Liquidity Choice, Productivity, and Efficiency*

Job Market Paper

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Abstract

This paper studies the effects of investment choice between assets of varying liquidity. One type of investment is more liquid, i.e., its return is more pledgeable, and the other is more productive. I characterize competitive equilibria and study how exogenous changes in liquidities of investments affect interest rate and investment composition. Relative change in liquidities rather than the absolute changes determines the nature of effect on interest rate and investment composition. The steady state equilibrium interest rate decreases (increases) when the liquidity of the productive (liquid) type goes up. I discuss an application to the effect of changes in contracting technology via securitization on U.S. real interest rate. Competitive equilibria may be constrained inefficient and the inefficiency is due to the endogeneity of entrepreneurs' choice between liquidity and return. This endogeneity leads to a pecuniary externality in portfolio decisions of entrepreneurs. When liquidities of both types are low this externality leads to inefficiently liquid equilibria with negative interest rate, that is, one less than the economy's growth rate. For the inefficient equilibria, a negative interest rate indicates that the interest rate is in fact too high. In this case lowering the interest rate further can make Pareto improvement. I show how government regulation of the investment portfolios can achieve efficiency. Introducing government bonds does not help, even lowering steady state welfare, when they raise the interest rate. Government bonds make Pareto improvement only when they do not change the interest rate.

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1 Introduction

Effects of illiquidity[1] on allocation of resources, both in the short and long run, is a major theme in macroeconomics. It encompasses a wide spectrum of issues, from positive effects of financial development on investment and interest rate to normative consequences of public liquidity provision policies. Critical in all these is how the aggregate level of liquidity and its allocation respond to various types of institutional changes or macroeconomic policies. Hence a key factor is how much resources private agents allocate to liquid as opposed to illiquid investments given the prices and possibly other macroeconomic conditions.

This paper studies a model of liquidity choice where private agents invest in projects with varying liquidity. Agents face a trade off between liquidity and return to their investment portfolios. I analyze competitive equilibria and show that the decentralized allocation may become inefficiently liquid.[2] Inefficiency arises because prices influence liquidity choice of agents in need of external finance, whose portfolios have limited pledgeability, by affecting how much they end up pledging to outside investors. I investigate whether and how a planner can improve social welfare by regulating the private sector or by other means such as introducing government bonds when the competitive equilibrium is inefficient. Competitive equilibria are shown to have positive properties that provide new insights on how financial development affects interest rate and investment in an economy. Nature of these effects depends crucially on relative changes in liquidity of different types of investments, rather than the absolute changes.

The model economy consists of overlapping generations of entrepreneurs who live for three periods: young, middle aged and old. There is a continuum of each generation present in each period and there are no aggregate or idiosyncratic risks in the economy. When young, entrepreneurs receive a fixed endowment of perishable consumption good (and nothing thereafter) which cannot be stored. Middle aged entrepreneurs have an opportunity to invest in a portfolio of investments. There are two types of constant return to scale investment technologies: one is more productive and has a higher *return per unit of investment* (productive type) and the other has a higher *pledgeable return per unit of investment* (liquid

[1] The term "liquidity" here refers to the ability to transfer wealth across different time periods by pledging the returns to a real or financial investment. In other words, my focus is on an intertemporal notion of liquidity or, using the terminology of Brunnermeier and Pedersen (2009), funding liquidity. I use the term "liquidity" to mean "pledgeability" throughout the paper, unless otherwise stated.

[2] Throughout, the term efficiency means constrained efficiency. "Constrained" means that the social planner faces the same constraints, i.e. limited pledgeability, as the private agents do.

type), that is, a bigger part of its return can be credibly used as collateral.[3] Middle aged uses her wealth which is the principal and interest on the loan she made when young to the middle aged in the previous period (current old) plus funds she borrows from the young in the current period. There is a competitive credit market in every period in which borrowing by the middle aged is constrained by the total amount of pledgeable return to her investment portfolio. Higher wealth as well as a more liquid portfolio allows the middle aged to borrow more from the young. Hence a liquid portfolio allows for a bigger investment size while a productive one raises the average return to the investment. Finally, entrepreneurs consume only when old.

I characterize the competitive equilibrium and the steady state and study their positive properties. More specifically, I do comparative statics with respect to liquidities of the two types of investments to see how they affect equilibrium prices and quantities. Contracting technology, contract enforcement, corporate governance and bankruptcy laws affect liquidity of investments. Hence these comparative statics have positive implications about how changes in contracting technology, contract enforcement, corporate governance and bankruptcy laws affect aggregate macroeconomic variables. I show that an increase in the liquidity of the less (more) liquid type of investment leads to a lower (higher) steady state interest rate. This asymmetric interest rate effect is precisely due to heterogeneity as in the benchmark model without heterogeneity, an increase in the liquidity of the investment always leads to a higher steady state interest rate. This interest rate effect can be understood as follows. An increase in the liquidity of the productive type has two effects. First, for any given investment portfolio, it increases the liquidity of that portfolio which, in turn, raises the investment demand and the interest rate. Second, an increase in the liquidity of the productive type makes the productive type more attractive to investors. This leads the investors to substitute away from the liquid type into the productive but *still* less liquid type which reduces the interest rate. It turns out that the second effect, i.e. the substitution effect, dominates the first one and so the steady state interest rate falls as the liquidity of the productive type increases.

This interest rate effect is consistent with the decline of the real interest rates in U.S. since 1990s as documented by Caballero et al. (2008). A series of financial innovations and regulatory reforms that took effect in late 1980s and early 1990s, led to a dramatic expansion

[3] I use the concept of pledgeability of return in the sense used in Farhi and Tirole (2009, 2010a,b). This concept is also close to the extent of frictions in bilateral commitments in Kiyotaki and Moore (2002, 2005, 2008).

in securitization of residential mortgages and sale of mostly illiquid loans in the secondary market. The growth in securitization, as a new contracting technology, made productive but less liquid investments more liquid which, through the lens of this model, depressed the interest rate.[4] Caballero et al. (2008) provide a prominent explanation for the decline of the real rates in U.S. after 1990s. This model provides a complementary explanation to theirs.

The trade off between liquidity and return presented in this paper is, to the best of my knowledge, a novel source of inefficiency in private liquidity provision which has not been explored and studied explicitly in previous works. I show that the trade off between liquidity and return gives rise to an externality in the private portfolio choices of the entrepreneurs that works through the interest rate. When entrepreneurs have relatively low initial wealth, they might end up investing too much in the liquid type of investment while the efficient allocation requires investing only in the productive type. Investment in the liquid type when entrepreneurs have low initial wealth does not bid up the equilibrium interest rate sufficiently to make this inefficient investment unprofitable. In general equilibrium, however, the initial wealth of each generation of investing entrepreneurs is endogenously determined. Low initial wealth is an equilibrium outcome when liquidities of both types are relatively low since a low fraction of the returns to investment in any period can be invested by future entrepreneurs. Hence, when liquidities are low this externality can lead to *inefficiently liquid* equilibria.

Similar to Lorenzoni (2008) this externality is pecuniary and acts through the borrowing constraints. One difference is that the externality in this paper works through an intertemporal price, i.e., the interest rate. Moreover the inefficient sale of productive assets in an environment with aggregate uncertainty, is the key in Lorenzoni (2008) that leads to an externality, while here, it is the demand for investible resources in an economy without any uncertainty that entails an inefficient outcome.

I further study some important properties of inefficient equilibria. I show that in an inefficiently liquid steady state, interest rate is non positive (not higher than the growth rate). This is because, the inefficiently liquid steady state equilibria arise when both types of investment have relatively low liquidity. There are three differences between the type of inefficiency discussed in this paper and the inefficiency in the traditional overinvestment models of overlapping generations, e.g. Diamond (1965). First, there exist inefficiently liquid equilibria with a zero interest rate in steady state. In contrast, an equilibrium with zero interest rate cannot be inefficient in the traditional models. Second, a negative[5] (lower

[4]For a discussion of financial innovations, regulatory reforms and their effects on the mortgage market look at Gerardi et al. (2010).

[5]Note that there is no growth in either the endowments or population of the agents in this model. However,

than the growth rate) interest rate in the traditional models is indicative of inefficiency while an equilibrium with negative interest rate can be efficient in this model. Third, and most importantly, a Pareto improvement in an inefficiently liquid equilibrium induces an even more negative interest rate. In traditional overinvestment models an interest rate below the growth rate of the output must be raised to make a Pareto improvement. In contrast, a negative interest rate in an inefficiently liquid equilibrium indicates that the interest rate is in fact *too high*. Hence, a Pareto improvement leads to an even more negative interest rate. This implies that the level of the interest rate can be a misleading indicator in determining the type of inefficiency in the economy and the direction of the policy response.

Finally, I study the role of macroeconomic policy. I show that, given an inefficient equilibrium, a planner can achieve efficiency, i.e. a point on the Pareto frontier, by regulating the fraction of resources that is invested in the liquid type by entrepreneurs. I also study the welfare effects of an important class of public liquidity, i.e. government bonds. I assume that government bonds are fully liquid due to the ability of government to tax. I show that government bonds can make Pareto improvement only for inefficient equilibria where there is strictly positive investment in both types in the long run. In this case, government bonds crowd out the liquid type and crowd in the productive type so that the demand for funds and hence the interest rate remains unchanged in equilibrium. Entrepreneurs can substitute fully liquid government bonds for the liquid type to gain extra amount of pledgeable return which can be used to borrow more funds that can be invested in the productive type. Since the interest rate does not increase, entrepreneurs' return goes up while their debt payment stays the same. This increases entrepreneurs' consumption and leads to Pareto improvement.

The trade off between liquidity and return can be observed in both real and financial sectors. First, consider investments in the real sector. Bigger and more mature firms tend to have lower cost of external financing, i.e., their investment is more liquid, due, for instance, to their reputation and higher value of their collateral while smaller firms within the same industry grow faster and face costlier external financing. High tech startups, e.g., IT ventures, can provide another example in this regard; they have higher returns relative to less knowledge intensive firms but are subject to many kinds of agency problems because of the very new and advanced nature of their technology which generally result in a very low borrowing capacity.[6] The same trade off can be observed when a firm can invest in different

all results of the paper survive a positive growth in endowments or population.

[6] Of course ventures have much riskier payoff as well, but note that riskiness per se does not lead to illiquidity of the investment.

types of projects: to build a new plant, i.e., accumulate more physical capital, or to invest more in the specific human capital in the same plant, i.e., accumulate more organizational capital. The latter option creates less collateralizable assets[7] and is therefore less liquid but may have a higher return to investment. Financial sector can also provide examples of this trade off. A bank that can make different types of loans which need different levels of specific monitoring skills to collect their payments is an example. Loans that needs higher skills become less liquid. This is because monitoring makes these loans more specific to the bank and makes it more difficult to sell them in the secondary market. Moreover, more intensive monitoring may generate private information about the type of investment or investors for the bank. Private information can lead to adverse selection and hence lower liquidity of the loan. [8]

1.1 Related Literature

Farhi and Tirole (2010a) is closely related to this paper and is used as a benchmark for the analysis. Model structure is the same and the only difference is that in this paper entrepreneurs have access to different types of investments with different liquidity. Heterogeneity in investment types leads to different positive and normative results that are complementary to the ones obtained in Farhi and Tirole (2010a). The way limited pledgeability is modeled in this paper is similar to Farhi and Tirole (2009, 2010b) and Matsuyama (2007) while it has a close connection to Holmstrom and Tirole (1998) and Kiyotaki and Moore (2002, 2005, 2008).

Normative results in this paper are in contrast with those of Woodford (1990) and Holmstrom and Tirole (1998). Low liquidity generated by the private sector is at the heart of inefficiency in Woodford (1990) and Holmstrom and Tirole (1998). Their policy recommendation, therefore, is to create more liquidity by introducing government bonds into the economy to achieve efficiency. In this paper, too much liquidity is what makes the decentralized allocation inefficient. Hence government bonds have a very different, potentially an opposite, effect on the welfare. The nature of inefficiency also differs from that of Diamond (1965). A negative interest rate, i.e., one that is less than economy's growth rate, in Diamond (1965) implies that the interest rate is too low. In contrast, a negative interest rate in

[7] This can be due to more specificity of organizational capital or because of the inalienability of the human capital as suggested by Hart and Moore (1994).

[8] For the role of loan specificity see Diamond and Rajan (2001). For an explicit model on monitoring that leads to adverse selection see Parlour and Plantin (2008).

an inefficient equilibrium in this paper indicates that the interest rate is in fact too high.

Similar to this paper, Kehoe and Levine (1993) and Lorenzoni (2008) feature pecuniary externality as the source of inefficiency in the credit market. Pecuniary externality in these models arise because prices, e.g., asset prices, spot prices or interest rate, appear in constraints different from the budget constraint.

There is a substantive literature on the macroeconomic implications of liquidity provision by the private sector. Most of this literature ignores heterogeneous assets with differing liquidity and liquidity choice by private agents. In models without this kind of heterogeneity, as in Farhi and Tirole (2010a) and Woodford (1990), an exogenous increase in the liquidity of investment leads to higher interest rate. In this model, an increase in the liquidity of the productive (illiquid) investment can lower the interest rate. Public liquidity, e.g, government bonds, also has different effects on the private investment and the interest rate in the two models.

Matsuyama (2007) studies a model with heterogenous assets of different liquidity. In contrast to this paper, Matsuyama (2007) assumes that each investment project has a fixed size and each entrepreneur can invest only in one type of project. Matsuyama (2007) focuses on the dynamics of aggregate credit and capital stock when investment composition plays an important role for given returns and liquidities of investment. The goal of this paper, however, is to study normative implications of heterogeneity in investment liquidity as well as the effects of exogenous changes in liquidity of investment on the economy. There are also two different types of assets with different liquidity in Giglio and Severo (2010), namely tangible and intangible capital. The paper develops a model to study how an increasing importance of intangibles in production in developed countries can lead to asset price bubbles. Only tangible capital can be used as a collateral for investment which leads to overaccumulation of physical capital and a low interest rate. In this environment asset price bubbles can be sustained. The liquid and illiquid capitals in Giglio and Severo (2010) are used in a Cobb-Douglas production technology. This implies that there is a high degree of complementarity between the two. In this paper, however, the liquid and illiquid types are perfect substitutes. Hence Giglio and Severo (2010) is closer to an economy with one type of asset, e.g., Farhi and Tirole (2010a), than this paper.

The paper proceeds as follows. Section 2 describes the model and characterize competitive equilibria and the steady states. Section 3 discusses positive properties of equilibria, its interpretations and applications. Section 4 studies the efficiency of competitive equilibria and how a planner can Pareto improve the competitive equilibrium allocation when it

is inefficient. In Section 5, I introduce government bonds and analyze their positive and normative implications. Finally, Section 6 concludes.

2 Model

2.1 Agents, Preferences and Technology

The model economy is an overlapping generations of entrepreneurs without any kind of uncertainty. Each individual lives for three periods and there is a unit measure of each of young, middle aged and old cohorts in each period. Each entrepreneur receives a fixed endowment $e > 0$ of non-storable and homogenous consumption good when young and no endowment thereafter and consumes only when she is old.

The choice of overlapping generations is mainly for simplicity and tractability. One can think of the agents in this economy as firms in the real or financial sectors facing alternating investment opportunities and borrowing constraints. The main feature of the model is the ability of these firms to choose a portfolio of investment projects while pledging the return to their portfolio to outside investors, i.e., non investing firms with otherwise idle resources. In this sense, this model is similar to Kiyotaki and Moore (2002) and Kiyotaki and Moore (2005) and has a close connection with Woodford (1990).

At any period the middle aged has the opportunity to invest in two types of investments which pay off in the next period. Investments differ in their return and liquidity. Investment of type $j \in \{1, 2\}$ has a constant return to scale of R_j and an amount $\theta_j R_j$ of the return can be pledged to the investors. Limited pledgeability of return can arise in many contexts and for a number of different reasons including moral hazard and limited commitment. Following Kiyotaki and Moore (2002, 2005, 2008) and Farhi and Tirole (2009, 2010a,b), I summarize all these frictions in the variable $\theta_j, j \in \{1, 2\}$. I refer to $\theta_j R_j$ as "liquidity" of type j investment. However when (R_1, R_2) are fixed and within the appropriate region of parameters, I refer to $\theta_j, j \in \{1, 2\}$ as well as $\theta_j R_j, j \in \{1, 2\}$ as liquidity of type j investment. I make the following assumption about the return and liquidity of investments:

Assumption 1. *$R_1 > R_2 > 1$ and $\theta_1 R_1 < \theta_2 R_2 < 1$.*

Assumption 1 is made for two purposes. First, it guarantees the existence of the competitive equilibrium and the stability of the steady state. Second, it captures the trade off between liquidity and return across the two types of investments; type one investment is

more productive but less liquid while type two is more liquid and less productive. It is helpful for future analysis to define a *benchmark economy* in which there is no trade off between the two types:

Definition 1. *The* ***Benchmark Economy*** *is an economy where* $R_1 > R_2 > 1$ *and* $1 > \theta_1 R_1 > \theta_2 R_2$.

Note that in the benchmark economy, type one investment dominates type two both in terms of liquidity and return. This implies that entrepreneurs never invest in type two in equilibrium and so the economy collapses to one with a single type of investment similar to Farhi and Tirole (2010a). The benchmark economy is used throughout to provide better understanding of the positive and normative analysis.

2.2 Problem of the Middle Aged Entrepreneurs

In each period a competitive credit market opens up where young and middle aged entrepreneurs can lend and borrow. Every young born at period $t > 0$ inelastically supplies all her endowments in the capital market. The middle aged at time t who has transferred funds from period $t-1$ by investing in the projects of middle aged at $t-1$, demands additional funds from the young constrained by the limited pledgeability of her investment portfolio. She chooses her optimal investment portfolio given the ongoing interest rate r_t and resources that has been transferred from period $t-1$ to period t.

Let x_{1t} and x_{2t} denote investments in types 1 and 2 and i_t denotes the new funds raised by the middle aged at t using the resources of young entrepreneurs in period t. Given the interest rate r_t, a middle aged entrepreneur at t solves the following problem:

$$\begin{aligned} c^o_{t+1} \equiv \max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1+r_t) i_t \\ s.t. \quad & x_{1t} + x_{2t} \leq (1+r_{t-1})e + i_t \,, \\ & (1+r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \,. \end{aligned} \tag{I}$$

The first constraint in the maximization above is the resource constraint of the middle aged entrepreneur. $(1+r_{t-1})e$ is transferred from period $t-1$ to t by the middle aged entrepreneur through investing her endowment e, in the projects of middle aged of period $t-1$. The second term, i_t, is the total external funds that the middle aged at t raises by borrowing from the young entrepreneurs at t. The second constraint is the manifestation of

the limited pledgeability of the investments; the middle aged entrepreneur can not borrow more than what she can credibly commit to pay in period $t+1$. For type $j \in \{1,2\}$ the maximum that can be credibly promised to the lenders is $\theta_j R_j x_{jt}$ and so the total amount of pledgeable return is given by the right hand side of the second constraint. Finally, c^o_{t+1} denotes the consumption of the old entrepreneur in period $t+1$.

The resource constraint is always binding in I. If the interest rate is not too high, the borrowing constraint has to be binding as well. In this case one can eliminate x_{1t} and x_{2t} in the above problem and reach the following reduced form:

Lemma 1. *In any competitive equilibrium where $1+r_t < R_1$ for all t, the borrowing constraint of the middle aged entrepreneur binds in every period. Moreover the problem of the middle aged entrepreneur can be written in the following form:*

$$\begin{aligned} \max_{i_t} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) i_t + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) e \\ s.t. \quad & \left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e \leq i_t \leq \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\,, \end{aligned} \tag{II}$$

where,

$$\begin{aligned} \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) &\equiv \left(\frac{(\theta_2-\theta_1)R_1R_2}{\theta_2R_2-\theta_1R_1}\right) - \left(\frac{(1-\theta_1)R_1-(1-\theta_2)R_2}{\theta_2R_2-\theta_1R_1}\right)(1+r_t)\,, \\ \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) &\equiv \left(\frac{(\theta_2-\theta_1)R_1R_2}{\theta_2R_2-\theta_1R_1}\right)(1+r_{t-1})\,. \end{aligned}$$

The bold symbols $(\boldsymbol{\theta}, \boldsymbol{R})$ is the vector of liquidities and returns of the two types of investments, i.e., $(\theta_1, \theta_2, R_1, R_2)$.

In Lemma 1 the term Λ is the net marginal (and average) return to increasing investment size i_t when borrowing constraint is binding. The two bounds in the constraint of problem II corresponds to the two limits; when i_t hits the lower bound the entrepreneur invests only in type one or the productive investment and when it hits the upper bound only in type two which is the liquid investment, depending on the sign of Λ. Define $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ as the (gross) interest rate in period t that makes Λ equal to zero:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2-\theta_1)R_1R_2}{(1-\theta_1)R_1-(1-\theta_2)R_2}\,. \tag{1}$$

Then the entrepreneurs' optimal liquidity demand is characterized as follows:

$$\begin{cases} i_t = \left(\frac{\theta_2 R_2(1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\,, & \text{if } r_t < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ i_t \in \left[\left(\frac{\theta_1 R_1(1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e, \left(\frac{\theta_2 R_2(1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\right]\,, & \text{if } r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ i_t = \left(\frac{\theta_1 R_1(1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e\,, & \text{if } r_t > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,. \end{cases} \tag{2}$$

Figure 1 is an illustration of middle aged demand for funds given by 2 and the inelastic supply of funds by the young entrepreneurs at time t. Wealth of the middle aged at t is $w_{t-1} = (1 + r_{t-1})e$ which is determined in period $t - 1$. As the figure shows, middle aged entrepreneurs specialize in type one or two when the supply curve crosses the demand on its left or right arms respectively. Entrepreneurs mix if the intersection happens on the flat part of the demand curve, characterized by $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. A higher period $t - 1$ interest rate, i.e. higher w_{t-1}, makes the two arms of the demand curve shift to the right but has no effect on its flat segment.

2.3 Competitive Equilibrium

In each period there is a fixed supply of funds e. Market clearing condition dictates:

$$i_t = e, \quad \forall t \geq 0\,. \tag{3}$$

Combining market clearing 3 and the optimal investment decision of the middle aged entrepreneur 2, I derive the optimal path of the interest rates at any competitive equilibrium:

$$r_t = \begin{cases} \theta_2 R_2(2 + r_{t-1}) - 1 & \text{if } \theta_2 R_2(2 + r_{t-1}) - 1 < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ \theta_1 R_1(2 + r_{t-1}) - 1 & \text{if } \theta_1 R_1(2 + r_{t-1}) - 1 > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{otherwise} \end{cases} \tag{4}$$

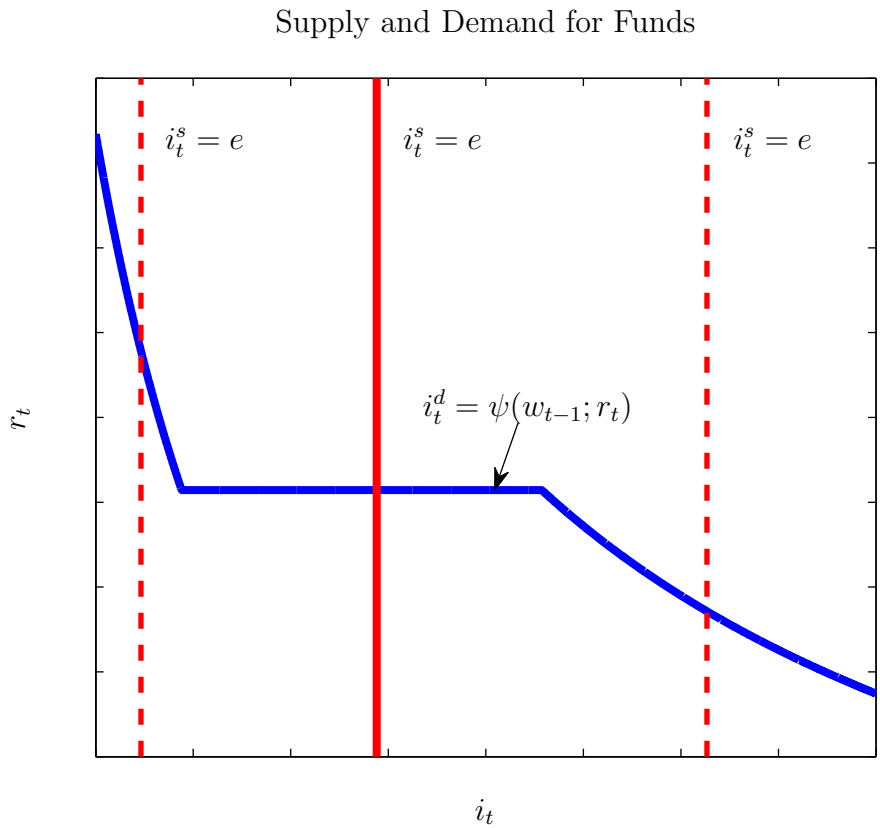

Figure 1: Supply (red) and demand (blue) for funds at any period t as a function of the interest rate. w_{t-1} denotes the wealth of the middle aged, i.e. $(1 + r_{t-1})e$. The two arms on the demand curve correspond to investing only in type 1 or 2. The flat segment in between corresponds to $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where entrepreneurs mix.

Using the middle condition in 2, the dynamic upper and lower bounds on the interest rate is:

$$\theta_1 R_1(2 + r_{t-1}) - 1 \leq r_t \leq \theta_2 R_2(2 + r_{t-1}) - 1\,. \tag{5}$$

Now, I can define a competitive equilibrium:

Definition 2. *A competitive equilibrium is a sequence $\{i_t, x_{1t}, x_{2t}, r_t\}_{t=0}^{\infty}$ of investments and interest rates and an initial value of r_{-1} that satisfy conditions 1 to 5, x_{1t} and x_{2t} solve problem I and $1 + r_t < R_1$ for all $t > 0$.*

Using market clearing 3, the optimal investment decisions in 2 and the borrowing constraint, one can compute the composition of the aggregate investment portfolio of the middle

aged entrepreneurs at any date as follows:

$$\begin{cases} x_{1t} = 0, x_{2t} = (2 + r_{t-1})e & \text{if } \; r_t < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ \\ x_{1t} = \left(\frac{\theta_2 R_2(2+r_{t-1}) - (1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}{\theta_2 R_2 - \theta_1 R_1}\right) e, & \\ x_{2t} = \left(\frac{(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})) - \theta_1 R_1(2+r_{t-1})}{\theta_2 R_2 - \theta_1 R_1}\right) e & \text{if } \; r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ \\ x_{1t} = (2 + r_{t-1})e, x_{2t} = 0 & \text{if } \; r_t > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,. \end{cases} \tag{6}$$

Entrepreneurs specialize in the productive type when interest rate is relatively high and invest only in the liquid type when the interest rate is relatively low. When $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ entrepreneurs mix in which case individual's investment composition is indeterminate but the aggregate investment composition is determined.

Before showing the existence and uniqueness of the competitive equilibrium it is useful to first characterize the steady state equilibria. To this end, let me define the following three regions of the parameter space:

Definition 3. *Define F as the set of $(\boldsymbol{\theta}, \boldsymbol{R})$ that satisfies Assumption 1 and also $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$. Then the three regions of F are defined as follows:*

Liquid Region *is defined as* $F_\ell = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) \leq (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\}$.

Mixed Region *is defined as* $F_m = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

Illiquid Region *is defined as* $F_i = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) \leq \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

Notice that all three regions, F_ℓ, F_m and F_i have nonempty interiors. In the definition above I require that elements of F satisfy an additional condition namely, $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$. This condition is to ensure that the steady state interest rate in the *illiquid region* is strictly below R_1 so that the borrowing constraint is binding in the steady state.[9] This particular partition of F is important because each region has a unique and different steady state equilibrium with different characteristics as suggested by the following lemma:

[9] When $\frac{\theta_1 R_1}{1-\theta_1 R_1} \geq R_1$, an steady state equilibria exists in the illiquid region where the borrowing constraint does not bind. In this steady state equilibrium $1 + r_i^{ss} = R_1$.

Lemma 2. *Each of the three regions in Definition 3 has a unique and stable steady state equilibrium. More specifically:*

$$\begin{cases} r_\ell^{ss} = \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) - 1 & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell\,. \\ \\ r_m^{ss} = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_m\,. \\ \\ r_i^{ss} = \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) - 1 & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_i\,. \end{cases}$$

Moreover at the steady state, the entrepreneurs specialize in the liquid and productive type of investments in regions F_ℓ and F_i respectively. Entrepreneurs invest in both types in F_m where the amounts of each type is given by 6.

Having the steady state equilibria characterized in Lemma 2, I close this subsection by the following proposition that establishes the existence and uniqueness of competitive equilibrium:

Proposition 1. *Given any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, and an initial condition $1 + r_{-1} < R_1$, there exists a unique competitive equilibrium that converges to the steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$, given by Lemma 2.*

3 Properties of Equilibria

In this subsection, I study positive properties of competitive equilibria. I analyze the three regions in Lemma 2 and do comparative statics in F with respect to $\boldsymbol{\theta} = (\theta_1, \theta_2)$ for a given vector of returns $\boldsymbol{R} = (R_1, R_2)$. Given $\boldsymbol{R}$, values of $\boldsymbol{\theta}$ correspond to different liquidities of the two investment types which reflect different legal and institutional environments, i.e., contract enforcement, contracting technology, bankruptcy laws, corporate governance etc.

Figure 2 is an illustration of the three regions for $R_1 = 4$ and $R_2 = 3$ and different values of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. The light gray, medium gray and dark gray indicate F_ℓ, F_m and F_i respectively. The white area below the positively sloped straight line is the region where Assumption 1 is violated, that is, $\theta_2 R_2 < \theta_1 R_1$. This area corresponds to the *benchmark economy* in Definition 1.

The figure shows that close to the origin, i.e., when θ_1 and θ_2 are both very low, one can

have all three kinds of steady state equilibria but this is not the case when liquidities are high. In the latter case no steady state equilibrium lies in the liquid region. The following lemma shows that in fact Figure 2 represents more general properties of the three regions for any given vector of returns:

PROPOSITION 2. *For a given vector of returns $\boldsymbol{R}$ satisfying Assumption 1, when $\boldsymbol{\theta}$ is small enough (close to the origin) one can have all three types of steady state equilibria. Given the vector of the returns $\boldsymbol{R}$, for any $\boldsymbol{\theta}$ in the liquid region $\boldsymbol{\theta} \leq (\frac{1}{1+R_1}, \frac{1}{1+R_1})$. For any value of θ_1 the values of θ_2 for which (θ_1, θ_2) belongs to the liquid region lies strictly above the respective values of θ_2 for which (θ_1, θ_2) belongs to the illiquid region. Moreover the boundary of the liquid region is a non-monotonic curve cutting $\theta_1 = 0$ line twice; once at the origin and again at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$. In contrast the inner boundary of the illiquid region is a strictly increasing and convex function of θ_1 which reaches the maximum possible of $\theta_2 = \frac{1}{R_2}$. Finally, the top right corner of F in the space of liquidities, that is, $\boldsymbol{\theta} = (\frac{1}{1+R_1}, \frac{1}{R_2})$ belongs to the illiquid region.*

A few observations are in order. Suppose for the moment that θ_2 is fixed at $\theta_2 = \bar{\theta}_2$, where $\bar{\theta}_2 \leq \frac{1}{1+R_1}$. As one moves along θ_1 dimension in Figure 2, one first passes the liquid region at lower values of θ_1 and then enters the mixed region for medium values of θ_1 and finally leaves the mixed region to enter into the illiquid region. This part is very intuitive as for low values of θ_1 the liquidity cost of investing in type one is very high. As the value of θ_1 increases and gets closer to its upper bound the liquidity gains of investing in type two diminishes further and so it becomes very costly, in terms of the forgone returns to investment, to invest in type two.

Let the thresholds of θ_1 after which one enters the mixed and illiquid regions be $\theta_1^m(\bar{\theta}_2)$ and $\theta_1^i(\bar{\theta}_2)$. By Lemma 2, $\theta_1^i(\bar{\theta}_2)$ is increasing in $\bar{\theta}_2$ which is intuitive since higher values of $\bar{\theta}_2$ implies higher cost of forgone pledgeable return per unit of investment in type one. By Lemma 2, however, $\theta_1^m(\bar{\theta}_2)$ is not monotonic. When $\bar{\theta}_2$ is raised, the threshold to enter the mixed region increases at first but beyond some value of $\bar{\theta}_2$, $\theta_1^m(\bar{\theta}_2)$ begins to decrease and even for $\bar{\theta}_2 \geq \frac{1}{1+R_1}$ the economy fails to have any liquid steady state equilibria.

Now let me do the same comparative statics fixing $\theta_1 = \bar{\theta}_1$. There is no liquid steady states for high values of $\bar{\theta}_1$ since type one investment has enough liquidity itself. For these values of $\bar{\theta}_1$, low θ_2 leads to an illiquid steady state but as one increases θ_2 and after some threshold, one enters the mixed region where the entrepreneurs invest in both types in the

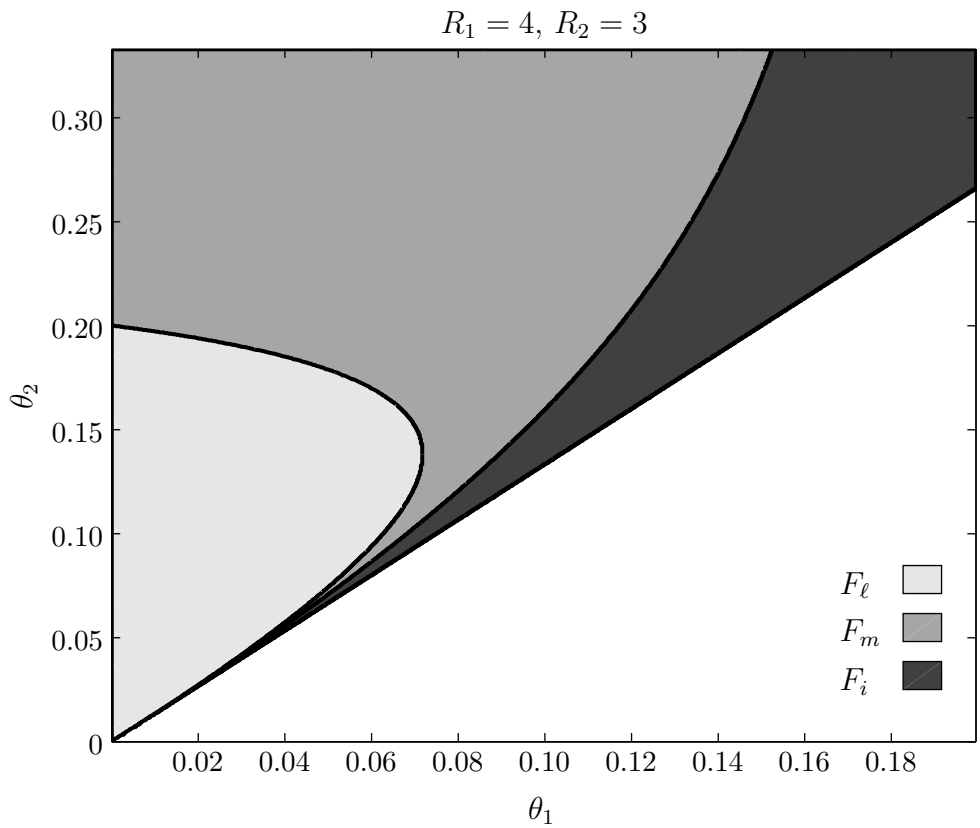

Figure 2: Image of F_ℓ, F_m and F_i for $R_1 = 4$ and $R_2 = 3$ over the space of (θ_1, θ_2).

steady state. This is because higher θ_2 makes type two investment more and more attractive relative to type one. The threshold to enter the mixed region is increasing in $\bar{\theta}_1$ by Lemma 2 due to the fact that it gets more difficult for type two to compete with type one when type one has higher pledgeable returns.

For lower values of $\bar{\theta}_1$ where liquid steady states exist, when θ_2 is increased, one first passes the illiquid region and then enters the mixed and liquid regions subsequently similar to the previous case. If one increases θ_2 further, one enters the mixed region for the second time and entrepreneurs start investing in type one again. One can notice a similar pattern for high $\theta_1 = \bar{\theta}_1$ at the bottom of Figure 3 where the ratio of liquid investment to the total in the mixed region is a humped shaped curve. This pattern is stated formally in the following lemma:

Lemma 3. *Given any $\boldsymbol{R}$ satisfying Assumption 1, the ratio of investment in the liquid type to the total investment at the steady state, i.e. $\frac{x_2^{ss}}{x_1^{ss}+x_2^{ss}}$, is non monotone in θ_2 and has an*

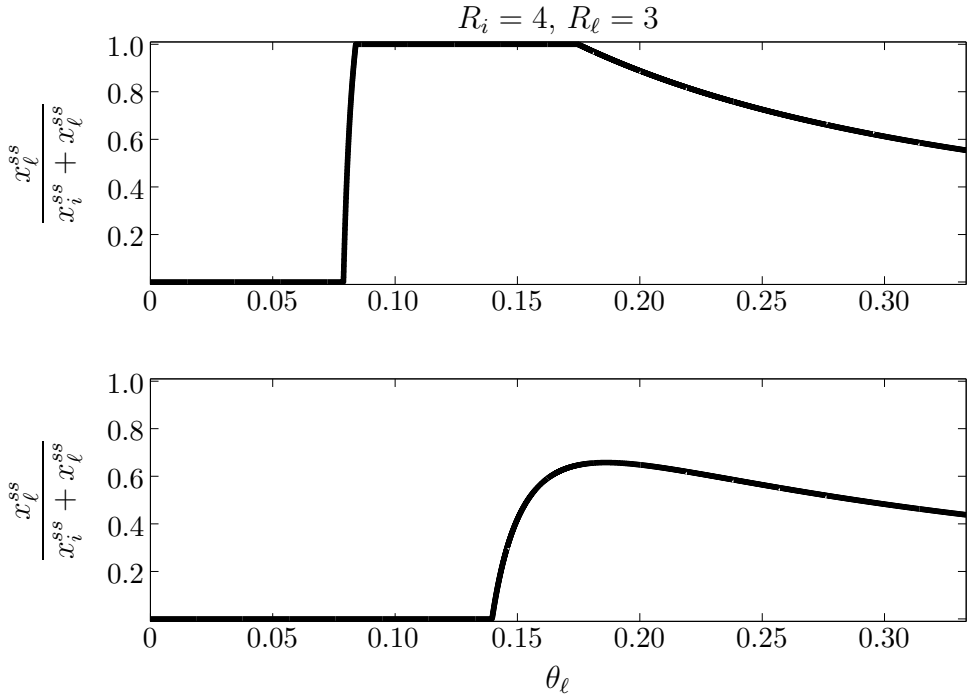

Figure 3: Fraction of liquid investment to the total investment at the steady state for two values of θ_1, and different values of θ_2. The top and the bottom figures correspond respectively to $\theta_1 = 0.055$ and $\theta_1 = 0.09$.

interior maximum for relatively low values of θ_1. In contrast, this ratio is always weakly decreasing in θ_1 and strictly decreasing in θ_1 in F_m. Moreover, the steady state interest rate, i.e. $1 + r_z^{ss}$ for $z \in \{\ell, m, i\}$, is non monotone in θ_1, strictly decreasing in F_m and strictly increasing in F_i, while it is weakly increasing in θ_2 and strictly increasing in θ_2 in F_m.

Suppose θ_2 increases while θ_1 is held constant. On one hand, this increase makes the liquid type investment more attractive and so *encourages* the middle aged entrepreneurs to raise its share in total investment at any given interest rate. On the other hand, this increase in liquidity of the liquid investment tends to raise the interest rate at the steady state which, in turn, *discourages* the entrepreneurs from investing in the liquid type. By Lemma 3, for high enough values of θ_2, the second effect dominates the first one. In this case, any further increase in θ_2 bids up the interest rate so much that entrepreneurs are forced to lower the share of liquid type in their portfolios.

The mixed region, F_m, is the interesting region to look at how interest rate changes with θ_1, since share of liquid investment is simply zero or one in the other two regions. Observe that:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} = \frac{(1-\theta_2)R_1R_2(R_2-R_1)}{\Big((1-\theta_1)R_1-(1-\theta_2)R_2\Big)^2} < 0\,, \tag{7}$$

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(1-\theta_1)R_1R_2(R_1-R_2)}{\Big((1-\theta_1)R_1-(1-\theta_2)R_2\Big)^2} > 0\,. \tag{8}$$

8 is intuitive but 7 is less straightforward. An increase in θ_1 raises the average liquidity of any portfolio and so raises the demand for funds at a given interest rate. This tends to bid up the steady state equilibrium interest rate. In contrast, an increase in θ_1 makes the productive type more attractive for a given interest rate. This, in turn, encourages the entrepreneurs to substitute productive for liquid investment. Since productive type is still the less liquid one, this tends to lower the interest rate. It turns out that the second effect dominates the first one in the mixed region which results in a strictly lower interest rate.

Figure 4 confirms the above intuition in a partial equilibrium setting. It shows how demand for funds at t changes when θ_1 changes unexpectedly to a higher value θ_1'. If the old interest rate at θ_1 is very high, the new demand for funds at the old interest rate will be higher. This bids up the interest rate to clear the market and leads to a higher new interest rate at θ_1'. When the interest rate at θ_1 is somewhere between the two mixed region interest rates (the two flat segments of the demand curves), however, a change in θ_1 to θ_1' makes the entrepreneurs switch from a purely liquid to a purely illiquid portfolio, hence a lower demand for funds. This leads, in turn, to a lower interest rate than the old one.

Figure 5 is a contour plot of the steady state interest rate in the three regions where steady state interest rates are constant on the isolines. Suppose that there are small increases in liquidities of the two types, say $\Delta\theta_1$ and $\Delta\theta_2$, at a point inside the mixed region. The direction of change in the interest rate and investment composition depends only on the relative change in the liquidities $\frac{\Delta\theta_2}{\Delta\theta_1}$ rather than the absolute changes. If $\frac{\Delta\theta_2}{\Delta\theta_1}$ is less (more) than the slope of the isoline at that particular point, the interest rate decreases (increases). In this sense, the model highlights the importance of *relative change in liquidities* for analyzing the effects of financial development on the economy.

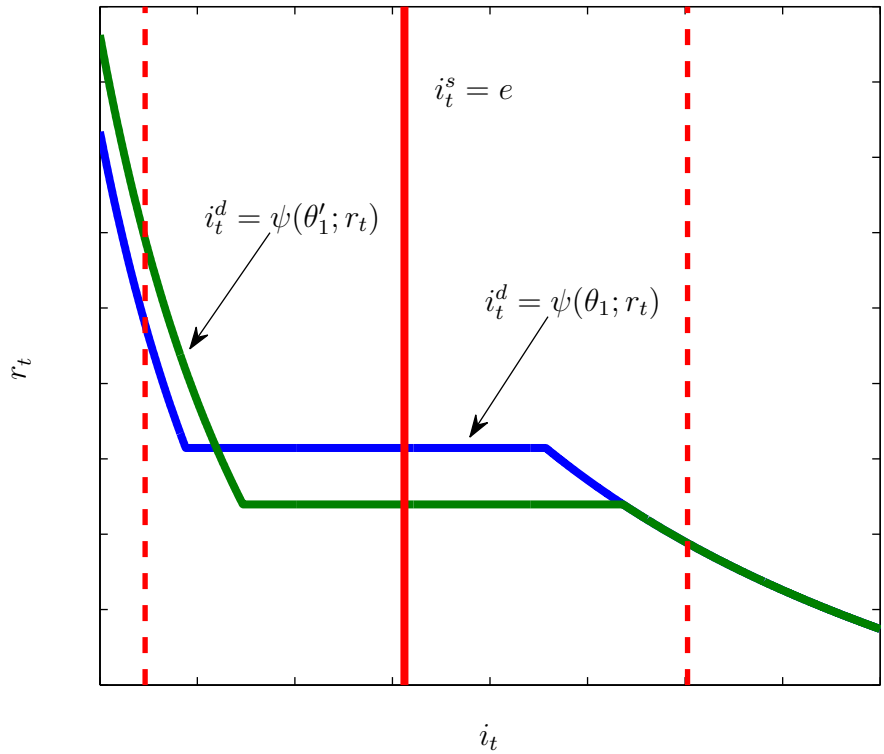

Figure 4: Change in demand for funds at time t when θ_1 increases to θ_1'.

3.1 Interpretations and Applications

For fixed values of returns, liquidity of the two types $\boldsymbol{\theta} = (\theta_1, \theta_2)$ depend, among other things, on the quality of contract enforcement, contracting technology, bankruptcy laws, corporate governance as well as technological features of the investment types. The two types can represent different investment opportunities within a country. For instance, liquid type (productive type) can be investment in more (less) tangible assets, e.g., manufacturing versus services, or more (less) redeployable assets, e.g., established versus newly formed industries, respectively.[10] Financial development in the form of an improvement in contract enforcement, contracting technology, bankruptcy laws or corporate governance raises the liquidity of the two types $\boldsymbol{\theta}$. Lemma 3 predicts that the direction of change in the equilibrium interest rate and the aggregate composition of investments in the two types depend crucially on the *relative changes in liquidity* of the two types where there are positive investment in both

[10]For the effect of tangibility on liquidity see Qian and Strahan (2007) and for the effect of redeployability of collateral see Shleifer and Vishny (1992) and Benmelech and Bergman (2009).

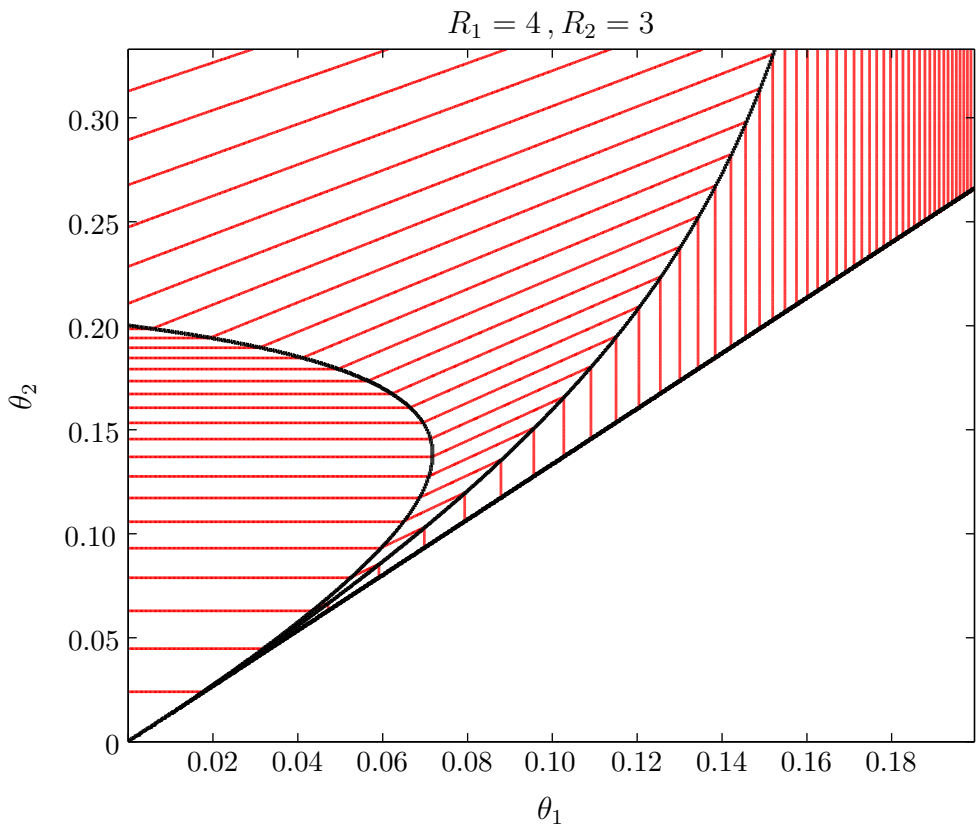

Figure 5: Contour plot of the steady state interest rates (red lines) for the three regions. Interest rate is highest at the top left corner.

types, i.e., in F_m. An increase only in liquidity of the liquid type, i.e., more tangible or more redeployable investment, unambiguously translates into a higher interest rate. It increases the share of investment in the liquid type if the liquid type is currently not very liquid and reduces its share otherwise. In contrast, an increase in the liquidity of the productive type reduces the interest rate and unambiguously increases the share of investment in the productive type.

One may also think of this model as description of a global economy. Liquid and productive types correspond to investment in financially developed countries with low growth and low return to investment, e.g., U.S. and Euro area, and those with underdeveloped financial institutions, high growth and high return to investment, e.g., China and India, respectively. To be more precise, suppose that there are global banks or other financial intermediaries which can invest costlessly in any country's corporate sector. They borrow funds from global investors who can lend to the global banks from their country of origin. In this case, (θ, R) denote the liquidities and returns to investment in the two types of

countries' corporate sectors and r is the interest rate paid by the global banks to the global investors. Note that both global banks and global investors have to be small enough to act competitively. As before, if the mixed region is a better description of the world economy, what matters for the direction of change in the interest rate and the investment composition is the relative changes in the liquidities of the two types, i.e., relative changes in quality of legal and financial institutions in the two types of countries.

3.1.1 Securitization and U.S. Real Interest Rate

One particular application of Lemma 3 concerns the effect of securitization on the real interest rate in U.S. The fact that steady state interest rate is strictly decreasing in θ_1 in the mixed region can provide a candidate explanation for the decline of the real interest rates in U.S. after 1990s. Caballero et al. (2008) provide a prominent explanation for the decline of the real rates in U.S. after 1990s. They argue that savings in some countries, e.g., China, have gone up while the financial markets in those countries have been unable to create enough stores of value to absorb this shift in savings. Hence, the excess savings have been invested in the more financially developed countries such as U.S. and have depressed the real interest rates. The explanation offered in this paper should be seen as a complement to theirs rather than a substitute.

The illiquid investment in this model can represent residential mortgages and similar types of loans that used to be highly illiquid before the growth of securitization in recent decades. The liquid type can represent other types of investments by financial intermediaries such as Treasury bonds, investment-grade corporate bonds, commercial paper and common stocks. The growth in securitization and the creation of a large and liquid secondary market for different types of loans and specially residential mortgages can be interpreted as an increase in θ_1. Moreover, despite the increase in the liquidity of residential mortgages, they are still less liquid than liquid financial assets such as Treasury bonds, investment-grade corporate bonds or commercial paper, that is, $\theta_1 R_1 < \theta_2 R_2$. Note that θ_2 need not be fixed in order to have the steady state interest rate decreasing in θ_1. It is evident in Figure 5 that as long as increases in θ_1 and θ_2 happen in appropriate proportions, the steady state interest rate decreases.[11]

Loutskina (2011) reports that the amount of home mortgages outstanding has grown from \$1.5 trillion at the beginning of 1986 to \$11.1 trillion by the end of 2007. Securitized

[11] In general within the mixed region and at any steady state interest rate r^*, as long as $\frac{\partial \theta_2}{\partial \theta_1} < \frac{(R_2-(1+r^*))R_1}{(R_1-(1+r^*))R_2}$, one has $dr^* < 0$.

home mortgages has grown 16-fold from \$410 billion to \$6.4 trillion within the same time period. These figures show the importance of residential mortgage market and significance of mortgage securitization. Gerardi et al. (2010) and McConnell and Buser (2011) discuss in some details the regulatory reforms and subsequent financial innovations that led to the explosive growth in securitization of residential mortgages and other types of loans. Numerous regulatory reforms contributed to the growth of securitization in the mortgage market. They include changes in accounting rules which allowed originators to sell their mortgages without booking a large accounting loss and led to creation of a liquid secondary market. Other reforms preempted state laws that constrained the types of mortgage products originators could offer or solved technical problems facing mortgage-backed securities. These regulatory reforms that facilitated the process of securitization and sale of mortgages in the secondary market became effective in the late 1980s and early 1990s.

Another implication of Lemma 3 is that the fraction of resources invested in the productive but less liquid type increases by θ_1 in F_m. This effect is also consistent with the available evidence. Loutskina (2011) presents evidence that while the liquidity of the loan portfolio of banks has increased significantly, the fraction of liquid securities held by banks has shrunk during the same time period. She constructs two measures: one measure for liquidity of loan portfolios and another for liquidity held by banks. Liquidity of a loan portfolio is measured as a weighted average of the bank's potential to securitize each type of loan where the weights are the share of that loan type in the loan portfolio. Liquidity held by a bank is measured as the share of liquid assets in the total assets. Liquid assets includes items such as U.S. Treasury securities, U.S. government agency and corporate obligations, all other bonds stocks and securities and Fed funds sold and securities purchased under agreements to resell.[12] Loan portfolio and liquidity held by a bank can be thought of as the productive type and the liquid type in this model. Note that residential mortgages are a big fraction of total loans outstanding. According to Table.1 in Loutskina (2011), share of residential mortgages in the total loans outstanding has grown from 35% in 1986 to about 60% by 2007. Top panel of Figure.2 in Loutskina (2011) shows the time trends in the two measures of liquidity held by banks and liquidity of loan portfolios. A secular increase in the liquidity of the loan portfolio, i.e., increase in θ_1, has been accompanied by a significant decline, since early 1990s, in the liquidity held by banks, i.e., share of liquid type in the total investment.

[12]She uses different items for different time periods to measure liquidity on the bank's balance sheet.

4 Welfare and Efficiency

In this section I study the welfare properties of competitive equilibria in the three regions of Definition 3. It is important to be explicit about the welfare measures based on which I want to assess the efficiency of equilibria:

Definition 4. *An allocation in the overlapping generations economy is called* ***Constrained Pareto Efficient*** *if a social planner cannot reallocate the resources to make at least one entrepreneur strictly better off while keeping all others at least as well off and the reallocation respects the pledgeability*[13] *constraint in I. More formally, an allocation* $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ *is constrained Pareto efficient if it is feasible, i.e. it satisfies the following series of constraints for all* $t \geq 0$*:*

$$\begin{cases} c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e\,, \\ \\ x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e\,. \end{cases} \tag{9}$$

and there does not exist any feasible allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ *such that* $c_t \geq c_t^*$ *for all* $t \geq 0$ *with at least one strict inequality, given initials* $x_{j,-1} = x_{j,-1}^*$ *for* $j \in \{1, 2\}$*.*

Equipped with the above definition, I can evaluate the equilibrium welfare of the model economy for different parameter values. The *benchmark economy* helps to understand the results in a more subtle manner:

Proposition 3. *Any competitive equilibrium in the benchmark economy is constrained Pareto efficient.*

4.1 Efficiency of Competitive Equilibria

In this section I study constrained Pareto efficiency of competitive equilibria in the overlapping generations economy. Classical models of overlapping generations in which Pareto inefficiency arises include Samuelson (1958) and Diamond (1965). Diamond (1965) presents an environment where overinvestment in capital takes place and shows that it leads to inefficiently high capital stock. A celebrated test of inefficiency due to overinvestment, at least in theory, is whether the prevailing interest rate is less than the growth rate of the output.

[13] I use the term "pledgeability constraint" instead of "borrowing constraint" in the analysis of the social planner problem.

Investing too much reduces the return to capital to a level below the growth rate of aggregate output. Similar to Samuelson (1958), Pareto inefficiency induced by overinvestment implies an interest rate that is too low. A Pareto improvement by the social planner that reallocates part of the inefficiently accumulated capital stock to consumers would raise the interest rate.

In contrast with the *benchmark economy* where there is only one type of investment and in the absence of the liquidity and return trade off, I show that competitive equilibria can be constrained Pareto inefficient in regions of parameters where Assumption 1 is satisfied. I further show that the interest rate in a constrained Pareto inefficient equilibrium *is too high despite being negative* and, in contrast with the conventional type of inefficiency, a Pareto improvement lowers the already negative interest rate.

For the moment consider only steady state equilibria. Let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$, where investment in the liquid type is strictly positive in steady state. Suppose the planner reduces the aggregate debt payments of all middle aged entrepreneurs in every generation to the young, $(1 + r_z^{ss})e, z \in \{m, \ell\}$, by an amount $\delta > 0$ by substituting the productive for liquid investment. Let the increase in x_1 be $\epsilon > 0$, then the resource constraint implies that x_2 has to be reduced by $\epsilon + \delta$ to make this increase possible. Given δ, the maximum possible ϵ is determined when the pledgeability constraint binds:

$$\begin{aligned}
\delta &= (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta \,, \\
\epsilon &= \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1} \delta \,.
\end{aligned}$$

The two sides of the equation above is simply changes in the two sides of the pledgeability constraint. The change in the consumption level of the old is $R_1\epsilon - (\epsilon + \delta)R_2 + \delta$ and hence for $z \in \{m, \ell\}$ is equal to:

$$\Delta V_z^{ss} = \left(\frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1} R_1 - (1 + \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}) R_2 + 1 \right) \delta \,. \tag{10}$$

Note that the change in consumption of the initial middle aged is always strictly positive:

$$\Delta V_0 = \left(1 + \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1} \right) \delta > 0 \,. \tag{11}$$

The initial middle aged is strictly better off because the planner reduces her debt payments to the young by substituting productive type for the liquid type while, in contrast with

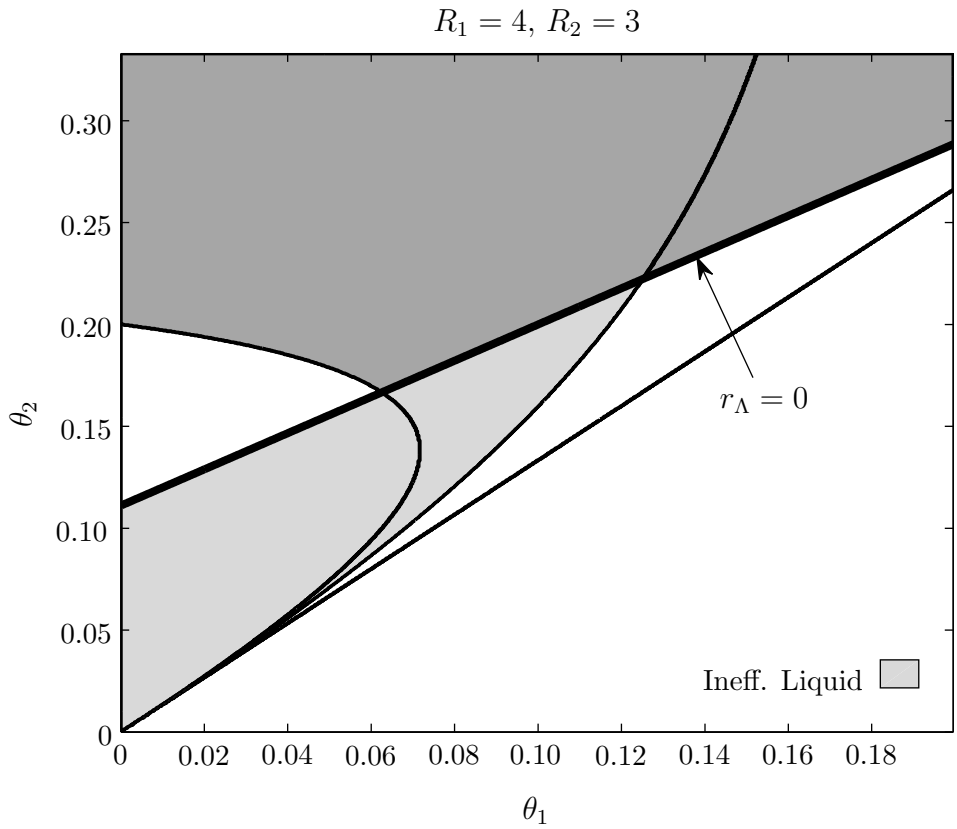

Figure 6: An illustration of inefficiently liquid and inefficiently illiquid equilibria and the line corresponding to $r_\Lambda = 0$, for $R_1 = 4$ and $R_2 = 3$.

future middle aged, her receipts from the initial old do not change.[14] Therefore if $\Delta V_z^{ss} \geq 0$, the steady state allocation is constrained Pareto inefficient. Now one has:

$$\Delta V_z^{ss} \geq 0 \;\Leftrightarrow\; r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0\,. \tag{12}$$

Hence, the region of $(\boldsymbol{\theta}, \boldsymbol{R})$ where this reallocation leads to a higher consumption for at least one generation (every generation when $\Delta V_z^{ss} > 0$) coincides with the steady state equilibria of $F_\ell \cup F_m$ where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, that is, the light gray region in Figure 6.

The above reallocation can also work outside the steady state. In any competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$, condition 12 determines whether a similar reallocation can lead to a Pareto improvement. This is of course not the case for $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_i$ because by Lemma 2, any competitive equilibrium converges to an illiquid steady state where

[14]Note that the initial old pays off her debt, which is given as an initial condition and is not changed by the planner, to the middle aged at $t = 0$, consumes and dies.

there is no investment in the liquid type, i.e., $x_2^{ss} = 0$. I can summarize the results in the following proposition:

Proposition 4. *Consider any competitive equilibrium with liquidities and returns given by* $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$. *If* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, *the competitive equilibrium is constrained Pareto inefficient. Moreover, the equilibrium interest rate at the steady state is strictly negative when* $(\boldsymbol{\theta}, \boldsymbol{R})$ *lies in the interior of the inefficient region, i.e.* $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0\}$, *and zero on part of its boundary that lies in* F_m.

This is in sharp contrast with the efficiency properties of the *benchmark economy*. In an economy where the private entrepreneurs have access only to one type of investment, say the first type (θ_1, R_1), there will be no inefficiency by Proposition 3. In this paper, I have added an endogenous choice between liquidity and return. The results above shows that this endogeneity can lead to inefficient private allocations with too much liquidity. These inefficient competitive equilibria, therefore, are in fact *inefficiently liquid*. I show below that typically there are subregions in both liquid and mixed regions where competitive equilibria are constrained Pareto efficient.

The social planner can also do the opposite by increasing the aggregate debt payments by $\delta > 0$ when there is strictly positive investment in the productive type along the equilibrium path. In that case condition 12 will be reversed to $\Delta V_z^{ss} > 0, z \in \{m, i\}$, if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. When $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i$ and $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, this reallocation increases the utility of every generation starting from $t = 1$ but is not a Pareto improvement because the initial middle aged would suffer.

Note, however, that the overlapping generations economy can be reinterpreted as follows. There is a continuum of each of three types of individuals living forever with the same technology and asynchronicity between receiving endowment, production and consumption opportunities. More explicitly type $j \in \{0, 1, 2\}$ receives endowment $e > 0$ in periods $(j + 1 \bmod 3) + 3k$, produces in $(j + 2 \bmod 3) + 3k$ and consumes in periods $(j \bmod 3) + 3k$ for all $k \in \{0, 1, 2, ...\}$. Let all individuals discount future consumption at a constant rate $\beta \in (0, 1)$ so that the utility function of type $j \in \{0, 1, 2\}$ is:

$$U_j = \sum_{k=0}^{\infty} \beta^k c_{j+3k}$$

This economy is similar, though not isomorphic, to the one in Kiyotaki and Moore (2002) and

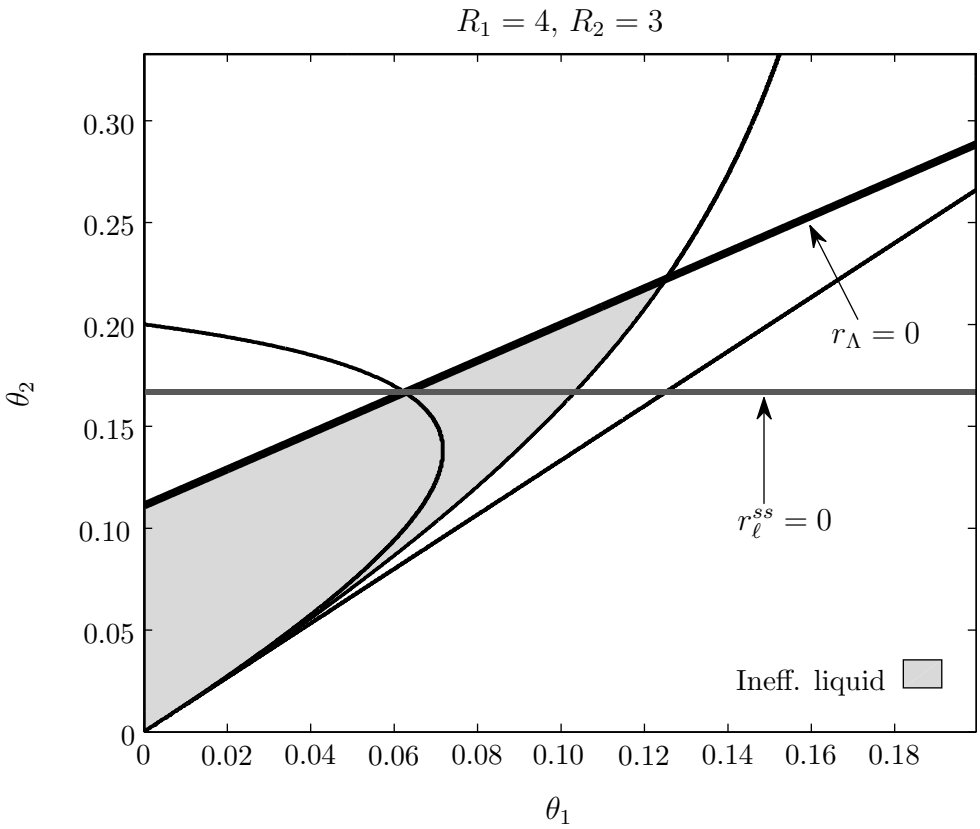

Figure 7: The insufficiency of the interest rate for determining the constrained steady state efficiency.

Woodford (1990).[15] With this interpretation, one can show that the proposed reallocation above would be a Pareto improvement, i.e., at least one type is made strictly better off while the other two are at least as well off, for a high enough discount factor β. The set $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0\}$ is the region which lies above $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ line in Figure 6 with dark gray color. In contrast with the notion of inefficiency in this model, the inefficiency under the above reinterpretation is similar to the one in Woodford (1990) in that there is insufficient liquidity and the interest rate is too low while being positive.

Sign of the interest rate, as a standard indicator of inefficiency, is not a sufficient determinant of the inefficiency in this model. The bold gray line in Figure 7 corresponds to the zero steady state interest rate, i.e., $r_\ell^{ss} = 0$. This line cuts the liquid region into two parts with negative and positive interest rates in steady state equilibrium. The inefficiently liquid equi-

[15] In Woodford (1990) it is shown that the infinite horizon model is equivalent to the overlapping generations presented in Diamond (1965) along the relevant dimensions.

libria in F_ℓ is a proper subset of F_ℓ with strictly negative interest rate. There is a nonempty subset of F_ℓ featuring negative interest rate in steady state that are not inefficiently liquid.[16]

Figure 6 suggests that for every $\boldsymbol{R} = (R_1, R_2)$, the set of inefficient equilibria is nonempty. The following lemma summarizes some of the properties of the inefficient region in F:

Lemma 4. *For any $\boldsymbol{R}$ satisfying Assumption 1 the following are correct. The sets of inefficiently liquid competitive equilibria in F are nonempty with strictly positive measure. There are inefficiently liquid equilibria in any arbitrarily small neighborhood of the origin. The set of inefficiently liquid equilibria in F_ℓ is a proper subset of F_ℓ if and only if $\frac{R_1-R_2}{R_2-1} \leq 1$. Finally, let S_i denote the unique intersection of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ with the boundary of F_i. Then all $(\boldsymbol{\theta}, \boldsymbol{R})$ which correspond to inefficiently liquid equilibria have liquidities, i.e., $\boldsymbol{\theta}$, less than S_i.*

As Figure 6 and Proposition 4 suggest, the economy becomes too liquid when liquidities of investment types are relatively low.[17] Another implication of Proposition 4 is that when the difference between returns to the two types is very high, all equilibria in F_ℓ become inefficiently liquid. This is illustrated in Figure 8. In other words, an increase in the difference between returns to the two types expands the inefficiently liquid region. I close this section by characterizing the set of constrained Pareto efficient allocation in the following proposition.

Proposition 5. *Let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, any allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$ is constrained Pareto efficient. Consequently, any competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$ is constrained Pareto efficient. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, any allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$* and *has $x_{2t} = 0, t \geq T$ for some $T \geq 0$ is constrained Pareto efficient. Hence, any competitive equilibrium in F_i is constrained Pareto efficient.*

4.2 Regulated Economy

In this section, I study a regulation that can implement the Pareto improving reallocation proposed in Section 4.1. This regulation dictates the fraction of resources that can be

[16] One other point in Figure 7 is the fact that the zero interest rate line, $r_\ell^{ss} = 0$, the line of zero interest rate in the mixed region, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$, and the boundary of the liquid region intersect at a point which is not a coincidence.

[17] Note that in one sense, liquidities, i.e., $\boldsymbol{\theta}$, are low for all $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and that is why the borrowing constraint is binding. Hence "high" and "low" should be understood in relative terms inside F.

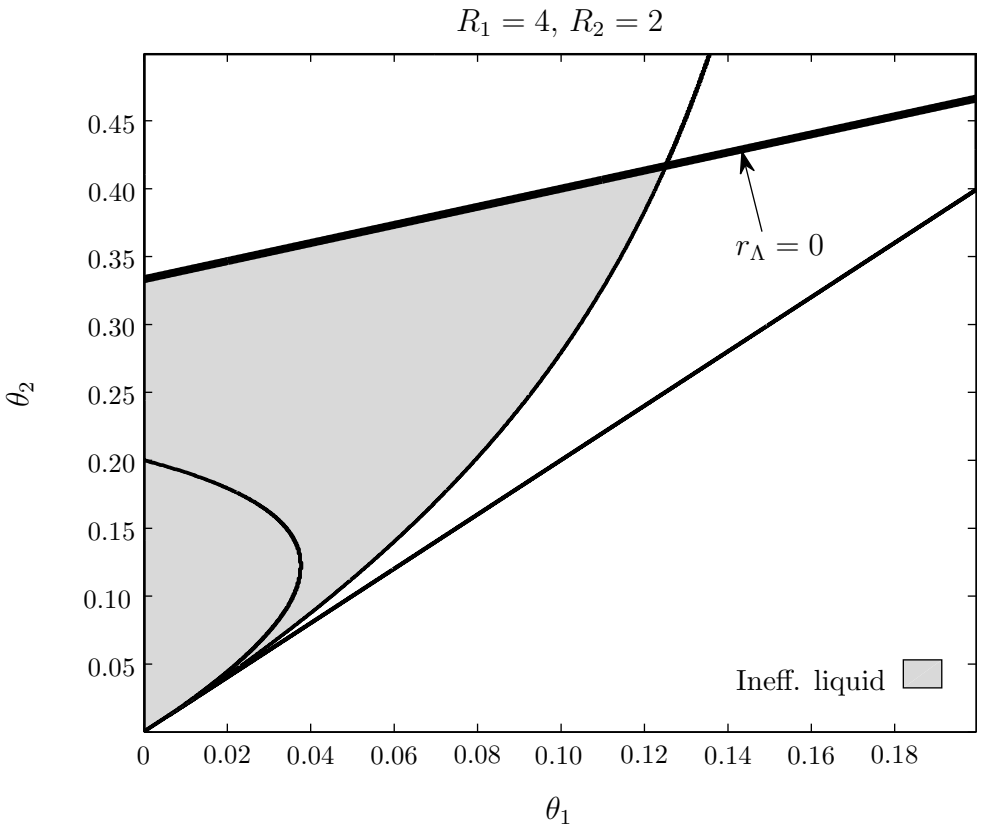

Figure 8: This figure shows the expansion of the inefficiently liquid region when $\boldsymbol{R} = (4, 2)$. In contrast with the case of $\boldsymbol{R} = (4, 3)$, all competitive equilibria in liquid region F_ℓ are constrained Pareto inefficient.

invested in the liquid type. Regulations that require financial institutions to keep a minimum liquid asset ratio[18] have been used in many countries as a monetary or macroprudential instrument.[19] In contrast, the regulation discussed in this section sets a maximum to the total liquid assets held by the entrepreneurs since it is aimed at correcting a different type of market failure in liquidity provision. A similar regulation, though with a different motivation, is a minimum required *"skin in the game"* in securitization of bank loans. It implies that banks may not securitize more than a certain fraction of their stock of loans and hence puts

[18]The base may include liquid liabilities, total liabilities, total assets, cash outflows in a given period of time etc, depending on the context.

[19]For historical record and motivation behind liquidity requirements see Gulde et al. (1997). This type of requirements are still in place in many developing countries and they have been a cornerstone in BASEL III. Look at: http://www.frbsf.org/publications/banking/asiafocus/2011/march.pdf and http://www.bis.org/publ/bcbs188.pdf.

a cap on liquidity of bank's portfolio.[20]

Note that an implicit assumption is that the planner can observe and monitor investments by the entrepreneurs in the two types. This is important because one can show that a maximum liquidity requirement cannot be implemented via a combination of (total) debt and wealth taxes in this model.[21]

Consider any competitive equilibria of the three regions and suppose that a social planner (e.g. government) can regulate investment portfolios of the middle aged entrepreneurs. More precisely suppose that the social planner can regulate the fraction $\alpha_{\ell t}$ of total investment, $(1+r_{t-1})e+i_t$, at the hands of middle aged entrepreneurs at any time $t \geq 0$, that is invested in the liquid type. Therefore:

$$\alpha_{\ell t} = \frac{x_{2t}}{i_t + (1+r_{t-1})e}\,.$$

In this case the entrepreneur only chooses the level of new funds raised i_t and so the maximization problem of the middle aged entrepreneurs takes the following form:

$$\begin{aligned} \max_{i_t \geq 0} \quad & ((1-\alpha_{\ell t})R_1 + \alpha_{\ell t}R_2)\,(i_t + (1+r_{t-1})e) - (1+r_t)i_t \\ s.t. \quad & (1+r_t)i_t \leq (\theta_1(1-\alpha_{\ell t})R_1 + \theta_2\alpha_{\ell t}R_2)\,(i_t + (1+r_{t-1})e) \end{aligned} \tag{IV}$$

The only constraint in the above problem is the borrowing constraint when the choice of liquidity-return is regulated by the planner. Let $R_{\alpha t} = (1-\alpha_{\ell t})R_1 + \alpha_{\ell t}R_2$ and $\gamma_{\alpha t} = \theta_1(1-\alpha_{\ell t})R_1 + \theta_2\alpha_{\ell t}R_2$ be the return and liquidity of the regulated portfolio at time t. Problem IV is the maximization problem of an entrepreneur that has access only to one type of investment project with return $R_{\alpha t}$ and liquidity of $\gamma_{\alpha t}$. The optimal solution to IV is:

$$\begin{cases} i_t = \left(\frac{(1+r_{t-1})\gamma_{\alpha t}}{(1+r_t)-\gamma_{\alpha t}}\right)e & \text{if } R_{\alpha t} \geq 1+r_t\,, \\ i_t = 0 & \text{if } R_{\alpha t} < 1+r_t\,. \end{cases}$$

Notice that $\gamma_{\alpha t} < 1$ by Assumption 1. Also note that for any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, there exists an $\epsilon > 0$ such that $\left(\frac{\gamma_{\alpha t}}{1-\gamma_{\alpha t}}\right) < R_{\alpha t}$ for all $\alpha_{\ell t} \in [0, \epsilon)$. This is because the inequality holds for $\alpha_{\ell t} = 0$ according to the definition of F and so by continuity it holds in a neighborhood of

[20] A minimum required *"skin in the game"* is part of the Credit Risk Retention in the Dodd-Frank act: http://www.fdic.gov/regulations/reform/dfa_selections.html#941.

[21] I do not provide the proof in this paper. With a debt and wealth tax the middle aged should pay $(1+\tau_d)(1+r_t)i_t$ next period using her initial wealth of $(1-\tau_w)(1+r_{t-1})e$. This implies that to implement this liquidity regulation the planner must have more instruments such as taxation of each investment types.

zero. If $\{\alpha_{\ell t}\}_{t=0}^{\infty}$ are all set to α_ℓ and this value is in the neighborhood above, the steady state equilibrium of the regulated economy is:

$$1 + r_\alpha^{ss} = \left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right). \tag{13}$$

Where variables without time subscript correspond to α_ℓ. The following proposition gives the steady state utility levels for the steady states of the regulated economy as well as the three regions of the original (unregulated) economy:

Proposition 6. *The steady state level of utility for any values of α_ℓ for which $\left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right) < R_\alpha$ is given by:*

$$V_\alpha^{ss} = \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right) e = \left(\frac{(1-\alpha_\ell)(1-\theta_1)R_1 + \alpha_\ell(1-\theta_2)R_2}{(1-\alpha_\ell)(1-\theta_1 R_1) + \alpha_\ell(1-\theta_2 R_2)}\right) e\,.$$

The steady state utility levels for the three regions in Definition 3 are, $V_\ell^{ss} = \left(\frac{(1-\theta_2)R_2}{1-\theta_2 R_2}\right) e$, $V_i^{ss} = \left(\frac{(1-\theta_1)R_1}{1-\theta_1 R_1}\right) e$ and:

$$V_m^{ss} = \left(\frac{(\theta_2-\theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)}\right) e\,.$$

Moreover suppose $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and that $\widetilde{\alpha}_\ell = \left(\frac{x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{x_1^{ss}(\boldsymbol{\theta},\boldsymbol{R})+x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}\right)$. Then in the regulated economy corresponding to $\widetilde{\alpha}_\ell$ one has:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 1 + r_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)\theta_1 R_1 + \widetilde{\alpha}_\ell \theta_2 R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right),$$

$$V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = V_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)(1-\theta_1)R_1 + \widetilde{\alpha}_\ell(1-\theta_2)R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) e\,.$$

Consider a steady state of the mixed region where entrepreneurs invest in both types of investment projects. Suppose the social planner sets the fraction of liquid investment α_ℓ equal to the ratio of the liquid investment to total investment that prevails in the steady state competitive equilibrium given by 6. Then, by the above proposition, the resulting interest rates and utility levels of the steady state equilibrium and the regulated economy will be the same. In this sense, there is a close relationship between the ratio of liquid investment to

the total investment, the interest rate and utility level at any steady state equilibrium.

Note that the numerator and denominator of V_α^{ss} are weighted averages of those of V_ℓ^{ss} and V_i^{ss}. This implies that V_α^{ss} always lies between the two values of V_ℓ^{ss} and V_i^{ss}. The following lemma provides conditions under which a social planner can achieve a higher level of steady state consumption by regulating the mixture of liquid and productive types of investments in each of the three regions:

Lemma 5. *For any values of $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, the following statements are correct. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ and $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ have the same sign. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is negative if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$.*[22]

Lemma 5 shows a close connection between sign of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and the possibility of raising the steady state utility via regulation. This is in fact what one could expect from the results obtained in Section 4.1. For instance suppose that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and so one must have $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$. Let $\widetilde{\alpha}_\ell$ be the ratio of liquid investment to the total funds invested for the particular $(\boldsymbol{\theta}, \boldsymbol{R})$. Then the planner can set the fraction of liquid investment to total equal to $\widetilde{\alpha}_\ell - \epsilon$ for $\epsilon > 0$ and raise the steady state utility of the entrepreneurs.

Notice that when $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$, the interest rate in the regulated economy will be lower than the competitive equilibrium. The reason is that the planner reduces the liquidity of the aggregate investment portfolio which reduces the demand for funds by the middle aged entrepreneurs. In the opposite case where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and hence $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$, the planner can raise the fraction of liquid investments $\widetilde{\alpha}_\ell$, to $\widetilde{\alpha}_\ell + \epsilon$ for $\epsilon > 0$ to raise the steady state utility which raises the steady state interest rate.

To see how the regulation affects welfare in the competitive equilibria and if it can make Pareto improvements, one has to look at the dynamics induced by changing $\alpha_{\ell t}$'s as well. The following proposition shows that, taking the dynamics into account, the above intuition about the steady state utilities and interest rates generalizes to any competitive equilibria:

Proposition 7. *Any Pareto improving reallocation of the type analyzed in Section 4.1, when δ is small enough in absolute value, can be implemented by regulating the investment portfolios of the entrepreneurs. In an inefficiently liquid equilibrium, planner chooses a lower liquid investment to total investment ratio, the steady state interest rate is* non positive *and*

[22] The expressions of the type $Z(\boldsymbol{\theta}, \boldsymbol{R})$ for variable Z should be considered solely as functions of parameters $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ in Lemma 5 which may or may not be an equilibrium utility level or interest rate for the particular vector $(\boldsymbol{\theta}, \boldsymbol{R})$.

the regulated interest rate is lower *than in the unregulated equilibrium. Moreover, given any inefficiently liquid equilibria where* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ *or one where* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ *in the mixed region, this regulation can implement a Pareto improvement reallocation that results in a constrained Pareto efficient allocation.*

For the inefficiently liquid equilibrium, interest rate is *non positive* and the planner can make Pareto improvement by *reducing* the equilibrium interest rate to an even lower level. This is in contrast with the conventional type of inefficiency in the overlapping generations models (e.g. Samuelson (1958), Diamond (1965)), where a negative interest rate has to be *raised* by a planner to achieve efficiency. Hence, the level of interest rate can be a misleading factor in determining and understanding of the nature of inefficiency present in the economy.

4.3 Discussion

The reallocations considered in this section clearly show that the aggregate resources that are transferred between entrepreneurs from different generations may not be socially optimal. Entrepreneurs may transfer too much resources to the next generation by investing, more than optimally, in liquid type. The inefficiency of this intertemporal resource transfer is confirmed by the observation in Section 4.2 that a social planner can make a Pareto improvement by changing the investment composition. To see how portfolio choices can entail an externality consider the problem of the middle aged entrepreneurs again:

$$\begin{aligned} \max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1 + r_t) i_t \\ s.t. \quad & x_{1t} + x_{2t} \leq w_{t-1} + i_t \,, \\ & (1 + r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \,. \end{aligned}$$

$w_{t-1} = (1 + r_{t-1})e$ is initial wealth of the middle aged which is determined at $t-1$. Suppose for the moment $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ and let $w_{t-1} < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))e$. In other words, entrepreneurs have a relatively low initial wealth in period t. Given these assumptions on model parameters and initial wealth of the middle aged, the competitive equilibrium in the credit market at t

is:

$$\begin{cases} i_t = e \\ 1 + r_t = \theta_2 R_2 (2 + r_{t-1}) \\ V = (1 - \theta_2) R_2 (w_{t-1} + e) \end{cases}$$

In this equilibrium all middle aged invest only in type two, i.e., the liquid type. V is the maximized value of the objective function of the middle aged at t. Now consider a government which prohibits any investment in type two, i.e., it forces the middle aged to set $x_{2t} = 0$. It is easy to see that given the same parameters and initial condition on wealth of the middle aged, the competitive equilibrium in the regulated economy is given by:

$$\begin{cases} \widetilde{i}_t = e \\ 1 + \widetilde{r}_t = \theta_1 R_1 (2 + r_{t-1}) \\ \widetilde{V} = (1 - \theta_1) R_1 (w_{t-1} + e) \end{cases}$$

Since $(1-\theta_1)R_1 > (1-\theta_2)R_2$ by Assumption 1, the regulation raises the consumption of the middle aged, i.e. $\widetilde{V} > V$. Absent regulation, middle aged entrepreneurs have more flexibility in their investment but this flexibility in portfolio decisions can lead to an excessively high level of interest rate.

As shown in Section 4.1, when $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, the equilibrium is constrained Pareto inefficient if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$. By Proposition 5, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ is equivalent to $\frac{(1-\theta_1)R_1}{1-\theta_1 R_1} > \frac{(1-\theta_2)R_2}{1-\theta_2 R_2}$, where the two sides of the latter inequality are steady state utility levels when entrepreneurs are allowed to invest only in productive and liquid types respectively. Under Assumption 1, one always has $(1 - \theta_1)R_1 > (1 - \theta_2)R_2$. One needs, however, a stronger assumption, i.e., $\frac{(1-\theta_1)R_1}{1-\theta_1 R_1} > \frac{(1-\theta_2)R_2}{1-\theta_2 R_2}$, to have inefficiency in the fully dynamic setting because one should consider the welfare of future middle aged at $t+1, t+2, ...$ as well.

In general equilibrium the initial wealth w_{t-1} is determined endogenously. A low initial wealth is an equilibrium outcome when liquidities of the two types are relatively low which is what Figure 6 and Proposition 4 suggest. Low liquidities of investment types are also the reason the inefficiently liquid steady states have non positive interest rate.

The above observation suggests that countries with low level of financial development

but high growth opportunities may be more prone to this type of inefficiency.[23] These economies are thought to suffer from shortage of stores of value due to low liquidity of return to investment. Investment in real state and foreign safe assets, e.g., U.S. treasuries, are among the liquid but unproductive investments that have served as important stores of value in these countries. The analysis above suggests that these countries may be investing too much in these liquid but unproductive assets (liquid type), as opposed to illiquid but highly productive projects, e.g., industries with high growth potential, (productive type).[24] If these economies are inefficiently liquid, the policy implication will be a maximum liquidity requirement for the real and financial sectors of these countries which is different from both standard prescriptions and common practice.[25] Also in this case the efficient interest rate is lower than the prevailing rate in these countries.

5 Public Liquidity

In this section I study the effects of public liquidity provision in the form of government bonds. Theoretically, as it is argued in Holmstrom and Tirole (1998), government bonds are liquid due to the exclusive ability of the government to tax the private sector. Krishnamurthy and Vissing-Jorgensen (2010) provide convincing evidence that the low yield on government treasuries is due to their liquidity services to the private agents.

I analyzed the competitive equilibria of a model without any government in previous sections. I studied positive properties of equilibria and showed that for certain regions in the space of parameter values, the competitive outcome fails to achieve efficiency. I want to see how the introduction of publicly supplied liquidity by government affects equilibrium prices and quantities and whether it can improve the welfare of the competitive equilibria in different regions of F. There are at least two objectives for this exercise. First, I want to compare the effects of public liquidity on welfare in this paper to environments without heterogenous investment opportunities, i.e. Farhi and Tirole (2010a), Holmstrom and Tirole (1998) and Woodford (1990). This comparison helps one better understand the unconventional nature of the Pareto inefficiency of the competitive equilibria in this paper. Second,

[23] This model is not quantitative and so it is difficult to rule out the possibility that countries with developed financial sectors may suffer from the same type of inefficiency. Nonetheless, countries with underdeveloped financial sectors are certainly the more likely candidates.

[24] Proposition 4 shows that as the difference between the returns to the investment types increases, the inefficient region expands. This means that a higher growth potential makes this type of inefficiency more likely.

[25] For a theoretical justification of liquidity requirement see Caballero and Krishnamurthy (2006).

the positive implications of introducing government bonds can be contrasted with those of models without heterogeneity as a way to empirically distinguish between the two types of models.

Intuitively, one may not expect any benefits to public liquidity provision in an inefficiently liquid equilibrium. This is simply because in an inefficiently liquid equilibrium, the private sector already provides more liquidity than the socially optimal level. This intuition turns out to be correct for the inefficient equilibria in F_ℓ but matters are different in the mixed region, F_m.

5.1 Competitive Equilibrium with Bonds

Consider the model in Section 2 with only one difference: the young and middle aged entrepreneurs at any time $t \geq 0$ can purchase a one period and risk free government bond sold at par, denoted by b_t^y and b_t^m. A unit of bond purchased at time t is a promise by the government to deliver one unit of consumption good plus the interest in period $t+1$. Since I have stated all the steps to solve the maximization problem of the middle aged in Section 2, I only show the final form of the maximization with government bonds as long as $1+r_t < R_1$ that can be compared to II:

$$\begin{aligned} \max_{i_t, b_t^m \geq 0} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t)(i_t - b_t^m) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1})e - \tau_{t+1}^o \\ s.t. \quad & \left(\frac{\theta_1 R_1(1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e \leq (i_t - b_t^m) \leq \left(\frac{\theta_2 R_2(1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\,. \end{aligned} \tag{II$_b$}$$

where Φ is determined at time $t-1$ and Λ is given by:

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) \equiv \left(\frac{(\theta_2-\theta_1)R_1R_2}{\theta_2R_2-\theta_1R_1}\right) - \left(\frac{(1-\theta_1)R_1-(1-\theta_2)R_2}{\theta_2R_2-\theta_1R_1}\right)(1+r_t)\,.$$

Problem II$_b$ is very similar to II. The main difference is that the middle aged entrepreneur have a new opportunity to invest in b_t^m units of government bond. The important assumption here is that the total return to investment in bonds is pledgable so that bond purchases essentially reduce the total debt payment by the same amount to $(1+r_t)(i_t - b_t^m)$. The middle aged entrepreneur also receives the principal and interests on her bond purchases in the previous period namely $(1+r_{t-1})b_{t-1}^y$. Finally τ_{t+1}^o denotes the lump sum tax that is

levied on the old entrepreneurs before consumption takes place.[26] I suppose that government balances its budget every period so that for all $t \geq 0$:

$$(1 + r_t)b_t = b_{t+1} + \tau^o_{t+1} \,. \tag{14}$$

Market clearings dictate that for all $t \geq 0$:

$$i_t + b^y_t = e \,, \tag{15}$$

$$b^m_t + b^y_t = b_t \,. \tag{16}$$

In order to ensure the existence of competitive equilibrium in which borrowing constraint is still binding, one needs to restrict the supply of bonds. Let $\sigma_t = \frac{b_t}{e}$ be the normalized supply of bonds for all $t \geq 0$[27]; then one needs the following assumption:

Assumption 2. *$\sigma_t < \min(1 - \theta_2 R_2, 1 - \frac{\theta_1 R_1}{1-\theta_1})$ for all $t \geq 0$ and $\sigma = \lim_{t\to\infty} \sigma_t < \min(1 - \theta_2 R_2, 1 - \frac{\theta_1 R_1}{1-\theta_1})$ exists.*

Note that by Assumption 2, $\sigma < \min(1 - \theta_2 R_2, 1 - \frac{\theta_1 R_1}{1-\theta_1})$. Let $\Sigma \equiv \{\sigma_t\}_{t=0}^{\infty}$, then one can redefine the three regions as follows:

Definition 5. *Define $F(\Sigma)$ as the set of $(\boldsymbol{\theta}, \boldsymbol{R})$ that satisfies Assumption 1 and Assumption 2, given the sequence $\Sigma = \{\sigma_t\}_{t=0}^{\infty}$. Then the three regions of $F(\Sigma)$ are defined as:*

Liquid Region*: $F_\ell(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) \leq (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\}$.*

Mixed Region*: $F_m(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.*

Illiquid Region*: $F_i(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) \leq \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.*

Assumption 2 has the following interpretation. First, σ has to be smaller than $1 - \theta_2 R_2$, otherwise no steady state with positive investment in type one or two exists. Second, one also needs $\sigma < 1 - \frac{\theta_1 R_1}{1-\theta_1}$ to ensure that the borrowing constraint is binding in steady state.

[26] I choose to levy the tax on the old for two reasons. First, if I impose the tax on young and middle aged entrepreneurs, the value of the tax collected would affect the equilibrium conditions in a more subtle way. I want to avoid this since effects of tax policy are not the focus of my study here and so I want to have the taxation as neutral as possible. Second, imposing the tax on the young and middle aged complicates the problem; for example if I impose it on the young I would obtain multiple steady state equilibria.

[27] From now on, I use the terms "supply" and "normalized supply" of bonds interchangeably.

The following lemma shows the existence and uniqueness of competitive equilibria and the steady states:

LEMMA 6. *For any $(\boldsymbol{\theta}, \boldsymbol{R})$ and Σ satisfying Assumption 1 and Assumption 2 and for any given initial condition $1 + r_{-1} < R_1$, there is a unique competitive equilibrium that converges to a unique and stable steady state corresponding to the region in Definition 5 containing $(\boldsymbol{\theta}, \boldsymbol{R})$. The steady state interest rates for the three regions are:*

$$\begin{cases} 1 + r_\ell^{ss}(\Sigma) = \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell(\Sigma)\,, \\ 1 + r_m^{ss}(\Sigma) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_m(\Sigma)\,, \\ 1 + r_i^{ss}(\Sigma) = \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_i(\Sigma)\,. \end{cases}$$

Moreover at the steady state, the entrepreneurs specialize in the liquid and productive type of investments in regions $F_\ell(\Sigma)$ and $F_i(\Sigma)$ respectively but invest strictly positive amounts in both types in $F_m(\Sigma)$.

A change in Σ which involves an increase in σ, i.e. long run supply of government bonds, has two effects. First, given a fixed $\boldsymbol{R}$ and as can be seen in Definition 5, it rotates the boundaries of $F_i(\Sigma)$ and $F_\ell(\Sigma)$ counterclockwise around the origin in the space of $\boldsymbol{\theta}$. Looking at Figure 6, it is easy to see that as a result of an increase in σ, both $F_\ell(\Sigma)$ and the inefficiently liquid region become smaller. $F_i(\Sigma)$ expands into the previously mixed region and may or may not shrink at its top right and upper boundaries (due to Assumption 2).[28] In this sense, government bonds eliminate part of the inefficiently liquid region as well as part of the liquid region. Second, by Lemma 6, an increase in σ increases the steady state interest rate in $F_i(\Sigma)$ and $F_\ell(\Sigma)$, as it does in the benchmark economy with one type of investment, for any given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma)$ but has no effects on the interest rate in $F_m(\Sigma)$ which is fixed at $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$.

The effects of government bonds on the steady state level of private investment also differ markedly in liquid and mixed regions. Let $i_x^{ss} \equiv x_1^{ss} + x_2^{ss}$ denotes the total amount of resources invested in the two types by entrepreneurs at the steady state. Using Lemma 6 and

[28] It depends on whether change in Σ entails a change in $max\{\sigma_t\}_{t=0}^{\infty}$. Notice also that I am using the fact that the locus of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$, which is the straight line that defines the upper boundary of inefficiently liquid region does not depend on the value of σ.

market clearings 15, one can compute the effect of a small increase in the long run supply of bonds σ on i_x^{ss}:

$$\frac{\partial i_x^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_\ell} = \left(\left(\frac{\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)^2 - 1\right) e\,, \tag{17}$$

$$\frac{\partial i_x^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_m} = -e\,. \tag{18}$$

17 is the same as the result obtained in the *benchmark economy*. An increase in the long run supply of public liquidity σ crowds out private investment i_x^{ss} when public liquidity is scarce, i.e., σ is low, and crowds in private investment when public liquidity is abundant, i.e., σ is high. Note also that the marginal effect of public liquidity on private spending is strictly increasing in the level of public liquidity, that is, $\frac{\partial^2 i_x^{ss}}{\partial \sigma^2} > 0$.

On the other hand, 18 shows that government bonds always crowds out private investment *one for one* in the mixed region and that $\frac{\partial^2 i_x^{ss}}{\partial \sigma^2} = 0$. Using II_b at the steady state, it is easy to show that:

$$\frac{\partial x_1^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_m} = \frac{1 + r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\, e > 0\,, \tag{19}$$

$$\frac{\partial x_2^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_m} = -\frac{1 + r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) - \theta_1 R_1}{\theta_2 R_2 - \theta_1 R_1}\, e < -e\,. \tag{20}$$

Public liquidity, therefore, crows out the liquid type and crowds in the productive type in the mixed region. The crowding out of the liquid type happens more than proportionately so that the demand for funds and consequently the interest rate remain unchanged.

5.2 Welfare Effects of Government Bond

I showed in the previous section that an increase in supply of government bonds raises the interest rate in $F_i(\Sigma)$ and $F_\ell(\Sigma)$ but has no effects on the interest rate in the mixed region. I further showed in Section 4.2 and Section 4.1 that a Pareto improvement would reduce the interest rate in the inefficiently liquid region. This observation suggests that government bonds may not help much to improve welfare in the inefficiently liquid equilibria. It is not clear, a priori, how government bonds affect the welfare in the mixed region. To study the welfare effects in competitive equilibria one needs to look at the steady state first:

LEMMA 7. *Let $V_z^{ss}(\Sigma)$ denotes the steady state utility level for region $z \in \{\ell, m, i\}$ given $(\boldsymbol{\theta},\boldsymbol{R}) \in F_z(\Sigma)$, when the long run supply of government bonds is σ. Then the welfare effects*

of an increase in σ are given as follows:

$$\frac{\partial V_\ell^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} = \left(\frac{R_2 - 1}{1 - \theta_2 R_2}\right) r_\ell^{ss} e\,,$$
$$\frac{\partial V_i^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} = \left(\frac{R_1 - 1}{1 - \theta_1 R_1}\right) r_i^{ss} e\,,$$
$$\frac{\partial V_m^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} = -r_m^{ss} e\,.$$

r_z^{ss} denotes the steady state interest rate for region $z \in \{\ell, m, i\}$ when there is no government bonds in the economy.

Suppose Σ is null, i.e., $\sigma_t = 0$ for all $t \geq 0$, so that $F_\ell(\Sigma) = F_\ell$, $F_i(\Sigma) = F_i$ and $F_m(\Sigma) = F_m$. I am interested in the effect of a small increase in the supply of government bonds from zero to $\sigma = \epsilon > 0$, i.e., Σ changes to Σ' where $\sigma'_t = \epsilon$ for $t \geq 0$. By Lemma 8, the utility gain is positive in either of F_ℓ or F_i, if and only if the equilibrium interest rate is *positive*. This immediately implies that government bonds reduce the steady state utility in the inefficiently liquid part of F_ℓ. This is because by Proposition 7, the interest rate is strictly negative in the inefficiently liquid part of F_ℓ.

Therefore, effects of publicly supplied liquidity in F_ℓ is in conformance with the intuition. Lemma 7 immediately implies that if a sequence of small enough government supplied bonds Σ together with $(\boldsymbol{\theta}, \boldsymbol{R})$ satisfy Assumption 2, then Σ cannot Pareto improve any inefficiently liquid competitive equilibria without bonds corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$. Note that a necessary condition for such a Pareto improvement is that the resulting steady state utility be no less than the equilibria without bonds. Government bonds can potentially be beneficial when the private sector generates less than socially optimal pledgable returns. This happens in the type of inefficiencies studied in Woodford (1990) and Holmstrom and Tirole (1998). On the other hand, government bonds can be harmful to the welfare, when the private sector generates more than the socially optimal pledgable returns, since government bond would encourage the private sector to generate even more pledgable returns. This is the case in an inefficiently liquid steady state, as observed in Section 4.1.

The situation is rather different in the mixed region. Lemma 8 implies that the supply of government bonds enhances the steady state utility if and only if the prevailing interest rate is *negative*. By Proposition 7, this implies that the steady state utility gain is positive in the inefficiently liquid part of F_m. This is an indication that government bonds might be

able to improve welfare in an inefficiently liquid competitive equilibrium. This is stated in the following proposition:

PROPOSITION 8. *Suppose a sequence of small enough government supplied bonds Σ together with liquidities and returns $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ satisfy Assumption 2. If $(\boldsymbol{\theta}, \boldsymbol{R})$ corresponds to inefficiently liquid equilibria with a strictly negative steady state interest rate, Σ cannot Pareto improve the competitive equilibrium allocation. More precisely, there exists $\epsilon > 0$ such that for any Σ with a long run supply of bonds no more than ϵ, i.e., $\sigma \leq \epsilon$, Σ cannot improve welfare for a competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, as long as Assumption 2 is satisfied and the steady state interest rate is strictly negative. Moreover for any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ corresponding to inefficiently liquid equilibria and also inefficient equilibria corresponding to the unique point $(\boldsymbol{\theta}^*, \boldsymbol{R}^*) \in F_\ell$ where $r(\boldsymbol{\theta}^*, \boldsymbol{R}^*) = 0$, i.e., steady state interest rate is zero, there exists a small enough sequence of government supplied bonds Σ which Pareto improves the competitive equilibrium allocation.*

As discussed in the previous subsection, public liquidity crowds out the liquid investment more than proportionately to keep the demand for funds and interest rate unchanged. This crowding out effect is also the reason why government bonds can make Pareto improvement in the mixed region. By substituting one unit of investment in government bonds for one unit in the liquid type, entrepreneurs can pledge more than before and so the borrowing constraint becomes slack. They can use that extra amount of pledgeability to invest in the productive type while interest rate does not increase. This raises their consumption and results in a Pareto improvement.

6 Conclusion

This paper studies an economy in which agents choose a portfolio of two assets with different liquidity and return. It highlights the positive and normative importance of an endogenous liquidity choice by showing how it leads to starkly different conclusions from a model without such a choice. The decentralized allocation may not be efficient and a planner can Pareto improve an inefficient allocation by regulating the investment composition of private agents. Effect of financial development, i.e., improvement in contract enforcement, contracting technology, bankruptcy laws etc, on interest rate, total investment and investment composition depends on the relative change in liquidity of investments rather than the absolute changes. As a result, financial development that affects illiquid investment disproportionately more

than liquid ones can reduce the interest rate: the decline of U.S. real interest rate after 1990s can be understood as a consequence of highly illiquid assets, i.e., residential mortgages, becoming more liquid through securitization. Finally, government bonds are introduced and shown to have limited ability to improve inefficient allocations. They can make Pareto improvements only when they do not change the interest rate. In this case, government bonds crowds out the liquid investment and crowds in the illiquid one.

During the recent financial crisis, Federal Reserve implemented various measures in order to raise the liquidity of financial institutions.[29] Holmstrom and Tirole (1998) and related papers provide some theoretical foundations for these liquidity measures, that is, at the time of an aggregate liquidity shock private sector may not supply a socially optimal level of liquidity. While giving critical insights, this theoretical justification largely ignores the fact that liquidity of investment is an endogenous choice. Although this paper is not about the episodes of financial crisis, since it has no uncertainty, the model provides a framework for thinking about how liquidity choice of the private sector responds to changes in supply of public liquidity or financial shocks, i.e., exogenous changes in liquidity of investment.[30] It is fairly easy to introduce shocks to endowments or to liquidity of investment types $\boldsymbol{\theta}$, in the model and study the implications of liquidity choice under aggregate uncertainty.

Using Kiyotaki and Moore (2008) framework, one can introduce infinitely lived agents that can accumulate different types of capital with different liquidity. Various kinds of shocks, e.g., productivity shocks, financial shocks etc, can be added to better understand the quantitative importance of liquidity choice over the business cycle. Moreover, productive (illiquid) type of capital in such a model can represent investment in knowledge intensive and less tangible assets, in newly formed industries or in sectors that are more labor intensive. Hence, it can be used to study the effects of long term changes in the quality of contract enforcement, contracting technology, bankruptcy laws and corporate governance on structural change and TFP growth. In this application, different liquidities $\boldsymbol{\theta} = (\theta_1, \theta_2)$ may correspond to different stages of financial development. With a global economy interpretation of this model, as noted in Section 3.1, model's implications for international capital flows and interest rates can be studied in a similar vein.

[29] See http://www.federalreserve.gov/newsevents/speech/bernanke20080513.htm

[30] Eisfeldt and Rampini (2006) provide evidence that the costs of capital reallocation are countercyclical. This can be due to countercyclicality of market liquidity and funding liquidity over the business cycle.

References

Benmelech, E. and N. K. Bergman (2009, March). Collateral pricing. *Journal of Financial Economics 91*(3), 339–360.

Brunnermeier, M. K. and L. H. Pedersen (2009, June). Market liquidity and funding liquidity. *Review of Financial Studies 22*(6), 2201–2238.

Caballero, R., E. Farhi, and P.-O. Gourichas (2008, March). An equilibrium model of global imbalances and low interest rates. *American Economic Review 98*(1), 358–393.

Caballero, R. J. and A. Krishnamurthy (2006, January). Bubbles and capital flow volatility: Causes and risk management. *Journal of Monetary Economics 53*(1), 35–53.

Diamond, D. W. and R. G. Rajan (2001, April). Liquidity risk, liquidity creation, and financial fragility: A theory of banking. *Journal of Political Economy 109*(2), 287–327.

Diamond, P. A. (1965, December). National debt in a neoclassical growth model. *American Economic Review 55*(5), 1126–1150.

Eisfeldt, A. L. and A. A. Rampini (2006, April). Capital reallocation and liquidity. *Journal of Monetary Economics 53*(3), 369–399.

Farhi, E. and J. Tirole (2009, May). Leverage and the central banker's put. *American Economic Review, Papers and Proceedings 99*(2), 589–593.

Farhi, E. and J. Tirole (2010a). Bubbly liquidity. *Unpublished Manuscript*.

Farhi, E. and J. Tirole (2010b). Collective moral hazard, maturity mismatch and systemic bailouts. *Forthcoming in American Economic Review*.

Gerardi, K. S., H. S. Rosen, and P. S. Willen (2010, February). The impact of deregulation and financial innovation on consumers: The case of the mortgage market. *Journal of Finance 65*(1), 333–360.

Ghate, A. and R. L. Smith (2005). Infinite linear programs: I. characterizing extreme points through basic variables. *Unpublished manuscript*.

Giglio, S. and T. Severo (2010). Intangible capital, relative asset shortages and bubbles. *Working Paper*.

Gulde, A.-M., J. C. Nascimento, and L. M. Zamalloa (1997, October). Liquidity asset ratios and financial sector reform. *IMF Working Paper*.

Hart, O. and J. Moore (1994, November). A theory of debt based on the inalienability of human capital. *Quarterly Journal of Economics 94*(4), 841–879.

Holmstrom, B. and J. Tirole (1998, February). Private and public supply of liquidity. *Journal of Political Economy 106*(1), 1–40.

Kehoe, T. J. and D. K. Levine (1993, October). Debt-constrained asset markets. *Review of Economic Studies 60*(4), 865–888.

Kiyotaki, N. and J. Moore (2002, May). Evil is the root of all money. *American Economic Review, Papers and Proceedings 92*(2), 62–66.

Kiyotaki, N. and J. Moore (2005, April-May). Financial deepening. *Journal of the European Economic Association 3*(2-3), 701–713.

Kiyotaki, N. and J. Moore (2008). Liquidity, business cycles and monetary policy. *Unpublished Manuscript*.

Krishnamurthy, A. and A. Vissing-Jorgensen (2010). Aggregate demand for tresury debt. *Unpublished manuscript*.

Lorenzoni, G. (2008, July). Inefficient credit booms. *Review of Economic Studies 75*(3), 809–833.

Loutskina, E. (2011, June). The role of securitization in bank liquidity and funding management. *Journal of Financial Economics 100*(3), 663–684.

Matsuyama, K. (2007, March). Credit traps and credit cycles. *American Economic Review 97*(1), 503–516.

McConnell, J. J. and S. A. Buser (2011, December). The origins and evolution of the market for mortgage-backed securities. *Annual Review of Financial Economics 3*, 173–192.

Parlour, C. A. and G. Plantin (2008, June). Loan sales and relationship banking. *Journal of Finance 63*(3), 1291–1314.

Qian, J. and P. E. Strahan (2007, December). How laws and institutions shape financial contracts: The case of bank loans. *Journal of Finance 62*(6), 2803–2834.

Samuelson, P. A. (1958, December). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy 66*(6), 467–482.

Shleifer, A. and R. W. Vishny (1992, September). Liquidation values and debt capacity: A market equilibrium approach. *Journal of Finance 47*(4), 1343–1366.

Woodford, M. (1990, May). Public debt as private liquidity. *American Economic Review 80*(2), 382–388.

A Appendix: Proofs

Proof of Lemma 1. In any equilibrium the resource constraint binds and so I can solve for the value of x_{2t} in the above and rewrite the problem as:

$$\begin{aligned} \max_{i_t, x_{1t} \geq 0} \quad & (R_1 - R_2)x_{1t} + (R_2 - (1 + r_t))\, i_t + R_2(1 + r_{t-1})e \\ s.t. \quad & (1 + r_t - \theta_2 R_2)i_t \leq (\theta_1 R_1 - \theta_2 R_2)x_{1t} + \theta_2 R_2(1 + r_{t-1})e\,, \\ & 0 \leq x_{1t} \leq (1 + r_{t-1})e + i_t\,. \end{aligned}$$

One can immediately see from the above that $1 + r_t > \theta_2 R_2$. Otherwise it must be that $1 + r_t \leq \theta_2 R_2 < R_2$ in which case i_t can be raised unboundedly and there would not be any maximum to the objective function. One must also have $1 + r_t \leq R_1$, otherwise the optimal solution to the problem requires that $i_t = 0$. To see why rewrite the above with x_{2t} in the objective function. If $1 + r_t > R_1$ then by Assumption 1 both coefficients of x_{2t} and i_t are strictly negative and so the best an entrepreneur can do is to set both to zero. This cannot be an equilibrium since market clearing in the capital market cannot be satisfied.

Now suppose that the borrowing constraint does not bind for some $t \geq 0$. Then since the coefficient of x_{1t} in the objective function of problem above, which is $R_1 - R_2$, is strictly positive by Assumption 1, x_{1t} must be at the highest possible value which is $(1 + r_{t-1})e + i_t$. At this value the objective function can be written as $(R_1 - (1 + r_t))i_t + D_{t-1}$ where D_{t-1} is determined at $t - 1$. Moreover the borrowing constraint at this value of x_{1t} is:

$$(1 + r_t - \theta_1 R_1)i_t < \theta_1 R_1(1 + r_{t-1})e\,.$$

Since the constraint is not binding one can raise i_t by an small amount $\epsilon > 0$ so that the constraint is still satisfied and the value of objective function is increased by $(R_1 - (1 + r_t))\epsilon$. This contradiction shows that the borrowing constraint has to be always binding. The rest of the lemma is straight forward by using the borrowing constraint to eliminate x_{1t}. ■

Proof of Lemma 2. First, I show that these are the only steady state equilibria for the three regions. Suppose that r_z^{ss} is a steady state interest rate for $z \in \{\ell, m, i\}$. Consider an steady state of the liquid region. If $\frac{\theta_2 R_2}{1 - \theta_2 R_2} < (1 + r_\ell^{ss})$, by 5 both of the upper and lower bounds on the next period interest rate will be strictly smaller than r_ℓ^{ss}. It cannot be that $(1 + r_\ell^{ss}) < \frac{\theta_2 R_2}{1 - \theta_2 R_2}$ either. In that case using 5, the upper bound for the next period interest

rate $\theta_2 R_2(2+r_\ell^{ss})$, will be strictly bigger than $1+r_\ell^{ss}$ but strictly less than $\frac{\theta_2 R_2}{1-\theta_2 R_2}$ and so strictly less than $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (since $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$). Hence by 4 the next period interest rate will be the upper bound itself which is a contradiction given that it is strictly bigger than $1+r_\ell^{ss}$. Hence one must have $(1+r_\ell^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. In a similar fashion I can show that if there exists an steady state for F_i, it must be $(1+r_i^{ss}) = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Finally, suppose that $(1+r_m^{ss}) < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$ in the mixed region, then using 4 and the fact that the economy is at the steady state, one must have $(1+r_m^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$ which gives $\frac{\theta_2 R_2}{1-\theta_2 R_2} < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$ that is a contradiction given that the economy is in F_m. Similarly one cannot have $(1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < (1+r_m^{ss})$ and so $(1+r_m^{ss}) = (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$.

It only remains to check that these steady states exist. As I showed above the trajectories of the interest rates are consistent with the equilibrium conditions 4 and 5 given an initial interest rate $1+r_{-1}$ equal to the steady state. The values of i_t are exogenously given and equal to e and the values of x_{1t} and x_{2t} can be derived from 6. The only condition that remains is that $1+r_t < R_1$ for all $t \geq 0$. To see this note that under Assumption 1:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2)\,.$$

Hence, the remaining condition is satisfied for the liquid and mixed regions. The condition is also satisfied in the illiquid region since I assumed that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$ in Definition 3. Local stability of the steady states in F_ℓ and F_i, follows from the fact that $\theta_1 R_1 < \theta_2 R_2 < 1$ by Assumption 1. If $1+r_z^{ss} \neq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$, suppose without loss of generality that $1+r_z^{ss} - \epsilon < 1+r_t < 1+r_z^{ss}$. For small enough $\epsilon > 0$, the whole interval $[1+r_z^{ss} - \epsilon, 1+r_z^{ss}]$ is either strictly below or above $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. In either case, 4 and 5 imply $1+r_t < 1+r_{t+1} = \theta_z R_z(2+r_t) < 1+r_z^{ss}$, and hence by Assumption 1, the interest rates starting from a point in the interval $[1+r_z^{ss} - \epsilon, 1+r_z^{ss}]$, converge to the steady state value. For the case where $1+r_z^{ss} = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$ or the mixed region, F_m, where the steady state interest rate is $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, suppose without loss of generality that $1+r_{t-1} < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (the proof for the case $1+r_{t-1} > 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar). If $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \in [\theta_1 R_1(2+r_{t-1}), \theta_2 R_2(2+r_{t-1})]$, then 4 gives $1+r_t = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Otherwise suppose that $\theta_2 R_2(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Then 5 implies $1+r_{t-1} < 1+r_t = \theta_2 R_2(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, $1+r_{t+k}, k = 1, 2, 3, ...$ converges to $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. The proof is very similar when $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_1 R_1(2+r_{t-1})$ and so this completes the proof. ■

Proof of Lemma 3. Claims about the steady state interest rate are established by 7 and 8 in the text. Let $s_j(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{x_j^{ss}}{x_1^{ss}+x_2^{ss}}$ be share of type $j \in \{1, 2\}$ in total investment at the steady state. By 6:

$$s_1(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\theta_2 R_2 - (1-\theta_2 R_2)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))},$$

$$s_2(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{(1-\theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}.$$

Now one can rewrite $s_1(\boldsymbol{\theta}, \boldsymbol{R})$ as:

$$s_1(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\frac{1}{2+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})} - (1-\theta_2 R_2)}{\theta_2 R_2 - \theta_1 R_1}.$$

Numerator of the above is strictly increasing in θ_1 by 7 and the denominator is strictly decreasing in θ_1. This implies that $s_1(\boldsymbol{\theta}, \boldsymbol{R})$ is strictly increasing in θ_1 when $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and hence monotone in θ_1 in all three regions. For $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ one has:

$$\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(\theta_2 R_2 - \theta_1 R_1)\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} - \big((1-\theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1\big) R_2 (2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{\Big((\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\Big)^2}.$$

Arranging terms in the numerator, the above can be written as:

$$\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{\big(a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)\big) R_2}{\Big((\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))((1-\theta_1)R_1 - (1-\theta_2)R_2)\Big)^2}.$$

where $a(\theta_1)$, $b(\theta_1)$ and $c(\theta_1)$ are:

$$\begin{aligned}
a(\theta_1) =& - R_1 R_2^2 (1 + R_1)(1 - (1 + R_1)\theta_1), \\
b(\theta_1) =& R_1 R_2^2 (1 + R_1) + R_1 R_2 \big((R_1 - R_2)(1 + 2R_1) - R_1(R_1 R_2 - 1)\big)\theta_1 \\
& - R_1^2 R_2 (1 + R_1)(2 + R_2)\theta_1^2, \\
c(\theta_1) =& R_1 R_2 (R_1 - R_2) - R_1 R_2 (R_1(2 + R_1) - R_2)\theta_1 + R_1^2 (1 + R_2)(R_1 R_2 - R_1 + R_2)\theta_1^2 \\
& + R_1^3 (1 + R_1)\theta_1^3.
\end{aligned}$$

To show that $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum, it is enough to show that given any θ_1, $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root as a quadratic polynomial of θ_2 inside

F. By Proposition 2, $\theta_1 \leq \frac{1}{1+R_1}$ and therefore $a(\theta_1) \leq 0$. In the next step, I show that $c(\theta_1) > 0$ for all θ_1 by proving that $\widetilde{c}(\theta_1) = R_1^{-}1\big(c(\theta_1) - R_1^3(1+R_1)\theta_1^3\big) > 0$ inside F. If $\widetilde{c}(\theta_1)$ has no roots then $\widetilde{c}(\theta_1) > 0$ since $\widetilde{c}(0) > 0$. Therefore suppose θ_1^* is the smallest root of $\widetilde{c}(\theta_1) = 0$:

$$\begin{aligned}\theta_1^* =& \frac{R_2(R_1(2+R_1) - R_2) - \sqrt{\Delta}}{2R_2(R_1 - R_2)}\,,\\ \Delta =& (R_2(R_1(2+R_1) - R_2))^2 - 4R_1R_2(R_1 - R_2)(1+R_2)(R_1R_2 - R_1 + R_2)\,.\end{aligned}$$

Now one has:

$$\begin{aligned}&\theta_1^* = \frac{R_2(R_1(2+R_1) - R_2) - \sqrt{\Delta}}{2R_2(R_1 - R_2)} > \frac{1}{1+R_1} \;\Leftrightarrow\\ &(1+R_1)^2\Big(R_2(R_1(2+R_1) - R_2))^2 - 4R_1R_2(R_1 - R_2)(1+R_2)(R_1R_2 - R_1 + R_2)\Big) <\\ &\Big(R_2(1+R_1)(R_1(2+R_1) - R_2)) - 2R_2(R_1 - R_2)\Big)^2 \;\Leftrightarrow\\ &(1+R_1)\Big(R_1(1+R_1)(1+R_2)(R_1(R_2-1) + R_2) - R_2(R_1(2+R_1) - R_2)\Big) > -R_2(R_1 - R_2)\,.\end{aligned}$$

The last inequality holds since:

$$\begin{aligned}&R_1(1+R_1)(1+R_2)(R_1(R_2-1) + R_2) - R_2(R_1(2+R_1) - R_2) >\\ &R_1R_2(1+R_1)(1+R_2) - R_1R_2(2+R_1) > R_1^2R_2 > 0 > -R_2(R_1 - R_2)\,.\end{aligned}$$

Hence $\theta_1^* > \frac{1}{1+R_1}$ and since $\theta_1 \leq \frac{1}{1+R_1}$ in F one must have $c(\theta_1) > 0$ in F. Now since $a(\theta_1) \leq 0$ and $c(\theta_1) > 0$ in F, at least one root of $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ for any given θ_1 has to be non positive. Therefore $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root in F for any θ_1 and consequently $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum. Note that when $\boldsymbol{\theta}$ is on the boundary of F_m and F_i, $s_2(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and hence $\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} > 0$ given any θ_1. Now suppose $\widetilde{\theta}_1$ is the value for which the vertical line $\theta_1 = \widetilde{\theta}_1$ is tangent to the boundary of F_ℓ. Observe that when θ_2 increases along $\theta_1 = \widetilde{\theta}_1$ line, $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ reaches the maximum of one at the point of tangency. Therefore beyond the point of tangency $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ must be strictly decreasing in θ_2. This implies that for the particular value of $\theta_1 = \widetilde{\theta}_1$, there is a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$. Hence by continuity, there must be a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ over the range of θ_2 given any θ_1 in a neighborhood of $\widetilde{\theta}_1$ which completes the proof. ■

Proof of Proposition 1. First I show that for all $t \geq 0$, $1+r_t < R_1$. Suppose $1+r_{t-1} < R_1$ for some $t \geq 0$. Consider the window defined by 5, where $1 + r_t \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$. If $\theta_2 R_2(2+r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, 4 implies $1+r_t = \theta_2 R_2(2+r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. The last inequality holds by Assumption 1. If $\theta_1 R_1(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2+r_{t-1})$, by 4 I get $1+r_t = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. Finally, consider the case $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_1 R_1(2+r_{t-1})$. 4 implies that $1 + r_t = \theta_1 R_1(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_1 R_1}{1-\theta_1 R_1}) < R_1$. The first inequality holds because the value of $\theta_1 R_1(2 + r_{t-1})$ is always between $1 + r_{t-1}$ and $\frac{\theta_1 R_1}{1-\theta_1 R_1}$. The second inequality is obtained by the assumption that $1+r_{t-1} < R_1$ and definition of F. By induction, $1 + r_{-1} < R_1$ implies $1 + r_t < R_1$ for all $t \geq 0$. This proves the necessary condition for the interest rates in the competitive equilibrium.

In the second step, I prove the existence and uniqueness. I show that given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and the initial condition $1+r_{-1}$, a unique path of interest rates are defined by 4 and 5. Note that given the path of interest rates I can simply solve for (x_{1t}, x_{2t}, i_t) for all $t \geq 0$ using 3 and 6 in each period. Suppose I have determined the unique interest rate $1 + r_{t-1}$ for $t - 1$. Consider the window, defined by 5, where $1 + r_t \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$. If $\theta_2 R_2(2 + r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq \theta_1 R_1(2+r_{t-1})$, using 4 gives $1+r_t = \theta_2 R_2(2+r_{t-1})$ and $1+r_t = \theta_1 R_1(2 + r_{t-1})$ respectively. Finally, suppose $\theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2 + r_{t-1})$. Then, if $1 + r_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1 + r_t = \theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Similarly if $1 + r_t < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1 + r_t = \theta_2 R_2(2 + r_{t-1}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. The two contradictions show that one must have $1+r_t = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, I showed that given $1+r_{t-1}$, there is a uniquely determined interest rate at time t, that is, $1+r_t$. Therefore, by induction, I have shown that given an initial condition $1 + r_{-1}$, there is a unique path of interest rates for all $t \geq 0$.

In the third and final step, I show that the unique equilibrium path of the interest rates defined in step two, converges to the unique steady state characterized in Lemma 2, for given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and an initial condition $1 + r_{-1}$. Consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ first. Note that if $1 + r_{t-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, using 4 and 5 implies $1 + r_t = \theta_2 R_2(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, if $1 + r_{-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the path of interest rates is defined as $1 + r_t = \theta_2 R_2(2 + r_{t-1})$ for all $t \geq 0$. This path is clearly convergent to $1+r_\ell^{ss} = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Now suppose $1+r_{-1} > 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ which implies $1+r_{-1} > \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Define the series $\{1 + \bar{r}_t\}_{t=-1}^{\infty}$ as $1 + \bar{r}_t = \theta_2 R_2(2 + \bar{r}_{t-1})$ for all $t \geq 0$ and $1 + \bar{r}_{-1} = 1 + r_{-1}$. If $1 + r_{t-1} \leq 1 + \bar{r}_{t-1}$, 5 implies $1 + r_t \leq \theta_2 R_2(2 + r_{t-1}) \leq \theta_2 R_2(2 + \bar{r}_{t-1}) = 1 + \bar{r}_t$. Hence by induction one must have $1 + r_t \leq 1 + \bar{r}_t$ for all $t \geq 0$. Since by Assumption 1, $\theta_2 R_2 < 1$, this immediately implies that, there is a finite t_0 for which $1+r_{t_0} \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, this

case is similar to the previous part of the proof and so convergence is established.

Now consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ where Definition 3 implies that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Without loss of generality suppose $1 + r_{-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Define the series $\{1 + \underline{r}_t\}_{t=-1}^{\infty}$ as $1 + \underline{r}_t = \theta_1 R_1(2 + \underline{r}_{t-1})$ for all $t \geq 0$ and $1 + \underline{r}_{-1} = 1 + r_{-1}$. It is easy to see that there is a finite and unique $t_0 \geq 0$ such that $1 + \underline{r}_{t_0} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1 + \underline{r}_{t_0-1}$. Now one notes that if $1 + r_{t-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ for some $t \geq 0$, one must have $\theta_2 R_2(2 + r_{t-1}) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and therefore 4 and 5 give $1 + r_t = \max(\theta_1 R_1(2 + r_{t-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$. Using this observation and by induction, for $-1 \leq t \leq t_0 - 1$ one must have $1 + r_t = 1 + \underline{r}_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and so $\theta_2 R_2(2 + r_t) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Using 4 and the definition of t_0, this implies that $1 + r_{t_0} = \max(\theta_1 R_1(2 + r_{t_0-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, the path of interest rates converges to the steady state interest rate, $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, in finite periods. The proof for the case $1 + r_{-1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar. Finally, if $1 + r_{-1} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the economy is already in the steady state and all future interest rates will be the same.

The proof for the illiquid region is very similar to the case of liquid region and so I do not provide it here. ■

Proof of Proposition 2. First, I compute the boundaries of the illiquid and liquid regions as functions of θ_1. For the illiquid region the defining boundary is characterized by:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_1 R_1}{1 - \theta_1 R_1}\right).$$

Using 1 and solving the above as a function of θ_1 I get:

$$\theta_2^i(\theta_1) = \left(\frac{\theta_1 R_1(1 - \theta_1(1 + R_2))}{R_2(1 - \theta_1(1 + R_1))}\right).$$

This function is strictly increasing and convex in θ_1 since $\theta_1 R_1$ is increasing and:

$$\frac{d}{d\theta_1}\left(\frac{(1 - \theta_1(1 + R_2))}{(1 - \theta_1(1 + R_1))}\right) = \frac{R_1 - R_2}{(1 - \theta_1(1 + R_1))^2} > 0\,.$$

Also observe that $\theta_2^i(0) = 0$ and so no matter how close to the origin, there are illiquid equilibria in any neighborhood of the $\boldsymbol{\theta} = \mathbf{0}$. Now the characterizing equation for the liquid

region is:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_2 R_2}{1 - \theta_2 R_2} \right) .$$

Collecting terms involving θ_1 or θ_2 on different sides I obtain two distinct curves:

$$\overline{\theta}_2^\ell(\theta_1) = \left(\frac{(\theta_1 R_1(1+R_2) + R_2) + \sqrt{(\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)} \right) ,$$

$$\underline{\theta}_2^\ell(\theta_1) = \left(\frac{(\theta_1 R_1(1+R_2) + R_2) - \sqrt{(\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)} \right) .$$

Note that obviously $\underline{\theta}_2^\ell(\theta_1) \leq \overline{\theta}_2^\ell(\theta_1)$ and $\underline{\theta}_2^\ell(0) = 0$ and so the lower boundary characterizing the liquid region passes through the origin. This means that there are liquid steady state equilibria at any neighborhood of the origin.

Now let $\Delta(\theta_1) \equiv (\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)$. Then the two curves $\underline{\theta}_2^\ell(\theta_1)$ and $\overline{\theta}_2^\ell(\theta_1)$ touch each other when $\Delta(\theta_1) = 0$. This equation has two roots:

$$\overline{\theta}_1 = \left(\frac{R_2 \left((2(1+R_1) - (1+R_2)) + \sqrt{4(1+R_1)(R_1 - R_2)} \right)}{R_1(1+R_2)^2} \right) ,$$

$$\underline{\theta}_1 = \left(\frac{R_2 \left((2(1+R_1) - (1+R_2)) - \sqrt{4(1+R_1)(R_1 - R_2)} \right)}{R_1(1+R_2)^2} \right) .$$

The smaller root is less than $\frac{1}{1+R_1}$ since:

$$\begin{aligned} \underline{\theta}_1 < \frac{1}{1+R_1} &\Leftrightarrow (1+R_1)R_2 \left((2(1+R_1) - (1+R_2)) - \sqrt{4(1+R_1)(R_1 - R_2)} \right) \\ &< R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1) - (1+R_2)) - R_1(1+R_2)^2 \\ &< (1+R_1)R_2\sqrt{4(1+R_1)(R_1 - R_2)} \Leftrightarrow (R_1 - R_2)(2R_1R_2 + R_2 - 1) < \\ &\qquad (1+R_1)R_2\sqrt{4(1+R_1)(R_1 - R_2)} . \end{aligned}$$

If I square both sides and cancel $R_1 - R_2$ and collect the terms, I get:

$$\Leftrightarrow R_1 < 4R_1^2R_2^3 + 8R_1^2R_2^2 + 4R_1R_2^3 + 7R_1R_2^2 + R_2^3 + 4R_1^2R_2 + 2R_2^2 + 2R_1R_2 + R_2 .$$

This is obviously the case given Assumption 1. The bigger root is greater than $\frac{1}{1+R_1}$ since:

$$\begin{aligned}\overline{\theta}_1 > \frac{1}{1+R_1} &\Leftrightarrow (1+R_1)R_2\left((2(1+R_1)-(1+R_2))+\sqrt{4(1+R_1)(R_1-R_2)}\right)\\ &> R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1)-(1+R_2))-R_1(1+R_2)^2\\ &> -(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)} \Leftrightarrow (R_1-R_2)(2R_1R_2+R_2-1) >\\ &\qquad -(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)}\,.\end{aligned}$$

The last inequality is obvious given that one term is positive and the other is negative. Therefore the point at which the two curves $\overline{\theta}_2^{\ell}(\theta_1)$ and $\underline{\theta}_2^{\ell}(\theta_1)$ touch each other inside F is $\underline{\theta}_1$ and also the fact that $\underline{\theta}_1 < \frac{1}{1+R_1}$ proves that for high θ_1 there is no liquid steady state.

Next, I prove that $\overline{\theta}_2^{\ell}(\theta_1)$ is strictly decreasing and $\underline{\theta}_2^{\ell}(\theta_1)$ is strictly increasing.The derivatives are:

$$\begin{aligned}\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2)+(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2)-2R_1R_2(1+R_1))\,\Delta(\theta_1)^{-\frac{1}{2}}\right)\,,\\ \frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2)-(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2)-2R_1R_2(1+R_1))\,\Delta(\theta_1)^{-\frac{1}{2}}\right)\,.\end{aligned}$$

C_0 is just a constant. It is easy to see that the term in parenthesis just before $\Delta(\theta_1)^{-\frac{1}{2}}$ is always negative for $\theta_1 \leq \underline{\theta}_1$. Hence, $\frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ should be strictly positive. Now for the other case:

$$\begin{aligned}&\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} < 0\\ \Leftrightarrow\ & R_1^2(1+R_2)^2\Delta(\theta_1) < \left(R_1^2(1+R_2)^2\theta_1 - R_1R_2(2(1+R_1)-(1+R_2))\right)^2\\ \Leftrightarrow\ & (1+R_2)^2\Delta(\theta_1) < \left(R_1(1+R_2)^2\theta_1 - R_2(2(1+R_1)-(1+R_2))\right)^2\\ \Leftrightarrow\ & (1+R_2) < 2(1+R_1)-(1+R_2)\,.\end{aligned}$$

The last statement is correct given Assumption 1. In the last step I used the definition of $\Delta(\theta_1)$ to cancel out all terms. What I proved show that for any $\boldsymbol{\theta} \in F_\ell$ one must have $\boldsymbol{\theta} \leq (\frac{1}{1+R_1}, \frac{1}{1+R_1})$. This is because I showed that $\underline{\theta}_1 < \frac{1}{1+R_1}$ and that $\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ is strictly decreasing while $\overline{\theta}_2^{\ell}(\theta_1)$ stays above $\underline{\theta}_2^{\ell}(\theta_1)$ and intersects with $\theta_1 = 0$ at $\frac{1}{1+R_1}$.

In the next step I want to prove that the liquid region lies above the illiquid region. First,

I observe the following:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(1-\theta_1)(R_1 - R_2)R_1R_2}{((1-\theta_1)R_1 - (1-\theta_2)R_2)^2} > 0\,.$$

Now suppose that $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \geq \frac{\theta_2 R_2}{1-\theta_2 R_2}$ and $r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \leq \frac{\theta_1 R_1}{1-\theta_1 R_1}$ where $(\theta_1, \theta_2, \boldsymbol{R})$ and $(\theta_1, \theta_2', \boldsymbol{R})$ are in F. Then if $\theta_2 \leq \theta_2'$, by the derivation above $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \leq r_\Lambda(\theta_1, \theta_2', \boldsymbol{R})$ and hence:

$$\frac{\theta_2 R_2}{1-\theta_2 R_2} \leq r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \leq r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \leq \frac{\theta_1 R_1}{1-\theta_1 R_1}\,.$$

This is not possible since it implies that $\theta_2 R_2 \leq \theta_1 R_1$ and hence $(\theta_1, \theta_2, \boldsymbol{R})$ cannot be in F. In the last step of the proof, I show that $(\frac{1}{R_2}, \frac{1}{1+R_1}) \in F_i$. First, note that:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} = \frac{(1-\theta_2)(R_2 - R_1)R_1R_2}{((1-\theta_1)R_1 - (1-\theta_2)R_2)^2} < 0\,.$$

Second, observe that $\theta_2^i(\theta_1)$ is strictly increasing, passes through the origin and also converges to infinity when θ_1 gets close to $\frac{1}{1+R_1}$. This means that $\theta_2^i(\theta_1)$ cuts the horizontal border of F that is $\theta_2 = \frac{1}{R_2}$ at an interior point, say, $(\bar{\theta}_1, \frac{1}{R_2})$ where $\bar{\theta}_1 < \frac{1}{1+R_1}$. At this point $\theta_2^i(\bar{\theta}_1) = \frac{\bar{\theta}_1 R_1}{1-\bar{\theta}_1 R_1}$. But since I proved above that $\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} < 0$, for any $\theta_1 \in (\bar{\theta}_1, \frac{1}{1+R_1})$ I obtain:

$$\theta_2^i(\theta_1) < \theta_2^i(\bar{\theta}_1) = \frac{\bar{\theta}_1 R_1}{1-\bar{\theta}_1 R_1} < \frac{\theta_1 R_1}{1-\theta_1 R_1}\,.$$

This means that $\theta_1 \in F_i$ for $\theta_1 \in (\bar{\theta}_1, \frac{1}{1+R_1})$.

■

Proof of Proposition 3. When $R_1 > R_2 > 1$ and $1 > \theta_1 R_1 > \theta_2 R_2$, entrepreneurs only invest in type one since type two is dominated both in terms of liquidity and return. Hence, this economy collapses to the economy in Farhi and Tirole (2010a) with only one investment type, (θ_1, R_1), and no bubbles or outside liquidity. Farhi and Tirole (2010a) show in their Proposition 5 that under the assumption that $R_1 > 1$, all competitive equilibria are Pareto efficient and hence constrained Pareto efficient as well. ■

Proof of Proposition 4. By Proposition 2, the competitive equilibrium converges to a unique steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$. This implies that there exist $T \geq 0$ and $\varepsilon > 0$ such that $x_{2t} \geq \varepsilon$ for $t \geq T$. Suppose one reduces x_{2t} for $t \geq T+1$ by $\delta + \epsilon$

and increases x_{1t} for $t \geq T+1$ by ϵ and for time T, reduces x_{2T} and increases x_{1T} both by $\frac{1}{\theta_2 R_2 - \theta_1 R_1}\delta$. Moreover, $\delta > 0$, $\epsilon > 0$ are such that $\epsilon + \delta < \varepsilon$ and:

$$\begin{aligned}\delta &= (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta\,,\\ \epsilon &= \frac{1-\theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\,\delta\,.\end{aligned}$$

Similar to what is shown in the text, this reallocation reduces the debt payments of each generation from T onward by δ and leaves all middle aged at or after T strictly better off when $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ the reallocation does not affect the utility of middle aged after T but increases utility of middle aged at T. This proves that the competitive equilibrium is constrained Pareto inefficient whenever $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$.

If $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ then by definition $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ implies a strictly negative interest rate at the steady state. If $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, by Lemma 5, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ implies:

$$\frac{\theta_2 R_2}{1-\theta_2 R_2} < \frac{\theta_1 R_1}{1-\theta_1 R_1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1\,.$$

Hence by Lemma 2, the steady state interest rate is strictly negative which completes the proof. ■

Proof of Lemma 4. The straight line corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ is $\theta_2^\Lambda(\theta_1) = \left(\frac{R_1(R_2-1)}{R_2(R_1-1)}\right)\theta_1 + \left(\frac{R_1-R_2}{R_2(R_1-1)}\right)$. This line intersects horizontal line $\theta_1 = 0$ at $\theta_2^\Lambda(0) = \frac{R_1-R_2}{R_2(R_1-1)}$ which implies $\theta_2^\Lambda(0) > 0$. Hence, Proposition 2 and Proposition 4 imply that an strictly positive neighborhood of the origin, i.e., $\boldsymbol{\theta} = 0$, corresponds to inefficiently liquid equilibria. Since by Proposition 2, any neighborhood of the origin contains liquid equilibria, by Proposition 4, it follows that there are inefficiently liquid equilibria in any small enough neighborhood of the origin. Note that by Proposition 2, the boundary of F_ℓ cuts the vertical axis $\theta_1 = 0$ at the origin and $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$ and also the upper part of the boundary is negatively sloped in (θ_1, θ_2) plane. Therefore, it follows that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ line passes through F_ℓ if and only if its intersection with $\theta_1 = 0$, that is $(0, \theta_2^\Lambda(0)$, lies below or at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$. This is the case whenever $\frac{R_1-R_2}{R_2-1} \leq 1$.

For the last part, let S_i denote the unique intersection of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ with the boundary of F_i. Observe that by Proposition 2, the inefficiently liquid region lie above the convex inner boundary of F_i and below $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$. This completes the proof. ■

Proof of Proposition 5. For part of the proof, I use some of the results in Ghate and Smith (2005), specially their Theorem 2.6. This theorem shows that complementary slackness conditions are sufficient for optimality in a linear programming with infinite variables and infinite number of constraints, when feasible points, constraints and objective functions of both primal and dual problems are elements of appropriate spaces. A necessary condition for this result is that the feasible points of the primal problem, i.e. feasible allocations $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$, lie in ℓ_∞. To see this note that by 9 and Assumption 1:

$$x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \leq \theta_1 R_1 (x_{1t-1} + x_{2t-1}) + e$$

This together with 9 give:

$$\begin{cases} x_{1t} + x_{2t} \leq i_{-1} + \frac{e}{1-\theta_1 R_1}\,, \\ \\ c_t \leq R_1 (i_{-1} + \frac{e}{1-\theta_1 R_1}) + e\,. \end{cases}$$

i_{-1} is total investment at $t = -1$ which is an initial condition to the problem. The above proves that $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty} \in \ell_\infty$ for any feasible allocation.

Now let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and consider an allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$. If there exists a series of strictly positive weights $\{\lambda_t\}_{t=0}^{\infty} \in \ell_1$ such that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ solves:

$$\begin{aligned} \max_{\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \lambda_t c_t \\ s.t. \quad & c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e \\ & x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \\ & c_t \geq 0\,,\ x_{1t} \geq 0\,,\ x_{2t} \geq 0 \end{aligned}$$

then $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient. Let $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ be the Lagrange multipliers for resource constraint, borrowing constraint and non negativity constraints on x_{1t}, x_{2t} and c_t respectively. As discussed above any feasible allocation is bounded. Hence

the sufficient conditions for $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ to be a maximum are:

$$\begin{cases} \lambda_t - \eta_t + \delta_{ct} = 0\,, \\ (R_1\eta_{t+1} - \eta_t) + (\theta_1 R_1 \gamma_{t+1} - \gamma_t) + \delta_{1t} = 0\,, \\ (R_2\eta_{t+1} - \eta_t) + (\theta_2 R_2 \gamma_{t+1} - \gamma_t) + \delta_{2t} = 0\,, \\ \eta_t \geq 0\,,\ \gamma_t \geq 0\,,\ \delta_{1t} \geq 0\,,\ \delta_{2t} \geq 0\,,\ \delta_{ct} \geq 0\,, \\ \delta_{1t}x_{1t} = 0\,,\ \delta_{2t}x_{2t} = 0\,,\ \delta_{ct}c_t = 0\,. \end{cases} \tag{SC}$$

for $t \geq 0$, provided that $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty} \in \ell_1$. First consider the case where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. In this case, if I set $\{\delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ to zero, solving the first three series of equations in SC, I obtain the following for $t \geq 0$:

$$\eta_t = \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1}\,,$$

$$\lambda_{t+2} = \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\lambda_{t+1}\,,$$

$$\lambda_1 = \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2 - \theta_1)R_1 R_2}\,(\lambda_0 + \gamma_0)\,.$$

The coefficient in the second difference equation above is $(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1}$. Therefore for any positive λ_0 and γ_0, λ_1 is given by the above and:

$$\lambda_t = \big(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\big)^{-(t-1)}\,\lambda_1\,.$$

Since $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, the resulting $\{\lambda_t\}_{t=0}^{\infty}$ and consequently all $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1. Therefore all the conditions above which are sufficient for optimality are satisfied and $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient.

Now, let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. If one sets $\{\delta_{1t}, \delta_{ct}\}_{t=0}^{\infty}$ to zero, the first three set of sufficient conditions in SC give the following for $t \geq 0$:

$$\eta_t = \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1} - \frac{1}{\theta_2 R_2 - \theta_1 R_1}\,\delta_{2t}\,,$$

$$\lambda_{t+2} = \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2-\theta_1)R_1R_2}\,\lambda_{t+1} + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\delta_{2t+1} - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\delta_{2t}\,,$$

$$\lambda_1 = \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2-\theta_1)R_1R_2}\,(\lambda_0 + \gamma_0 - \delta_{20})\,.$$

Given any positive λ_0 and γ_0, suppose one sets $\delta_{2t} = 0$ for $0 \le t \le T-1$. This implies $\lambda_t = \rho^{t-1}\lambda_1$ for $1 \le t \le T$, where $\rho = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1} > 1$. Moreover, let $\delta_{2T} = \alpha'\lambda_T$ and $\delta_{2t} = \alpha\lambda_t$ for $t \ge T+1$ where α and α' are positive constants to be determined. For $t \ge T+2$, the above equations lead to the following difference equation:

$$\lambda_{t+1} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha\big)\,\lambda_t - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha\lambda_{t-1}\,.$$

This difference equation has a solution of the form $\lambda_{t+1} = m\lambda_t$ where m is the smallest root of the characteristic equation:

$$m = \frac{1}{2}\Big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2} - \sqrt{\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\big)^2 - \frac{4\alpha}{(\theta_2-\theta_1)R_1R_2}}\,\Big)\,.$$

It is easy to see that:

$$m < 1 \quad \Leftrightarrow \quad \alpha < \frac{(\theta_2-\theta_1)R_1R_2(\rho-1)}{1-\theta_2 R_2}\,.$$

Hence if α is small enough and given the appropriate initial condition, i.e. $\lambda_{T+2} = m\lambda_{T+1}$, one can generate $\{\lambda_t\}_{t=0}^{\infty} \in \ell_1$. For time $T+1$ and $T+2$ the difference equation becomes:

$$\lambda_{T+1} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\big)\,\lambda_T\,,$$

$$\lambda_{T+2} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha\big)\,\lambda_{T+1} - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\lambda_T\,.$$

and therefore $\lambda_{T+2} = m\lambda_{T+1}$ if and only if:

$$\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha - m\big)\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\big) = \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\,.$$

The above equation is linear in α'. Note that one always has $\theta_2 R_2 \rho < 1$ and hence for small

enough α there is a strictly positive solution for α'. Therefore a small enough $\alpha > 0$ defines unique values of $0 < m < 1$ and $\alpha' > 0$ such that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ are in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient.

Finally, let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. Setting $\{\delta_{1t}, \delta_{ct}\}_{t=0}^{\infty}$ and $\{\delta_{2t}\}_{t=0}^{T-1}$ to zero implies $\lambda_t = \lambda_1$ for $1 \leq t \leq T$. Using SC for $k \geq 1$ one can obtain:

$$\lambda_{T+k} = \lambda_T - \zeta\Big(\sum_{j=0}^{k-2} \delta_{2T+j}\Big) + \nu\delta_{2T+k-1}\,.$$

where $\zeta = \frac{1-\theta_2 R_2}{(\theta_2-\theta_1)R_1 R_2}$ and $\nu = \frac{1}{(\theta_2-\theta_1)R_1 R_2}$. To satisfy the above condition, for any $j \geq 0$ define:

$$\delta_{2T+j} = \Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j+1}\,,$$

$$\lambda_{T+j} = (\lambda_T + \zeta + \nu)\Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j}\,.$$

It is easy to see that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient and completes the proof. ■

Proof of Proposition 6. Using the values of steady state interest rates in Lemma 2, market clearings and the objective function in II, deriving V_ℓ^{ss}, V_i^{ss} and V_m^{ss} is straight forward. For the regulated economy remind that by IV the objective function when the social planner sets $\alpha_\ell = \alpha$ will be:

$$\begin{aligned} V_\alpha^{ss} &= (((1-\alpha)R_1 + \alpha R_2)(2 + r_\alpha^{ss}) - (1 + r_\alpha^{ss}))e \\ &= (R_\alpha(1 + \frac{\gamma_\alpha}{1-\gamma_\alpha}) - \frac{\gamma_\alpha}{1-\gamma_\alpha})e \\ &= \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right)e\,. \end{aligned}$$

The first part above uses 13. For last part of the proposition observe that 6 gives:

$$\widetilde{\alpha}_\ell = \frac{(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}\,.$$

The interest rate is:

$$1 + r^{ss}_{\widetilde{\alpha}_\ell} = \frac{\gamma_{\widetilde{\alpha}_\ell}}{1-\gamma_{\widetilde{\alpha}_\ell}} =$$
$$\frac{((1+r_\Lambda) - \theta_1 R_1(2+r_\Lambda))\theta_2 R_2 + (\theta_2 R_2(2+r_\Lambda) - (1+r_\Lambda))\theta_1 R_1}{((1+r_\Lambda) - \theta_1 R_1(2+r_\Lambda))(1-\theta_2 R_2) + (\theta_2 R_2(2+r_\Lambda) - (1+r_\Lambda))(1-\theta_1 R_1)} =$$
$$\frac{(\theta_2 R_2 - \theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,.$$

Therefore the utility levels at the steady state should be the same. Note that $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \leq R_{\widetilde{\alpha}_\ell}$ by Assumption 1 and so the proof is complete. ■

Proof of Lemma 5. First, consider $V^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R}) - V^{ss}_i(\boldsymbol{\theta}, \boldsymbol{R})$. Note that:

$$V^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R}) > V^{ss}_i(\boldsymbol{\theta}, \boldsymbol{R}) \Leftrightarrow \frac{(1-\theta_2)R_2}{1-\theta_2 R_2} > \frac{(1-\theta_1)R_1}{1-\theta_1 R_1} \Leftrightarrow$$
$$\frac{R_2 - 1}{1-\theta_2 R_2} > \frac{R_1 - 1}{1-\theta_1 R_1} \Leftrightarrow (R_2 - 1)(1-\theta_1 R_1) > (R_1 - 1)(1-\theta_2 R_2) \Leftrightarrow$$
$$(R_2 - \theta_1 R_1 R_2 + \theta_1 R_1) > (R_1 - \theta_2 R_1 R_2 + \theta_2 R_2) \Leftrightarrow$$
$$(\theta_2 - \theta_1) R_1 R_2 > (1-\theta_1) R1 - (1-\theta_2) R_2 \Leftrightarrow$$
$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{(1-\theta_1) R1 - (1-\theta_2) R_2}\right) > 1\,.$$

Now define the following terms:

$$\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{(\theta_2 - \theta_1) R_1 R_2 - ((1-\theta_1) R1 - (1-\theta_2) R_2)}{(\theta_2 R_2 - \theta_1 R_1)}\right) e\,,$$

$$\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2 R_2((1-\theta_1) R1 - (1-\theta_2) R_2) - (1-\theta_2 R_2)(\theta_2 - \theta_1) R_1 R_2}{(1-\theta_2 R_2)((1-\theta_1) R1 - (1-\theta_2) R_2)}\,.$$

Notice that the denominators of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ are strictly positive. Moreover it is easily seen that the numerator of $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 1$ and the numerator of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2} = 1 + r^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ or equivalently $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the welfare gains per unit of reduction in x_1 of investing the freed resources in x_2 and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the maximum amount of reduction in x_1

that can possibly occur (see Section 4.1). By Proposition 6:

$$V_m^{ss} = \left(\frac{(\theta_2 - \theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)} \right) e \,.$$

Now I want to compute and simplify $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{DEN}{NUM}$. The common denominator and the numerator are:

$$\begin{aligned}
DEN =& (1-\theta_2 R_2)((1-\theta_1)R1 - (1-\theta_2)R_2)(\theta_2 R_2 - \theta_1 R_1)\,, \\
NUM =& (1-\theta_2 R_2)(\theta_2 - \theta_1)R_1 R_2 \Big((\theta_2 - \theta_1)R_1 R_2 \\
& - \Big((\theta_2 - \theta_1)R_1 R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2) \Big) \Big) \\
& + \theta_2 R_2 \left((1-\theta_1)R1 - (1-\theta_2)R_2 \right) \Big((\theta_2 - \theta_1)R_1 R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2) \Big)\,, \\
=& ((1-\theta_1)R1 - (1-\theta_2)R_2)) \Big((\theta_2 - \theta_1)R_1 R_2 + \theta_2 R_2^2 - \theta_2 R_1 R_2 - \theta_2 R_2 (\theta_2 R_2 - \theta_1 R_1) \Big)\,, \\
=& ((1-\theta_1)R1 - (1-\theta_2)R_2))(1-\theta_2)R_2(\theta_2 R_2 - \theta_1 R_1)\,.
\end{aligned}$$

Therefore:

$$\begin{aligned}
V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) &= \frac{\theta_2 R_2}{1-\theta_2 R_2} = V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R})\,, \Rightarrow \\
V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) &= \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})\,.
\end{aligned}$$

By last equation it is obvious that the sign of $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. For the last case define:

$$\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{((1-\theta_1)R1 - (1-\theta_2)R_2) - (\theta_2 - \theta_1)R_1 R_2}{(\theta_2 R_2 - \theta_1 R_1)} \right) e\,,$$

$$\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(1-\theta_1 R_1)(\theta_2 - \theta_1)R_1 R_2 - \theta_1 R_1((1-\theta_1)R1 - (1-\theta_2)R_2)}{(1-\theta_1 R_1)((1-\theta_1)R1 - (1-\theta_2)R_2)}\,.$$

Note that $\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) = -\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$. Similar simplifications lead to:

$$V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = \Omega_i(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R})\,.$$

Hence $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$. ■

Proof of Proposition 7. First, I prove that the proposed regulation can implement Pareto improving reallocations used in Proposition 4 and Lemma **??**. Consider an inefficiently liquid equilibrium corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$. Similar to the proof of Proposition 4, there exists $T \geq 0$ and $\varepsilon > 0$ such that $x_{2t} \geq \varepsilon$ for $t \geq T$. Suppose one reduces x_{2t} for $t \geq T+1$ by $\delta + \epsilon$ and increases x_{1t} for $t \geq T+1$ by ϵ and for time T, reduces x_{2T} and increases x_{1T} both by $\frac{1}{\theta_2 R_2 - \theta_1 R_1}\delta$. Moreover, $\delta > 0$, $\epsilon > 0$ are such that $\epsilon + \delta < \varepsilon$ and:

$$\begin{aligned}\delta &= (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta\,,\\ \epsilon &= \frac{1-\theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\,\delta\,.\end{aligned}$$

This reduces the debt payments of generations on or after T exactly by δ. It has already been shown in the text that the above reallocation is a Pareto improvement. Let $\{\delta_t, \kappa_t, \nu_t\}_{t=0}^{\infty}$ be defined as the decreases or increases in $(1+r_t)e$, x_{1t} and x_{2t} respectively as above. Then one has:

$$\begin{aligned}\delta_t &= \theta_2 R_2 \nu_t - \theta_1 R_1 \kappa_t\,,\\ \delta_{t-1} &= \nu_t - \kappa_t\,.\end{aligned}$$

Let $\{\alpha_{\ell t}\}_{t=0}^{\infty}$ be the fraction of the liquid type in total investment for the original competitive equilibrium. Now define $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$ as follows:

$$\widetilde{\alpha}_{\ell t} = \frac{x_{2t} - \nu_t}{x_{1t} + x_{2t} - \delta_{t-1}}\,.$$

Suppose the planner regulates the portfolios according to $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$. Let $\{\widetilde{x}_{1t}, \widetilde{x}_{2t}, \widetilde{r}_t\}_{t=0}^{\infty}$ be the prices and quantities in the regulated equilibrium. Define $r^*_{-1} = r_{-1}$ and $\{r^*_t\}_{t=1}^{\infty}$ recursively:

$$1 + r^*_t = (\widetilde{\alpha}_{\ell t}\theta_2 R_2 + (1 - \widetilde{\alpha}_{\ell t})\theta_1 R_1)(2 + r^*_{t-1})\,.$$

By IV and market clearing, r^*_t is an upper bound for $\widetilde{r}_t$ for all t. Now suppose $(1 + r^*_{t-1})e = \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} - \delta_{t-1}$ which is true for $t = 0$ by assumption. Then one has:

$$\widetilde{\alpha}_{\ell t}\theta_2 R_2 + (1 - \widetilde{\alpha}_{\ell t})\theta_1 R_1 = \frac{\theta_2 R_2 (x_{2t} - \nu_t) + \theta_1 R_1 (x_{1t} + \kappa_t)}{x_{1t} + x_{2t} - \delta_{t-1}}\,.$$

But by resource constraint of the original competitive equilibrium and using recursive equations above defining $\{\delta_t, \kappa_t, \nu_t\}$:

$$\theta_2 R_2(x_{2t} - \nu_t) + \theta_1 R_1(x_{1t} + \kappa_t) = \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} - \delta_t \,,$$
$$x_{1t} + x_{2t} - \delta_{t-1} = \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} - \delta_{t-1} + e \,.$$

Hence it must be that $(1 + r_t^*)e = \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} - \delta_t$ and so by induction this holds for all $t \geq 0$. Now notice that in the original competitive equilibrium one must have $1 + r_t < R_{\alpha,t} \equiv (1 - \alpha_{\ell t})R_1 + \alpha_{\ell t} R_2$ for all t. Whether entrepreneurs specialize in liquid type or mix at time t, the interest rate has to be no bigger than $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Using Assumption 1, one has $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_2$ and so:

$$1 + r_t \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_2 \leq R_{\alpha\ell,t} \,.$$

Observe that $1 + r_t^* < 1 + r_t$ and $R_{\widetilde{\alpha},t} \geq R_{\alpha\ell,t}$. The latter is true because $\widetilde{\alpha}_{\ell t} \geq \alpha_{\ell t}$ by construction. Hence $1 + r_t^* < R_{\widetilde{\alpha},t}$ for all t which immediately implies that the borrowing constraints are binding in the regulated equilibrium and that $\widetilde{r}_t = r_t^*$ for all t and so the allocation induced by the regulation coincides with the Pareto superior allocation given at the beginning.

In the second part, I show that this type of regulation can make a Praeto improvement that reaches the Pareto frontier given by Proposition 5. Consider an inefficiently liquid equilibrium corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$. Since it converges to the steady state by Proposition 1, for an arbitrarily small $\epsilon > 0$, one can choose T such that the differences between equilibrium values of interest rate, investments in the two types and utility of the old generations and their steady state values be all less than ϵ for $t \geq T$. Similar to the first part, suppose $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$, $\{\widetilde{x}_{1t}, \widetilde{x}_{2t}, \widetilde{r}_t\}_{t=0}^{\infty}$ be the liquid fraction of investment, the prices and quantities in the regulated equilibrium. Define $\widetilde{\alpha}_{\ell t} = \alpha_{\ell t}$ if $t \leq T - 1$ and $\widetilde{\alpha}_{\ell t} = 0$ if $t \geq T$. In words, lets replicate the original competitive equilibrium allocation up to time $T - 1$ and then shut down investment in the liquid type completely on or after T.

Similar to the first part it is easy to show that borrowing constraints are binding in the regulated equilibrium and that it converges to a new steady state with $\widetilde{x}_1^{ss}$, $\widetilde{x}_2^{ss}$ and $\widetilde{r}^{ss}$. The new steady state interest rate $1 + \widetilde{r}^{ss} = \frac{\theta_1 R_1}{1 - \theta_1 R_1}$ is below $\{1 + \widetilde{r}_t\}_{t=T}^{\infty}$ and strictly so at least for $1 + \widetilde{r}_T$ for small enough ϵ. The reason is that the original steady state interest rate is either $1 + r^{ss} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or $1 + r^{ss} = \frac{\theta_2 R_2}{1 - \theta_2 R_2}$ and in either case strictly bigger than

$1+\widetilde{r}^{ss} = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Since ϵ is small, the whole sequence of $\{1+\widetilde{r}_t\}_{t=T}^{\infty}$ has to lie above $1+r^{ss}$ and strictly so at least for $t=T$.

This implies that utilities of middle aged for $t \geq T$ have to be above their new steady state value, i.e., $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$, and strictly above for $t=T$, in the regulated economy because initial wealth of the middle aged are all bigger than the steady state level. By Lemma 5, if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ one has $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_z^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ for $z \in \{m, \ell\}$, where $V_z^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is the steady state utility for the original steady state. Hence if ϵ is small enough, all middle aged at or after T are better off, while middle aged before T are left as well off. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, then after some T, the original equilibrium reaches the steady state level and so it is still true that at least the middle aged at T is strictly better off while all others are at least as well off. ■

Proof of Lemma 6. I only show that given the assumptions on parameters and the initial value, competitive equilibrium is unique, borrowing constraint is always binding and the interest rate evolves according to a law of motion similar to 4. The rest of the proof is very similar to Proposition 1 and Lemma 2 and so is not provided here.

First, note that $1+r_t \leq R_1$ for any t. Suppose $1+r_t > R_1$, then middle aged do not invest in any of the two investment types since their returns are strictly less than the interest rate. The resource constraint dictates that $i_t - b_t^m = -(1+r_{t-1})e < 0$ while using 15, in equilibrium one must have $i_t - b_t^m = e - b_t$ which is strictly positive given Assumption 2.

Now I want to show $1+r_t < R_1$ using induction. This is true for $t=-1$ by assumption. Suppose $1+r_t = R_1$ while $1+r_{t-1} < R_1$. Then the entrepreneurs at t do not invest in the liquid type since they can always raise their consumption by reducing their investment in the liquid type. The resource constraint then gives $x_{1t} = (1+r_{t-1})e + (i_t - b_t^m)$. Substituting this into the borrowing constraint yields:

$$(1+r_t-\theta_1 R_1)(i_t - b_t^m) \leq \theta_1 R_1 (1+r_{t-1})e\,.$$

In equilibrium, by 15 and the fact that $1+r_t = R_1$ and $1+r_{t-1} < R_1$ one has:

$$(1-\theta_1)(e-b_t) \leq \theta_1(1+r_{t-1})e < \theta_1 R_1 e \Rightarrow \sigma_t > 1 - \frac{\theta_1 R_1}{1-\theta_1}\,.$$

which contradicts Assumption 2 and hence it must be that $1+r_t < R_1$. This implies that borrowing constraint has to be binding at any t, otherwise the entrepreneurs can raise their

consumption by increasing x_{1t} by an small amount. Using the borrowing and the resource constraints to express x_{1t} and x_{2t} in terms of $(1+r_{t-1})e$ and $i_t - b_t^m$ and then noting that $x_{1t} \geq 0$ and $x_{2t} \geq 0$, one gets:

$$\left(\frac{\theta_1 R_1(1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e \leq (i_t - b_t^m) \leq \left(\frac{\theta_2 R_2(1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\,.$$

15 implies $i_t - b_t^m = e - b_t$ and so one obtains:

$$\frac{\theta_1 R_1}{1-\sigma_t}(1+r_{t-1}) + \theta_1 R_1 \leq 1 + r_t \leq \frac{\theta_2 R_2}{1-\sigma_t}(1+r_{t-1}) + \theta_2 R_2\,.$$

The problem of the middle aged in II_b is similar to II. The only difference is that entrepreneurs are maximizing with respect to $i_t - b_t^m$ rather than i_t. The net gain of increasing the size of net investment $i_t - b_t^m$ is equal to $\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t)$ as before and hence the threshold interest rate at which entrepreneurs switch from one type to another is equal to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore the law of motion for interest rate is:

$$r_t = \begin{cases} \frac{\theta_2 R_2}{1-\sigma_t}(1+r_{t-1}) + \theta_2 R_2 - 1 & \text{if } \frac{\theta_2 R_2}{1-\sigma_t}(1+r_{t-1}) + \theta_2 R_2 - 1 < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ \frac{\theta_1 R_1}{1-\sigma_t}(1+r_{t-1}) + \theta_1 R_1 - 1 & \text{if } \frac{\theta_1 R_1}{1-\sigma_t}(1+r_{t-1}) + \theta_1 R_1 - 1 > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,, \\ \\ r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{otherwise} \end{cases}$$

Assumption 2 implies that there exists a $\epsilon > 0$ such that $\sigma_t < 1 - \theta_2 R_2 - \epsilon$ for all t. This implies that $1 - \sigma_t > \theta_2 R_2 + \epsilon$ for all t which guarantees that the above difference equations are stable and that there is a unique path of interest rates. The rest of the proof is similar to the proof of Proposition 1 and Lemma 2 and so is omitted. ■

Proof of Lemma 7. Using problem II_b and conditions 14 and 15 for $z \in \{\ell, m, i\}$, one observes that:

$$V_z^{ss}(\sigma) = \big((1-\sigma)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) - \sigma r_z^{ss}(\sigma)\big) e\,.$$

using the definitions of Λ and Φ, I can obtain a more explicit form of the objective function:

$$V_z^{ss}(\sigma) = \left(\frac{(1-\sigma)\big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_z^{ss}(\sigma))\big)}{\theta_2R_2-\theta_1R_1}\right)e$$
$$+\left(\frac{(1+r_z^{ss}(\sigma))(\theta_2-\theta_1)R_1R_2 - r_z^{ss}(\sigma)(\sigma)(\theta_2R_2-\theta_1R_1)}{\theta_2R_2-\theta_1R_1}\right)e\,.$$

Now using the expressions for $r_z^{ss}(\sigma)$ for $z \in \{\ell, m, i\}$ in Lemma 6 I can take the derivative for each $z \in \{\ell, m, i\}$. For $z = \ell$:

$$\frac{d(1+r_\ell^{ss}(\sigma))}{d\sigma} = \left(\frac{1}{1-\sigma}\right)^2(1+r_\ell^{ss}(\sigma)) = \left(\frac{\theta_2R_2}{(1-\sigma)-\theta_2R_2}\right)^2.$$

Now using above I get:

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{\big(-(\theta_2-\theta_1)R_1R_2 + ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\big)}{\theta_2R_2-\theta_1R_1}\right)e$$
$$+\left(\frac{\Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1 - (1-\theta_2)R_2)\Big)(1+r_\ell^{ss})^2}{(\theta_2R_2-\theta_1R_1)}\right)e$$
$$-\,r_\ell^{ss}e\,.$$

I simplify to:

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{\big((\theta_2-\theta_1)R_1R_2(2+r_\ell^{ss}) - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\big)}{\theta_2R_2-\theta_1R_1}\right)r_\ell^{ss}e$$
$$-\,r_\ell^{ss}e\,.$$

I note that $r_\ell^{ss} = \frac{\theta_2R_2}{1-\theta_2R_2}$ and so I can simplify to get:

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{R_2-1}{1-\theta_2R_2}\right)r_\ell^{ss}e\,.$$

The proof for $z = i$ is very similar. For $z = m$, one observes that $r_m^{ss}(\sigma) = r_m^{ss}$ for any (small enough) σ and so:

$$\begin{aligned} \frac{d(1 + r_m^{ss}(\sigma))}{d\sigma} &= 0\,, \\ \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_m^{ss}(\sigma)) &= 0\,. \end{aligned}$$

Hence:

$$\frac{dV_m^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = -r_m^{ss} e\,.$$

■

Proof of Proposition 8. The first part is obvious by Lemma 7, since for small enough $\sigma > 0$ the change in steady state welfare is strictly negative in inefficient equilibria of the liquid region because the steady state interest rate is strictly negative. Therefore for small enough $\sigma > 0$, a sequence of bonds Σ with a long term supply of σ cannot make Pareto improvement.

For the second part, let $T \geq 0$ be such that $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ for $t \geq T$. Consider $\Sigma = \{\sigma_t\}_{t=0}^{\infty}$, where $\sigma_t = 0$ for $t \leq T - 1$ and $\sigma_t = \epsilon$ for $t \geq T$ and $\epsilon > 0$ is small enough. Using problem II_b and conditions 14 and 15 one has:

$$V_t(\Sigma) = \big((1 - \sigma_t)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t(\Sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}(\Sigma)) - ((1 + r_t(\Sigma))\sigma_t - \sigma_{t+1})\big) e\,.$$

For small enough ϵ, one has $r_t(\Sigma) = r_t$ for all t, where r_t are the interest rates in the competitive equilibrium without government bonds. Hence consumption of middle aged at $t \leq T - 1$ does not change. For $t \geq T + 1$, consumption increases exactly by $-r_t\epsilon \geq 0$, since competitive equilibrium is inefficient. Finally the change in consumption of middle aged at T is $\epsilon > 0$ and therefore Σ makes a Pareto improvement. ■