+1} + K_{42}R_{21}x_t^1 + K_{42}R_{22}x_{t-1}^2 + K_{42}S_2 z_t + M_4 z_t \\
= \; & H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}R_{21}P_{11}x_t^1 + J_{42}R_{21}P_{12}x_{t-1}^2 + J_{42}R_{21}\Psi_1 z_t + J_{42}R_{21}\Theta_1 z_{t+1} \\
& + J_{42}R_{22}P_{21}x_t^1 + J_{42}R_{22}P_{22}x_{t-1}^2 + J_{42}R_{22}Q_2 z_t + J_{42}S_2 z_{t+1} + K_{41}R_{11}x_t^1 + K_{41}R_{12}x_{t-1}^2 \\
& + K_{41}\Gamma_1 z_t + K_{41}\Lambda_1 z_{t+1} + K_{42}R_{21}x_t^1 + K_{42}R_{22}x_{t-1}^2 + K_{42}S_2 z_t + M_4 z_t \\
= \; & \left[H_{41} + J_{42}R_{21}P_{11} + J_{42}R_{22}P_{21} + K_{41}R_{11} + K_{42}R_{21}\right]x_t^1 \\
& + \left[H_{42} + J_{42}R_{21}P_{12} + J_{42}R_{22}P_{22} + K_{41}R_{12} + K_{42}R_{22}\right]x_{t-1}^2 \\
& + \left[J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 + K_{41}\Gamma_1 + K_{42}S_2 + M_4\right]z_t + \left[J_{42}R_{21}\Theta_1 + J_{42}S_2 + K_{41}\Lambda_1\right]z_{t+1} \\
= \; & [H_{41} + J_{42}R_{21}P_{11} + J_{42}R_{22}P_{21} + \\
& K_{41}\left(-K_{41}^{-1}H_{41} - K_{41}^{-1}J_{42}R_{21}P_{11} - K_{41}^{-1}J_{42}R_{22}P_{21} - K_{41}^{-1}K_{42}R_{21}\right) + K_{42}R_{21}]x_t^1 \\
& + [H_{42} + J_{42}R_{21}P_{12} + J_{42}R_{22}P_{22} \\
& + K_{41}\left(-K_{41}^{-1}H_{42} - K_{41}^{-1}J_{42}R_{21}P_{12} - K_{41}^{-1}J_{42}R_{22}P_{22} - K_{41}^{-1}K_{42}R_{22}\right) + K_{42}R_{22}]x_{t-1}^2 \\
& + \left[J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 + K_{41}\Gamma_1 + K_{42}S_2 + M_4\right]z_t + \left[J_{42}R_{21}\Theta_1 + J_{42}S_2 + K_{41}\Lambda_1\right]z_{t+1} \\
= \; & \left[H_{41} + J_{42}R_{21}P_{11} + J_{42}R_{22}P_{21} - H_{41} - J_{42}R_{21}P_{11} - J_{42}R_{22}P_{21} - K_{42}R_{21} + K_{42}R_{21}\right]x_t^1 \\
& + \left[H_{42} + J_{42}R_{21}P_{12} + J_{42}R_{22}P_{22} - H_{42} - J_{42}R_{21}P_{12} - J_{42}R_{22}P_{22} - K_{42}R_{22} + K_{42}R_{22}\right]x_{t-1}^2 \\
& + \left[J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 + K_{41}\Gamma_1 + K_{42}S_2 + M_4\right]z_t + \left[J_{42}R_{21}\Theta_1 + J_{42}S_2 + K_{41}\Lambda_1\right]z_{t+1} \\
= \; & \left[J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 + K_{41}\Gamma_1 + K_{42}S_2 + M_4\right]z_t + \left[J_{42}R_{21}\Theta_1 + J_{42}S_2 + K_{41}\Lambda_1\right]z_{t+1}
\end{aligned}
$$

where the first equality uses equations (3.25) and (3.24), the second equality uses the fact that $\Phi_{11} = R_{11}$, $\Phi_{12} = R_{12}$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, and $\Gamma_2 = S_2$, the third equality uses equations (3.22) and (3.23), the fourth equality uses the fact that $\Omega_{11} = P_{11}$, $\Omega_{12} = P_{12}$, $\Omega_{21} = P_{21}$, $\Omega_{22} = P_{22}$,

$\Psi_2 = Q_2$, where the sixth equality uses equations (9.10) and (9.11),

Observe that

$$\begin{aligned} J_{42}R_{21}\Theta_1 + J_{42}S_2 + K_{41}\Lambda_1 &= J_{42}R_{21}\Theta_1 + J_{42}S_2 - J_{42}R_{21}\Theta_1 - J_{42}S_2 \\ &= 0 \end{aligned} \tag{9.20}$$

where the first equality uses equation (3.32).

Also observe that

$$\begin{aligned} & J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 + K_{41}\Gamma_1 + K_{42}S_2 + M_4 \\ = \; & J_{42}R_{21}\Psi_1 + J_{42}R_{22}Q_2 - J_{42}R_{21}\Psi_1 - J_{42}R_{22}Q_2 - K_{42}S_2 - M_4 + K_{42}S_2 + M_4 \\ = \; & 0 \end{aligned} \tag{9.21}$$

where the first equality uses equation (3.33).

From equations (9.19), (9.20) and (9.21) it follows that equation (3.19) is satisfied.

It remains to show that equation (3.20) holds.

Applying conditional expectations to equations (3.22)-(3.25) and using the fact that $\Omega_{11} = P_{11}$, $\Omega_{12} = P_{12}$, $\Omega_{21} = P_{21}$, $\Omega_{22} = P_{22}$, $\Psi_2 = Q_2$, $\Phi_{11} = R_{11}$, $\Phi_{12} = R_{12}$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, $\Gamma_2 = S_2$, we have

$$E_t\left(x_{t+1}^1\right) = P_{11}x_t^1 + P_{12}x_{t-1}^2 + \left[\Psi_1 + \Theta_1 N\right] z_t, \tag{9.22}$$

$$E_t\left(x_t^2\right) = P_{21}x_t^1 + P_{22}x_{t-1}^2 + Q_2 z_t, \tag{9.23}$$

$$E_t\left(y_{t+1}^1\right) = R_{11}x_t^1 + R_{12}x_{t-1}^2 + \left[\Gamma_1 + \Lambda_1 N\right] z_t, \tag{9.24}$$

$$E_t\left(y_t^2\right) = R_{21}x_t^1 + R_{22}x_{t-1}^2 + S_2 z_t. \tag{9.25}$$

From equations (9.9) and (9.12) we have that

$$\begin{aligned} Q_1 &= -A_{21}^{-1}C_{21}S_1 \\ &= A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}Q_1 + A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2N \\ &\quad + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4 \end{aligned}$$

Hence,

$$\begin{aligned} \left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right] Q_1 &= A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 \\ &\quad + A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2N + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4 \end{aligned}$$

Using equations (3.34), (3.31) and (3.30) we then get that

$$\begin{aligned}
Q_1 &= \Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2N + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right] \\
&= \Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right] + \Upsilon A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2N \\
&= \Psi_1 + \Theta_1 N \qquad (9.26)
\end{aligned}$$

Also, using equations (3.33), (3.32), (9.26) and (9.12) we have that

$$\begin{aligned}
& \Gamma_1 + \Lambda_1 N \qquad (9.27) \\
= \; & -K_{41}^{-1}J_{42}R_{21}\Psi_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4 - K_{41}^{-1}J_{42}R_{21}\Theta_1 N - K_{41}^{-1}J_{42}S_2N \\
= \; & -K_{41}^{-1}J_{42}R_{21}\left[\Psi_1 + \Theta_1 N\right] - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4 - K_{41}^{-1}J_{42}S_2N \\
= \; & S_1
\end{aligned}$$

Using equations (9.26) and (9.27) we can then write equations (9.22)-(9.25) as follows:

$$\begin{aligned}
E_t\left(x_{t+1}^1\right) &= P_{11}x_t^1 + P_{12}x_{t-1}^2 + Q_1 z_t, \qquad (9.28) \\
E_t\left(x_t^2\right) &= P_{21}x_t^1 + P_{22}x_{t-1}^2 + Q_2 z_t, \qquad (9.29) \\
E_t\left(y_{t+1}^1\right) &= R_{11}x_t^1 + R_{12}x_{t-1}^2 + S_1 z_t, \qquad (9.30) \\
E_t\left(y_t^2\right) &= R_{21}x_t^1 + R_{22}x_{t-1}^2 + S_2 z_t. \qquad (9.31)
\end{aligned}$$

From equation (9.31) we have

$$E_{t+1}\left(y_{t+1}^2\right) = R_{21}x_{t+1}^1 + R_{22}x_t^2 + S_2 z_{t+1}$$

Using the Law of Iterated expectations and equations (9.28) and (9.29) we then get

$$\begin{aligned}
E_t\left(y_{t+1}^2\right) &= R_{21}E_t\left(x_{t+1}^1\right) + R_{22}E_t\left(x_t^2\right) + S_2E_t\left(z_{t+1}\right) \qquad (9.32) \\
&= R_{21}\left[P_{11}x_t^1 + P_{12}x_{t-1}^2 + Q_1 z_t\right] \\
&\quad + R_{22}\left[P_{21}x_t^1 + P_{22}x_{t-1}^2 + Q_2 z_t\right] + S_2 N z_t
\end{aligned}$$

Also, from equation (9.29) we have

$$E_{t+1}\left(x_{t+1}^2\right) = P_{21}x_{t+1}^1 + P_{22}x_t^2 + Q_2 z_{t+1}$$

Using the Law of Iterated expectations and equations (9.28) and (9.29) we then get

$$\begin{aligned}
E_t\left(x_{t+1}^2\right) &= P_{21}E_t\left(x_{t+1}^1\right) + P_{22}E_t\left(x_t^2\right) + Q_2E_t\left(z_{t+1}\right) \qquad (9.33) \\
&= P_{21}\left[P_{11}x_t^1 + P_{12}x_{t-1}^2 + Q_1 z_t\right] \\
&\quad + P_{22}\left[P_{21}x_t^1 + P_{22}x_{t-1}^2 + Q_2 z_t\right] + Q_2 N z_t
\end{aligned}$$

Observe that equation (3.20) evaluates as follows

$$
\begin{aligned}
& F_{52}E_t\left(x_{t+1}^2\right)+G_{52}E_t\left(x_t^2\right)+H_{51}x_t^1+H_{52}x_{t-1}^2+J_{52}E_t\left(y_{t+1}^2\right)+K_{51}E_t\left(y_{t+1}^1\right)+K_{52}E_t\left(y_t^2\right) \\
& +L_5Nz_t+M_5z_t \\
= \; & F_{52}P_{21}\left[P_{11}x_t^1+P_{12}x_{t-1}^2+Q_1z_t\right]+F_{52}P_{22}\left[P_{21}x_t^1+P_{22}x_{t-1}^2+Q_2z_t\right]+F_{52}Q_2Nz_t \\
& +G_{52}\left[P_{21}x_t^1+P_{22}x_{t-1}^2+Q_2z_t\right]+H_{51}x_t^1+H_{52}x_{t-1}^2+J_{52}R_{21}\left[P_{11}x_t^1+P_{12}x_{t-1}^2+Q_1z_t\right] \\
& +J_{52}R_{22}\left[P_{21}x_t^1+P_{22}x_{t-1}^2+Q_2z_t\right]+J_{52}S_2Nz_t \\
& +K_{51}\left[R_{11}x_t^1+R_{12}x_{t-1}^2+S_1z_t\right]+K_{52}\left[R_{21}x_t^1+R_{22}x_{t-1}^2+S_2z_t\right]+L_5Nz_t+M_5z_t \\
= \; & F_{52}P_{21}P_{11}x_t^1+F_{52}P_{21}P_{12}x_{t-1}^2+F_{52}P_{21}Q_1z_t+F_{52}P_{22}P_{21}x_t^1+F_{52}P_{22}P_{22}x_{t-1}^2+F_{52}P_{22}Q_2z_t \\
& +F_{52}Q_2Nz_t+G_{52}\left[P_{21}x_t^1+P_{22}x_{t-1}^2+Q_2z_t\right]+H_{51}x_t^1+H_{52}x_{t-1}^2 \\
& +J_{52}R_{21}\left[P_{11}x_t^1+P_{12}x_{t-1}^2+Q_1z_t\right]+J_{52}R_{22}\left[P_{21}x_t^1+P_{22}x_{t-1}^2+Q_2z_t\right]+J_{52}S_2Nz_t \\
& +K_{51}\left[R_{11}x_t^1+R_{12}x_{t-1}^2+S_1z_t\right]+K_{52}\left[R_{21}x_t^1+R_{22}x_{t-1}^2+S_2z_t\right]+L_5Nz_t+M_5z_t \\
= \; & \left[F_{52}P_{21}P_{11}+F_{52}P_{22}P_{21}+G_{52}P_{21}+H_{51}+J_{52}R_{21}P_{11}+J_{52}R_{22}P_{21}+K_{51}R_{11}+K_{52}R_{21}\right]x_t^1 \\
& +\left[F_{52}P_{21}P_{12}+F_{52}P_{22}P_{22}+G_{52}P_{22}+H_{52}+J_{52}R_{21}P_{12}+J_{52}R_{22}P_{22}+K_{51}R_{12}+K_{52}R_{22}\right]x_{t-1}^2 \\
& +\left[F_{52}P_{21}Q_1+F_{52}P_{22}Q_2+F_{52}Q_2N+G_{52}Q_2+J_{52}R_{21}Q_1+J_{52}R_{22}Q_2\right. \\
& \left.+\,J_{52}S_2N+K_{51}S_1+K_{52}S_2+L_5N+M_5\right]z_t \\
= \; & 0
\end{aligned}
$$

where the first equality uses equations (9.33), (9.32), (9.30) and (9.31), and the third equality uses equation (9.13). Thus equation (3.20) is satisfied ■

10 Mirrlees economy: Linearization

This appendix first lists each of the first order conditions for the Mirrlees RBC economy, describes the arguments involved in each first order condition, and states the total number of each first order condition. The appendix then classifies each of the variables involved into the vectors defined by equations (8.19)-(8.22), classifies each first order condition into one of the five types given by equations (3.16)-(3.20), and provides the total number of equations of each type as well as the total number of variables. Finally, the appendix shows how the general linearized model (3.16)-(3.21) simplifies in the Mirrlees RBC economy, as well as the corresponding transformation given by equations (3.30)-(3.34).

Similarly to equations (8.19)-(8.22), J denotes the total number of grid points used in the spline approximations. However, given that promised values must lie in a closed interval, the functions $w_{oLt}(v)$ and $w_{oLt}(v)$ may hit the limits of those intervals when v is close to those limits. In what follows, I denote by J_1 the lowest grid point for which $w_{oLt}(\bar{v}_j)$ is larger than the low limit and J_2 the largest grid point for which $w_{oLt}(\bar{v}_j)$ is smaller than the high limit. Similarly, I denote by J_3 the lowest grid point for which $w_{oHt}(\bar{v}_j)$ is larger than the low limit and J_4 the largest grid point for which $w_{oHt}(\bar{v}_j)$ is smaller than the high limit.

10.1 First order conditions

1) Equation

$$0 = \psi_L - \lambda_t \left[(1-\varphi) u_{yLt} + 1\right]^{\frac{1}{1-\varphi}-1} \psi_L + \lambda_t \xi_{yt}$$

becomes

$$0 = L_{u_{yL}} \left(\triangle u_{yL,t}, \triangle \ln \xi_{yt}, \triangle \ln \lambda_t\right)$$

Number of equations: 1

2) Equation

$$0 = \psi_H - \lambda_t \left[(1-\varphi) u_{yHt} + 1\right]^{\frac{1}{1-\varphi}-1} \psi_H - \lambda_t \xi_{yt}$$

becomes

$$0 = L_{u_{yH}} \left(\triangle u_{yH,t}, \triangle \ln \xi_{yt}, \triangle \ln \lambda_t\right)$$

Number of equations: 1

3) Equation

$$0=\alpha_L\psi_L-\lambda_t q_t\left[(1-\pi)\,n_{yLt}+1\right]^{\frac{1}{1-\pi}-1}\psi_L+\lambda_t\alpha_L\xi_{yt}$$

becomes

$$0=L_{n_{yL}}\left(\triangle n_{yL,t},\triangle\ln\xi_{yt},\triangle\ln\lambda_t,\triangle\ln q_t\right)$$

Number of equations: 1

4) Equation

$$0=\alpha_H\psi_H-\lambda_t q_t\left[(1-\pi)\,n_{yHt}+1\right]^{\frac{1}{1-\pi}-1}\psi_H-\lambda_t\alpha_L\xi_{yt}$$

becomes

$$0=L_{n_{yH}}\left(\triangle n_{yH,t},\triangle\ln\xi_{yt},\triangle\ln\lambda_t,\triangle\ln q_t\right)$$

Number of equations: 1

5) Equation

$$0=\beta\sigma\psi_L+\lambda_t\beta\sigma\xi_{yt}-\theta\lambda_{t+1}\sigma\psi_L\eta_{t+1}\left(w_{yL,t+1}\right)$$

becomes

$$0=L_{w_{yL}}\left(\triangle\ln\xi_{yt},\triangle\ln\lambda_t,\triangle\ln\lambda_{t+1},\left[\triangle\ln\eta_{t+1}\left(\bar{v}_j\right)\right]_{j=1}^{J},\triangle w_{yL,t+1}\right)$$

Number of equations: 1

6) Equation

$$0=\beta\sigma\psi_H-\lambda_t\beta\sigma\xi_{yt}-\theta\lambda_{t+1}\sigma\psi_H\eta_{t+1}\left(w_{yH,t+1}\right)$$

becomes

$$0=L_{w_{yH}}\left(\triangle\ln\xi_{yt},\triangle\ln\lambda_t,\triangle\ln\lambda_{t+1},\left[\triangle\ln\eta_{t+1}\left(\bar{v}_j\right)\right]_{j=1}^{J},\triangle w_{yH,t+1}\right)$$

Number of equations: 1

7) Equation

$$0=-\left[(1-\varphi)\,u_{oLt}(v)+1\right]^{\frac{1}{1-\varphi}-1}\psi_L+\xi_{ot}(v)+\eta_t(v)\,\psi_L$$

becomes

$$0=L_{u_{oL}(\bar{v}_i)}\left(\triangle u_{oL,t}\left(\bar{v}_i\right),\triangle\ln\xi_{ot}\left(\bar{v}_i\right),\triangle\ln\eta_t\left(\bar{v}_i\right)\right)$$

for $i=1,...,J$.

Number of equations: J

8) Equation

$$0=-\left[(1-\varphi)\,u_{oHt}(v)+1\right]^{\frac{1}{1-\varphi}-1}\psi_H-\xi_{ot}(v)+\eta_t(v)\,\psi_H$$

becomes

$$0 = L_{u_{oH}(\bar{v}_i)}\left(\triangle u_{oH,t}\left(\bar{v}_i\right), \triangle \ln \xi_{ot}\left(\bar{v}_i\right), \triangle \ln \eta_t\left(\bar{v}_i\right)\right)$$

for $i = 1, ..., J$.

Number of equations: J

9) Equation

$$0 = -q_t\left[\left(1-\pi\right) n_{oLt}\left(v\right) + 1\right]^{\frac{1}{1-\pi}-1} \psi_L + \alpha_L \xi_{ot}\left(v\right) + \eta_t\left(v\right) \alpha_L \psi_L$$

becomes

$$0 = L_{n_{oL}(\bar{v}_i)}\left(\triangle n_{oL,t}\left(\bar{v}_i\right), \triangle \ln \xi_{ot}\left(\bar{v}_i\right), \triangle \ln \eta_t\left(\bar{v}_i\right), \triangle \ln q_t\right)$$

for $i = 1, ..., J$.

Number of equations: J

10) Equation

$$0 = -q_t\left[\left(1-\pi\right) n_{oHt}\left(v\right) + 1\right]^{\frac{1}{1-\pi}-1} \psi_H - \alpha_L \xi_{ot}\left(v\right) + \eta_t\left(v\right) \alpha_H \psi_H$$

becomes

$$0 = L_{n_{oH}(\bar{v}_i)}\left(\triangle n_{oH,t}\left(\bar{v}_i\right), \triangle \ln \xi_{ot}\left(\bar{v}_i\right), \triangle \ln \eta_t\left(\bar{v}_i\right), \triangle \ln q_t\right)$$

for $i = 1, ..., J$.

Number of equations: J

11) Equation

$$0 = \lambda_t \beta \sigma \xi_{ot}\left(v\right) + \lambda_t \eta_t\left(v\right) \beta \sigma \psi_L - \theta \lambda_{t+1} \sigma \psi_L \eta_{t+1}\left[w_{oL,t+1}\left(v\right)\right]$$

becomes

$$0 = L_{w_{oL}(\bar{v}_i)}\left(\triangle \ln \xi_{ot}\left(\bar{v}_i\right), \triangle \ln \lambda_t, \triangle \ln \eta_t\left(\bar{v}_i\right), \triangle \ln \lambda_{t+1}, \left[\triangle \ln \eta_{t+1}\left(\bar{v}_j\right)\right]_{j=1}^{J}, \triangle w_{oL,t+1}\left(\bar{v}_i\right)\right)$$

for $i = J_1, ..., J_2$.

Number of equations: $J_2 - J_1 + 1$

12) Equation

$$0 = -\lambda_t \beta \sigma \xi_{ot}\left(v\right) + \lambda_t \eta_t\left(v\right) \beta \sigma \psi_H - \theta \lambda_{t+1} \sigma \psi_H \eta_{t+1}\left[w_{oH,t+1}\left(v\right)\right]$$

becomes

$$0 = L_{w_{oH}(\bar{v}_i)} \left(\triangle \ln \xi_{ot} (\bar{v}_i), \triangle \ln \lambda_t, \triangle \ln \eta_t (\bar{v}_i), \triangle \ln \lambda_{t+1}, [\triangle \ln \eta_{t+1} (\bar{v}_j)]_{j=1}^{J}, \triangle w_{oH,t+1} (\bar{v}_i) \right)$$

for $i = J_3, ..., J_4$.

Number of equations: $J_4 - J_3 + 1$

13) Equation

$$0 = u_{yLt} + \alpha_L n_{yLt} + \beta\sigma E_t [w_{yL,t+1}] - \{u_{yHt} + \alpha_L n_{yHt} + \beta\sigma E_t [w_{yH,t+1}]\}$$

becomes

$$0 = E_t [L_{IC^y} (\triangle u_{yL,t}, \triangle n_{yL,t}, \triangle w_{yL,t+1}, \triangle u_{yH,t}, \triangle n_{yH,t}, \triangle w_{yH,t+1})]$$

Number of equations: 1

14) Equation

$$0 = u_{oLt} (v) + \alpha_L n_{oLt} (v) + \beta\sigma E_t [w_{oL,t+1} (v)] - \{u_{oHt} (v) + \alpha_L n_{oHt} (v) + \beta\sigma E_t [w_{oH,t+1} (v)]\}$$

becomes

$$0 = E_t \left[L_{IC^o(\bar{v}_i)} (\triangle u_{oL,t} (\bar{v}_i), \triangle n_{oL,t} (\bar{v}_i), \triangle w_{oL,t+1} (\bar{v}_i), \triangle u_{oH,t} (\bar{v}_i), \triangle n_{oH,t} (\bar{v}_i), \triangle w_{oH,t+1} (\bar{v}_i)) \right]$$

for $i = 1, ..., J$.

Number of equations: J

15) Equation

$$0 = v - \{u_{oLt} (v) + \alpha_L n_{oLt} (v) + \beta\sigma E_t [w_{oL,t+1} (v)]\} \psi_L$$
$$- \{u_{oHt} (v) + \alpha_H n_{oHt} (v) + \beta\sigma E_t [w_{oH,t+1} (v)]\} \psi_H$$

becomes

$$0 = E_t \left[L_{PK^o(\bar{v}_i)} (\triangle u_{oL,t} (\bar{v}_i), \triangle n_{oL,t} (\bar{v}_i), \triangle w_{oL,t+1} (\bar{v}_i), \triangle u_{oH,t} (\bar{v}_i), \triangle n_{oH,t} (\bar{v}_i), \triangle w_{oH,t+1} (\bar{v}_i)) \right]$$

Number of equations: J

16) Equation

$$0 = q_t - e^{z_t} K_{t-1}^{\gamma} (1 - \gamma) H_t^{-\gamma}$$

becomes

$$0 = L_q (\triangle \ln q_t, \triangle \ln K_{t-1}, \triangle \ln H_t, \triangle z_t)$$

Number of equations: 1

17) Equation

$$0 = -\lambda_t + \theta E_t \left\{ \lambda_{t+1} \left[e^{z_{t+1}} \gamma K_t^{\gamma-1} H_{t+1}^{1-\gamma} + 1 - \delta \right] \right\}$$

becomes

$$0 = E_t \left[L_\lambda \left(\triangle \ln \lambda_t, \triangle \ln K_t, \triangle \ln \lambda_{t+1}, \triangle \ln H_{t+1}, \triangle z_{t+1} \right) \right]$$

Number of equations: 1

18) Equation

$$\begin{aligned} 0 &= (1-\sigma) \left[(1-\varphi) u_{yLt} + 1 \right]^{\frac{1}{1-\varphi}} \psi_L + (1-\sigma) \left[(1-\varphi) u_{yHt} + 1 \right]^{\frac{1}{1-\varphi}} \psi_H \\ &+ \int \left[(1-\alpha) u_{oLt}(v) + 1 \right]^{\frac{1}{1-\alpha}} \psi_L d\mu_t + \int \left[(1-\alpha) u_{oHt}(v) + 1 \right]^{\frac{1}{1-\alpha}} \psi_H d\mu_t \\ &+ I_t - e^{z_t} K_{t-1}^{\gamma} H_t^{1-\gamma} \end{aligned}$$

becomes

$$\begin{aligned} 0 = L_Y \Big(& \triangle u_{yL,t}, \triangle u_{yH,t}, \left[\triangle u_{oL,t}(\bar{v}_j) \right]_{j=1}^{J}, \left[\triangle u_{oH,t}(\bar{v}_j) \right]_{j=1}^{J}, \triangle \ln I_t, \triangle \ln K_{t-1}, \\ & \triangle \ln H_t, \triangle z_t, \left\{ \triangle w_{yL,t-n}, \triangle w_{yH,t-n}, \left[\triangle w_{oL,t-n}(\bar{v}_j) \right]_{j=J_1}^{J_2}, \left[\triangle w_{oH,t-n}(\bar{v}_j) \right]_{i=J_3}^{J_4} \right\}_{n=0}^{N} \Big) \end{aligned}$$

Observe that this linear approximation must be done numerically using Monte Carlo simulations.

Number of equations: 1

19) Equation

$$\begin{aligned} 0 &= (1-\sigma) \left\{ 1 - \left[(1-\pi) n_{yLt} + 1 \right]^{\frac{1}{1-\pi}} \right\} \psi_L + (1-\sigma) \left\{ 1 - \left[(1-\pi) n_{yHt} + 1 \right]^{\frac{1}{1-\pi}} \right\} \psi_H \\ &+ \int \left\{ 1 - \left[(1-\pi) n_{oLt}(v) + 1 \right]^{\frac{1}{1-\pi}} \right\} \psi_L d\mu_t + \int \left\{ 1 - \left[(1-\pi) n_{oHt}(v) + 1 \right]^{\frac{1}{1-\pi}} \right\} \psi_H d\mu_t - H_t \end{aligned}$$

becomes

$$\begin{aligned} 0 = L_H \Big(& \triangle n_{yL,t}, \triangle n_{yH,t}, \left[\triangle n_{oL,t}(\bar{v}_j) \right]_{j=1}^{J}, \left[\triangle n_{oH,t}(\bar{v}_j) \right]_{j=1}^{J}, \triangle \ln H_t, \\ & \left\{ \triangle w_{yL,t-j}, \triangle w_{yH,t-n}, \left[\triangle w_{oL,t-n}(\bar{v}_j) \right]_{j=J_1}^{J_2}, \left[\triangle w_{oH,t-n}(\bar{v}_j) \right]_{j=J_3}^{J_4} \right\}_{n=0}^{N} \Big) \end{aligned}$$

Observe that this linear approximation also must be done numerically, using Monte Carlo simulations.

Number of equations: 1

20) Equation

$$0 = K_t - (1-\delta) K_{t-1} - I_t$$

becomes

$$L_I (\triangle \ln K_t, \triangle \ln K_{t-1}, \triangle \ln I_t)$$

Number of equations: 1

21) Equation

$$\begin{aligned}
&\left[\triangle w_{yL,t+1-n}, \triangle w_{yH,t+1-n}, [\triangle w_{oL,t+1-n}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t+1-n}(\bar{v}_j)]_{j=J_3}^{J_4}\right]_{n=0}^{N} \\
&= \triangle w_{yL,t+1}, \triangle w_{yH,t+1}, [\triangle w_{oL,t+1}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t+1}(\bar{v}_j)]_{j=J_3}^{J_4}, \\
&\left[\triangle w_{yL,t-n}, \triangle w_{yH,t-n}, [\triangle w_{oL,t-n}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t-n}(\bar{v}_j)]_{j=J_3}^{J_4}\right]_{n=0}^{N-1}
\end{aligned}$$

becomes

$$0 = L_\mu \left(\begin{array}{c} \triangle w_{yL,t+1}, \triangle w_{yH,t+1}, [\triangle w_{oL,t+1}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t+1}(\bar{v}_j)]_{j=J_3}^{J_4}, \\ \left[\triangle w_{yL,t-n}, \triangle w_{yH,t-n}, [\triangle w_{oL,t-n}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t-n}(\bar{v}_j)]_{j=J_3}^{J_4}\right]_{n=0}^{N-1} \end{array} \right)$$

Number of equations: $(N+1)[2 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1)]$

10.2 Classification of variables and equations

Classify the variables as follows:

$$x_t^1 = \left\{\triangle w_{yL,t-n}, \triangle w_{yH,t-n}, [\triangle w_{oL,t-n}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t-n}(\bar{v}_j)]_{j=J_3}^{J_4}\right\}_{n=0}^{N} \tag{10.1}$$

$$x_{t-1}^2 = \{\triangle \ln K_{t-1}\} \tag{10.2}$$

$$y_{t+1}^1 = \left\{\triangle w_{yL,t+1}, \triangle w_{yH,t+1}, [\triangle w_{oL,t+1}(\bar{v}_j)]_{j=J_1}^{J_2}, [\triangle w_{oH,t+1}(\bar{v}_j)]_{j=J_3}^{J_4}\right\} \tag{10.3}$$

$$\begin{aligned}
y_t^2 = \Big\{&\triangle u_{yL,t}, \triangle u_{yH,t} \triangle n_{yL,t}, \triangle n_{yH,t} \triangle \ln \xi_{yt}, \triangle \ln \lambda_t, \triangle \ln q_t, [\triangle \ln \eta_t(\bar{v}_j)]_{j=1}^{J}, \\
&[\triangle u_{oL,t}(\bar{v}_j)]_{j=1}^{J}, [\triangle u_{oH,t}(\bar{v}_j)]_{j=1}^{J}, [\triangle n_{oL,t}(\bar{v}_j)]_{j=1}^{J}, [\triangle n_{oH,t}(\bar{v}_j)]_{j=1}^{J}, \\
&[\triangle \ln \xi_{ot}(\bar{v}_j)]_{j=1}^{J}, \triangle \ln H_t, \triangle I_t\Big\}
\end{aligned} \tag{10.4}$$

The following Table classifies the different equations into five types. Types 1-5 correspond to equations (3.16)-(3.20), respectively.

Equation #	Equation name	# of equations	Type (1,2,3,4 or 5)
1	$L_{u_{yL}}$	1	5
2	$L_{u_{yH}}$	1	5
3	$L_{n_{yL}}$	1	5
4	$L_{n_{yH}}$	1	5
5	$L_{w_{yL}}$	1	4
6	$L_{w_{yH}}$	1	4
7	$L_{u_{oL}(\bar{v}_i)}$	J	5
8	$L_{u_{oH}(\bar{v}_i)}$	J	5
9	$L_{n_{oL}(\bar{v}_i)}$	J	5
10	$L_{n_{oH}(\bar{v}_i)}$	J	5
11	$L_{w_{oL}(\bar{v}_i)}$	$J_2 - J_1 + 1$	4
12	$L_{w_{oH}(\bar{v}_i)}$	$J_4 - J_3 + 1$	4
13	L_{IC^y}	1	5
14	$L_{IC^o(\bar{v}_i)}$	J	5
15	$L_{PK^o(\bar{v}_i)}$	J	5
16	L_q	1	5
17	L_λ	1	5
18	L_Y	1	1
19	L_H	1	1
20	L_I	1	3
21	L_μ	$(N+1)\left[2+(J_2-J_1+1)+(J_4-J_3+1)\right]$	2

Total number of equations:

$$12 + 6J + (J_2 - J_1 + 1) + (J_4 - J_3 + 1) + (N+1)\left[2+(J_2-J_1+1)+(J_4-J_3+1)\right]$$

Total number of Type 1 equations: 2

Total number of Type 2 equations: $(N+1)\left[2+(J_2-J_1+1)+(J_4-J_3+1)\right]$

Total number of Type 3 equations: 1

Total number of Type 4 equations: $2 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1)$

Total number of Type 5 equations: $7 + 6J$

The following is the dimensionality of the different variables:

$$\dim\left(x_{t-1}^{1}\right) = (N+1)\left[2 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1)\right]$$

$$\dim\left(x_{t-1}^{2}\right) = 1$$

$$\dim\left(y_{t}^{1}\right) = 2 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1)$$

$$\dim\left(y_{1}^{2}\right) = 9 + 6J$$

$$\dim\left(z_t\right) = 1$$

Total number of endogenous variables: $(N+1)\left[2 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1)\right] + 12 + (J_2 - J_1 + 1) + (J_4 - J_3 + 1) + 6J$ (same as number of equations).

10.3 Simplified linear system

Under the classifications of the previous section, the linearized model given by equations (3.16)-(3.21) simplifies to the following:

$$\begin{aligned}
0 &= B_{11}x_t^1 + B_{12}x_{t-1}^2 + C_{12}y_t^2 + D_1 z_t \\
0 &= A_{21}x_{t+1}^1 + B_{21}x_t^1 + C_{21}y_{t+1}^1 \\
0 &= A_{32}x_t^2 + B_{32}x_{t-1}^2 + C_{32}y_t^2 \\
0 &= J_{42}y_{t+1}^2 + K_{41}y_{t+1}^1 + K_{42}y_t^2 \\
0 &= E_t\left\{G_{52}x_t^2 + H_{52}x_{t-1}^2 + J_{52}y_{t+1}^2 + K_{51}y_{t+1}^1 + K_{52}y_t^2 + L_5 z_{t+1} + M_5 z_t\right\}
\end{aligned}$$

and the transformation given by equations (3.30)-(3.34) simplifies to:

$$\Theta_1 = \Upsilon A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2, \tag{10.5}$$

$$\Psi_1 = \Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2\right], \tag{10.6}$$

$$\Lambda_1 = -K_{41}^{-1}J_{42}R_{21}\Theta_1 - K_{41}^{-1}J_{42}S_2, \tag{10.7}$$

$$\Gamma_1 = -K_{41}^{-1}J_{42}R_{21}\Psi_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}K_{42}S_2, \tag{10.8}$$

where

$$\Upsilon = \left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]^{-1} \tag{10.9}$$