Financial Leverage, Corporate Investment and Stock Returns

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Abstract

This paper presents a dynamic model of the firm with limited capital irreversibility and incomplete debt contracts in order to analyze the effects of financial leverage on investment and explain the cross-sectional differences in equity returns. I find that the debt capacity of the firm exceeds the collateral value of its capital even if debt is risk-free. Moreover, in the absence of corporate profit taxes, debt has no effect on total firm value and investment decisions whereas taxes cause the firm to invest and disinvest at lower levels of productivity and its value to rise with debt. Finally, I show how the capital structure affects risk exposure. In particular, financial leverage amplifies the effect of capital irreversibility and operating leverage on the cross-section of stock returns and hence can account for the cross-sectional variation of returns even with limited capital irreversibility.

1 Introduction

Stocks with high ratio of book value of equity to market value of equity, referred to as value stocks, earn higher expected returns than growth stocks that have low book-to-market equity ratio. However, conventional wisdom tells us that growth stocks, which derive their market value from unrealized growth options, must earn higher returns than value stocks , which derive their value from assets in place. As Grinblatt and Titman (2002, p.392) point out growth options depend on future economic conditions and therefore must be riskier than assets in place:

> Consider Wal-Mart, for example. The value of this firm's assets can be regarded as the value of the existing Wal-Mart outlets in addition to the value of any outlets that Wal-Mart may open in the future. The option to open new

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stores is known as a growth option. Because growth options tend to be most valuable in good times and have implicit leverage they contain a great deal of systematic risk.

To add insult to injury, Fama and French (1992) show that portfolios of stocks with different book-to-market ratios have similar riskiness as measured by the standard Capital Asset Pricing Model (CAPM) of Sharpe (1964), Litner (1965) and Black (1972). This phenomenon is coined as the "value premium puzzle" and helped the Fama and French model replace the CAPM as the benchmark model in asset pricing literature.

This paper extends the investment model of Abel and Eberly (1996) with limited capital irreversibility and incomplete debt contracts in order to analyze the effects of financial leverage on investment and explain the differences of value and growth stocks. In this model, the firm tries to maximize the discounted value of its cash flows net of operating and financial costs by choosing its investment and financing plans. Capital irreversibility exists because the resale price of capital is a fraction of its purchase price due to specialized machinery[1]. Because the firm faces linear investment and disinvestment costs, its optimal investment behavior is characterized by a two-trigger policy in which the firm purchases capital to prevent the marginal value of capital to the shareholders from rising above the purchase price and sells capital to prevent the marginal value of capital from falling below the resale price. When the marginal value of capital is between the resale and purchase price, the firm is in the inaction region where the net investment is zero. As a result of this trigger policy, the equity value will not only depend on the cash flows generated by the existing assets but also on the value derived from future investment (growth) or disinvestment opportunities[2].

The financing decisions in this model are similar, but not identical, to a trade-off model. According to the trade-off theory of capital structure, a firm chooses its financing policy by balancing the costs of bankruptcy and benefits of debt, such as tax shields due to interest payments[3]. In my model, firms benefit from tax shield of debt as in the trade-off theory but the amount of debt is limited by the lenders rather than endogenously chosen by the firm. In particular, lenders agree to provide the firm with a credit line at risk-free rate up to a certain fraction of capital.

The credit line of the firm is determined endogenously in the following way: Since interest payment is tax deductible, the firm prefers debt financing to equity financing and it would rather have infinite amount of debt. However, this leads to negative equity value in some states of the world and the firm would rather go bankrupt instead of paying its debt. Therefore, for debt to remain risk-free, lenders will limit the amount of debt. They can do so by accepting the resale value of capital as collateral and ensuring that this value is not lower than the amount of debt so that they can recover their money in case of liquidation of assets. Alternatively, lenders may limit the amount of debt in

[1] Other justifications for capital irreversibility in the literature include installation/detachment costs or limited second-hand market for capital due to asymmetric information regarding the quality of capital, i.e. lemon's problem.

[2] This decomposition of the firm value is closely related to intangible sources of capital in macroeconomic models as discussed by McGrattan and Prescott (2000), Hall (2001), Atkeson and Kehoe (2005) and Hansen, Heaton and Li (2005).

[3] See Hennessy and Whited (2005) for a recent application of this theory.

order to ensure that the market value of equity is always non-negative and bankruptcy is suboptimal for the firm. Because lenders are indifferent between these two policies they are ready to accept the least restricting one for the firm. I show that the equity value is always non-negative and hence that bankruptcy is suboptimal even if the resale price of capital is less than the amount debt. The reason is that firms with a lot of debt are also stuck with an excess of operating capital relative to their productivity. Because operating and financial costs increase with capital, these firms would rather disinvest and reduce their capital and debt at the same time and continue their operations instead of going bankrupt if lenders choose the correct limit on leverage. As a result, the lenders follow the latter lending policy with no-bankruptcy condition and are ready to lend the firms more than the collateralizable value of their capital at risk-free rate even when the firm is very unproductive. Moreover, since leverage has been determined by the no-bankruptcy condition bankruptcy costs do not play any role in the determination of capital structure unlike the standard trade-off models.

I show that the expected stock returns are determined by the elasticity of market value of equity with respect to changes in productivity because the productivity shocks are the main source of systematic risk and risk heterogeneity across firms. Value firms have higher book value of equity relative to market value of equity because their capital is high relative to their productivity and it is not optimal to lower the capital level immediately because of disinvestment costs. Due to this excess of capital, their cash flows, and hence their firm value, are more responsive to productivity shocks than growth firms. As a result they earn higher expected returns.

In this model, financial leverage affects stock returns directly through its effect on risk structure a la Modigliani and Miller (1958) and indirectly through its effect on investment decisions. The former effect dominates the latter. In particular, financial leverage amplifies the effect of capital irreversibility on the cross-section of stock returns and hence can account for the value premium without relying on a high degree of capital irreversibility.

I also show that value premium is countercyclical as predicted by Zhang (2005) and later confirmed by Chen, Petkova and Zhang (2008). Whereas countercyclical price of risk plays a central role in the countercyclicality of value premium in Zhang's model, my model produces countercyclical value premium due to the convex relationship of risk exposures and book-to-market values as shown in Figure 1. The premium is smaller after subsequent good aggregate shocks because most of the firms are concentrated at lower levels of book-to-market where the relationship of book-to-market and risk exposure is flatter. Moreover, it increases after subsequent bad aggregate shocks because more firms are concentrated at higher book-to-market levels. Hence, the premium decreases in good times and increases in bad times.

Analysis of investment decisions reveals that in the absence of corporate profit taxes, debt has no effect on total firm value and investment decisions whereas profit taxes cause the firm to invest and disinvest at lower levels of productivity. In particular, debt has two counteracting effects on investment and disinvestment boundaries. First, it reduces the marginal value of capital to the firms because firms have to pay interest on debt and firms with higher capital also have more debt. Therefore, firms reach their investment and disinvestment boundaries at a higher level productivity. Second, due to financial leverage, the proportion of investment that should be financed out of shareholders' pocket decreases

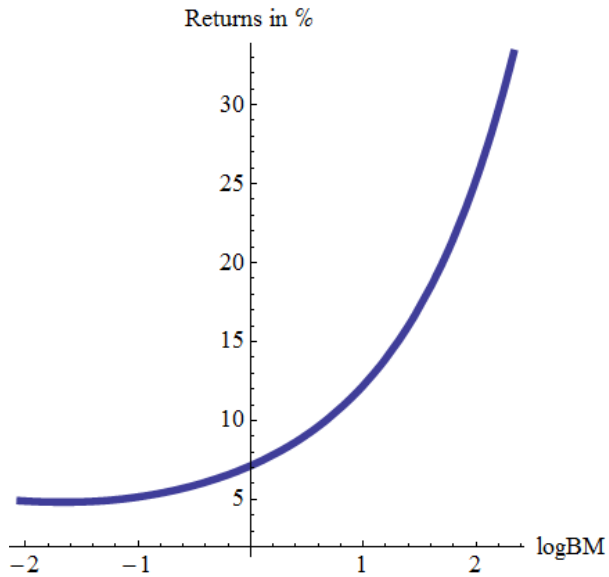

Figure 1: Expected returns vs. book-to-market ratio for benchmark parameter values.

whereas in case of disinvestment most of the proceeds goes to debtholders because of debt repayment. This pushes the investment boundaries in the opposite direction. In the absence of taxes, these two effects offset each other whereas in the presence of taxes, the first effect becomes weaker because interest is tax deductible. Hence, taxes cause the firm to invest and disinvest at lower levels of productivity.

2 Literature Review

This paper is closely related with a recent line of literature that ties value premium to capital irreversibility in production based asset pricing models[4]. But these papers ignore debt financing. One justification for this is that the Modigliani-Miller theorem holds in frictionless economies so that financing decisions do not matter for the firm value[5]. However, financing does matter for equity returns even in the absence of frictions: In Modgiliani-Miller world, unlevered (all-equity financed) firm's return is a weighted average of levered firm (equity) return and return on debt. Since these papers focus on the unlevered firms they do not actually provide an explanation of the value premium for the levered firms although the evidence provided by Fama and French (1992, 1995, 1996) is for the latter.

Unfortunately, because these papers are silent regarding the choice of leverage they do not provide any guideline about how to relever the unlevered firm's stock returns to facilitate comparison with Fama-French evidence. One way to solve this problem is to find appropriate measures for market leverage and debt returns to unlever the observed

[4]See Carlson, Fisher and Giammarino (2004), Zhang (2005), Cooper (2006) and Gala (2006).

[5]The basic Modigliani and Miller (1958) theorem states that, in the absence of taxes, bankruptcy costs, and asymmetric information, and in an efficient market, the value of a firm, its debt and equity combined, is unaffected by how that firm is financed.

equity returns and compare them with the unlevered firm returns implied by the model. However, this daunting task will distance us from the original aim of explaining value premium in levered returns and there is no evidence that ties unlevered returns to total market to book ratio. On the contrary, after accounting for the debt component of the firm return, Hecht (2000) finds that many of the cross sectional determinants of expected equity and debt returns, including market-to-book value, are nonexistent at the level of the firm. Instead, this paper models the debt of companies explicitly and focuses on the levered stock returns.

Moreover, the papers in this literature rely on a very high degree of irreversibility to obtain a significant variation among different stocks' returns. This requirement is satisfied through different assumptions. In a setting with linear and fixed adjustment costs and operating leverage, Carlson, Fisher and Giammarino (2004) assume that the net proceeds from capital sales is zero. In a similar setting, Cooper (2006) suggests that capital is irreversible, i.e. disposing of capital is prohibitively costly, so that firms rather go bankrupt instead of disinvesting. Zhang (2005) assumes quadratic and asymmetric capital adjustment costs where disinvestment is ten times more costly than investment. Gala (2006) extends Zhang's model to a general equilibrium framework and prohibits capital disposal as in Cooper (2006).

This paper contributes to the literature by relaxing the degree of capital irreversibility. This step is justified by two studies. First of all, Hall (2004) estimates the adjustment cost parameter for capital and finds that adjustment costs are relatively small and are not an important part of the explanation of the large movements of company values. Second, the degree of irreversibility assumed by the literature implies that the net value generated by disinvestment is non-positive after adjustment costs are included[6]. However, if adjustment costs are this large then the proceeds from liquidation of discontinued operations would also be very low. Otherwise, the firms would have preferred to liquidate their assets and recreate another firm with smaller size instead of trying to adjust their capital at the margin[7]. In contrast to this implication of high adjustment costs, Berger et al. (1996) find that a dollar of book value yields, on average, 55 cents for inventory, and 54 cents for fixed assets all of which are significantly positive. Therefore, we conclude that the degree of capital irreversibility should be much lower than what the current theoretical literature assumes. My model still generates significant cross-sectional variation of returns because financial leverage amplifies the effect of irreversibility on equity returns.

This paper is also part of a growing literature on dynamic quantitative models investigating the implications of firms' financing decisions for asset returns. Some recent papers along these lines include Livdan, Sapriza and Zhang (2008), Gomes and Schmid (2008), Obreja (2006) and Garlappi and Yan (2008). Livdan, Sapriza and Zhang (2008) study

[6] In a linear adjustment cost model this would formally mean that the net resale value of capital is non-positive. In case of convex (quadratic) adjustment costs the net proceeds from disinvestment depends on the rate of disinvestment. Zhang's parameterization leads to 1.4% average annual rate of disinvestment which implies that the cost of disinvestment is slightly higher than the price of the capital leading to a negative net value from disinvestment.

[7] This argument is even stronger in the absense of fixed costs of adjustment. The papers that include those fixed costs, such as Carlson, Fisher and Giammarino (2004) and Cooper (2006) claim that the models without fixed adjustment costs fit Fama-French evidence better.

the quantitative effects of firms' financing constraints and leverage on stock returns in a model with exogenously given collateral constraints without default or taxes. Gomes and Schmid (2008) introduce taxes and default to look at the relationship between leverage and returns. Obreja (2006) looks at whether financial leverage gives any information regarding returns beyond the information captured by firm size and book-to-market ratio. Finally, Garlappi and Yan (2007) examine the link between distress risk and equity returns[8]. My paper adds to this literature by connecting the value premium to market leverage and decomposing the effects of capital irreversibility and financial leverage on the cross-section of stock returns.

The debt structure in my model allows me to capture several empirical facts about capital structure of value and growth firms in a parsimonious and tractable way. First, book leverage is the same for firms with different book-to-market ratios because the debt agreement pins down the book leverage. Moreover, value firms have higher market leverage because constant book leverage, high book-to-market ratio and the equivalence of book value and market value of risk-free debt imply high market leverage. These implications fit to the results of Fama and French (1992, Table IV) who show that the book leverage of the firms in different book-to-market portfolios is fairly stable around 0.65 whereas value firms have significantly higher market leverage. This phenomenon cannot be captured by a trade-off model, including the ones mentioned above. In trade-off models, profitable firms with low book-to-market ratio have higher book leverage because debt is less costly for them as they are less likely to go bankrupt.

Second, because the level of debt is constant in the inaction region when the firm does not invest the firm's market debt-equity ratio varies closely with fluctuations in its own stock prices. This implication of the model is supported by Welch's (2004) finding that the U.S. corporations do little to counteract the influence of stock price changes on their capital structures. Standard dynamic trade-off models cannot capture this fact because they tie the leverage of the firm to its profitability that is constantly changing[9].

3 Model

This section presents the problem and the solution of an individual firm in continuous time setting. The firm tries to maximize the discounted value of cash flows to shareholders by choosing investment and financing plans. Investment is subject to partial irreversibility, i.e. the purchase price of one unit of capital is 1 and the resale price is $\eta < 1$. I assume that the discrepancy between purchase and resale price is due to specialized machinery. In other words, the firm has to incur a cost of $1 - \eta$ in order to make each unit of its capital usable by another firm[10].

[8] These papers also assume a very high degree of capital irreversibility and are subject to the same criticism above.

[9] Another way to capture both of these facts is to introduce capital structure adjustment costs that makes changing debt harder such as in Fisher, Heinkel and Zechner (1989) and Leary and Roberts (2005). However, this makes the analysis much harder without providing any additional intuition.

[10] Other justifications for capital irreversibility in the literature inculde installation/detachment costs or limited second-hand market for capital due to asymmetric information regarding the quality of capital, i.e. lemon's problem.

The cash flow of the firm is given by the operating cash flows net of cost of maintenance and cash flows to debtholders plus tax shields from depreciation and interest payment. I model debt as risk-free debt extended through a credit line where the debtholders agree to finance a certain fraction, b, of firm's operating capital. This fraction is determined endogenously by "no bankruptcy" restriction: The lenders will limit the amount of debt so that the firm value is always non-negative and bankruptcy is suboptimal. Otherwise, the debt would not be risk-free because the resale value of capital would not be enough to cover the face value of debt. I will show later although $\eta > b$ is a sufficient no bankruptcy condition that can be considered as a collateral constraint as in Lidvan, Sapriza and Zhang (2008), it is unnecessary and overrestrictive. The reason is that firms that have a lot of debt are also the firms that are stuck with a lot of capital. These firms would rather disinvest and reduce their capital and debt at the same time and continue their operations instead of going bankrupt if debtholders choose the correct limit on leverage.

As a result of this credit line the firm will invest when the marginal value of capital to equity holders is $1 - b$ as this is the fraction of new investment that should be financed with equity. Moreover, the firm will disinvest when the marginal value of capital is $\eta - b$ because the firm gets η for each unit of capital sold but has to give back b to debtholders in order to keep the book leverage constant according to the debt agreement. As will be shown later, the level of b does not depend on state variables and hence is time invariant.

We can model the firm in two steps. In the first step, debtholders and the firm agrees on the level of the credit line and set the level of b. In the second step, the firm takes b as given and chooses its investment strategy. This two stage approach allows me to directly model the problem of the firm without the bankruptcy option in the second stage because debtholders will make sure that the firm will never go bankrupt.

I will start with the problem of the firm without tax and hence without debt-tax shield to show that the marginal value of debt for this firm is zero, i.e. that the firm is indifferent between any level of debt. Therefore, we will prove Modigliani-Miller capital structure irrelevance proposition under partial irreversibility. This will help us establish the conclusion that the marginal value of debt is positive when there is tax shield due to interest payment. As a result, the demand for risk-free debt is infinite and the actual level of debt is determined by the no bankruptcy condition.

3.1 Firm without debt-tax shield - Second Stage

Each firm produces output at time t using capital K_t and takes the level of profitability X_t and stochastic discount factor, S_t, as exogenously given. Both X_t and S_t follow geometric Brownian motions

$$\begin{aligned} \frac{dX_t}{X_t} &= \mu_X dt + \sigma_A dw_A + \sigma_i dw_i = \mu_X dt + \sigma dw \\ \frac{dS_t}{S_t} &= -rdt - \sigma_S dw_A \end{aligned}$$

where $E_t[dS_t/S_t] = -rdt$ is the interest rate and σ_S is the risk price. The Brownian increments dw_A and dw_i represent systematic and idiosyncratic shocks respectively and are independent of each other.

The net cash flows of the firm is given by operating profits net of maintenance costs, which we can interpret as operating leverage, and interest payments[11]

$$\pi\left(K_t, X_t\right) = \frac{h}{1-\gamma} X_t^{\gamma} K_t^{1-\gamma} - \delta K_t - rbK_t = \frac{h}{1-\gamma} X_t^{\gamma} K_t^{1-\gamma} - mK_t$$

where $h > 0$, $0 < \gamma < 1$, δ is the depreciation rate and $m = (\delta + rb)$. The level of b is predetermined according to the credit line agreement between debtholders and the firm. Note that as long as $\delta \gg r$ and $b < 1$ the operating leverage has a much greater impact on cash flows than financial leverage. However, due to credit line agreement, the financial leverage affects the marginal cost of capital faced by the firm and hence the stock returns significantly. As a result, financial frictions amplify the effect of capital irreversibility dramatically.

The firm can purchase capital at a unit price and can sell it at a constant price $\eta < 1$. If we let U_t and L_t denote respectively total capital purchases and total capital sales up to time t we can write net change in the stock of capital as

$$dK_t = dU_t - dL_t$$

where $dU_t \geq 0$ and $dL_t \geq 0$. Note that there is no depreciation term in the evolution of capital since I assume that the firm covers for the depreciation of capital via maintenance. As a result we can express the present value of the firm as

$$W(K_t, X_t, S_t) = \max_{\{dU_{t+s}, dL_{t+s}\}} E_t \left\{ \int_0^{\infty} \frac{S_{t+s}}{S_t} \left[\pi\left(K_{t+s}, X_{t+s}\right) ds - (1-b) dU_{t+s} + (\eta - b)\, dL_{t+s}\right] \right\}$$

where $\int_0^{\infty} dU_{t+s}$ and $\int_0^{\infty} dL_{t+s}$ are Stieltjes integrals. Since both the objective function and law of motion of state variables are linear in the stochastic discount factor we can define $J\left(K_t, X_t\right) \equiv W(K_t, X_t, S_t)/S_t$ as the current value of the firm[12]. Because the costs of investment and disinvestment are linear functions of the change in capital the resulting problem is a singular control problem. Thus, its solution is given by an inaction region enclosed by an upper boundary $X_U\left(K\right)$ along which the firm invests and a lower boundary $X_L\left(K\right)$ along which the firm disinvests[13]. The Hamilton-Jocabi-Bellman (HJB) equation for the firm valuation in the inaction region is

$$rJ\left(K, X\right) = \pi\left(K, X\right) + \mu X J_X\left(K, X\right) + \frac{1}{2}\sigma^2 X^2 J_{XX}\left(K, X\right) \tag{1}$$

where $\mu = \mu_X - \sigma_S \sigma_A$ is the risk-adjusted drift of the profitability process. Since this equation holds identically in K we can take the derivative of both sides with respect to K to get

$$rJ_K\left(K, X\right) = \pi_K\left(K, X\right) + \mu X J_{KX}\left(K, X\right) + \frac{1}{2}\sigma^2 X^2 J_{KXX}\left(K, X\right)$$

[11] This functional form nests a Cobb-Douglas production function with an isoelastic demand curve and geometric Brownian motion technology process in which variable inputs, such as labor, have been opmized out. This is why I call X_t as level of profitability rather than demand or technology explicitely.

[12] This is essentially the same as substituting the stochastic discount factor with risk-free rate and taking the expectations under risk-neutral measure.

[13] See Dixit (1993), Dixit and Pindyck (1994).

Since all terms in firm's problem is homogenous of degree one in X and K the value of the firm should also be homogenous of degree one[14] in X and K. As a result the marginal value of capital should be homogenous of degree zero in X and K. Therefore, we can define $y \equiv X/K$ and $q(y) \equiv J_K(X, K)$ to express the last equation as

$$rq(y) = hy^{\gamma} - m + \mu y q'(y) + \frac{1}{2}\sigma^2 y^2 q''(y) \tag{2}$$

This reduces the original HJB equation to an ordinary differential equation of which solution involves finding two constants of integration and and boundary values for y.

The function $q(y)$ should also satisfy the boundary conditions defined as smooth pasting and value matching conditions[15] which imply that the net investment is positive when y reaches an upper bound where the marginal value of capital equals $1 - b$ and negative when y reaches a lower bound where the marginal value of capital equals $\eta - b$. These conditions are given below

$$q(y_L) = \eta - b \text{ and } q'(y_L) = 0 \tag{3}$$
$$q(y_U) = 1 - b \text{ and } q'(y_U) = 0 \tag{4}$$

Because y_U and y_L are constants the optimal investment policy in the (X, K) plane is characterized by two lines, $X_U(K) = y_U K$ and $X_L(K) = y_L K$, that separate investment, disinvestment and inaction regions[16]. The following proposition shows that the boundaries and hence the investment decisions are not affected by capital structure in the absence of debt tax shield.

Proposition 1 *In the absence of debt tax shield the boundaries y_U and y_L and the constants of integration do not depend on leverage, b.*

Proof. Define $\tilde{q}(y) \equiv q(y) + m/r$. Therefore, equations (2), (3) and (4) can be rewritten as $r\tilde{q}(y) = hy^{\gamma} + \mu y \tilde{q}'(y) + \frac{1}{2}\sigma^2 y^2 \tilde{q}''(y)$, $\tilde{q}(y_L) = (\eta + \delta/r)$, $\tilde{q}'(y_L) = 0$, $\tilde{q}(y_U) = (1 + \delta/r)$, $\tilde{q}'(y_U) = 0$ none of which depends on b. Therefore, the boundaries y_U and y_L and the constants of integration do not depend on leverage, b. Also, $\tilde{q}(y)$ does not depend on b. ■

The following corollary establishes that the firm value, debt and equity combined, will remain the same in the absence of taxes.

Corollary 2 *In the absence of taxes firm value, debt and equity combined, is independent of leverage at all times.*

Proof. The firm value is given by $J(X, K) + bK = \int q(X/K)\, dK + bK = \int \tilde{q}(X/K)\, dK - \frac{m}{r}K + bK = \int \tilde{q}(X/K)\, dK - \frac{\delta}{r}K$. We know from previous proposition that $\tilde{q}$ does not depend on b. Therefore, the firm value does not depend on b. ■

[14] See Abel and Eberly (1996).
[15] See Dumas (1991), Dixit (1993) and Abel and Eberly (1996).
[16] See Figure 1.

3.2 Firm without debt-tax shield - First Stage

This section establishes the result that in the absence of debt tax shield the firm is indifferent between different levels of leverage.

Proposition 3 *Denote the equity value by $J(K, X; b)$. In the absence of debt tax shield $J_b(K, X; b) = -K$.*

Proof. The equity value is given by $J(X, K; b) = \int q(X/K)\, dK = \int \tilde{q}(X/K)\, dK - \left(\frac{\delta}{r} + b\right) K$. By previous proposition $\int \tilde{q}(X/K)\, dK$ does not depend on b. As a result direct differentiation yields $J_b(K, X; b) = -K$. This result can also be confirmed using the HJB equation (1) because this equation holds both in the inaction region and at the boundaries (see footnote 2 in Abel and Eberly (1996)). So, we can take the derivative of both sides of equation (1) with respect to b. The resulting equation holds for $J_b = -K$. ■

Proposition 4 *In the absence of debt tax shield the firm is indifferent regarding the choice of the leverage.*

Proof. At its foundation the net value of the firm to its shareholders is the market value of equity minus the cost of equity that is given by $J(K, X; b) - (1 - b) K$. The main aim of the firm in the first stage is to maximize this value subject to the restriction $J(X, K) \geq 0$. This restriction can be rewritten as $b \leq \bar{b}$ where $\bar{b}$ is the highest level of leverage that debtholders will agree[17]. Regardless of the choice of initial capital the first order condition of this net value with respect to leverage is given by $J_b(K, X; b) + K \geq 0$ with equality if $b < \bar{b}$. By previous proposition we know that $J_b(K, X; b) = -K$ and therefore the first order condition holds for all admissible values of b. As a result the firm is indifferent between different leverage choices that are admissible. Since this result is independent of the value of K, X or the admissible value of b the indifference between financing choices is valid at all times. ■

An immediate corollary, which will be proven in the next section, is that the firm prefers debt financing to equity financing under corporate income tax due to interest tax shield. This implies that amount of risk-free debt should be limited by the supply via a no bankruptcy condition. The next section discusses this condition and presents the firm's problem under debt tax shield.

3.3 Firm with debt-tax shield - Second Stage

In this section I will include corporate taxes to the picture and omit personal taxes as the latter does not provide any additional intuition. Corporate income is taxed at the rate τ with full loss-offset provisions. Then the HJB equation that governs the firm value becomes

$$rJ = \bar{\pi}(K, X) + \mu X J_X(K, X) + \frac{1}{2}\sigma^2 X^2 J_{XX}(K, X) \tag{5}$$

[17] The details regarding the determination of this upper limit on leverage is discussed in the next section to keep the analysis here simple.

where $\bar{\pi}(K_t, X_t) = (1-\tau)\left(\frac{h}{1-\gamma}X_t^{\gamma}K_t^{1-\gamma} - \delta K_t - rbK_t\right)$ is the net income after interest, depreciation allowances and taxes. Following the analysis in the previous section I can rewrite the differential equation for the marginal value of capital as

$$rq(y) = \bar{h}y^{\gamma} - \bar{m} + \mu y q'(y) + \frac{1}{2}\sigma^2 y^2 q''(y) \tag{6}$$

where $\bar{h} = (1-\tau)h$ and $\bar{m} = (1-\tau)(\delta + rb)$. The boundary conditions at the upper and lower bounds are the same as before, i.e.

$$q(y_L) = (\eta - b) \text{ and } q'(y_L) = 0 \tag{7}$$
$$q(y_U) = (1 - b) \text{ and } q'(y_U) = 0 \tag{8}$$

where I assume implicitly that the accounting salvage value of the capital is the same as the actual salvage value for the sake of simplification so that the firm does not pay any taxes on resale price of capital.

Let us redefine $\tilde{q}(y) \equiv q(y) + \bar{m}/r$. Then we can rewrite equations (6), (7) and (8) as

$$\begin{aligned} r\tilde{q}(y) &= \bar{h}y^{\gamma} + \mu y\tilde{q}'(y) + \frac{1}{2}\sigma^2 y^2 \tilde{q}''(y) \\ \tilde{q}(y_L) &= (\eta - b + \bar{m}/r) \equiv b_L \text{ and } \tilde{q}'(y_L) = 0 \\ \tilde{q}(y_U) &= (1 - b + \bar{m}/r) \equiv b_U \text{ and } \tilde{q}'(y_U) = 0 \end{aligned}$$

which are analogous to the equations (6), (7) and (8) in Abel and Eberly (1996). So, I follow their paper for the characterization of the solution.

Let's define the following functions

$$\begin{aligned} \rho(x) &= -\frac{1}{2}\sigma^2 x^2 - \left(\mu - \frac{1}{2}\sigma^2\right)x + r \\ \theta(x) &= \frac{x^{\alpha_P} - x^{\gamma}}{x^{\alpha_P} - x^{\alpha_N}} \\ \phi(x) &= \frac{1}{\rho(\gamma)}\left\{1 - \frac{\gamma}{\alpha_N}\theta(x) - \frac{\gamma}{\alpha_P}[1 - \theta(x)]\right\} \end{aligned}$$

where α_P and α_N are the roots of the quadratic equation $\rho(x) = 0$ and satisfy $\alpha_P > 1 > \gamma > 0 > \alpha_N$. Let $H(\gamma) \equiv \bar{h}/\rho(\gamma)$ and $G \equiv y_U/y_L$. Then the solution of the differential equation for $\tilde{q}(y)$ is given by

$$\tilde{q}(y) = H(\gamma)\, y_L^{\gamma}\left[\left(\frac{y}{y_L}\right)^{\gamma} - \frac{\gamma}{\alpha_P}[1 - \theta(G)]\left(\frac{y}{y_L}\right)^{\alpha_P} - \frac{\gamma}{\alpha_N}\theta(G)\left(\frac{y}{y_L}\right)^{\alpha_N}\right]$$

where G is the solution of

$$\frac{b_U}{b_L}\phi(G) - G^{\gamma}\phi\left(G^{-1}\right) = 0$$

and the values of boundaries are given by

$$\bar{h}y_U^{\gamma} = \frac{b_U}{\phi(G^{-1})} \text{ and } \bar{h}y_L^{\gamma} = \frac{b_L}{\phi(G)}$$

These results can be verified by plugging them into the differential equation and boundary conditions for $\tilde{q}(y)$.

The following proposition establishes the conclusion that firms prefer debt financing over equity financing when there are taxes due to debt tax shield.

Proposition 5 *In the presence of debt tax shield, debt financing is strictly preferable to equity financing at all times.*

Proof. The proof is exactly the same as we have done for the case without taxes except that we have to multiply h, δ, and r with $(1-\tau)$ to get their tax adjusted counterparts. Following the same steps results in $J_b(K,X;b) = -(1-\tau)K$ which implies $J_b(K,X;b) + K > 0$ for all admissible values of b. Again we can confirm this result directly taking the derivative of both sides of (5) with respect to b. Hence debt is strictly preferred to equity financing at any state (K,X). ■

3.4 Firm with debt-tax shield - First Stage: No Bankruptcy Condition

In the presence of debt tax shield, debt financing is strictly preferred to equity financing at all times. Therefore the amount of risk-free debt is limited by the supply via no bankruptcy condition. This condition guarantees that debt repayment is not a suboptimal policy compared to bankruptcy even when $\eta < b$, i.e. if the resale value of firm's assets does not cover the face value of debt. The following propositions shows that the market value of equity is always positive for $\eta \geq b$ and therefore establishes that the firm will never go bankrupt as long as $\eta \geq b$. It follows immediately that the level of leverage implied by no bankruptcy condition should satisfy $\eta < b$, i.e. the debt capacity of the firm exceeds the collateral value of its capital. In other words, collateral constraints as in Lidvan, Sapriza and Zhang (2008) are sufficient to keep debt risk-free but they are overrestrictive.

Proposition 6 *The no bankruptcy condition is guaranteed for all values of y if it holds at $y = y_L$.*

Proof. Given $\tilde{q}(y)$, we can write the market value of as $J(K,X) = \int \tilde{q}(X/K)\,dK - \frac{\bar{m}}{r}K$. This leads to

$$\begin{aligned} & J(K,X) \\ = \; & H(\gamma)\, y_L^{\gamma}\left[\frac{1}{1-\gamma}\frac{X^{\gamma}K^{1-\gamma}}{y_L^{\gamma}} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}\frac{X^{\alpha_P}K^{1-\alpha_P}}{y_L^{\alpha_P}} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\frac{X^{\alpha_N}K^{1-\alpha_N}}{y_L^{\alpha_N}}\right] - \frac{\bar{m}}{r}K \\ = \; & \left\{H(\gamma)\, y_L^{\gamma}\left[\frac{1}{1-\gamma}\left(\frac{y}{y_L}\right)^{\gamma} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}\left(\frac{y}{y_L}\right)^{\alpha_P} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\left(\frac{y}{y_L}\right)^{\alpha_N}\right] - \frac{\bar{m}}{r}\right\}K \end{aligned} \tag{9}$$

which can be verified by direct substitution into HJB equation (5). Therefore we can write the condition $J(K,X) \geq 0$ as $V(y) \equiv J(K,X)/K \geq 0$, i.e.

$$V(y) = H(\gamma)\, y_L^{\gamma}\left[\frac{1}{1-\gamma}\left(\frac{y}{y_L}\right)^{\gamma} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}\left(\frac{y}{y_L}\right)^{\alpha_P} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\left(\frac{y}{y_L}\right)^{\alpha_N}\right] - \frac{\bar{m}}{r} \geq 0 \tag{10}$$

For a given level of capital the term in the above equality is increasing in X since $J_X > 0$. Moreover, y increases in X. It immediately follows that $V(y)$ is increasing in y. Therefore, if the last inequality holds for y_L it will hold for $y \geq y_L$. ■

Corollary 7 *The market value of equity is positive as long as $\eta \geq b$. Therefore, the leverage implied by no bankruptcy condition should satisfy $\eta < b$. Therefore, the debt capacity of the firm exceeds the collateral value of its capital.*

Proof. We can write the market value of equity as $J(K, X) = J(X/y_U, X) + \int_{X/y_U}^{K} q(X/k)\, dk$. From previous proposition we know that $V(y) \equiv J(K, X)/K$ has its greatest value at $y = y_U$ and therefore $J(X/y_U, X) \geq 0$. Since $0 \leq (\eta - b) < q$ we have $J(K, X) > 0$ for almost all X and K. ■

The following corollary pins down the no bankruptcy condition.

Corollary 8 *The no bankruptcy condition reduces to $V(y_L) = 0$, i.e.*

$$H(\gamma)\, y_L^{\gamma} \left[\frac{1}{1-\gamma} - \frac{\gamma}{\alpha_P} \frac{1-\theta(G)}{1-\alpha_P} - \frac{\gamma}{\alpha_N} \frac{\theta(G)}{1-\alpha_N} \right] = \frac{\bar{m}}{r}$$

Therefore, the leverage written on debt covenant is state-independent.

3.5 Time Invariance of Credit Line

To summarize, the boundary values that govern the investment decisions and the level of leverage are given by the following equations

$$\begin{aligned}
0 &= \frac{b_U}{b_L} \phi(G) - G^{\gamma} \phi\left(G^{-1}\right) \\
b_L &\equiv (\eta - b + \bar{m}/r) \text{ and } b_U \equiv (1 - b + \bar{m}/r) \\
\bar{h} y_U^{\gamma} &= \frac{b_U}{\phi(G^{-1})} \text{ and } \bar{h} y_L^{\gamma} = \frac{b_L}{\phi(G)} \\
\frac{\bar{m}}{r} &= \frac{b_L}{\rho(\gamma)\, \phi(G)} \left[\frac{1}{1-\gamma} - \frac{\gamma}{\alpha_P} \frac{1-\theta(G)}{1-\alpha_P} - \frac{\gamma}{\alpha_N} \frac{\theta(G)}{1-\alpha_N} \right]
\end{aligned}$$

Since none of the state variables appear in these equations the level of leverage that the firm and the lenders agree does not depend on the state variables. Therefore, whenever both parties want to revise the terms of the credit line the new credit line will have the same level of leverage. As a result, the credit line that restricts debt to a certain fraction of total assets will have the same terms. In other words, b is the time invariant level of the credit line where debt remains riskless.

One seeming caveat in this argument is that the debtholders might initially agree to provide leverage that is above the value implied by no bankruptcy condition because there is no immediate threat of bankruptcy and there is the possibility of renegotiating the covenant in favor of lower leverage once bankruptcy becomes more likely. However, since debt is the preferred form of financing the firm will not agree with the lower leverage in renegotiations. This exposes the debtholders to the risk of bankruptcy. Foreseeing this, the debtholders will never agree to provide leverage above the one implied by no debt overhang condition.

r	σ_S	δ	μ_X	σ_A	σ_i	γ	η	τ
0.02	0.43	0.12	0.033	0.0781	0.2375	0.15	0.45	0.35

Table 1: Calibration of model parameters.

4 Analysis

4.1 Calibration

Campbell (1999) and Campbell and Cochrane (1999) report the average risk-free rate to be 1.8% which I round up to 2%. The value of σ_S is set to 0.43 in order to match the long-run Sharpe ratio of the market return as presented in Campbell and Cochrane (1999). The value of depreciation rate, δ, is set as 0.12 implying 1% monthly depreciation as in Zhang (2005) and Cooper (2006). For the other parameters, I follow Cooper (2006) and set the variance and capital share parameter as $\sigma_A = 0.0781$, $\sigma_i = 0.2375$ and $\gamma = 0.15$. Finally, I set the growth rate of productivity to $\mu_X = 0.033$ in order to match long-run growth rate of S&P earnings that are reported on Shiller's webpage. This implies that the risk-adjusted growth rate is $\mu = 0.033 - 0.0781 * 0.43 = -0.001 < r$ and hence guarantees the convergence of the optimization problem of the firm.

Rajan and Zingales claim that the debt is 66% in the US. This estimate is also in line with Table IV in Fama and French (1992)[18]. In order to capture this with limited capital irreversibility, I set the resale price of capital to $\eta = 0.45$, a number close to 0.55 reported by Berger, et. al. (1996), which implies that the book leverage of the firms is equal to 69%. The corporate tax rate is set equal to $\tau = 0.35$ in line with the IRS data provided by Taylor (2003).

Finally, since the parameter h has only a level effect on firm value but does not effect firm's policies and returns I normalize its value to 1. Table 1 summarizes these values.

4.2 Effect of Leverage on Investment Policy

We have seen in the previous section that the investment policy of the firm can be characterized as an inaction region bounded by two lines $X_U(K) = y_U K$ and $X_L(K) = y_L K$ and that net investment is positive once the state (X, K) hits $X_U(K)$ and negative once the state (X, K) hits $X_L(K)$. These curves are linear and intersect at the origin of (X, K) space because of the homogeneity assumption. Figure 2 illustrates these points.

I have shown previously that the capital structure does not affect investment decisions in the absence taxes. However, this no longer holds when taxes are included into the picture. On the one hand, the levered firm distributes cash flows to the debtholders in proportion to its debt and hence to its capital. As a result, its marginal value of capital is lower compared to the all equity financed firm at any given state. Hence the levered firm shall invest and disinvest at a higher level of productivity for a given level of capital which pushes both boundaries upwards. This is illustrated as effect 1 in figure 3. On the other hand, the levered firm has lower marginal value of capital at its investment

[18] According to Fama and French (1992) $\log(A/BE) = \log 1/(1-b) = 0.7$ for stocks in different book-to-market portfolios. This implies $b = 0.65$.

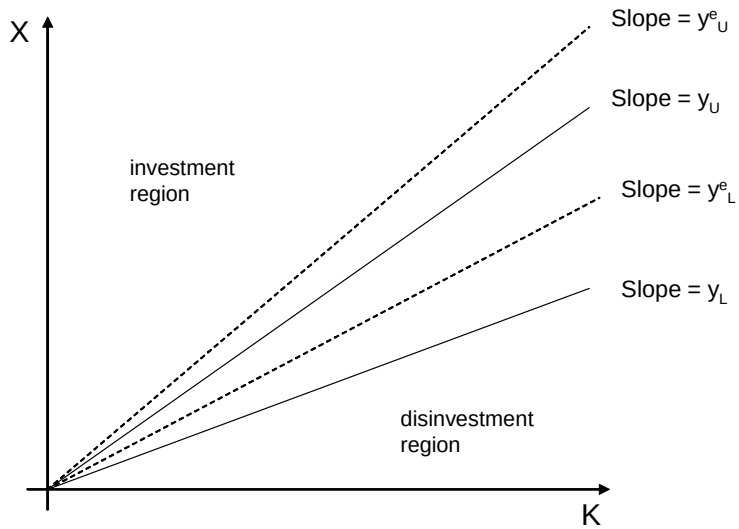

Figure 2: Investment, disinvestment and inaction regions for the levered firm (solid line) and for the all equity financed firm (dashed line), denoted by superscript e. The firm invests immediately once its state enters investment region that is above the line with slope y_U (y_U^e). The firm disinvests immediately once its state enters disinvestment region that is below the line with slope y_L (y_L^e). The inaction region is the region encompassed by the investment and disinvestment boundaries.

boundaries compared to marginal value of capital at the investment boundaries of the all equity financed firm, i.e. $1-b$ vs 1 and $\eta - b$ vs η. This effect pushes the boundaries downwards. This is illustrated as effect 2 in figure 3.

In the absence of taxes these two effects offset each other. However, in the presence of taxes, the debt tax shield weakens the first effect and hence the second effect dominates. As a result, the investment and disinvestment boundaries of the levered firm lies above the corresponding boundaries of the all equity financed firm. Figure 2 maps this discussion on (X, K) plane. The bottom line is that the levered firm invests and disinvests at a lower level of profitability for a given level of capital compared to all equity financed firm.

4.3 Effect of Taxes on Financial Leverage

We know from the previous section that the investment and financing policy are given by the following equations.

$$0 = \frac{b_U}{b_L}\phi(G) - G^{\gamma}\phi\left(G^{-1}\right) \tag{11}$$

$$\frac{\bar{m}}{r} \equiv (1-\tau)\left(b + \frac{\delta}{r}\right) = \frac{\bar{h}}{\rho(\gamma)} y_L^{\gamma}\left[\frac{1}{1-\gamma} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\right] \tag{12}$$

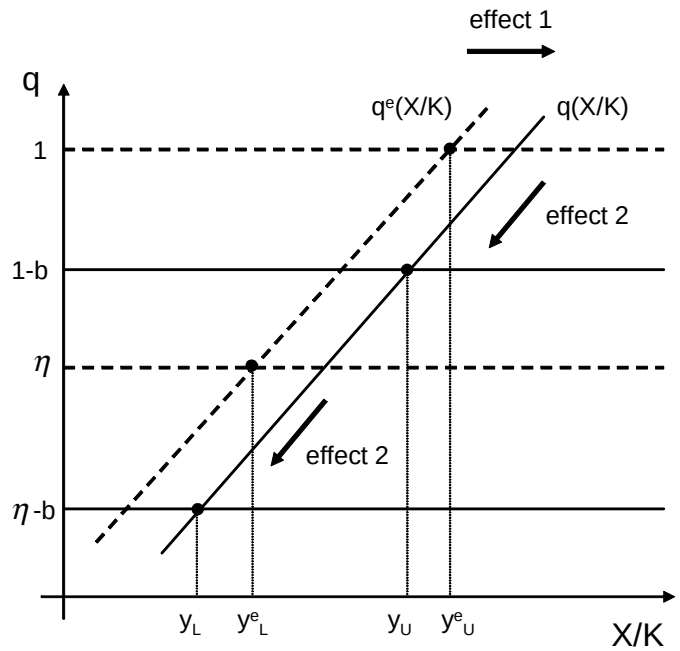

Figure 3: The effects of debt on investment and disinvestment boundaries. The solid line is for the levered firm, the dashed line is for the all equity financed firm.

where

$$\begin{aligned} b_L &\equiv (\eta - b + \bar{m}/r) \text{ and } b_U \equiv (1 - b + \bar{m}/r) \\ \bar{h} y_U^{\gamma} &= \frac{b_U}{\phi\left(G^{-1}\right)} \text{ and } \bar{h} y_L^{\gamma} = \frac{b_L}{\phi\left(G\right)};\ G \equiv \frac{y_U}{y_L} \end{aligned}$$

Equation (11) is analogous to the equation that determines the investment policy in Abel and Eberly (1996) with $b_U/b_L = 1/\eta$. In contrast, the ratio in this model

$$\frac{b_U}{b_L} = \frac{1 - b + (1-\tau)(b + \delta/r)}{\eta - b + (1-\tau)(b + \delta/r)}$$

is adjusted by financial and operating leverage. First of all, since the firm finances its investment partially with debt the effective cost of capital to the shareholders at the time of purchase is $1 - b$ whereas the net proceeds at the time of resale is $\eta - b$ because the firm has to pay back its debt in order not to exceed its debt capacity. These costs are further adjusted by the term $(1-\tau)(b + \delta/r)$ which is the present value of operating and financial costs to be incurred for a marginal increase in value of capital. Therefore, we can call b_U/b_L the "effective ratio of resale and purchase costs". An increase in this effective ratio implies a higher degree of irreversibility of capital from shareholder's perspective which makes frequent adjustment of capital less desirable for the firm. In order to decrease the frequency of capital adjustment the firm increases the wedge between upper and lower boundaries for adjustment, i.e. $G \equiv y_U/y_L$. This can be confirmed directly from equation (11) because $G^{\gamma}\phi\left(G^{-1}\right)/\phi\left(G\right)$ is increasing in G as shown by Abel and Eberly (1995).

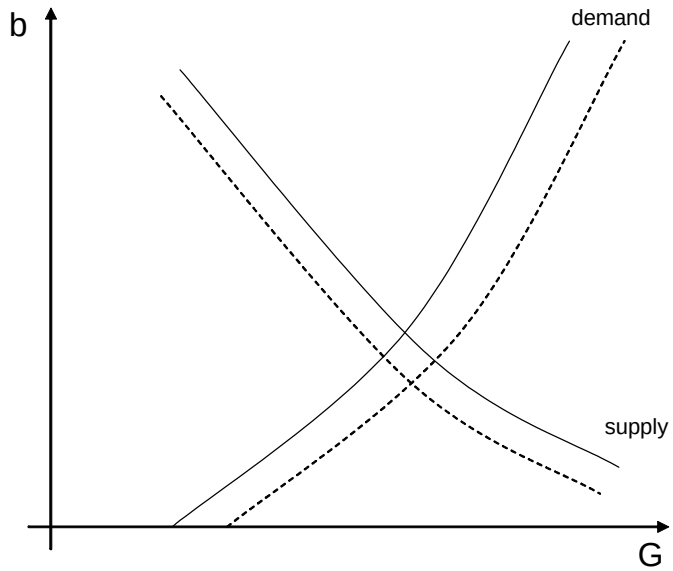

Figure 4: Determination of financing and investment policy and the effect of taxes. Solid line: before tax increase; dashed line: after tax increase.

Both equations (11) and (12) can be drawn on a (b, G) plane as in Figure 4. We can reinterpret the model as if the lenders provide the firm with a menu of investment and financial policies (b, G) that they are ready to support (equation 12) whereas the firm chooses the policy that maximizes its value (equation 11). In this regard, equation (11) can be interpreted as a demand curve that gives the magnitude of financial leverage, b, desired by the firm in order to sustain a particular investment policy, G. This curve is upward sloping because an increase in b causes an increase in effective ratio both due to taxes and capital irreversibility[19]. Equation (12) is, on the other hand, the supply curve for leverage and gives us the financial leverage lenders are ready to provide for a given investment policy in order to guarantee that the firm will never go bankrupt. The Appendix shows that this curve is downwards sloping once we substitute the formula for y_L back into the equation (12) and rewrite it as

$$(1-\tau)\left(b+\frac{\delta}{r}\right)=\frac{\eta-b+(1-\tau)(b+\delta/r)}{\rho(\gamma)}\frac{1}{\phi(G)}\left[\frac{1}{1-\gamma}-\frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}-\frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\right]$$

This equation shows that the value of equity decreases at the disinvestment boundary y_L and hence risk of bankruptcy increases. To eliminate this risk lenders provide lower leverage to the firm.

4.3.1 Effect of Tax Increase

Analysis of equation (12) shows that the supply of leverage decreases as the equity value decreases due to taxes which increases the bankruptcy risk at the upper boundary. There-

[19] The demand curve would be a parallel line to the b-axis if $\eta = 1$, $\tau = 0$ or the investment expenditure out of shareholder's pocket would be tax deductable.

fore, the supply curve shifts down. Analysis of equation (11) shows that the demand curve shifts down too because higher taxes increase the effective ratio of resale and purchase cost of capital: Higher taxes reduce the corporate income, the present value of financial and operating costs, and hence the marginal revenue product of capital, but not the direct cost of purchasing or selling the capital[20]. This makes capital irreversibility relatively more important compared to other costs and gives the firm an incentive to adjust its capital stock less frequently to which the firm responds by increasing G for a given level of b as discussed in the previous section. The final result is that higher corporate income tax implies lower leverage. The Appendix provides the comparative statics analytically.

Although the model is similar to trade-off theory in spirit its implications regarding the response of leverage to an increase in tax is quite different. Traditional trade-off theory suggests that firms should optimize their capital structure considering the tax benefits of debt and bankruptcy costs. Since higher taxes imply greater opportunity for tax shield they increase the marginal tax benefit of debt relative to bankruptcy costs and hence debt should increase. In my model, debt is limited by the supply and because the lenders want to minimize their exposure to bankruptcy risk they decrease the amount of debt in response to a tax increase. This intuition would carry over also to the standard trade-off theory with risky debt since higher taxes reduce the value of the equity and increase bankruptcy risk so that the firms can borrow less for a given coupon rate[21]. On the demand side, it is true that marginal benefit of debt increases with taxes. However, unlike the standard trade-off theory, taxes affect investment decisions through effective costs of purchasing and selling capital so that the firms can sustain a given investment policy with less leverage.

4.4 Stock Returns

The individual stock returns are given by

$$dR_i = \frac{\bar{\pi}\left(K_i, X_i\right) + dJ\left(K_i, X_i\right)}{J\left(K_i, X_i\right)} = \left[r + \sigma_S \sigma_A \frac{X_i J_{iX}\left(K_i, X_i\right)}{J_i\left(K_i, X_i\right)}\right] dt + \frac{X_i J_{iX}\left(K_i, X_i\right)}{J_i\left(K_i, X_i\right)} \sigma dw \tag{13}$$

where σ_S is the price of risk, $\sigma_A \frac{X_i J_{iX}(K_i, X_i)}{J_i}$ is the risk exposure and $\sigma dw = \sigma_A dw_A + \sigma_i dw_i$. The second equality follows from the HJB equation (5).

Equation (13) tells us that the expected returns are determined by the elasticity of the equity value with respect to productivity shocks, $\frac{X_i J_{iX}(K_i, X_i)}{J_i}$. The difference in the expected returns of value and growth firms come from their differences in this elasticity. In particular, high book-to-market firms are bogged with a lot of capital that they do not get rid off because of low resale price and debt repayment in case of disinvestment. Therefore, their cash flows, and hence their value, is more responsive to productivity shocks than growth firms.

[20] If the corporate taxes would affect the direct purchase and resale value of capital financed by equity, i.e. $1 - b$ and $\eta - b$, in the same proportion as they affect the profits the tax rate would cancel in the numerator and denominator of b_U / b_L and would not affect the demand side.

[21] The simple dynamic versions of trade-off theory such as Leland (1994) ignores the effect of financial decisions on the total value of the firm. A more modern version with the consideration of investment policy presented by Hennessy and Whited (2005) imposes a risk-free rate restriction through a collateral constraint.

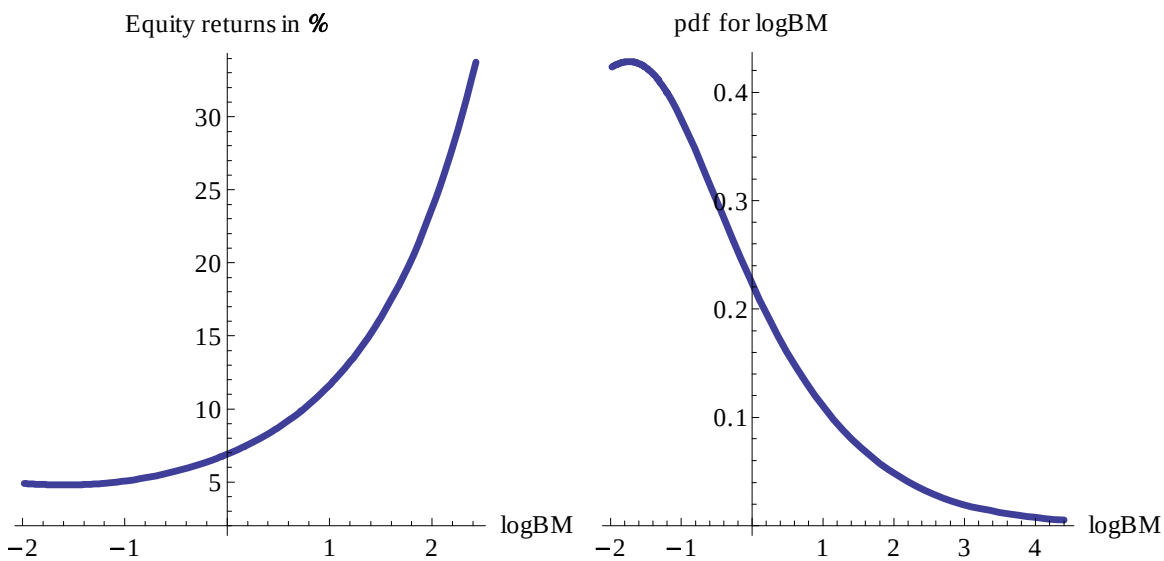

Figure 5: Expected returns vs. book-to-market ratio and the long run distribution of book-to-market ratios

We can express the elasticity term as a function of $y \equiv X/K$ by defining $V(y) \equiv J/K$ and rewriting $\frac{XJ_X(K,X)}{J(K,X)} = \frac{V'(y)y}{V(y)}$. Therefore, we can write the stock returns, after dropping the firm index, as

$$dR = \left[r + \sigma_S \sigma_A \frac{V'(y)y}{V(y)} \right] dt + \frac{V'(y)y}{V(y)} \sigma dw$$

The book-to-market equity ratio is given by $(1-b)K/J = (1-b)/V(y)$ and hence is decreasing in y. Therefore, the relationship between stock returns and book-to-market ratio depends on whether the elasticity $\frac{V'(y)y}{V(y)}$ is increasing or decreasing in y. Figure 5 illustrates the relationship between book-to-market ratio and expected returns along with the long-run distribution of book-to-market value[22].

The convex shape of this relationship tells us that the value premium should be countercyclical as predicted by Zhang (2005) and later confirmed by Chen, Petkova and Zhang (2008): The premium is smaller or even slightly negative after subsequent good aggregate shocks because most of the firms are concentrated at lower levels of book-to-market. Moreover, it increases after subsequent bad aggregate shocks because more firms are concentrated at higher book-to-market levels. Hence, the premium decreases in good times and increases in bad times. Contrary to Zhang (2005), my model captures this countercyclical relationship without relying on countercyclical price of risk.

Using the long-run distribution, I can calculate long-run expected returns of different book-to-market portfolios under the assumption that the portfolios are sorted at each instant according their book-to-market ratios as opposed to yearly sorting in Fama and

[22] The details for the derivation of this distribution is given in the Appendix. This long run distribution should be considered as a long-run average rather than a stationary distribution because the distribution of B/M also depends on the aggregate state which varies over time.

Portfolio	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Return(%)	4.85	4.81	4.83	4.94	5.17	5.57	6.25	7.50	10.47	35.46
ln(BE/ME)	-1.86	-1.63	-1.39	-1.15	-0.89	-0.59	-0.25	0.18	0.80	2.14

Table 2: Long-run average of annualized expected returns of firms in 10 portfolios sorted according to their book-to-market ratios. Portfolio returns are calculated as equally-weighted returns as in Fama and French (1992)

French (1992). Table 2 provides the result for the yearly returns which can be considered as a sneak peek of the simulation results[23].

In the following, I will first discuss how different risk components affect stock returns. Then I will move on to the effects of leverage on stock returns.

4.4.1 Decomposition of Expected Returns

To see how the risk exposure of growth stocks and value stocks respond to changes in book-to-market values it is useful to decompose $V(y)$ using equation (10), into its three components: assets-in-place, growth options, i.e. the call option to purchase capital, and disinvestment option, i.e. the put option to sell capital[24]

$$\begin{aligned} V(y) &= H(\gamma)\, y_L^{\gamma}\left[\frac{1}{1-\gamma}\left(\frac{y}{y_L}\right)^{\gamma} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}\left(\frac{y}{y_L}\right)^{\alpha_P} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\left(\frac{y}{y_L}\right)^{\alpha_N}\right] - \frac{\bar{m}}{r} \\ &= V_{AP}(y) + V_G(y) + V_D(y) - \frac{\bar{m}}{r} \end{aligned}$$

where the term $\bar{m}/r$ captures the effect of operating and financial leverage on firm value. The components captured by growth and disinvestment options can be considered as intangibles in the spirit of Hall (2001) and Hansen, Heaton and Li (2005).

We can also decompose the risk exposure as the weighted average of the three components in a similar fashion. This gives us

$$\begin{aligned} \sigma_A \frac{V'(y)y}{V(y)} &= \sigma_A\left[\frac{V_{AP}(y)}{V(y)}\frac{V'_{AP}(y)\,y}{V_{AP}(y)} + \frac{V_G(y)}{V(y)}\frac{V'_G(y)\,y}{V_G(y)} + \frac{V_D(y)}{V(y)}\frac{V'_D(y)\,y}{V_D(y)}\right] \\ &= \sigma_A\left[\gamma\frac{V_{AP}(y)}{V(y)} + \alpha_P\frac{V_G(y)}{V(y)} + \alpha_N\frac{V_D(y)}{V(y)}\right] \end{aligned}$$

where γ, α_P and α_N are the elasticities of the three components with respect to productivity shocks. Because these elasticities are constant the differences in stock returns are explained by the relative importance of these three components in determining the firm

[23] This long-run distribution can also be used for sensitivity analysis to capture the effect of changes in parameters. This analysis will be added to the updated version.

[24] Intuitively, the term associated with $\alpha_P > 0$ is the growth option term because as capital decreases for a given level of productivity the firm is more likely to invest and this term becomes larger. The term associated $\alpha_N < 0$ is the disinvestment option term because as capital increases for a given level of productivity the firm is more likely to disinvest and this term becomes larger.

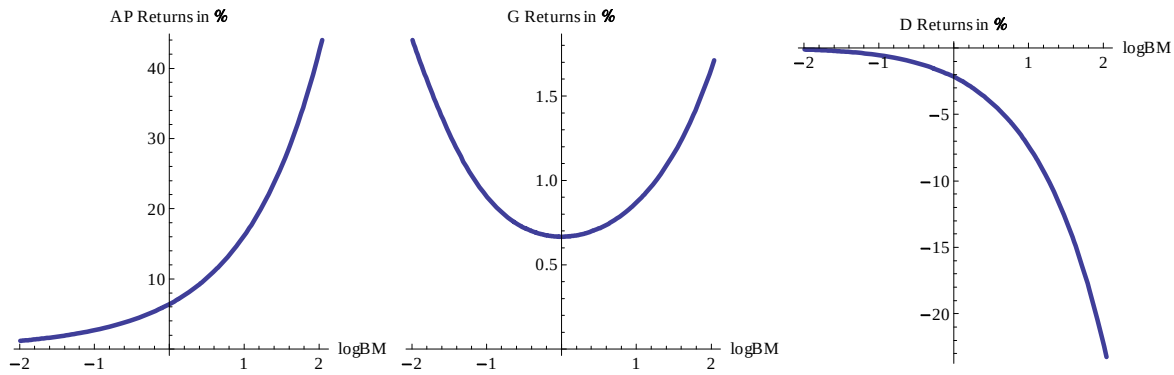

Figure 6: The decomposition of yearly excess equity returns into its three components: assets-in-place (AP), growth option (G) and disinvestment option (D).

value. Note that none of the numerators in this decomposition depends on $\bar{m}/r$. Therefore, contrary to Carlson, Fisher and Giammarino (2004, p. 2579, 2582), the operating leverage, i.e. maintenance costs, affects the returns not through the "numerator" of the exposure terms but by magnifying and affecting the shape of the denominator, $V(y)$.

Figure 6 shows the contribution of each component to expected returns. The shapes of the curves in this figure are worth a special discussion. First of all, we observe that assets-in-place become riskier as book-to-market values increase because marginal productivity of capital decreases due to concavity of production function and financial and operating leverage. Second, the contribution of disinvestment options become larger as book-to-market values increase because disinvestment options constitute a larger share of the firm value for higher book-to-market firms that are more likely to disinvest. Finally, we see that the riskiness of growth options is not monotone which is puzzling: Conventional wisdom tells us that growth options depend on future economic conditions and are very risky. Moreover, they should constitute a greater share of the firm value of growth firms and hence increase their riskiness. This suggests that the effect of growth options should be monotonically decreasing in book-to-market values. However, figure 6 reveals that this monotonic relationship is valid only at the lower range of book-to-market values. For firms with higher book-to-market values this relationship reverses due to operating and financial leverage: Due to leverage $V(y)$ is a concave function that increases faster for high book-to-market firms whereas $V_G(y)$ is a convex function that increases faster for low book-to-market firms. This reveals another channel through which operating and financial leverage affect returns that is not apparent in previous studies.

The horse race between the riskiness of option components and assets in place determine the equity returns. The equity risk due to growth options dominates for low book-to-market firms that are very productive and more likely to invest. For higher book-to-market firms, the risk due to assets-in-place and disinvestment options become more significant and the former dominates the latter. As a result, capital irreversibility and financial leverage increases the riskiness of assets in place particularly for high book-to-market firms. This effect, in combination with operating and financial leverage, makes assets-in-place riskier than growth and disinvestment options.

4.4.2 Leverage and stock returns

Leverage effects stock returns directly through its effect on the risk structure of the firm a la Modigliani and Miller (1958, 1963) and indirectly through its effect on investment decisions.

In order to understand these effects separately it is useful to distinguish equity returns and firm returns, its debt and equity combined. For the all-equity firm these two returns are the same and are given by

$$dR_e = \left[r + \sigma_S \sigma_A \frac{J_X^e X}{J^e} \right] dt + \frac{J_X^e X}{J^e} \sigma dw$$

where $\sigma dw = \sigma_A dw_A + \sigma_i dw_i$ and the superscript e is for all-equity-financed. For the levered firm, the firm return is the market-value-weighted average of debt and equity return, i.e.

$$\begin{aligned} dR_F &= \left[r + \sigma_S \sigma_A \frac{J_X X}{J + bK} \right] dt + \frac{J_X X}{J + bK} \sigma dw \\ &= \frac{bK}{J + bK} r dt + \frac{J}{J + bK} dR_i \end{aligned}$$

where J is the market value of equity including debt tax shield and $dR_i = \left[r + \sigma_S \sigma_A \frac{J_X X}{J} \right] dt + \frac{J_X X}{J} \sigma dw$. We have seen before that the firm value, debt and equity combined, is independent of leverage in the absence of taxes, i.e $J^e = J + bK$. Therefore, firm returns remain the same in the absence of taxes. However, because $\frac{J_X X}{J} > \frac{J_X^e X}{J^e} = \frac{J_X X}{J + bK}$, we conclude that leverage increases the expected returns and volatility of the returns. This constitutes the direct effect of debt a la Modigliani and Miller.

Next comes the indirect effect of leverage through investment channel. This channel is opened because financial leverage affects investment policy once taxes are included. For this purpose, figure 7 compares total firm returns for all equity financed firm and the levered firm after inclusion of taxes. Two striking conclusions arise from this figure: First, the effect of leverage through investment channel has very little impact on cross-sectional distribution of firm returns beyond its effect on realized book-to-market values. This is the direct result of limited capital irreversibility. Second, whereas the book-to-market effect is slightly negative for levered firm returns, equity returns are sharply increasing implying that the main source of book-to-market effect comes from direct Modigliani-Miller channel. This finding is in line with the paper by Hecht (2000) who finds that many of the cross-sectional determinants of expected equity returns, including book-to-market value, are non-existent at the level of the firm and that the cross-sectional variation in expected firm returns is small relative to expected equity returns. In the cross-section, firms with high weighted expected equity returns tend to have low weighted expected debt returns. Thus, the components of expected firm returns (weighted expected equity and debt returns) effectively cancel each other out, diminishing the cross-sectional variation in expected firm returns. Figure 7 also shows that the required cross-sectional variation in stock returns is not possible to obtain if we only rely on limited capital irreversibility.

Figure 8 reveals that the firm risk due to growth and disinvestment options decrease with book-to-market value whereas the risk of assets in place increases. In particular we

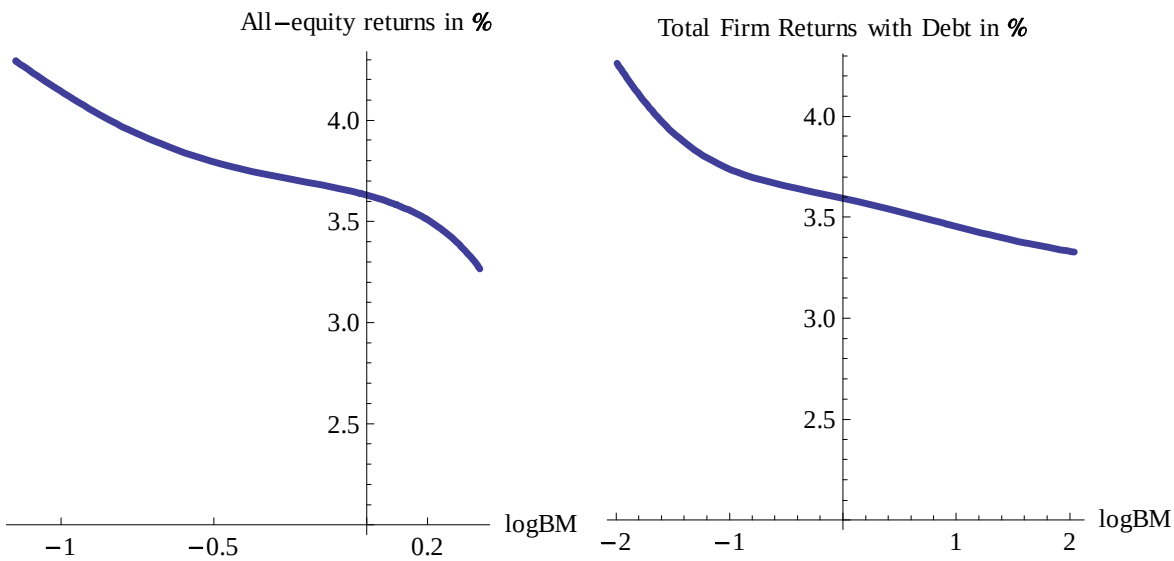

Figure 7: Total yearly expected firm returns, debt and equity combined, for the all-equity firm returns and levered firm returns.

observe that the effect of assets-in-place and growth options cancel each other implying that the slightly negative slope in figure 7 is due to disinvestment options.

4.5 Market Return and Conditional CAPM

The market return is given by

$$dR_m = rdt + \frac{\int_i X_i J_{iX}(K_i, X_i)\, di}{\int_i J_i(K_i, X_i)\, di}[\sigma_S \sigma_A dt + \sigma_A dw_A]$$

Using the formulae for stock returns I can write the conditional CAPM beta for a

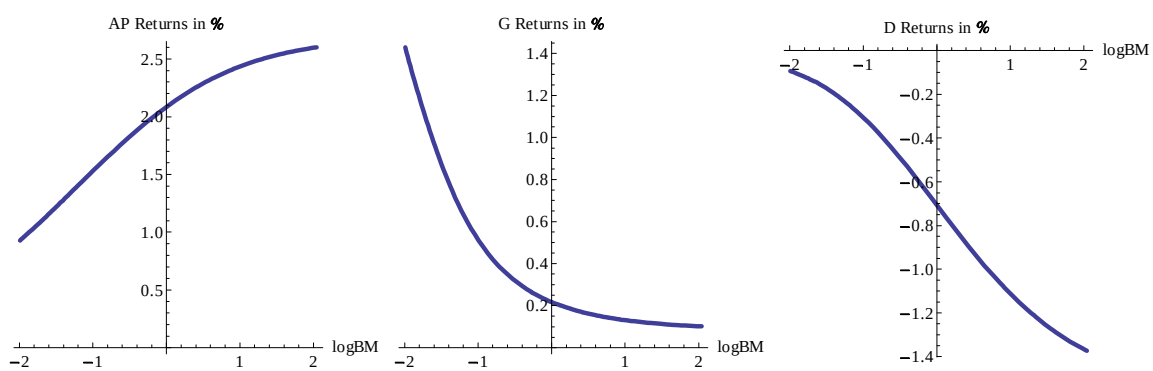

Figure 8: The decomposition of excess firm returns in its its three components: assets-in-place (AP), growth option (G) and disinvestment option (D).

Portfolio	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Return(%)	4.75	4.77	4.84	5.15	5.25	5.70	6.33	7.74	32.12	55.3
ln(BE/ME)	-1.88	-1.62	-1.38	-1.14	-0.89	-0.61	-0.29	0.11	0.69	2.1
CAPM Beta	0.35	0.36	0.36	0.37	0.40	0.45	0.54	0.68	1.23	22.47
CAPM Implied	3.97	3.97	4.00	4.09	4.26	4.52	4.98	5.79	8.31	124.6
FF Beta	2.22	2.19	2.17	2.16	2.17	2.19	2.26	2.49	3.14	6.14
FF Implied	14.69	14.45	14.22	14.02	14.04	14.02	14.24	15.33	18.53	34.73

Table 3: Simulation results with benchmark parameters, 25 simulations, 4000 firms, 2500 periods. The first 1500 periods have been discarded to allow the system to converge its steady state. The portfolios are sorted every month as done by Cooper (2006) instead of every year as in Fama and French (1992). Yearly sorting does not change the results in a significant way. The results are averages across simulations.

portfolio p as

$$\beta_p = \frac{\Delta_p}{\Delta_m}$$

where

$$\begin{aligned} \Delta_p &= \frac{\int_{i\in p} X_i J_{iX}(K_i, X_i)\, di}{\int_{i\in p} J_i(K_i, X_i)\, di} \\ \Delta_m &= \frac{\int_{i\in m} X_i J_{iX}(K_i, X_i)\, di}{\int_{i\in m} J_i(K_i, X_i)\, di} \end{aligned}$$

Therefore, $E(dR_p) - rdt = \beta_p [E(dR_m) - rdt]$ and hence conditional CAPM holds. This is a common result in single factor models including those discussed in the introduction.

5 Simulation

Using the parameter values in Table 1 I simulate the model to obtain the statistics for different book-to-market portfolios a la Fama and French (1992). Table 3 presents the simulation results and Table 4 reproduces Table IV in Fama and French (1992).

The comparison of returns and book-to-market ratios in Tables 2 and 3 shows that the analytical results for long-run averages of returns and book-to-market values are a good approximation to simulation results except for the highest deciles. Since the price of risk, σ_S, is constant the differences in returns comes only from their risk exposures that moves the variance of returns in the same direction as the excess returns. Hence firms with high book-to-market ratios have both higher expected returns and higher return variance and the simulation results for their expected returns is more prone to error. As a result, the returns are measured less precisely for higher book-to-market firms which explains the discrepancy in simulated and analytical returns for high book-to-market firms.

A comparison of Tables 3 and 4 show that the model fits Fama and French evidence qualitatively: Firms in higher book-to-market portfolios earn higher returns and the portfolio betas calculated a la Fama and French (1992) do not vary much across portfolios

Portfolio	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Return	5.82	10.44	11.64	12.48	14.04	15.6	17.3	18.0	19.08	22.50
ln(BE/ME)	-1.87	-1.09	-0.75	-0.51	-0.32	-0.14	0.03	0.21	0.41	0.83
Beta	1.35	1.32	1.30	1.28	1.27	1.27	1.27	1.27	1.29	1.34

Table 4: Fama and French (1992) Table IV with returns adjusted for inflation. Inflation is calculated using the monthly CPI data on Robert Shiller's webpage, http://www.econ.yale.edu/ shiller/data.htm

except the highest book-to-market portfolio[25]. On the other hand, the CAPM βs vary significantly which is in line with the findings of Zhang (2005) over the period 1927-2001.

I am currently working on a method of moments estimation of parameters using the analytical long-run distribution of book-to-market values in order to improve the fit of the model to data. Moreover, the simulation results can be further improved by limiting the volatility of returns. One way of doing this without changing the structure of the model significantly is to assume that firms are forced to bankruptcy while their value is still positive due to legal and fixed costs of bankruptcy. This will put a lower bound to the numerator of the expression in the stock return volatility and hence an upper bound to the volatility of stock returns.

6 Conclusion

I have presented a dynamic model of the firm with limited capital irreversibility and incomplete debt contracts in order to analyze the effects of financial leverage on investment and explain the cross-sectional differences in equity returns. This model can capture several regularities in corporate finance and asset pricing literature in a parsimonious and tractable way.

Introducing debt into production based asset pricing models has also other advantages. For example, the model presented here can be extended with time varying interest rates in a similar framework to Merton's (1973) intertemporal capital asset pricing model (ICAPM). This will serve for two purposes. First, it will decrease the explanatory power of conditional market beta for stock returns and will get us one step closer to solving the value premium puzzle. Second, because firms with high book-to-market ratio also have higher leverage they will have greater exposure to interest rate shock which further reinforces the value premium. I hope that this paper will stimulate future research in this direction.

7 References

- A. Abel, J. Eberly, A unified model of investment under uncertainty, American Economic Revie2 84 (1994), 1369-1384.

[25] Fama and French calculate the betas of portfolios by averaging the betas of the firms after they have been ranked into size-β portfolios rather than estimating CAPM β directly.

- Abel and Eberly (1995): Optimal Investment with Costly Reversibility, NBER Working Paper No. 5091.
- Abel and Eberly (1996): Optimal Investment with Costly Reversibility, REStud, pp. 581-593.
- Abel and Eberly (2002): Investment and q with Fixed Costs: An Empirical Analysis, Mimeograph, University of Pennsylvania
- Atkeson, A. and P. J. Kehoe (2005). Measuring organizational capital. Federal Reserve Bank of Minneapolis Research Department Staff Report 291.
- Berger, P., E. Ofek, and I. Swary. 1996. Investor Valuation and Abandonment Option. Journal of Financial Economics 42:257–87.
- Campbell (1999): Asset prices, consumption, and the business cycle, Handbook of Macroeconomics, Volume 1C
- Campbell, John Y., and John H. Cochrane, 1999, By force of habit:Aconsumption-based explanation of aggregate stock market behavior, Journal of Political Economy 107, 205–251.
- Carlson, Murray, Adlai Fisher, and Ron Giammarino, 2004, Corporate investment and asset price dynamics: Implications for the cross section of returns, Journal of Finance 59 (6), 2577–2603.
- Chen, L.; Ralitsa Petkova and Lu Zhang, The expected value premium, Journal of Financial Economics 87 (2008) 269–280
- Cochrane, John H., 2005, Asset pricing, Revised Edition (Princeton University Press, Princeton, NJ).
- Cooper (2006): Asset Pricing Implications of Nonconvex Adjustment Costs and Irreversibility of Investment, Journal of Finance
- Dixit, Avinash K. (1993): The Art of Smooth Pasting (Harwood Academic Publishers, Chur, Switzerland).
- Dixit, Avinash K. and Robert S. Pindyck (1994): Investment under Uncertainty, Princeton University Press
- Dumas, B. (1991), "Super Contact and Related Optimality Conditions", Journal of Economic Dynamics and Control, 15, 675-685.
- Edwin O. Fischer; Robert Heinkel; Josef Zechner, Dynamic Capital Structure Choice: Theory and Tests, The Journal of Finance, Vol. 44, No. 1. (Mar., 1989), pp. 19-40.
- Fama, Eugene F., and Kenneth R. French (1992): The cross-section of expected stock returns, Journal of Finance 47, 427–465.

- Fama, Eugene F., and Kenneth R. French (1993): Common risk factors in the returns on stocks and bonds, Journal of Financial Economics 33 (1993) 3-56.
- Fama, Eugene F., and Kenneth R. French (1995): Size and Book-to-Market Factors in Earnings and Returns, The Journal of Finance, Vol. 50, No. 1 (Mar., 1995), pp. 131-155
- Fama, Eugene F., and Kenneth R. French (1996): Multifactor Explanations of Asset Pricing Anomalies, The Journal of Finance, Vol. 51, No. 1 (Mar., 1996), pp. 55-84
- Garlappi, Lorenzo and Hong Yan, 2008, Financial distress and the cross section of equity returns, working paper, University of Texas at Austin.
- Gomes, Joao F. and Lukas Schmid, 2008, Levered returns, working paper, University of Pennsylvania.
- Grinblatt, Mark, and Sheridan Titman, 2001, Financial Markets and Corporate Strategy, 2nd Edition (McGraw-Hill).
- Hall, R. E. (2001). The stock market and capital accumulation. American Economic Review 91, 1185-1202.
- Hall, Robert E., 2004, Measuring Factor Adjustment Costs, Quarterly Journal of Economics, pp. 899-927
- Hansen, L., J. Heaton, and N. Li (2005): "Intangible Risk?," in Measuring Capital in New Economy Corrodo, C., and et. al. Eds.
- Hecht, Peter A. (2000): The Cross Section of Expected Firm (not equity) Returns, The University of Chicago Working Paper and AFA 2001 New Orleans. Available at SSRN: http://ssrn.com/abstract=235108
- Hennessy, Christopher A. and Whited, Toni M., The Journal of Finance, Volume 60 Issue 3 (2005), pp. 1129 - 1165
- Dmitry Livdan, Horacio Sapriza, and Lu Zhang(2008): Financially Constrained Stock Returns, Journal of Finance, forthcoming
- Leary, Mark T. and Michael R. Roberts, The Journal of Finance, Volume 60 Issue 6 (2005), pp. 2575 - 2619
- Leland, H.E., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, Journal of Finance
- McGrattan, E. R. and E. C. Prescott (2000). Is the stock market overvalued? Federal Reseve Bank of Minneapolis Quarterly Review 24, pp. 20-40.
- Merton, Robert C., "An Intertemporal Capital Asset Pricing Model", 1973, Econometrica

- Franco Modigliani and Merton H. Miller (1958): The Cost of Capital, Corporation Finance and the Theory of Investment, The American Economic Review, Vol. 48, No. 3 (Jun., 1958), pp. 261-297
- Miller, M.; Modigliani, F. (1963). "Corporate income taxes and the cost of capital: a correction". American Economic Review 53 (3): 433–443.
- Obreja, Iulian, 2006, Financial leverage and the cross-section of stock returns, working paper, Carnegie-Mellon University.
- Rajan, R. G. and L. Zingales, 1995, What do we know about capital structure? Some evidence from international data, Journal of Finance 50(5), 1421-1460.
- Taylor, Jack (2003): Corporation income tax brakets and rates, 1909-2002, Statistics Income Bulletin, Fall. (http://www.irs.gov/pub/irs-soi//99corate.pdf)
- Welch, Ivo, Capital Structure and Stock Returns, Journal of Political Economy, 2004, vol. 112, no. 1, pp. 106-131
- Zhang, Lu, 2005, The value premium, Journal of Finance 60 (1), 67–103.

8 Appendix

8.1 Supply curve for debt

As shown in the body of the paper the supply curve for leverage is given by the following equation

$$\frac{(1-\tau)(b+\delta/r)}{\eta-b+(1-\tau)(b+\delta/r)}=\frac{1}{\rho(\gamma)\phi(G)}\left[\frac{1}{1-\gamma}-\frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P}-\frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\right]$$

It is straightforward to show that the left side of this equation is increasing in b. To show that the right side is decreasing in G note that we can rewrite it as

$$\frac{1}{\rho(\gamma)}\frac{\left(\frac{1}{1-\gamma}-\frac{\gamma}{\alpha_P(1-\alpha_P)}\right)+\left(\frac{\gamma}{\alpha_P(1-\alpha_P)}-\frac{\gamma}{\alpha_N(1-\alpha_N)}\right)\theta(G)}{\left(1-\frac{\gamma}{\alpha_P}\right)+\left(\frac{\gamma}{\alpha_P}-\frac{\gamma}{\alpha_N}\right)\theta(G)}=\frac{c+d*\theta(G)}{e+f*\theta(G)}$$

where $c>e>0$, $f>d$ and $f>0$ and $\theta'(G)>0$. Direct differentiation with respect to G yields

$$\frac{d*e-c*f}{(e+f*\theta(G))^2}\theta'(G)<0$$

As a result higher G implies lower b on supply side and hence the supply curve is downward sloping.

8.2 Comparative Statics on (b, G) plane

We can write equations (11) and (12) as

$$\begin{aligned}
\frac{(1-\tau)(b+\delta/r)}{\eta-b+(1-\tau)(b+\delta/r)} - \frac{1}{\rho(\gamma)\phi(G)}\left[\frac{1}{1-\gamma} - \frac{\gamma}{\alpha_P}\frac{1-\theta(G)}{1-\alpha_P} - \frac{\gamma}{\alpha_N}\frac{\theta(G)}{1-\alpha_N}\right] &= 0 \\
\frac{1-b+(1-\tau)(b+\delta/r)}{\eta-b+(1-\tau)(b+\delta/r)} - G^{\gamma}\frac{\phi(G^{-1})}{\phi(G)} &= 0
\end{aligned}$$

which can be simplified as

$$\begin{aligned}
A(b,\tau,\delta) - R(G) &= 0 \\
B(b,\tau,\delta) - Q(G) &= 0
\end{aligned}$$

where it is straightforward to prove that $A_b > 0$, $B_b > 0$, $A_\tau > 0$, $B_\tau > 0$ given that $\eta > b$. Abel and Eberly (1995) shows in a long proof that $R'(G) > 0$ and $0 > Q'(G)$ is shown in the previous subsection. These equations give us unambiguously which direction the supply and demand curves shift after a change in taxes. In particular both supply and demand curves shift down after a tax increase whereas they shift up after an increase in fixed operating costs. Once we take the total derivative of these equations and solve for $db/d\tau$ we get

$$\frac{db}{d\tau} = \frac{A_\tau Q'(G) - B_\tau R'(G)}{-A_b Q'(G) + B_b R'(G)} < 0$$

which confirms our discussion analytically.

8.3 Long-run distribution of book-to-market values

We can calculate the cross-section of returns in the long-run by looking at the stationary distribution of y between two reflecting barriers, y_L and y_U. The law of motion for y is given by $dy/y = \mu_X dt + \sigma dw$. Let's define $z \equiv \log y$, $z_L \equiv \log y_L$ and $z_U \equiv \log y_U$ and let $g(z)$ be the long-run distribution of z. Then, $g(z)$ is given by the Kolmogorov forward equation (see Dixit (1993))

$$g''(z) = 2\frac{\left(\mu_X - \frac{1}{2}\sigma^2\right)}{\sigma^2} g'(z)$$

with the boundary conditions

$$\begin{aligned}
g'(\log y_U) &= 2\frac{\left(\mu_X - \frac{1}{2}\sigma^2\right)}{\sigma^2} g(\log y_U) \\
g'(\log y_L) &= 2\frac{\left(\mu_X - \frac{1}{2}\sigma^2\right)}{\sigma^2} g(\log y_L)
\end{aligned}$$

and the integral condition

$$\int_{z_L}^{z_U} g(z)\, dz = 1$$

After solving these equations and substituting y back we get the long-run distribution of y as[26]

$$\varphi(y) = 2\frac{\left(\mu_X - \frac{1}{2}\sigma^2\right)}{\sigma^2}\frac{y^{2\left(\mu_X - \sigma^2\right)/\sigma^2}}{y_U^{2(\mu_X - 0.5*\sigma^2)/\sigma^2} - y_L^{2(\mu_X - 0.5*\sigma^2)/\sigma^2}}$$

for $y_L \leq y \leq y_U$ and zero otherwise.

We can write book-to-market value as $(1-b)K/J = (1-b)/V(y)$. Once we define the function $\omega(y) = \log(1-b)/V(y)$ the long-run distribution of book-to-market values, bm, is given by

$$\zeta(bm) = \varphi\left(\omega^{-1}(bm)\right)\left|\frac{d\omega^{-1}(bm)}{d(bm)}\right|$$

for $\log(1-b)/V(y_U) \leq bm \leq \log(1-b)/V(y_L)$ and zero otherwise.

[26] The solution to these equations are $g(z) = Ae^{\gamma z}$ where $\gamma = 2\frac{\left(\mu_X - \frac{1}{2}\sigma^2\right)}{\sigma^2}$ and $A = \gamma/(y_U^\gamma - y_L^\gamma)$. Therefore, $\varphi(y) = g(\log y)/y = Ay^{\gamma-1}$.