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Simple Markov-Perfect Industry Dynamics

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Abstract

This paper develops a tractable model for the computational and empirical analysis of infinite-horizon oligopoly dynamics. It features aggregate demand uncertainty, sunk entry costs, stochastic idiosyncratic technological progress, and irreversible exit. We develop an algorithm for computing a symmetric Markov-perfect equilibrium quickly by finding the fixed points to a finite sequence of low-dimensional contraction mappings. If at most two heterogenous firms serve the industry, the resuilt is the unique "natural" equilibrium in which a high profitability firm never exits leaving behind a low profitability competitor. With more than two firms, the algorithm always finds a natural equilibrium. We present a simple rule for checking ex post whether the calculated equilibrium is unique, and we illustrate the model's application by assessing how price collusion impacts consumer and total surplus in a market for a new product that requires costly development. The results confirm Fershtman and Pakes' (2000) finding that collusive pricing can increase consumer surplus by stimulating product development. A distinguishing feature of our analysis is that we are able to assess the results' robustness across *hundreds* of parameter values in only a few minutes on an off-the-shelf laptop computer.

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1 Introduction

This paper supplies fast, effective, and simple computational methods for important special cases of Ericson and Pakes' (1995) model of dynamic oligopoly. These cases feature aggregate uncertainty, sunk entry costs, and stochastic firm-specific technological progress; but they exclude investment decisions other than entry and exit. This simplification facilitates a range of equilibrium characterization, existence, and uniqueness results that are not available for the more general framework. Moreover, it enables the development of algorithms that calculate equilibria by finding the fixed points of a finite sequence of low-dimensional contraction mappings. These results can be used to explore some key aspects of Ericson and Pakes' model with very low computational cost. This is often useful in itself, and can serve as a first stage of a richer analysis with a more complex specification.

Substantial methodological progress in the computation of Markov-perfect equilibria followed Ericson and Pakes' original presentation of their framework. Nevertheless, Doraszelski and Pakes (2007) note that these methodological developments are only in their infancy and applications remain rare. This paper contributes to this literature by developing relatively rich analytical results and effective computational methods for a comparatively simple model. It shares this approach with Abbring and Campbell's (2010) analysis of last-in first-out oligopoly dynamics. They consider a dynamic extension of Bresnahan and Reiss' (1990) static entry model that can naturally be applied to the empirical analysis of market level entry and exit data (Abbring, Campbell, and Yang, 2010). Timing and expectational assumptions simplify its equilibrium analysis: Otherwise homogeneous firms move sequentially, oldest first; and older firms never exit expecting to leave a younger firm behind. The present paper contributes more directly to the analysis of Ericson and Pakes' framework and its potential applications, because it allows for idiosyncratic technological progress in a model with simultaneously moving incumbent firms.

Our results leverage one key insight into the structure of payoffs in a symmetric Markov-perfect equilibrium: If any firm chooses to exit with positive probability, then all identically situated firms must have an expected continuation value of zero. This allows us to calculate firms' expected continuation values at some nodes of the game tree without knowing everything about how the game will proceed thereafter. Our results demonstrate how to use these initial calculations to recover all equilibrium payoffs and actions. For this task, it is very helpful to know beforehand that adding an active firm to an industry weakly reduces all other firms' continuation values. We prove that this intuitive property must hold if at most two firms can serve the industry at one time. For the more general oligopoly case, we show that if a Markov-perfect equilibrium with such monotonicity exists, then it is essentially unique. In this case, the algorithm we propose always computes it. If no such equilibrium exists, then our algorithm can be easily adapted to find all equilibria satisfying a desirable property we call "one-shot renegotiation proofness".

The remainder of this paper proceeds as follows. The next section presents the model's primitives. It also discusses the equilibrium concept used, natural Markov-perfect equilibrium. As in Cabral (1993), the restriction to "natural" equilibrium requires no firm with high flow profits to exit leaving a lower-profitability rival in the market.

Section 3 covers the special case of a market that can support at most two active firms. The proofs of equilibrium existence and uniqueness are constructive, and so they naturally generate an algorithm for equilibrium computation. Its central steps find the fixed points of a finite number of *low-dimensional* contraction mappings. We apply the results to a numerical analysis of the effects of relaxing short-term price competition on welfare-enhancing product development, earlier explored by Fershtman and Pakes (2000).

Section 4 begins with extending the algorithm from duopoly model to accommodate three or more potentially heterogeneous firms. We then show that *if* a natural equilibrium in which adding incumbent firms weakly lowers continuation values exists, then it is essentially unique and the algorithm computes it. Next, we illustrate with an example that it is possible for entry to *increase* an incumbent's expected discounted payoff. This counterintuitive effect of entry arises from the entry deterring effects of competition. Our analysis identifies two sources of equilibrium multiplicity, both of which require entry to raise an incumbent's equilibrium payoff at some point. One arises from the failure of incumbent firms to coordinate on survival when this is mutually beneficial. We propose to exclude such coordination failures by requiring equilibria to be "one-shot renegotiation proof". The other occurs when multiple mixed strategies leave incumbents indifferent between exit and continuation.

2 The Model

In Ericson and Pakes (1995), a countable number of firms with heterogeneous productivity levels serve a single industry. Entry requires the payment of a sunk cost, and exit allows firms to avoid per-period fixed costs of production. Surviving incumbent firms choose investments that stochastically improve their technologies. Exogenous stochastic increases in the knowledge stock outside the industry increase the quality of an outside good and, this way, decrease all incumbent firms' profits simultaneously. These outside knowledge shocks are embodied in potential entrants to the industry, and therefore do not affect their profits.

Two main changes to Ericson and Pakes' primitive assumptions facilitate our demonstration of Markov-perfect equilibrium uniqueness and our algorithm for its rapid computation. First, we assume that productivity evolves exogenously, instead of allowing firms to make costly investments in accelerating technological progress. Second, we replace the common

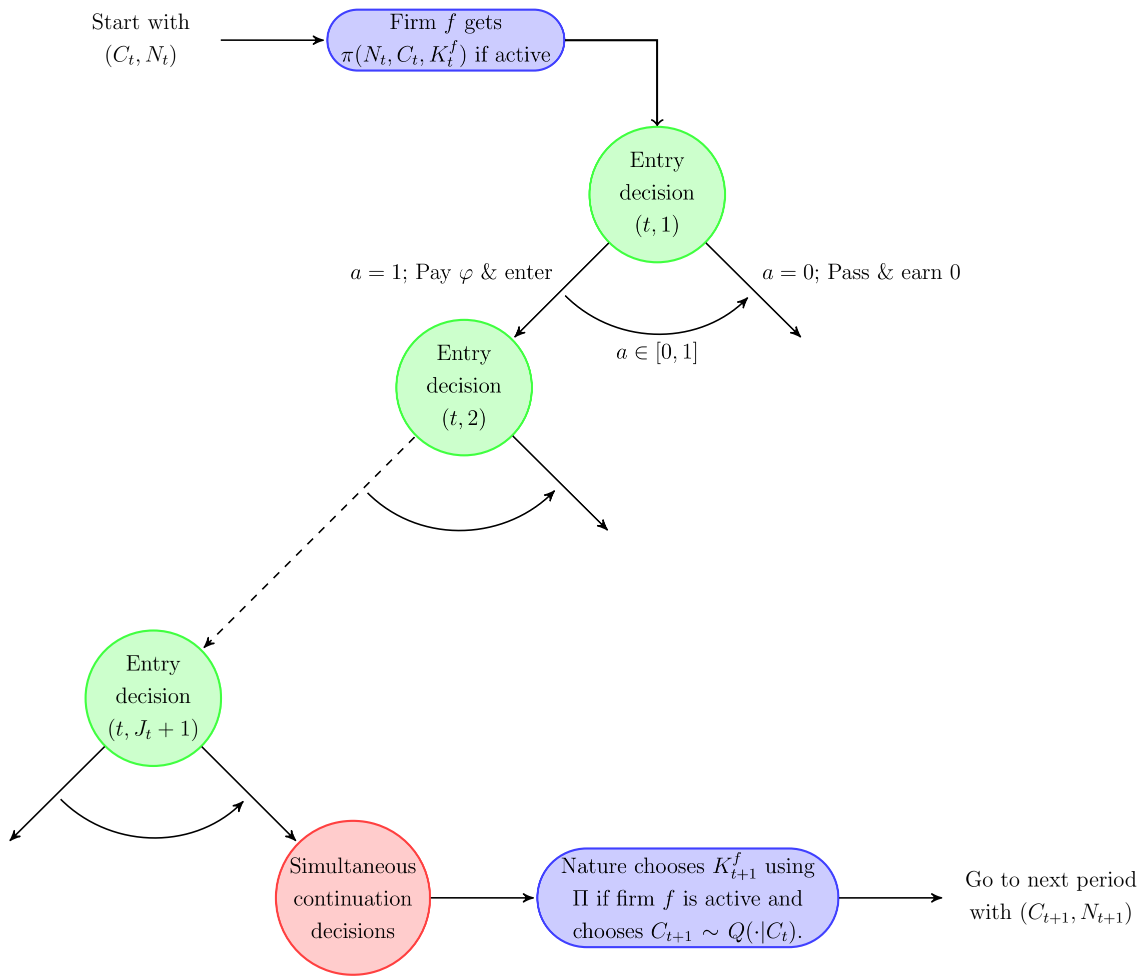


Figure 1: The Sequence of Events and Actions within a Period

negative shocks to the incumbents' profits by general aggregate demand shocks that equally affect the profits of incumbent firms and potential entrants.

2.1 Primitive Assumptions

The model consists of a single oligopolistic market in discrete time $t \in \mathbb{Z}_\star \equiv \{0, 1, \ldots\}$. A countable number of firms potentially serve the market. These are indexed by $f \in \mathbb{Z}_\star \times \mathbb{N}$. Below we refer to f as the firm's *name*. At a given time t, some of the firms are *active*, and the remaining producers are *inactive*. Each active firm f has an idiosyncratic productivity type K_t^f that takes values in $\mathbb{K} \equiv \{1, \ldots, \check{k}\}$. Stack the numbers of active firms with each productivity level at time t into the $\check{k} \times 1$ vector N_t, the *market structure*. Initially, no firms serve the market: N_0 equals a vector of zeros. Subsequently; entry, stochastic productivity improvement, and exit determine its evolution.

Figure 1 illustrates the sequence of events and actions within a period t. It begins with the inherited values of two state variables, N_t and a scalar index of demand $C_t \in [\hat{c}, \check{c}]$, with $\check{c} < \infty$. With these in place, the active participants receive their profits from serving the market. For a type K_t firm facing the market structure N_t, these equal $\pi(N_t, C_t, K_t)$. [1]

We assume that a firm's flow profit decreases with the number and productivity of its competitors and increases with its own productivity. For this assumption's formal statement, we use ι_k to denote a $\check{k} \times 1$ vector with a one in its kth position and zeros elsewhere, and set $\iota_0 \equiv 0$. This allows us to denote a market structure with at least one type k firm with $n + \iota_k$.

Assumption 1 (Monotone Producer Surplus)**.** For all productivity types $k \in \mathbb{K}$, demand states $c \in [\hat{c}, \check{c}]$, and market structures $n \in \mathbb{Z}_\star^{\check{k}}$:

1. $\pi(\iota_k + n, c, k) \leq \check{\pi} < \infty$ for all $c \in [\hat{c}, \check{c}]$
2. $\pi(\iota_k + n, c, k) \leq \pi(\iota_k + n, d, k)$ for all $c, d \in [\hat{c}, \check{c}]$ such that $c < d$.
3. $\pi(\iota_k + n + \iota_l, c, k) < \pi\left(\iota_k + n + \iota_{l-1}, c, k\right)$ for all $l \in \mathbb{K}$;
4. $\pi(\iota_k + n, c, k) \to -\kappa(k) < 0$ as the number of firms in n goes to infinity; and
5. $\pi(\iota_k + \iota_l + n, c, k) \leq \pi(\iota_k + \iota_l + n, c, l)$ for all $k, l \in \mathbb{K}$ such that $k < l$, with strict inequality hold for some $c \in [\hat{c}, \check{c}]$.

In item 4, $\kappa(k)$ represents a type k firm's per-period fixed cost. After production, firms with names in $\{t\} \times \mathbb{N}$ make entry decisions sequentially in the order of their names, starting with $(t, 1)$. These continue until a firm chooses to remain out of the industry. We denote the number of entrants in period t with J_t, so the name of the first potential entrant choosing to stay out of the market and thereby ending its entry stage is $(t, J_t + 1)$. The cost of entry is $\varphi > 0$. After paying this cost, the entrant immediately joins the set of active firms with productivity type 1.[2] A firm with an entry opportunity cannot delay its choice, so the payoff to staying out of the industry is zero.

After the entry decisions, all active firms— including those that just entered the market— decide simultaneously between survival and exit. Exit is irreversible but otherwise costless. It allows firms to avoid future periods' fixed production costs. Firms' entry and exit decisions maximize their expected profit streams discounted with a factor $\beta < 1$.

In the period's final stage, C_t and the firms' productivity types evolve. The demand index evolves exogenously according to a nonnegative first-order Markov process bounded between

[1] We leave this undefined if the K_t'th element of N_t equals zero.

[2] Since entrants' productivity types evolve before their first period of production, we can use the distribution of K_{t+1}^f given $K_t^f = 1$ to distribute new firms' types nontrivially. That is, the assumption that all entrants have $K_t^f = 1$ is not overly restrictive.

$\hat{c}$ and $\check{c}$. We denote the conditional distribution of C_{t+1} with $Q\left(c \,|\, C_t\right) \equiv \Pr\left(C_{t+1} \leq c \,|\, C_t\right)$, and the corresponding probability density function with $q(\cdot \mid C_t)$. Each firm's idiosyncratic productivity type follows an independent Markov chain with a common $(\check{k} \times \check{k})$ transition matrix Π. Its typical element is $\Pi_{k,k'} \equiv \Pr\left(K^f_{t+1} = k' \,\middle|\, K^f_t = k\right)$. Following Ericson and Pakes (1995), we assume that the idiosyncratic productivity types never regress:

Assumption 2 (No Productivity Regress)**.** Π is upper diagonal.

We further assume that K^f_{t+1} (weakly) stochastically increases with K^f_t.

Assumption 3 (Monotone Productivity Dynamics)**.** For all $k', k, l \in \mathbb{K}$ such that $k < l$,

$$\Pr\left(K^f_{t+1} \geq k' \,\middle|\, K^f_t = k\right) \leq \Pr\left(K^f_{t+1} \geq k' \,\middle|\, K^f_t = l\right).$$

This assumption gives high technology firms no worse advancement opportunities than low technology firms have.

2.2 Markov-Perfect Equilibrium

A *Markov-perfect equilibrium* is a subgame-perfect equilibrium in strategies that are only contingent on *payoff-relevant* variables. For a potential participant $f = (t, j)$ contemplating entry these are C_t and the market structure M^f_t just after f's possible entry. The latter is period t's initial market structure N_t augmented with j type 1 entrants: $M^f_t \equiv N_t + j\iota_1$. Denote the market structure after the period's final entry with $M_{E,t} \equiv N_t + \iota_1 J_t$. If firm f is contemplating survival in period t, the payoff-relevant variables are this market structure, the current demand index (C_t), and its productivity type (K^f_t).

A *Markov strategy* for firm f is a pair (a^f_E, a^f_S) of functions

$$a^f_E : \mathbb{Z}^{\check{k}}_\star \times [\hat{c}, \check{c}] \longrightarrow [0,1] \qquad \text{and} \qquad a^f_S : \mathbb{Z}^{\check{k}}_\star \times [\hat{c}, \check{c}] \times \mathbb{K} \longrightarrow [0,1].$$

This strategy's *entry rule* a^f_E assigns a probability of becoming active given an entry opportunity to each possible value of (M^f_t, C_t). Similarly, its *exit rule* a^f_S assigns a probability of being active in the next period given that the firm is currently active to each possible value of its payoff-relevant state $(M_{E,t}, C_t, K^f_t)$. Since calendar time is not payoff-relevant, we hereafter drop the t subscript from all variables. A symmetric equilibrium is an equilibrium in which all firms follow the same strategy (a_E, a_S). In the remainder of the paper, we focus on symmetric equilibria and drop the superscript f from the firms' common strategy.

Throughout the paper, we will focus on equilibria in which a high productivity firm never exits when a low productivity competitor survives. Such equilibria are natural, because a high productivity firm earns strictly higher flow profit in each period than a low productivity firm. Formally, we define a natural Markov-perfect equilibrium as follows:

Definition 1. *A* natural *Markov-perfect equilibrium is a symmetric Markov-perfect equilibrium in a strategy* (a_E, a_S) *such that for all* $k, l \in \mathbb{K}$ *such that* $k < l$; $m_k \geq 1$, $m_l \geq 1$, *and* $a_S(m, c, k) > 0$ *together imply that* $a_S(m, c, l) = 1$.

Cabral (1993) restricts attention to similar natural equilibria in a model with deterministic productivity progression.

Firms' expected discounted profits at each node of the game depend on that node's payoff-relevant state variables when they all use Markov strategies. The payoffs in two of each period's nodes are of particular interest, the *post-entry* value and the *post-survival* value. The post-entry value $v_E(M_E, C, K)$ equals the expected discounted profits of a type K firm in a market with demand index C and market structure M_E just after all entry decisions have been sequentially realized. Since it gives the payoffs to a potential producer from entering in each possible market structure that could arise from other players subsequent entry decisions, it determines optimal entry choices. The post-survival value $v_S(M_S, C, K)$ equals the expected discounted profits of a type K firm facing demand index C and market structure M_S just after all survival decisions have been realized. It gives the payoffs to a surviving firm in each possible market structure following firms' simultaneous continuation decisions, so it is central to the analysis of exit.

The value functions v_E and v_S satisfy

$$v_E(m_E, c, k) = a_S(m_E, c, k)\mathbb{E}\left[v_S(M_S, c, k) \,|\, M_E = m_E\right] \tag{1}$$

and

$$v_S(m_S, c, k) = \beta\mathbb{E}\left[\pi(N', C', K') + v_E(M_E', C', K') \,|\, M_S = m_S, C = c, K = k\right]. \tag{2}$$

Here and throughout, we denote the variable corresponding to X in the next period with X'. The conditional expectation in (1) is computed given that the firm of interest continues, and embodies the use of a_S by all other active firms. In fact, the only nontrivial randomness it embodies arises from firms' possible use of mixed strategies. The conditional expectation in (2) accounts for the use of a_E by all potential participants with entry opportunities in the next period as well as the exogenous evolutions of C and the firms' productivity types.[3]

For (a_E, a_S) to form a symmetric Markov-perfect equilibrium, it is necessary and sufficient that no firm can gain from a one-shot deviation from (a_E, a_S) (e.g. Fudenberg and Tirole, 1991, Theorem 4.2):

$$a_E(m, c) \in \arg\max_{a \in [0,1]} a \left\{\mathbb{E}\left[v_E\left(M_E, c, 1\right) \,|\, M = m\right] - \varphi\right\} \text{ and} \tag{3}$$

$$a_S(m_E, c, k) \in \arg\max_{a \in [0,1]} a \,\mathbb{E}\left[v_S(M_S, c, k) \,|\, M_E = m_E\right]. \tag{4}$$

[3]Section **??** of this paper's online appendix presents the two conditional distributions underlying the conditional expectations in (1) and (2) in detail.

The conditional expectations in (3) and (4) are computed like those in (1) and (2). For example, $\mathbb{E}\left[v_E\left(M_E, c, 1\right) \mid M = m\right]$ is the payoff, gross of the entry cost φ, that a potential participant in state (m, c) expects from entering if all firms with entry opportunities later in the period use the entry rule a_E and the value of ending up as a type 1 firm in a market with structure m_E and c consumers equals $v_E\left(m_E, c, 1\right)$.

Together, conditions (1)–(4) are necessary and sufficient for a strategy (a_E, a_S) to form a symmetric Markov-perfect equilibrium with payoffs v_E and v_S.

Before proceeding to examine the set of natural Markov-perfect equilibria, consider one uninteresting source of equilibrium multiplicity. With an equilibrium in hand, change one player's action at a particular node of the game. If this change gives the same payoff to the player in question and all other player's equilibrium actions at that node remain optimal, then this change forms a second equilibrium. In our model, this situation can arise when the payoff to entry equals zero and when the payoff to survival as the only firm of your type equals zero. To eliminate this difficulty, we require firms in such a situation to choose inactivity.

Definition 2. *A Markov strategy (a_S, a_E) with corresponding payoff v_E* defaults to inactivity *if*

- $a_S(m - m_k \times \iota_k + \iota_k, c, k) = 0$ *if* $v_S(m, c, k) = 0$
- $a_E(m, c) = 0$ *if* $v_E(m, c, 1) = \varphi$,

for all $k \in \mathbb{K}$ and all c.

The remainder of the paper restricts attention to equilibria with strategies that default to inactivity, unless otherwise mentioned.[4]

3 Duopoly

It is helpful to begin the model's analysis with one additional restriction: At most two firms can be active at once. Throughout this section, we represent duopoly market structures with $\iota_k + \iota_l$ with $k, l \in \mathbb{K} \cup \{0\}$. The following lemma arises from this simplification.

Lemma 1 (Monotone Payoffs in the Heterogenous-Duopoly Model)**.** *In a natural Markov-perfect equilibrium, for all $c \in [\hat{c}, \check{c}]$ and $k \in \mathbb{K}$, $v_E(2\iota_k, c, k) \leq v_E(\iota_k, c, k)$ and $v_S(2\iota_k, c, k) \leq v_S(\iota_k, c, k)$.*

Proof. See Appendix A. □

[4]Note that we do *not* restrict the game's strategy space to include only strategies that default to inactivity.

Figure 2: Reduced-form Representation of the Duopoly Continuation Game

	Survive	Exit
Survive	$v_S(2,c)$ $v_S(2,c)$	$v_S(1,c)$ 0
Exit	0 $v_S(1,c)$	0 0

Note: In each cell, the upper-left expression gives the row player's payoff. Please see the text for further details.

Lemma 1 states that a duopolist facing a rival of the same type always has a lower value than it would have without the rival present. With its help, we develop the duopoly model's analysis in three stages. First, we examine the special case without heterogeneity, $\check{k} = 1$. This introduces the model's most important moving parts without undue complication. We then generalize this slightly in Section 3.2 by adding a second productivity type and walking through the procedure for equilibrium calculation. Section Section 3.3 formalizes this procedure into an algorithm and then establishes equilibrium existence and uniqueness results. Finally, Section 3.4 uses this algorithm for numerical analysis of the effects of technological progress and demand uncertainty on industry dynamics. This illustration demonstrates that the natural Markov-perfect equilibrium of this model can be easily computed.

3.1 One Productivity Type

When firms have identical productivity types by assumption, the restriction to a natural equilibrium merely requires symmetry of players' strategies. Here, we construct a symmetric Markov-perfect equilibrium for this case in three steps. The type distribution is trivial, so we write $\pi(N, C, 1)$ as $\pi(N, C)$ and make the analogous substitution for the value functions throughout this example's development.

Step 1: Calculation of $v_E(2,\cdot)$, $v_S(2,\cdot)$, and $a_E(2,\cdot)$ The equilibrium construction begins with a characterization of the duopoly payoffs $v_E(2,\cdot)$ and $v_S(2,\cdot)$. In a Markov-perfect equilibrium, the survival rule $a_S(2,c)$ satisfies (4): Given c, it is a Nash equilibrium of the static simultaneous-move game with payoffs given by the expected continuation values. Figure 2 gives the reduced-form representation of this game with the two pure strategies "Survive" and "Exit". The upper-left expression in each cell is the row player's payoff. Both firms receive the duopoly post-survival payoff $v_S(2,c)$ following joint survival. This payoff adds the discounted duopoly flow payoff $\pi(2, C')$ to the the discounted duopoly post-entry payoff $v_E(2, C')$. Consequently, it satisfies a special case of Equation (2):

$$v_S(2, c) = \beta \mathbb{E}\left[\pi(2, C') + v_E(2, C') \,|\, C = c\right].$$

A firm that survives while its rival exits earns the monopoly post-survival value $v_S(1, c)$.

Suppose that $v_S(2, c) > 0$. Lemma 1 guarantees that $v_S(1, C) > 0$, so in this case "Survive" is a dominant strategy. If instead $v_S(2, c) < 0$, then a symmetric equilibrium strategy must put positive probability on "Exit". That pure strategy's payoff always equals zero. Since $v_E(2, c)$ equals the symmetric equilibrium payoff to this game, these facts together yield the following special case of Equation (1):

$$\begin{aligned} v_E(2, c) &= \max\{0, v_S(2, c)\} \\ &= \max\Big\{0, \beta \mathbb{E}\left[\pi(2, C') + v_E(2, C') \,|\, C = c\right]\Big\}. \end{aligned} \tag{5}$$

The right-hand side defines a contraction mapping, so this necessary condition uniquely determines $v_E(2, \cdot)$ and, using (2), $v_S(2, \cdot)$. This is the key technical insight that makes the calculation of the model's Markov-perfect industry dynamics simple. Although duopoly is *not* an absorbing state for the industry, we can calculate the equilibrium duopoly payoffs without knowledge of the firms' payoffs in possible future market structures. This is because firms' common post-entry value in a symmetric equilibrium equals zero unless joint continuation is individually profitable.

With the duopoly post-entry value in hand, we can proceed to the problem of a potential entrant facing a single incumbent. By Equation (3), this firm enters if $v_E(2, c) > \varphi$ and stays out of the market if $v_E(2, c) \leq \varphi$. For all c,

$$a_E(2, c) = I\left\{v_E(2, c) > \varphi\right\}.$$

Note that strategy defaults to inactivity. When C has an atomless distribution, this strategy almost surely prescribes the same action as any other entry strategy consistent with profit maximization that does not default to inactivity. For this reason, our requirement that the potential entrant default to inactivity has no substantial economic content.

Step 2: Calculation of $v_E(1, \cdot)$, $v_S(1, \cdot)$, $a_E(1, \cdot)$, and $a_S(1, \cdot)$ We proceed to consider the monopoly payoffs, a potential entrant's decision to enter an empty market, and an incumbent monopolist's survival decision. Because an incumbent monopolist choosing to survive will earn $v_S(1, c)$, the post-entry value to a monopolist in (1) reduces to

$$\begin{aligned} v_E(1, c) &= \max\{0, v_S(1, c)\} \\ &= \max\Big\{0, \beta \mathbb{E}\Big[\pi(1, C') + a_E(2, C') v_E(2, C') + (1 - a_E(2, C'))\, v_E(1, C') \;\Big|\; C = c\Big]\Big\}. \end{aligned}$$

Given $v_E(2,\cdot)$ and $a_E(2,\cdot)$ from Step 1, the right-hand side defines a contraction mapping that uniquely determines $v_E(1,\cdot)$ and, using Equation (2), $v_S(1,\cdot)$. It is not difficult to demonstrate that the $v_E(1,c)$ and $v_S(1,c)$ so constructed always weakly exceed, respectively, $v_E(2,c)$ and $v_S(2,c)$ from Step 1; so that the constructed value functions are consistent with the requirements of Lemma 1.

Just as with a potential duopolist, we select the unique entry rule for a potential monopolist that defaults to inactivity. Since $v_E(2,c) \leq v_E(1,c)$, this is

$$a_E(1,c) = I\left\{v_E(1,c) > \varphi\right\}.$$

By (4), a monopolist chooses survival in demand states c such that $v_S(1,c) > 0$ and exit if $v_S(1,c) < 0$. Our equilibrium construction uses the unique monopoly survival rule that defaults to inactivity:

$$a_S(1,c) = I\left\{v_S(1,c) > 0\right\}.$$

Step 3: Calculate $a_S(2,\cdot)$ The first two steps have determined the only possible post-entry and post-survival values, as well as an entry rule and a monopoly survival rule that are consistent with them. This last step completes the equilibrium strategy's construction by determining a duopoly survival rule that satisfies (4).

As we noted above in Step 1, equilibrium requires $a_S(2,c) = 1$ if $v_S(2,c) > 0$. All that remains undetermined is the survival rule when $v_S(2,c) \leq 0$. If profit maximization would require even a monopolist to exit (i.e. $v_S(1,c) \leq 0$), then both duopolists exit for sure and $a_S(2,c) = 0$. If instead $v_S(1,c) > 0$, then the reduced-form continuation game above has no pure strategy equilibrium. In its unique mixed-strategy equilibrium, each firm's survival probability leaves its rival indifferent between exiting (and getting a payoff of zero for sure) and surviving. That is, in demand states c such that $v_S(2,c) \leq 0$ and $v_S(1,c) > 0$, the indifference condition

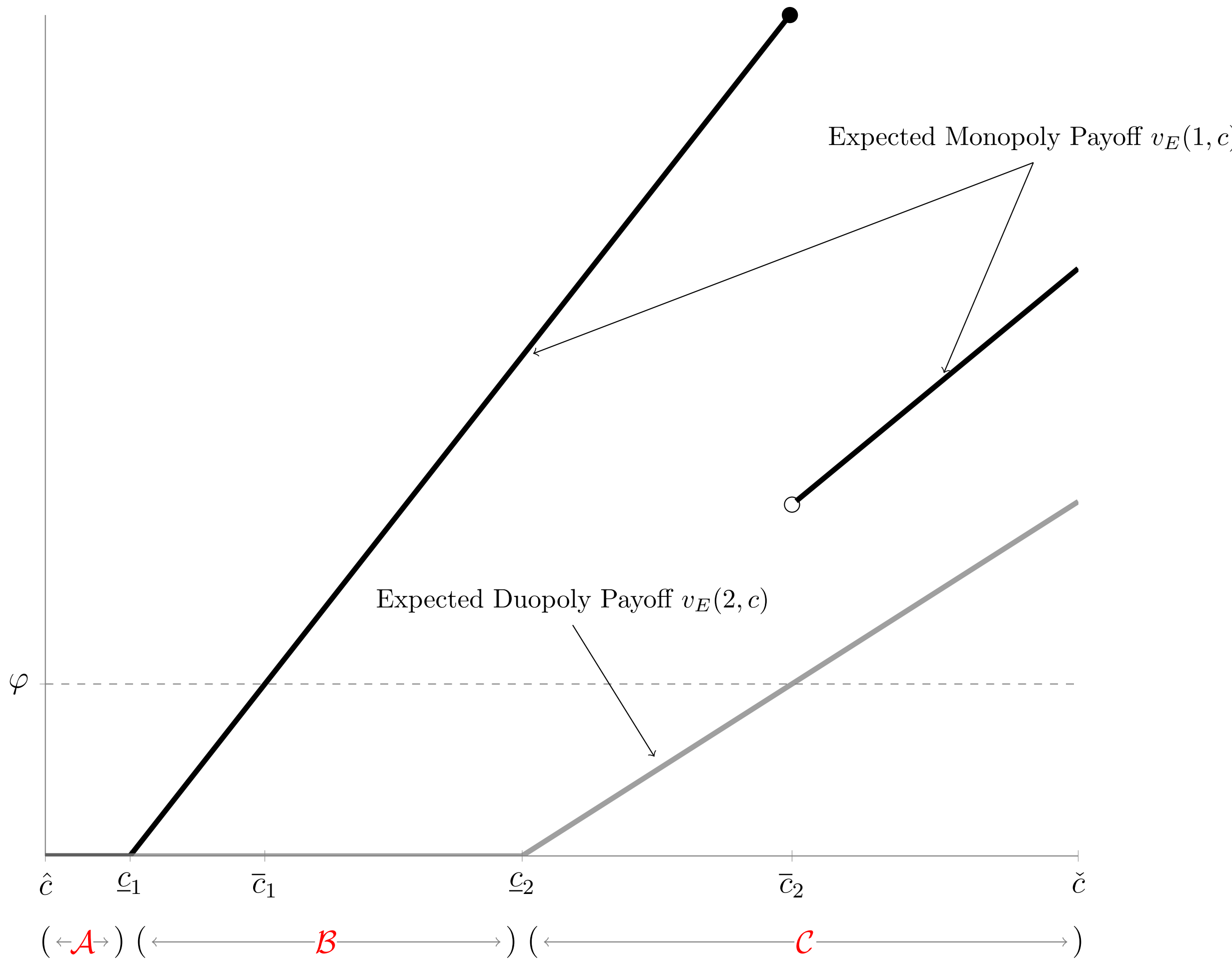
$$a_S(2,c)v_S(2,c) + \left(1 - a_S(2,c)\right)v_S(1,c) = 0$$

uniquely determines $a_S(2,c)$.[5]

Illustration of the Constructed Equilibrium The entry and survival rules so calculated form our equilibrium. Figure 3 plots the payoffs for a particular numerical example. In it $\pi(c,n) = c\varpi(n) - \kappa$ with $\kappa > 0$ and $\varpi(2) < \varpi(1)$. We also choose the stochastic process for C so that its current value has no influence on the equilibrium probability of a firm entering

[5]The mixed-strategy so derived prescribes that both firms exit for sure if $v_S(1,c) = 0$, as the required by our restriction to strategies that default to inactivity.

Figure 3: Equilibrium Payoffs in the Homogeneous Duopoly Example



or exiting in any future period. Specifically, $C' = c$ with probability $1 - \lambda$ and equals a draw from a uniform distribution over $[\hat{c}, \check{c}]$ with the complementary probability. This and the affine specification of $\pi(c, n)$ together guarantee that $v_E(1, c)$ and $v_E(2, c)$ are piecewise linear in c.

The lower (continuous) function in grey gives the duopoly post-entry value, $v_E(2, c)$. By construction, this is identical to the expected discounted profits of a duopolist facing a rival that will never exit first. It equals zero for $c \leq \underline{c}_2$. Thereafter it rises $\pi(2)/(1 - \beta(1 - \lambda))$ for each extra consumer. For $c > \overline{c}_2$, entry into a market with one incumbent is optimal.

The monopoly post-entry value $v_E(1, c)$ equals zero for demand levels $c \leq \underline{c}_1$, and it increases $\pi(1)/(1 - \beta(1 - \lambda))$ with each extra consumer until reaching $\overline{c}_2$. . When $c > \overline{c}_2$, the period's entry stage always ends with two firms, so the equilibrium payoff to a firm that began the period as a monopolist drops to the grey expected duopoly payoff. The disconnected line segment above this gives the expected payoff to a firm that finds itself as a monopolist after the period's entry stage is complete.[6] Given this value function, the equilibrium strategy for a potential entrant facing an empty market is $a_E(1,c) = I\{v_E(1,c) > \varphi\} \equiv I\{c > \overline{c}_1\}$, and the analogous continuation rule for an incumbent monopolist is $a_S(1,c) = I\{v_E(1,c) > 0\} = I\{v_S(1,c) > 0\} \equiv I\{c > \underline{c}_1\}$.

Duopolists' common continuation strategy corresponds to the unique Nash equilibrium of the game in Figure 2. Exit is a dominant strategy when $c \in \mathcal{A}$, and survival is dominant when $c \in \mathcal{C}$. When $c \in \mathcal{B}$ the firms mix over survival and exit.

3.2 Two Productivity Types

We now proceed to adapt the basic ideas presented above to the case with productivity heterogeneity. In the interest of expositional clarity, we denote the higher productivity type with the intuitive $\mathcal{H}$ (instead of 2) and the lower type with $\mathcal{L}$ (instead of 1). We construct this case's unique natural Markov-perfect equilibrium in six steps. Just as before, these steps take us through a finite partition of the state space. In each of the first five steps, we compute the equilibrium payoffs in the states considered by finding the unique fixed point of a contraction mapping. The results from the completed steps are used as inputs in the following steps. Figure 4 illustrates this sequence of computations. The construction ends by specifying the unique strategy that supports the equilibrium payoffs in the sixth step.[7]

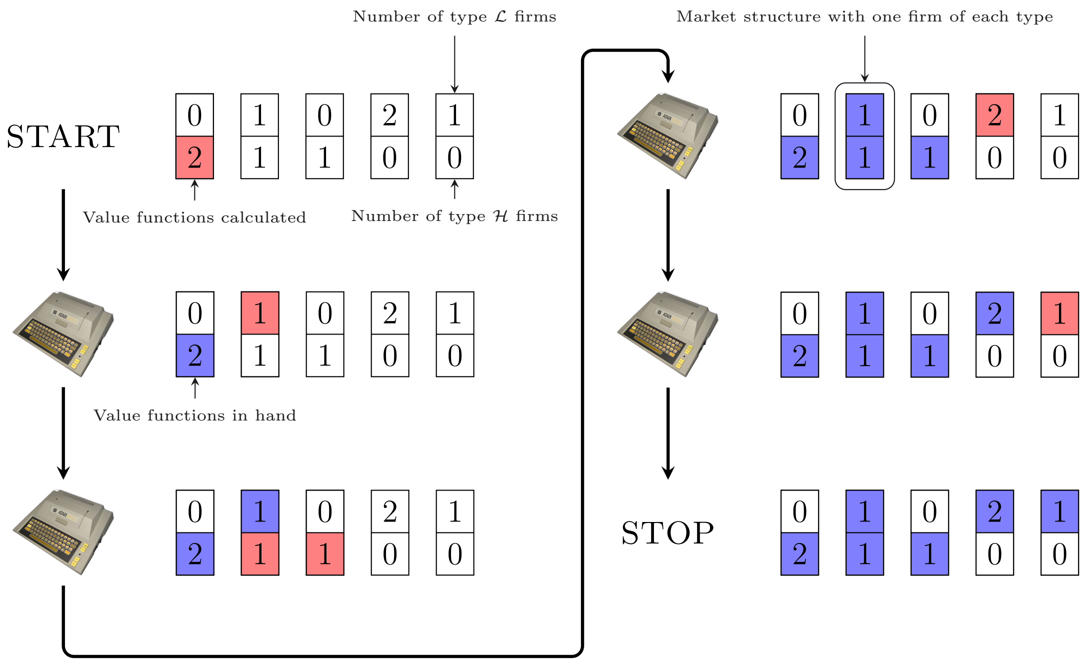
Step 1: Calculation of $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_S(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ As depicted by the upper-left panel in Figure 4, we start the equilibrium construction by considering a market populated by two type $\mathcal{H}$ firms. The analysis in this step is a carbon copy of the first step of the previous example. The simultaneous-move survival game between two type $\mathcal{H}$ firms is analogous to the one in Figure 2, and Lemma 1 guarantees that "Survive" is the dominant strategy if joint continuation gives both firms positive payoffs. Therefore, finding the fixed point of a contraction mapping analogous to that in (1) yields $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$. The continuation payoff $v_S(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ immediately follows.

Step 2: Calculation of $v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$, $v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$, $a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot)$, and $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$. A type $\mathcal{L}$ firm that chooses to survive advances to $\mathcal{H}$ with probability $\Pi_{\mathcal{L}\mathcal{H}}$ and remains unchanged with probability $\Pi_{\mathcal{L}\mathcal{L}} \equiv 1 - \Pi_{\mathcal{L}\mathcal{H}}$. In a natural MPE, the survival of the type $\mathcal{L}$ firm guarantees survival of any type $\mathcal{H}$ rival, so the continuation value $v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C, \mathcal{L})$ must

[6] This would require some potential entrant to deviate from the equilibrium strategy.

[7] The Atari 400 computer appearing in the illustration went on sale in November 1979 and had 8K of RAM. It has more than enough computing power to calculate the unique equilibrium.

Figure 4: Equilibrium Computation for a Duopoly with Two Productivity Types



Note: There are five possible duopoly market structures. Each divided rectangle represents one of them, and each collection of five rectangles displays the value functions being calculated (in red) and the value functions already in hand (in blue) at one stage of the algorithm (which is Section 3.3's Algorithm 1 with $\check{k} = 2$).

satisfy

$$\begin{aligned} v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) =& \max\Big\{0, v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L})\Big\} \\ =& \beta \max\Big\{0, \Pi_{\mathcal{L}\mathcal{L}}\mathbb{E}\left[\pi(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L}) + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L})|C = c\right] \\ & + \Pi_{\mathcal{L}\mathcal{H}}\mathbb{E}\left[\pi(2\iota_{\mathcal{H}}, C', \mathcal{H}) + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H})|C = c\right]\Big\}. \end{aligned}$$

Since $v_E(2\iota_{\mathcal{H}}, \cdot, \iota_{\mathcal{H}})$ is in hand from Step 1, this defines a contraction mapping in the desired value function. With its fixed-point in hand, we can then easily compute $v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{L})$ and

$$\begin{aligned} a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c) &= I\{v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) > \varphi\}, \\ a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) &= I\{v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) > 0\}. \end{aligned}$$

Step 3: Calculation of $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $a_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{H})$, and $a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{H})$. A market with a monopolist incumbent with type $\mathcal{H}$ attracts an entrant next period if and only if $a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C') = 1$, so $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_E(\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, \cdot, \mathcal{H})$

together satisfy

$$
\begin{aligned}
v_E(\iota_{\mathcal{H}}, c, \mathcal{H}) =& \max\Big\{0, v_S(\iota_{\mathcal{H}}, c, \mathcal{H})\Big\} \\
=& \beta \max\Big\{0, \mathbb{E}[\pi(\iota_{\mathcal{H}}, C', \mathcal{H}) + a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C') v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) \\
& \qquad + \{1 - a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C')\} v_E(\iota_{\mathcal{H}}, C', \mathcal{H}) | C = c]\Big\}.
\end{aligned}
\tag{6}
$$

Step 2 determined $a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot)$, so the only unknowns in (6) are the value functions. Since a type $\mathcal{H}$ duopolist facing a type $\mathcal{L}$ rival becomes a monopolist if and only if $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L}) = 0$, these value functions must also satisfy

$$
\begin{aligned}
& v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \iota_{\mathcal{H}}) \\
&= a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \iota_{\mathcal{H}}) + \{1 - a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L})\} v_E(\iota_{\mathcal{H}}, c, \mathcal{H}) \\
&= a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) \beta \Big\{\Pi_{\mathcal{L}\mathcal{L}} \mathbb{E}[\pi(\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, C', \mathcal{H}) + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) | C = c] \\
& \qquad + \Pi_{\mathcal{L}\mathcal{H}} \mathbb{E}[\pi(2\iota_{\mathcal{H}}, C', \mathcal{H}) + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H}) | C = c]\Big\} \\
& \quad + \{1 - a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L})\} v_E(\iota_{\mathcal{H}}, c, \mathcal{H}).
\end{aligned}
\tag{7}
$$

We have $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ from Step 1 and $a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{L})$ from Step 2, so together, (6) and (7) determine $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{H})$. Obtaining $v_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, \cdot, \mathcal{H})$ from these is straightforward. Since we seek a natural equilibrium, the survival strategies of interest must satisfy

$$
a_S(\iota_{\mathcal{H}}, c, \mathcal{H}) = a_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{H}) = I\{v_S(\iota_{\mathcal{H}}, c, \mathcal{H}) > 0\}.
$$

Step 4: Calculation of $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $a_E(2\iota_{\mathcal{L}}, \cdot)$, and $v_S(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$. Next, we consider a duopoly market with two type $\mathcal{L}$ firms. If both firms choose survival, then their idiosyncratic shocks could change the market structure to either of the duopoly structures considered in Steps 1-3 or leave it unchanged. Lemma 1 guarantees that if the value of simultaneous survival to either incumbent is positive, then joint continuation is the only Nash equilibrium outcome of their survival game. Therefore, $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$ satisfies

$$
\begin{aligned}
v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) =& \max\Big\{0, v_S(2\iota_{\mathcal{L}}, c, \mathcal{L})\Big\} \\
=& \beta \max\Big\{0, \Pi^2_{\mathcal{L}\mathcal{L}} \mathbb{E}\left[\pi(2\iota_{\mathcal{L}}, C', \mathcal{L}) + v_E(2\iota_{\mathcal{L}}, C', \mathcal{L}) | C = c\right] \\
& + \Pi_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}\mathcal{H}} \mathbb{E}\left[\pi(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L}) + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L}) | C = c\right] \\
& + \Pi_{\mathcal{L}\mathcal{H}} \Pi_{\mathcal{L}\mathcal{L}} \mathbb{E}\left[\pi(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) | C = c\right] \\
& + \Pi^2_{\mathcal{L}\mathcal{H}} \mathbb{E}\left[\pi(2\iota_{\mathcal{H}}, C', \mathcal{H}) + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H}) | C = c\right]\Big\}.
\end{aligned}
$$

The only unknown on its righthand side is $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, so we can use this Bellman equation to calculate it. With this in hand, we construct the rule for entry into a market with one type $\mathcal{L}$ incumbent as

$$a_E(2\iota_{\mathcal{L}}, c) = I\{v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) > \varphi\}.$$

Moreover, it is straightforward to determine $v_S(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$.

Step 5: Calculation of $v_E(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $v_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $a_E(\iota_{\mathcal{L}}, \cdot)$, and $a_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$. If a type $\mathcal{L}$ monopolist chooses survival, then one of four market structures will prevail in the next period, depending on the incumbent's idiosyncratic shock and on the decision of a potential entrant:

$$\begin{aligned}
v_E(\iota_{\mathcal{L}}, c, \mathcal{L}) &= \max\Big\{0, v_S(\iota_{\mathcal{L}}, c, \mathcal{L})\Big\} \\
&= \max\Big\{0, \Pi_{\mathcal{L}\mathcal{L}}\mathbb{E}\Big[\pi(\iota_{\mathcal{L}}, C', \mathcal{L}) + a_E(2\iota_{\mathcal{L}}, C')v_E(2\iota_{\mathcal{L}}, C', \mathcal{L}) \\
&\qquad\qquad + \{1 - a_E(2\iota_{\mathcal{L}}, C')\}\, v_E(\iota_{\mathcal{L}}, C', \mathcal{L})|C = c\Big] \\
&\qquad + \Pi_{\mathcal{L}\mathcal{H}}\mathbb{E}\Big[\pi(\iota_{\mathcal{H}}, C', \mathcal{H}) + a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C')v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) \\
&\qquad\qquad + \{1 - a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C')\}\, v_E(\iota_{\mathcal{H}}, C', \mathcal{H})|C = c\Big]\Big\}.
\end{aligned} \tag{8}$$

Given the quantities calculated in Steps 1–4, the righthand side of (8) defines a contraction mapping with $v_E(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$ as its fixed point. With this, it is straightforward to compute $v_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, which gives the survival rule

$$a_S(\iota_{\mathcal{L}}, c, \mathcal{L}) = I\{v_S(\iota_{\mathcal{L}}, c, \mathcal{L}) > 0\}.$$

Since $v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) \leq v_E(\iota_{\mathcal{L}}, c, \mathcal{L})$, the entry rule for a potential monopolist can be written as

$$a_E(\iota_{\mathcal{L}}, c) = I\left\{v_E(\iota_{\mathcal{L}}, c) > \varphi\right\}$$

Step 6: Calculation of $a_S(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $a_S(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$. Steps 1–5 have determined all equilibrium continuation values, entry strategies, and survival strategies for firms facing no identical rival. All that remains is to determine the exit strategies for duopolies of identical firms. Their construction parallels that from the case with homogeneous firms: Unless either survival or exit is a dominant strategy, both firms mix between the two pure actions to leave each other indifferent between them.

$$a_S(2\iota_{\mathcal{L}}, c, \mathcal{L}) = \begin{cases} 1 & \text{if } v_S(2\iota_{\mathcal{L}}, c, \mathcal{L}) > 0, \\ \frac{v_S(\iota_{\mathcal{L}}, c, \mathcal{L})}{v_S(\iota_{\mathcal{L}}, c, \mathcal{L}) - v_S(2\iota_{\mathcal{L}}, c, \mathcal{L})} & \text{if } v_S(2\iota_{\mathcal{L}}, c, \mathcal{L}) \leq 0 \text{ and } v_S(\iota_{\mathcal{L}}, c, \mathcal{L}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

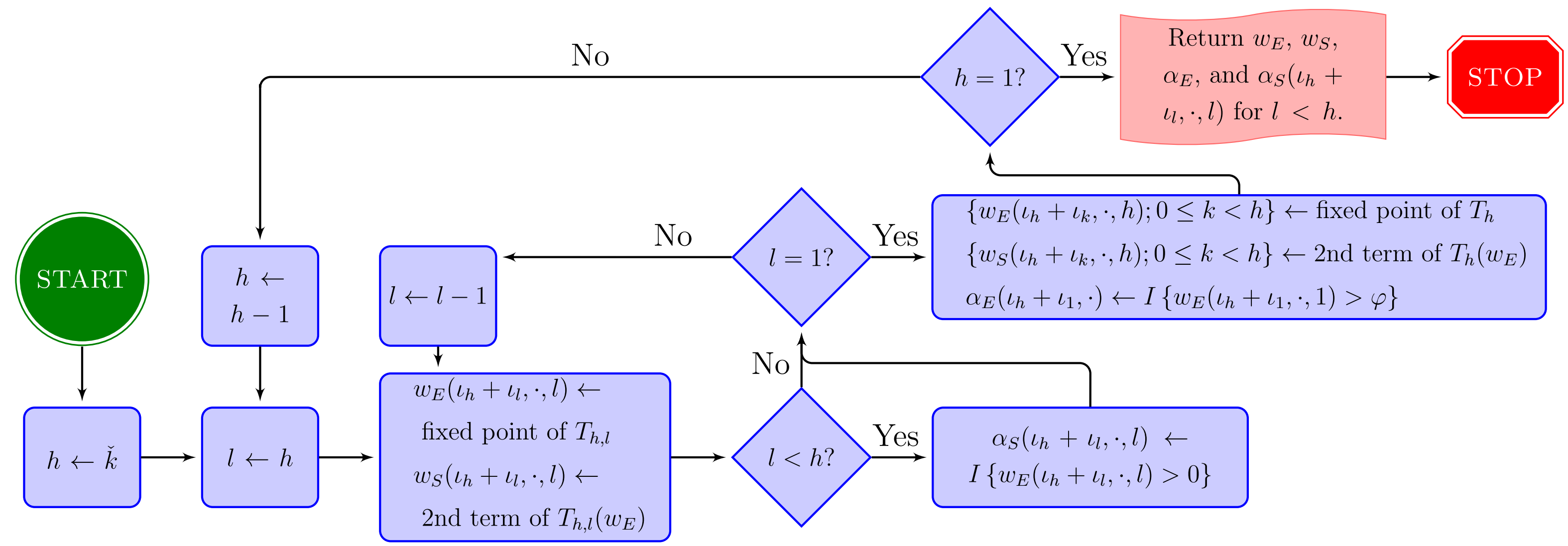
$$a_S(2\iota_{\mathcal{H}}, c, \mathcal{H}) = \begin{cases} 1 & \text{if } v_S(2\iota_{\mathcal{H}}, c, \mathcal{H}) > 0, \\ \frac{v_S(\iota_{\mathcal{H}}, c, \mathcal{H})}{v_S(\iota_{\mathcal{H}}, c, \mathcal{H}) - v_S(2\iota_{\mathcal{H}}, c, \mathcal{H})} & \text{if } v_S(2\iota_{\mathcal{H}}, c, \mathcal{H}) \leq 0 \text{ and } v_S(\iota_{\mathcal{H}}, c, \mathcal{H}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the equilibrium construction. Although adding productivity heterogeneity increased the number of steps required, calculating the fixed point of a contraction mapping on a low-dimensional function space remains the most computationally intensive required task.

3.3 Equilibrium Existence, Uniqueness, and Computation

We next extend the six-step calculation of duopoly equilibrium with $\check{k} = 2$ to allow for arbitrary $\check{k}$.The resulting algorithm consists of two procedures, which we present as flow charts in Procedures 1 and 2. The first computes all payoffs, survival strategies for duopolists facing strictly higher productivity types, and strategy for a potential entrant facing an incumbent. The second procedure calculates the survival strategies for duopoly incumbents with weakly higher productivity types and the strategy for a potential entrant facing an empty market.

In Procedure 1's flow chart, h indexes the productivity type for the weakly better firm, and l for the weakly worse firm. In the course of its execution, h decreases from $\check{k}$ to 1. For each level of h, l decreases from h to 1. For any pair of (h, l); the post-entry value w_E to the type l firm that faces a type h rival is computed as the fixed point of $T_{h,l}$. This functional operator is defined by the recursive condition for $w_E(\iota_l + \iota_h, \cdot, l)$. The type l firm has weakly lower productivity type and rationally expects its rival to remain whenever it continues with positive probability, so this firm's payoffs only depend on future states in which both firms survive. That is, evaluating $T_{h,l}$ only requires the continuation value being calculated and on $w_E(\iota_i + \iota_j, \cdot, j)$ for all $(i, j) \neq (h, l)$ such that $i \geq h$, $j \geq l$. Since Procedure 1 proceeds in descending order of (h, l), these post-entry values are in hand for the computation of $w_E(\iota_l + \iota_h, \cdot, l)$.



Required Functional Operators

$$
\begin{aligned}
T_{h,l}(f)(c) &= \max\left\{0, \beta\mathbb{E}\left[\sum_{i,j}\Pi_{hi}\Pi_{lj}\pi(\iota_i+\iota_j,C',j) + \sum_{(i,j)\neq(h,l)}\Pi_{hi}\Pi_{lj}w_E(\iota_i+\iota_j,C',j) + \Pi_{hh}\Pi_{ll}f(C') \,\middle|\, C=c\right]\right\} \\
T_h(f)(c,k) &= \max\left\{0, \beta\mathbb{E}\left[\alpha_S(\iota_h+\iota_k,c,k)\left(\sum_{i,j}\Pi_{hi}\Pi_{kj}\pi(\iota_i+\iota_j,C',i) + \sum_j \Pi_{hh}\Pi_{kj}f(C',j) + \sum_{i>h}\sum_j \Pi_{hi}\Pi_{kj}w_E(\iota_i+\iota_j,C',i)\right)\right.\right. \\
&\qquad + \{1-\alpha_S(\iota_h+\iota_k,c,k)\}\left(\sum_i \Pi_{hi}\pi(\iota_i,C',i) + \Pi_{hh}\{1-\alpha_E(\iota_h+\iota_1,C')\}f(C',0)\right. \\
&\qquad\qquad + \sum_{i>h}\Pi_{hi}\{1-\alpha_E(\iota_i+\iota_1,C')\}w_E(\iota_i,C',i) + \Pi_{hh}\alpha_E(\iota_h+\iota_1,C')f(C',1) \\
&\qquad\qquad \left.\left.\left. + \sum_{i>h}\Pi_{hi}\alpha_E(\iota_i+\iota_1,C')w_E(\iota_i+\iota_1,C',i)\right) \,\middle|\, C=c\right]\right\}, \quad \alpha_S(\iota_h+\iota_0,\cdot,0) \equiv 0, \Pi_{00} \equiv 1
\end{aligned}
$$

Procedure 1: Initial Equilibrium Calculations for the Heterogeneous Duopoly Model

When l reaches 1, the next step is to compute simultaneously the monopoly payoff for a type h firm and the duopoly payoff for a type h firm facing a type k (for all $k < h$) rival as the fixed point of T_h. Evaluating its right-hand side requires the value function being computed, the entry rule of a potential entrant facing an incumbent with productivity type no less than h ,the corresponding post-entry values for the incumbent, the survival strategies for rivals with types $k < h$, and the corresponding post-survival values. Again, previous computations using higher values of h, l determined these before this computation begins.

Procedure 2 complements Procedure 1 by determining the entry strategy for a potential monopolist; and the survival strategy for a firm with weakly better productivity type. All but one of these strategies are pure and refelct the values of entry and continuation as expcted. The survival strategy is mixed when both firms have the same type, the payoff to joint continuation is negative, and the payoff isolated continuation is positive. By construction, the resulting probability of survival lies in $(0, 1]$.

Algorithm 1 (Duopoly Equilibrium Calculation)**.** *Compute a candidate equilibrium strategy* (α_S, α_E) *and payoffs* w_S *and* w_E *in two steps:*

1. *Use Procedure 1 to compute* w_E*,* w_S*,* $\alpha_S(\iota_h + \iota_l, c, l)$ *and* $\alpha_E(\iota_h + \iota_1, c)$ *for all* $h, l \in \mathbb{K}, l < h$ *and* $c \in [\hat{c}, \check{c}]$*.*
2. *Use Procedure 2 to compute* $\alpha_E(\iota_1, c)$ *for all* $c \in [\hat{c}, \check{c}]$ *and to compute* $\alpha_S(\iota_h + \iota_l, c, h)$ *for all* $h \in \mathbb{K}$*,* $l \in \{0, \ldots, h\}$*, and* $c \in [\hat{c}, \check{c}]$*;*

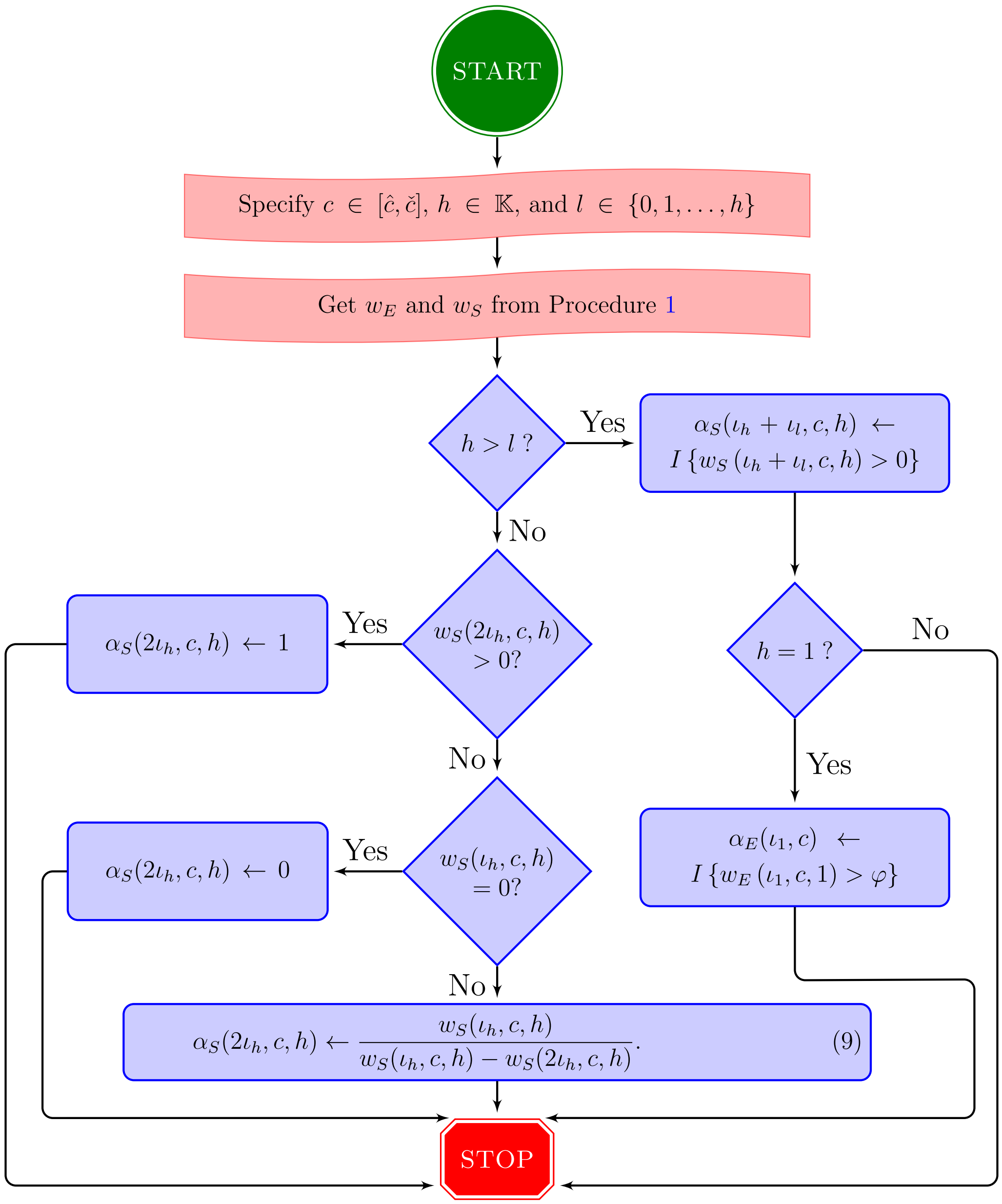
We can use Lemma 1 to prove that the constructed equilibrium is unique among all equilibria that default to inactivity.

Proposition 1 (Equilibrium in Heterogeneous Duopoly Model)**.** *There exists a unique natural Markov-perfect equilibrium. Algorithm 1 computes its payoffs and strategy. The equilibrium payoffs* $v_S = w_S$ *and* $v_E = w_E$*. The equilibrium strategy* $(a_S, a_E) = (\alpha_S, \alpha_E)$*.*

Proof. See Appendix A. □

3.4 Application

We apply our heterogenous-duopoly model to the welfare analysis of an R&D race game. Consider a market for some new good. In period t, C_t consumers populate the market. All of these consumers have the same utility function, which is quadratic in the quantity of the new good consumed. Consequently, total demand for the new good at time t and price p equals $C_t(a - p)/b$, for some parameters $a, b > 0$. A firm supplying q units of this good receives a surplus $C_t pq$.



Procedure 2: Calculation of Candidate Survival Rule for the Heterogeneous Duopoly Model

Firms must invent the good before they can supply it to the market. This requires that they enter the market, incurring an entry cost φ, and subsequently invest in R&D, at a fixed cost $\kappa(k)$. There are several milestone stages in the invention process, marked by $1, 2, \ldots, \check{k}$. New entrants start in stage 1 and, as long as they stay in the market and pay the fixed cost $\kappa(k)$ according to their current stage k, progress through the successive R& D stages according to a Markov chain with transition matrix Π. Once a firm reaches the final stage $\check{k}$, it has invented the product and can start selling it in the market. The fixed cost $\kappa(\check{k})$ still needs to be paid to produce the good.[8] An active firm may exit the market in any stage of the R&D race to avoid paying future fixed costs.

We assume that at most two firms are active in the market at any give time. If only one firm is active in stage $\check{k}$, it sells the good at the monopoly price. If two firms are selling the good, they set symmetric quantities to maximize $q^o p + \lambda q^r p$, where q^o and q^r denote the quantities set by the firm and its rival, respectively, and λ indexes the level of collusion. If $\lambda = 0$, these two firms are Cournot competitors. At higher values of λ, they collude more. If $\lambda = 1$, then they operate as if they are branches of a monopoly firm that split their joint monopoly revenues evenly.

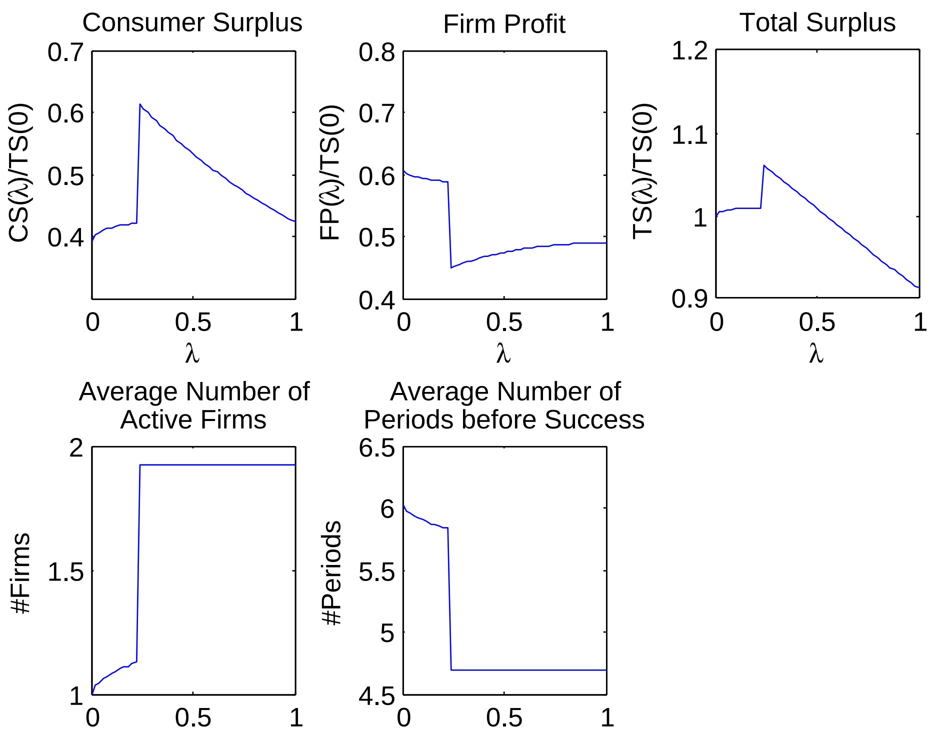
This game embodies Fershtman and Pakes' (2000) key "semi-collusion" assumption that firms may collude in setting quantities (or prices) but not when choosing R&D investment. Unlike Fershtman and Pakes, we take the level of collusion as exogenously given and ignore the intensive margin of the firms' strategic R&D investments. This focus on the (entry and exit) decisions to participate in the R&D race allow us to apply the heterogenous-duopoly model to analyze industry dynamics and welfare under different levels of collusion. We find that the model is sufficiently rich to replicate one of Fershtman and Pakes' main findings: Consumers may benefit from collusion, unlike in static models that take the industry structure as given. Intuitively, the direct negative effect of collusion on consumer welfare through weakened competition in the product market, well known from static models, is counteracted by a positive effect on R&D participation that increases product availability and product market competition.

To obtain this result, we first compute the model's unique natural Markov-perfect equilibrium for each value of λ between 0 and 1, with a 0.01 increment. Throughout, we specify $Q(\cdot|C)$ to approximate a random walk in the logarithm of C with innovation variance 0.3^2, reflected off of the state space's upper and lower boundaries, $\ln \hat{c} = -1.5$ and $\ln \check{c} = 1.5$. Also, we specify $\check{k} = 4$, $\beta = 0.95$, $\kappa(k) = 20$ for all k, $\varphi = 470$, demand parameters $a = 20$ and $b = 2$, and the Markov transition matrix Π for the R&D stages so that firms either progress one stage or remain put: $\Pi_{k,k} = \Pi_{k,k+1} = 0.5$ for all $k < \check{k}$ and $\Pi_{\check{k},\check{k}} = 1$.

[8] Alternatively, we can let the fixed cost decline with k. This only requires a minor adjustment of the model.

Subsequently, for each value of λ, we use the equilibrium strategy to simulate the market's evolution over 100 periods, starting from a fixed $c_0 = 2.718$, drawn from the demand process's ergodic distribution, and an empty market. We repeat the simulation 10,000 times, drawing new demand and type transitions in each simulation, but using the same random draws across the different values of λ. To analyze the impact of collusion on welfare, we compute, for each level of collusion λ, the discounted sum $FP(\lambda)$ of all firms' revenues net of all firms' fixed costs and entry costs over the 100 periods, and the discounted sum $CS(\lambda)$ of the consumer surplus over the 100 periods, both averaged over the 10,000 simulation runs. We assume that consumers have the same discount factor as firms. The total surplus $TS(\lambda) \equiv FP(\lambda)+CS(\lambda)$.

Figure 5: Welfare Analysis for Various Levels of Collusion



The upper-left and upper-middle panels of Figure 5 show $CS(\lambda)$ and $FP(\lambda)$ for each value of λ, as a proportion of the competitive market's total surplus $TS(0)$. First, if λ increases from 0, $CS(\lambda)$ gradually increases and $FP(\lambda)$ gradually decreases. Then, $CS(\lambda)$ jumps up and $FP(\lambda)$ jumps down. At higher levels of collusion, increases in λ decrease the consumer surplus and increase firms' profits.

Clearly, for low values of λ, the positive effect of increasing collusion on R&D investment dominates its direct weakening effect on product market competition. Figure 5's bottom-left panel sheds further light on this. It plots the number of active firms for each λ, averaged over the 100 periods and all the simulation runs. We observe a gradual increase and then an upward jump in the number of firms, paralleling the increase and jump in the consumer surplus. If λ is low; that is, with little or no collusion; no entrant facing a monopoly market can recover the sunk cost of entry, even when demand is at its highest level. Therefore, markets with little collusion are often monopoly markets. If λ increases, firms expect higher payoffs from a duopoly product market, and are more willing to participate in the R&D race, even if one firm is already in this race. The value of λ at which the number of firms and welfare jump is the level of collusion above which two firms enter immediately, in the initial demand state c_0.

This increase in the number of firms improves the consumer surplus in two ways. First, it improves product availability. Specifically, in this example, on average the first product reaches the market faster with higher levels of collusion (see Figure 5's bottom-middle panel). Second, it mitigates the anticompetitive effects of collusion, by ensuring that consumers are more often charged the (collusive) duopoly price, which, for all $\lambda < 1$, is lower than the monopoly price. At low levels of λ, the consumer welfare enhancing effects dominate the direct negative effects of increased collusion.

In contrast, as is clear from Figure 5's bottom panels, at higher levels of collusion, the market is often served by the maximum number of two firms. Consequently, further increases in λ have only small effects on the number of firms serving the market and the speed at which the good becomes available. Therefore, at higher levels of collusion, the direct effects of collusion dominate, and the consumer surplus gradually falls if λ increases. Nevertheless, the benefits from earlier consumption under full collusion ($\lambda = 1$) ensure that $CS(1) > CS(0)$.

The variation of $FP(\lambda)$ with λ mirrors the variation of $CS(\lambda)$. If λ crosses the level at which two firms immediately enter the market, instead of one, the total fixed cost incurred is doubled, but the total revenue is not. Consequently, $FP(\lambda)$ jumps down. For similar reasons, $FP(\lambda)$ falls gradually if λ increases at lower levels of collusion. In contrast, at higher levels of collusion, the market is usually a duopoly and the market structure does not change much with increases in λ. Consequently, the positive effects of such increases on the collusive duopoly price dominate, and $FP(\lambda)$ increases. Finally, $FP(1) < FP(0)$, because of scale savings: The monopoly price is usually charged at either collusive extreme, but two firms, instead of one, often incur fixed costs under full collusion.

Figure 5's upper-right panel plots the total surplus as a fraction of the competitive market's total surplus. At low levels of collusion, an increase in λ increases $TS(\lambda)$. In particular, the upward jump in the number of firms leads to an upward jump in the total surplus. At these levels of collusion, the positive effects of increased product market competition and earlier consumption on consumer welfare dominate its negative effects on firms through price decreases and fixed cost increases. At higher levels of collusion, the total surplus falls with increases in λ, because R&D activity is hardly affected and the negative welfare effects of collusion familiar from static models dominate.

In this specific example, as in static models that take the market structure as given, full collusion in the product markets lowers welfare below that in a competitive market: $TS(1) < TS(0)$. However, unlike in such static models, the competitive market is often served by only one firm and monopolistic pricing is common under both levels of collusion. Consequently, the result that full collusion lowers total welfare cannot be explained by the usual negative welfare effects of collusive pricing. Instead, it is due to the waste of fixed costs caused by excess entry of producers, which is not offset by the gains from earlier consumption.[9]

It is worth stressing that these results are obtained at a very low computational cost. For any particular value of λ, with 301 grid points for C and the parameter values of this section's experiment, we can solve the model within one second using Matlab on a PC. Even with $\beta = 0.995$ (monthly data) and $\check{k} = 10$, which implies a state space with over 33,000 points, we can solve the model in about 5–30 seconds on a PC.[10] This feature of our framework makes it a very useful complement to existing richer, but computationally more forbidding, frameworks for the analysis of industry dynamics. For example, Fershtman and Pakes' framework allows for a more detailed study of collusion dynamics by modeling, among other things, the intensive margin of R&D investment and endogenizing collusion. However, their framework's comparative richness comes at a substantial computational cost: It makes the replication of their results across different parameter values very hard. In contrast, our framework allows us to quickly examine the welfare implications of collusion for a wide range of parameter values.

4 The General Model

We now turn to the general model, with arbitrary finite $\check{m}$ and $\check{k}$. The central difficulty of the equilibrium analysis is that the equilibrium payoff function does not necessarily satisfy a monotonicity property analogous to the ones in Lemmas 1. In Section 4.1, we first analyze a type of equilibrium in which the payoffs still retain the monotonicity property: Adding an active firm into the industry weakly decreases other firms' payoffs. We can straightforwardly extend Algorithm 1 and use a sequence of contraction mappings to efficiently compute such an equilibrium, if it does exist. This monotonicity property is the key to establish the essential

[9]Obviously, this result can be reversed if consumers are impatient and/or have much larger weight in the total surplus than producers do.

[10]We use value function iteration to compute the fixed points of the contraction mappings, which simplifies our code, but results in a (slow) linear convergence rate in β. To cope with this issue, one can turn to more sophisticated approaches (see Judd, 1998, for a brief survey). For example, Ferris, Judd, and Schmedders's (2007) Newton-based method ensures global convergence with a quadratic convergence rate.

uniqueness of this type of equilibrium. However, since the monotonicity property does not always hold in the general model, this type of equilibrium may not exist. In Section 4.2, we discuss a simple example in which the monotonicity of equilibrium payoffs is violated and multiple equilibria emerge. In one class of those equilibria, if firms were allowed to renegotiate, they could strictly improve their payoffs by playing another equilibrium. We continue to focus on the type of equilibria that are renegotiation-proof and establish their existence. An extension of our algorithm can compute all such equilibria, if C has a discrete distribution.

4.1 Payoff-Monotone Equilibrium

We define an equilibrium to be *payoff-monotone* if the equilibrium payoffs satisfy conditions analogous to the ones in Lemma 1.

Definition 3. *A Markov-perfect equilibrium is payoff-monotone if its equilibrium payoff functions satisfies $v_S(m, c, k) \geq v_S(m + \iota_k, c, k)$ and $v_E(m, c, k) \geq v_E(m + \iota_k, c, k)$ for all (m, c, k).*

We showed in Section 3 that duopoly firms of the same type choose to continue if continuation guarantees a positive payoff, because the heterogenous duopoly model's equilibrium payoffs satisfy Lemma 1. This property allows us to construct a sequence of contraction mappings in Algorithm 1 to compute the unique natural Markov-perfect equilibrium. Similarly, in the general model, suppose that, for some parameter values, there exists a payoff-monotone natural Markov-perfect equilibrium. Then, if continuation renders the payoff to all firms of the same type positive, continuation is the dominant strategy for these firms. Following the argument leading to condition (5) in Section 3.1, we can establish similar necessary conditions on equilibrium payoffs. For instance, in the market with $\check{m}$ type $\check{k}$ firms, the payoff-monotone equilibrium payoff $v_E(\check{m}\iota_{\check{k}}, c, \check{k})$ necessarily satisfies

$$v_E(\check{m}\iota_{\check{k}}, c, \check{k}) = \max\{0, \beta\mathbb{E}[\pi(\check{m}\iota_{\check{k}}, c', \check{k}) + v_E(\check{m}\iota_{\check{k}}, c', \check{k})|C = c]\}. \tag{10}$$

The right hand side of (10) defines a Bellman operator that uniquely determines $v_E(\check{m}\iota_{\check{k}}, \cdot, \check{k})$.

Note that the heterogeneous duopoly model and the general model only differ in the number of firms and share essentially the same dynamic specification. Therefore, Algorithm 1 can be naturally extended to solve for the payoff-monotone natural equilibrium, by computed the fixed points of a sequence of contraction mappings. Similarly to Algorithm 1, we partition the state space, order the parts, and compute the equilibrium in a corresponding sequence of steps. Each step covers the computation on a single part of the state space. We order the steps so that all results that are needed in later steps are passed on from earlier steps.

The partition and its order are defined using an oriental lexicographic order.

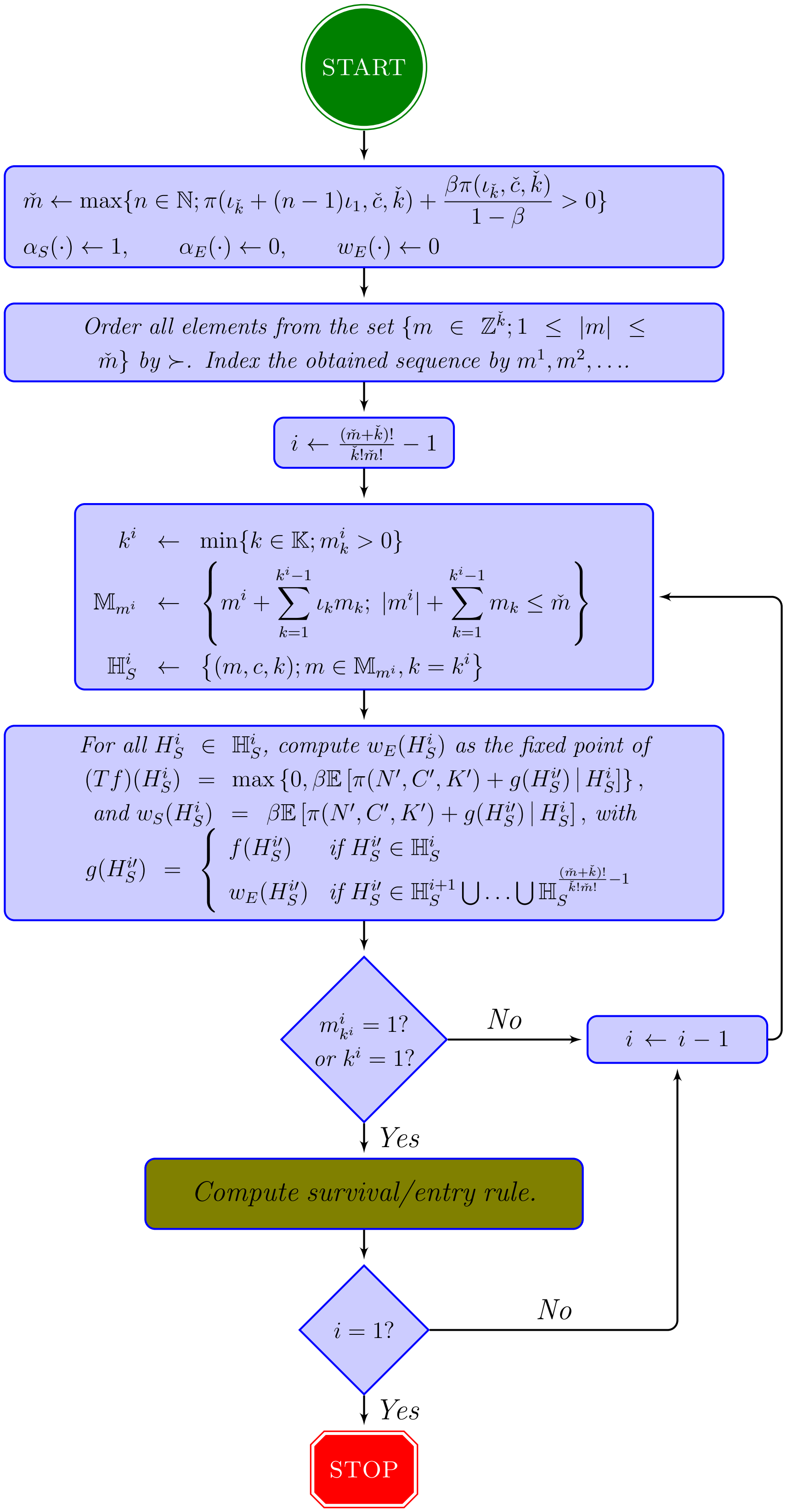
Definition 4. *Oriental lexicographical superiority (OLS) $\succ$ is a relation over $\mathbb{R}^n$. For any pair of vectors $x, y \in \mathbb{R}^n$, $x \succ y$ if $x_n > y_n$, or $(x_n = y_n$ and $x_{n-1} > y_{n-1})$, or $, \ldots,$ or $(x_n = y_n$ and $x_{n-1} = y_{n-1}$ and $\ldots$ and $x_1 > y_1)$.*

We use the phrase "oriental" because the vectors x, y are read from right to left when being compared, as in Arabic and Hebrew. In the previous sections, we have implicitly used an ordering based on OLS; the equilibrium payoff for an OLS market structure is always computed before the payoffs in any state it is superior to. For example, in Section 3.1's one productivity type example, $\succ$ is equivalent to $>$ on $\mathbb{R}$ and the payoff to a duopolist is computed first, followed by the payoff to a monopolist. In Section 3.2, the sequence of market structures considered was $\{2\iota_{\mathcal{H}}, \iota_{\mathcal{H}} + \iota_{\mathcal{L}}, \iota_{\mathcal{H}}, 2\iota_{\mathcal{L}}, \iota_{\mathcal{L}}\}$. Thus, we partitioned the state space into five parts and ordered them in decreasing OLS to compute the equilibrium payoffs and strategy. Furthermore, this ordering extends to Algorithm 1 as well; the index pair (h, l) in Procedure 1 is decreasing in OLS. This ordering ensures that equilibrium payoffs and entry/survival rules necessary for computation in later steps are calculated in earlier steps.

We construct the algorithm for the general model following the same ordering. For any $(\check{m}, \check{k})$ pair, there are $\binom{\check{m}+\check{k}}{\check{k}} - 1 = \frac{(\check{m}+\check{k})!}{\check{k}!\check{m}!} - 1$ possible non-empty markets. First, we partition the state space into $\frac{(\check{m}+\check{k})!}{\check{k}!\check{m}!} - 1$ parts, with each step of the algorithm computing the payoff on one of these parts. In step i, to see what the states in this part are, suppose the i-th ranked market structure in the OLS sequence is m^i. Let $k^i = \min\{k \in \mathbb{K}; m^i_k > 0\}$ be the lowest type of active firm in m^i and let a set $\mathbb{M}_{m^i}$ collect all the market structures that share the same number of type-$k^i, k^i + 1, \ldots, \check{k}$ firms as m^i. The part of the state space considered in this step is then $\{(m, c, k^i); m \in \mathbb{M}_{m^i}, c \in [\hat{c}, \check{c}]\}$. In other words, in the i-th step, we compute a type-k^i firm's payoff in every market structure in $\mathbb{M}_{m^i}$ for all c. Since this part of the state space are constructed from m^i, we say that it is *indexed* by m^i, and hence name m^i as the *indexing market structure.*

The first step of the algorithm is indexed by the most superior market structure, $\check{m}\iota_{\check{k}}$. Then, we proceed the algorithm and sequentially determine the payoffs and strategy for market structures in the order of decreasing OLS.

Algorithm 2 (Calculation of a Candidate Equilibrium for the General Model).



In Algorithm 2, when computing the candidate post-entry payoff w_E as the fixed point of the Bellman equation, the expectation relies on the relevant parts of other firms strategy[11]. Because of the algorithm's OLS ordering, for the firms with lower productivity types than the firm of interest and the potential entrants, these values have been computed in previous steps. The survival rules for firms with productivity types at least as good as the firm of interest are set to continuation. Also, the OLS ordering of the algorithm helps to ensure that the current states that the firm of interest is facing only evolves to states that have been covered in previous computation, providing that this firm continues. Hence, all relevant values of future post-entry payoff have been computed in previous steps. Then, T is always a contraction mapping with unique fixed point $w_E(H_S)$.

Procedure 3 is devoted to computing the survival/entry strategy. In this procedure, when firms are randomizing between survival and exit, the mixing probability is chosen to be one of possible probabilities that solves the indifference condition (11). If it is not profitable to unilaterally deviate from exit to survival given all other firms of the same type opt for exit, the mixing probability can also be set to 0.

Note that Algorithm 2 does not require w_E to be monotone as in Definition 3. After computing w_E with Algorithm 2, we can check whether it satisfies this monotonicity condition. If it does, we can show that the candidate equilibrium strategy (α_S, α_E) is unique. We can also verify that (α_S, α_E) forms a natural Markov-perfect equilibrium. Since the Bellman equation for w_E defines the necessary condition for any payoff-monotone natural Markov-perfect equilibrium payoff, if one such equilibrium exists, not only we are able to compute it using Algorithm 2, but also we can prove its essential uniqueness.

Proposition 2 (Payoff-Monotone Equilibrium in the General Model)**.** *If there exists a payoff-monotone natural Markov-perfect equilibrium, it is the unique such equilibrium and Algorithm 2 computes it. The post-entry equilibrium payoff function is w_E and the equilibrium strategy is (α_S, α_E).*

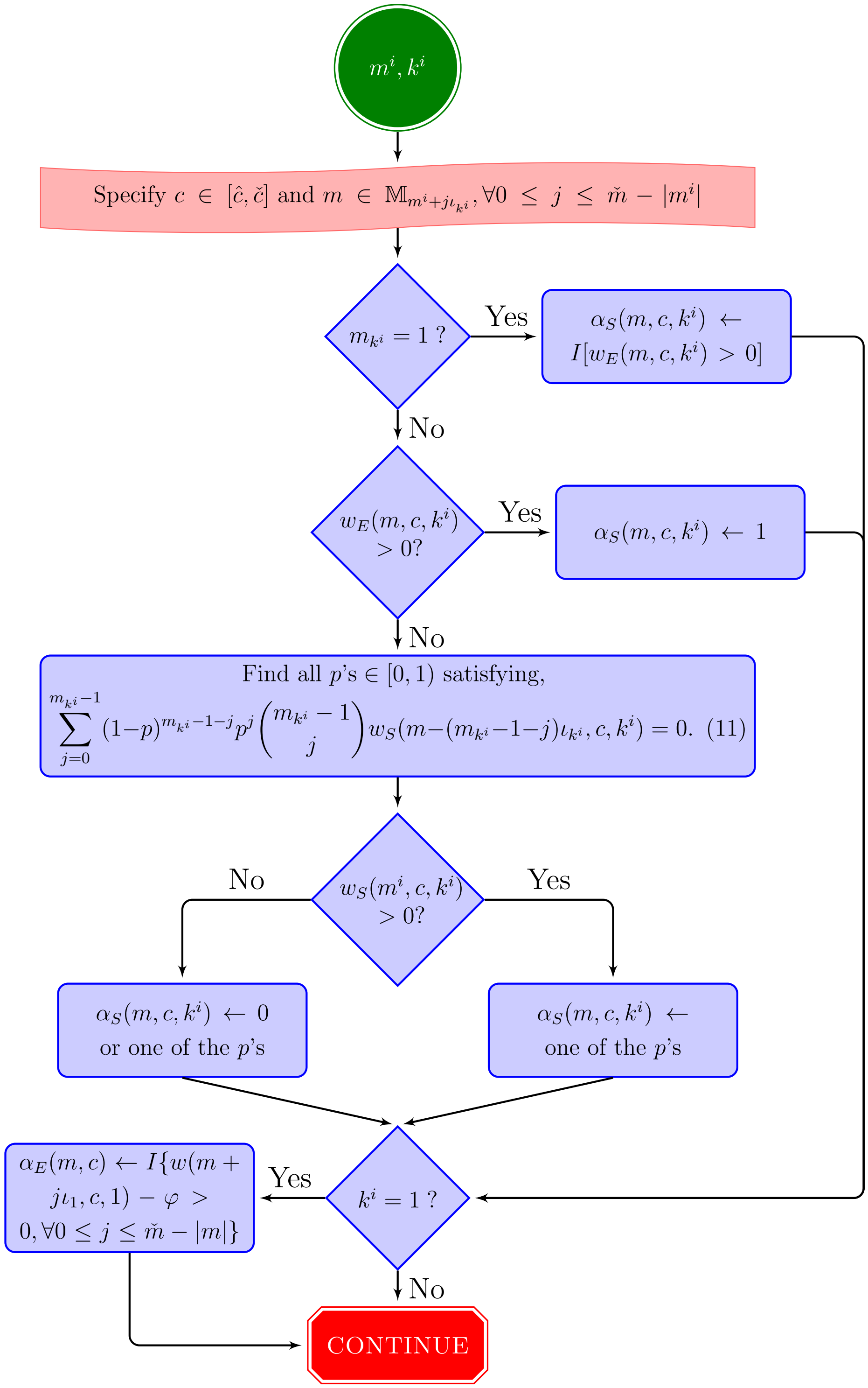
Proof. See Appendix B. □

4.2 Renegotiation-Proof Equilibrium

Proposition 2 implies that a payoff-monotone natural Markov-perfect equilibrium does not exist if w_E is not monotone as in Definition 3.

In Appendix C, we present a simple example in which equilibrium payoffs are not monotone and there are multiple natural Markov-perfect equilibria. In this example, we consider an industry with at most three active firms. We assume that firms can be type $\mathcal{H}$ or type

[11] Computing the market structure transition matrix conditional on firms' strategy is conceptually straightforward, but practically involved. We describe the details in Section ?? of the online appendix.



Procedure 3: Calculation of Candidate Entry/Survival Rule for the General Model

$\mathcal{L}$. For two type $\mathcal{H}$ duopoly firms contemplating survival, we create a situation that if these two firms jointly continue to next period, any type $\mathcal{L}$ potential entrant will never find it profitable to enter this market. This way, the type $\mathcal{H}$ duopolists deter any future entry by joint survival and enjoy a high duopoly surplus forever. Otherwise, if one of the firms exits, then two type-$\mathcal{L}$ firms will enter the market and remain active onwards. The survived type-$\mathcal{H}$ firm will only receive a low triopoly surplus thereafter. Connecting this example to the static survival game depicted in Figure 2, we construct the payoff matrix such that, for some c, the post-survival value satisfies $v_S(2\iota_{\mathcal{H}}, c, \mathcal{H}) > 0 > v_S(\iota_{\mathcal{H}}, c, \mathcal{H})$. Therefore, although "Survive, Survive" remains an equilibrium in this static game, "Exit, Exit" emerges as another equilibrium. Also, there could be equilibria involving mixed strategies. Indeed, we show in Appendix C that we do have three possible equilibrium actions at this particular point of the game tree. Namely, to survive for sure, to exit for sure, and to survive with some probability. We further demonstrate that when three firms are randomizing between survival and exit because joint survival is not profitable, the mixing probability can be multiple.

We distinguish two sources of equilibrium multiplicity using this example. One comes from the incumbents' failure to jointly continue if this is profitable. If the two type-$\mathcal{H}$ firms can coordinate on continuation, they can strictly improve their equilibrium payoffs. Since these two firms repeatedly interact, it seems reasonable to assume that they are able to "renegotiate" to joint continuation whenever this is profitable. Henceforth, we restrict attention to equilibria with the desirable property that firms cannot improve their payoffs by one-shot change of action. We call this property *renegotiation-proofness.*

Definition 5. *A natural Markov-perfect equilibrium is (one-shot) renegotiation-proof if, for any* (m, c) *pair, no one-shot agreement satisfying the following properties can be negotiated:*

- *all firms in the agreement change their survival actions once;*
- *the agreement is self-enforcing, so no firm in the agreement has incentive to unilaterally change the agreed action;*
- *if one type* k *firm is in the agreement, all type* k *firms are; and*
- *the payoffs to all firms* in the agreement *are strictly improved.*

In any equilibrium, a firm earns positive payoffs only when continuing for sure. Therefore, if all firms of a certain type can strictly improve their payoff by changing their actions, it must be the case that (i) the actions must be changed from exiting with non-negative probability to surviving with probability one (ii) the actions of joint continuation must give all firms in the agreement positive payoff. Therefore, this refinement has bite only when all incumbents of certain type(s) could coordinate on sure joint continuation and earn positive payoffs, but will not unilaterally continue if others do not. Note that in the duopoly model, Lemma 1 ensures that both incumbents of the same type continue for sure if joint continuation renders payoff positive. Therefore, no further improvement is possible via renegotiation. Consequently, the natural equilibrium in Section 3 is renegotiation-proof. Since the monotonicity in Definition 3 essentially functions in the same way as the monotonicity in Lemma1, the payoff-monotone equilibrium in the general model is also renegotiation-proof.

Recall that in Algorithm 2, $\alpha_S(m, c, k)$ is set to its initial value of one when computing $w_E(m, c, k)$. This implies that all type-k firms are "forced" to jointly continue if positive payoff is expected. Therefore, the Bellman equation for w_E is a necessary condition on w_E for a renegotiation-proof natural Markov-perfect equilibrium. When verifying that (α_S, α_E) forms a natural equilibrium, we also verify that it is a renegotiation-proof one. We can further show that Algorithm 2 always delivers some (α_S, α_E) as its outcome, which proves the existence of a renegotiation-proof natural Markov-perfect equilibrium.

Proposition 3 (Renegotiation-proof Equilibrium in the General Model)**.** *Algorithm 2 always computes some (α_S, α_E) and this strategy (α_S, α_E) forms a renegotiation-proof natural Markov-perfect equilibrium. So, a renegotiation-proof natural Markov-perfect equilibrium always exists.*

Proof. See Appendix B. □

The renegotiation-proof property helps to eliminate the equilibria involving exit and mixing actions when joint continuation is profitable. However, the other source of the multiplicity persists. As we have illustrated in the example in Appendix C, when joint survival is not profitable and more than two firms are randomizing between survival and exit, there can be multiple equilibrium mixing probabilities. The property of renegotiation-proofness is silent on which probability to select. Therefore, each distinct equilibrium mixing probability leads to a different equilibrium survival rule. Different combinations of the equilibrium survival rules result in different renegotiation-proof natural Markov-perfect equilibria. Since the Bellman equation for w_E defines the necessary condition for renegotiation-proof natural Markov-perfect equilibrium payoff, any such equilibrium should be an outcome of Algorithm 2, if the mixing probabilities that correspond to this equilibrium are used in the computation.

Recall that we do have proven in Proposition 2 that if a payoff-monotone equilibrium exists, the mixing probability is always unique, and such equilibrium is the unique outcome of Algorithm 2. This implies the following corollary to Proposition 2.

Corollary 1. *If there exists a payoff-monotone natural Markov-perfect equilibrium, it is also the unique renegotiation-proof natural Markov-perfect equilibrium.*

If there is no payoff-monotone equilibrium, the possible multiplicity of mixing probabilities may challenge our equilibrium computation. After all, each step of Algorithm 2 requires the unique input of payoffs and rules computed in the previous steps. (In Section 4.1, Algorithm 2 simply selects an arbitrary mixing probability to continue when the multiplicity arises.) In Section **??** of the online appendix, we prove that the number of renegotiation-proof natural Markov-perfect equilibria is finite if C is discrete. We also extend Algorithm 2 so that it computes all renegotiation-proof natural Markov-perfect equilibria, by creating parallel branches of Algorithm 2 every time the multiplicity arises, with each branch corresponding to a distinct choice of mixing probability.

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Appendices

A Proofs for Section 3

Proof of Lemma 1. First, we verify a property of the post-entry payoff function.

Property 1. For a given $k \in \mathbb{K}$, and for all x such that $0 \le x \le \check{k}$, if $v_E(\iota_k + \iota_x, \cdot, k)$ is weakly decreasing in x, we say that v_E satisfies Property 1 for k.

Intuitively, Property 1 requires that a type-k firm's post-entry payoff is weakly decreasing in its opponent's type. A sufficient condition for Lemma 1 is that in any natural equilibrium, v_E satisfies Property 1 for all $k \in \mathbb{K}$. In order to prove this sufficient condition, we also need to verify v_E satisfies three other interdependent properties for all $k \in \mathbb{K}$.

Property 2. For a given $k \in \mathbb{K}$, and for all x such that $0 \le x \le \check{k}$, if $v_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x, we say that v_E satisfies Property 2 for k.

Property 3. For a given $k \in \mathbb{K}$, and for all x such that $1 \le x \le k-1$, if $v_E(\iota_k + \iota_x, \cdot, x) \le v_E(\iota_{k-1} + \iota_x, \cdot, x)$, we say that v_E satisfies Property 3 for k.

Property 4. For a given $k \in \mathbb{K}$, if $v_E(\iota_k, \cdot, k) \ge v_E(\iota_{k-1}, \cdot, k-1)$, we say that v_E satisfies Property 4 for k[12].

To prove the three properties for all k, we first show that given any equilibrium strategy (a_S, a_E), v_E is computed as the unique fixed point of a contraction mapping. Then, we prove in turn that v_E satisfies Property 2, 1, 3, and 4 for $k = \check{k}$. Therefore, if we have a Banach space in which all elements are functions that satisfy the properties for $\check{k}$, the contraction maps this space into itself. Next, we iterate on $k = \check{k}-1, \dots, 1$. In each step, we start with a Banach space in which all elements are functions that satisfy the properties for $k+1, \dots, \check{k}$. We know from the previous steps that the contraction maps the space into itself. Then, we construct a smaller Banach space by requiring all its elements to additionally satisfy the properties for k. We further show that the contraction maps the smaller Banach space into itself. This procedure gives us a sequence of shrinking spaces, with the smallest one contains all functions that satisfy Properties 1-4 for all k. Because v_E is the unique fixed point of the contraction, it must be in the smallest space and satisfy these properties for all k. We leave all the algebraic details to Section **??** of the online appendix.

After Property 1 is verified for all k, $v_E(2\iota_k, c, k) \le v_E(\iota_k, c, k)$ for any $k \in \mathbb{K}$ follows immediately. Using the definition in equation (2) and Property 1, we can prove $v_S(2\iota_k, c, k) \le v_S(\iota_k, c, k)$ for any $k \in \mathbb{K}$. The details are in the online appendix. □

[12] For completeness, $v_E(\iota_0, \cdot, 0) \equiv 0$.

Proof of Proposition 1. We prove the proposition in two steps. First, we establish a lemma verifying that the candidate equilibrium computed by Algorithm 1 is indeed a natural Markov-perfect equilibrium. Then, we use Lemma 1 to prove that the constructed equilibrium is essentially unique.

Lemma 2. *The strategy (α_S, α_E) and payoff function w_E constructed by Algorithm 1 form a natural Markov-perfect equilibrium.*

Proof. The proof for Lemma 2 has two parts. First, note that Algorithm 1 already embodies the requirement in Definition 1, i.e., for $k^1 > k^2$, holding $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$ when computing the payoff and strategy for k^2 firm. We then need to verify that the candidate equilibrium payoff function w_E supports this heuristic, i.e., $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) > 0$ whenever $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) > 0$. Second, we show that (α_S, α_E) forms a natural equilibrium by proving that it is one-shot-deviation-proof.

Prove $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$. Given any (α_S, α_E), w_E is computed as the fixed points of the contraction mappings in Algorithm 1. This enables us to use the same trick as in the proof for Lemma 1. We focus on a Banach space in which all elements are functions that satisfy $f(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq f(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$. Then we prove that all the contractions computing w_E map the space into itself. The algebraic details are in the online appendix, Section **??**.

Verify one-shot deviation proofness. To verify one-shot deviation proofness for α_S, we need to show that for any k^1, k^2, c, $1 \leq k^1 \leq \check{k}$ and $0 \leq k^2 \leq \check{k}$,

$$\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) \in \arg\max_{a \in [0,1]} a\mathbb{E}[w_S(M', c, k^1) | M = \iota_{k^1} + \iota_{k^2}] \tag{12}$$

where

$$\mathbb{E}[w_S(M', c, k^1) | M = \iota_{k^1} + \iota_{k^2}] = \begin{cases} a_S(\iota_{k^1} + \iota_{k^2}, c, k^2) w_S(\iota_{k^1} + \iota_{k^2}, c, k^1) & \\ +(1 - a_S(\iota_{k^1} + \iota_{k^2}, c, k^2)) w_S(\iota_{k^1}, c, k^1) & \text{if } k^1 \geq k^2 \\ w_S(\iota_{k^1} + \iota_{k^2}, c, k^1) & \text{if } k^1 < k^2 \end{cases} \tag{13}$$

and w_S is defined analogously as v_S by equation 2. To verify (12), consider the following cases

1. For all c such that $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$, we know $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) > 0$. Then, we show that $\mathbb{E}[w_S(M', c, k^1) | M = \iota_{k^1} + \iota_{k^2}] > 0$.

   (a) If $k^1 \leq k^2$, then $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1)$ is computed by T_{k^2, k^1} and $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, w_S(\iota_{k^1} + \iota_{k^2}, c, k^1)\} > 0$. Then, from (13), $\mathbb{E}[w_S(M', c, k^1) | M = \iota_{k^1} + \iota_{k^2}] = w_S(\iota_{k^1} + \iota_{k^2}, c, k^1) > 0$.

(b) If $k^1 > k^2$, then $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1)$ is computed by T_{k^1} and $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, \mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}]\} > 0$, so $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1}+\iota_{k^2}] > 0$.

So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = \{1\} \ni \alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$.

2. For all c such that $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 0$, we know $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = 0$. Then, we show that $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = 0$.

(a) If $k^1 < k^2$, then $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1)$ is computed by T_{k^2,k^1} and $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, w_S(\iota_{k^1} + \iota_{k^2}, c, k^1)\} = 0$. Then, from (13), $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1}+\iota_{k^2}] = w_S(\iota_{k^1} + \iota_{k^2}, c, k^1) \leq 0$.

(b) If $k^1 \geq k^2$, then in natural equilibrium it must be that $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$, and hence (13) gives $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = w_S(\iota_{k^1}, c, k^1)$. Because $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1)$ is computed by T_{k^1} and $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, w_S(\iota_{k^1}, c, k^1)\} = 0$, $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] \leq 0$.

So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] \ni \alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 0$. Note that if we require default to inactivity, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = \{0\}$.

3. For all c such that $\alpha_S(2\iota_{k^1}, c, k^1)$ is determined by (9), then $k^1 = k^2$ and

$$\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1}+\iota_{k^2}] = \alpha_S(2\iota_{k^1}, c, k^1)w_S(2\iota_{k^1}, c, k^1)+(1-\alpha_S(2\iota_{k^1}, c, k^1))w_S(\iota_{k^1}, c, k^1) = 0.$$

The last equality is due to equation (9). So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = [0, 1] \ni \alpha_S(2\iota_{k^1}, c, k^1)$.

To verify one-shot-deviation-proofness for α_E, we need to show that for any k, c, $0 \leq k \leq \check{k}$,

$$\alpha_E(\iota_k + \iota_1, c) \in \arg\max_{a\in[0,1]} a(\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] - \varphi) \tag{14}$$

where

$$\mathbb{E}[w_E(M', c, 1)|M = \iota_k+\iota_1] = \begin{cases} w_E(\iota_k + \iota_1, c, 1) & \text{if } k > 0 \\ (1 - \alpha_E(2\iota_1, c))w_E(\iota_1, c, 1) + \alpha_E(2\iota_1, c)w_E(2\iota_1, c, 1) & \text{if } k = 0 \end{cases}$$

By construction, $\alpha_E(\iota_k + \iota_1, c)$ satisfies (14) except for $k = 0$. Thus, at this moment, we can assert that $\alpha_E(\iota_k + \iota_1, c)$ is a natural Markov-perfect equilibrium strategy for all $k > 0$. Since no operator T in Algorithm 1 depends on $\alpha_E(\iota_1, c)$, this result is sufficient to ensure that the fixed points, w_E is the natural Markov-perfect equilibrium payoff corresponding to (α_S, α_E). Lemma 1 then guarantees that w_E also exhibits $w_E(2\iota_1, c, 1) \leq w_E(\iota_1, c, 1)$. Thus,

1. when $\alpha_E(2\iota_1, c) = 1$, it must be the case that $w_E(\iota_1, c, 1) \geq w_E(2\iota_1, c, 1) > \varphi$ so $\alpha_E(\iota_1, c) = 1$. The right-hand-side of (14) is

$$\arg\max_{a\in[0,1]} a(\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] - \varphi) = \arg\max_{a\in[0,1]} a(w_E(2\iota_1, c, 1) - \varphi) = \{1\} \ni \alpha_E(\iota_1, c).$$

2. when $\alpha_E(2\iota_1, c) = 0$, $\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] = w_E(\iota_1, c, 1)$. So $\alpha_E(\iota_1, c) = I[w_E(\iota_1, c, 1) > \varphi]$ satisfies (14).

So, we conclude that (α_S, α_E) forms a natural Markov-perfect equilibrium and w_E, w_S are the associated payoffs. □

With Lemma 2 in hand, we can prove the uniqueness of natural Markov-perfect equilibrium following the order of Procedure 1, starting from $k_1^1 = \check{k}$. First, observe that in a symmetric equilibrium, if $v_S(2\iota_{\check{k}}, c, \check{k}) \leq 0$, then $v_E(2\iota_{\check{k}}, c, \check{k}) = 0$. Then, if $v_S(2\iota_{\check{k}}, c, \check{k}) > 0$, Lemma 1 ensures that in any equilibrium, $v_S(\iota_{\check{k}}, c, \check{k}) \geq v_E(2\iota_{\check{k}}, c, \check{k}) > 0$ for any $n < \check{m}$. This means that continuation dominates any other action when $v_S(2\iota_{\check{k}}, c, \check{k}) > 0$. Therefore, any equilibrium post-entry payoff must be a fixed point of $T_{\check{k},\check{k}}$ in Algorithm 1, $v_E(2\iota_{\check{k}}, \cdot, \check{k}) = w_E(2\iota_{\check{k}} + \iota_k, \cdot, k)$. Consequently, for any $k < \check{k}$, $w_E(\iota_{\check{k}} + \iota_k, \cdot, k)$ is determined as the unique natural Markov-perfect equilibrium payoff. Recall that in the proof for Lemma 2, the optimal strategy sets for type-k firm are singletons. Therefore, $\alpha_S(\iota_{\check{k}} + \iota_k, \cdot, k)$ and $\alpha_E(\iota_{\check{k}} + \iota_1, \cdot)$ is the unique natural Markov-perfect equilibrium strategy, which guarantees that $w_E(\iota_{\check{k}} + \iota_k, \cdot, \check{k})$ is the unique post-entry equilibrium payoff. The uniqueness of $w_S(\iota_{\check{k}} + \iota_k, \cdot, \check{k})$ readily follows and the uniqueness of $a_S(2\iota_{\check{k}}, \cdot, \check{k})$ is ensured by equation (9).

In the i-th steps in Procedure 1, suppose the pair of types considered is $(h, l) = (k_1, k_2)$. When $k^1 = k^2$, Lemma 1 and equilibrium symmetry again ensure that any equilibrium post-entry payoff $v_E(2\iota_{k^1}, \cdot, k^1)$ must be a fixed point of T_{k^1,k^1}. Hence, $v_E(2\iota_{k^1}, \cdot, k^1) = w_E(2\iota_{k^1}, \cdot, k^1)$ is the unique natural Markov-perfect equilibrium payoff. Then, for any $k^2 < k^1$, since T_{k^1,k^2} does not depend on any strategy, $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$ is determined as the unique post-entry equilibrium payoff and $\alpha_S(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$, $\alpha_E(\iota_{k^1} + \iota_1, \cdot)$ as the unique natural Markov-perfect equilibrium strategy that is consistent with payoff-maximization. Because T_{k^1} only depends on natural Markov-perfect equilibrium strategy that has been verified to be unique, $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1)$ is then uniquely determined as the post-entry equilibrium payoff. The uniqueness of $w_S(\iota_{k^1} + \iota_{k^2}, \cdot, k^1)$ and $w_S(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$ are then straightforwardly verified. Equation (9) ensures the uniqueness of $a_S(2\iota_{k^1}, \cdot, k^1)$. □

B Proofs for Section 4

To prove Propositions 3 and 2, we prove a useful lemma first.

Lemma 3. *Algorithm 2 always delivers some (α_S, α_E) as outcome. Furthermore, (α_E, α_S) forms a natural Markov-perfect equilibrium, with payoffs w_E and w_S.*

Proof. We follow four steps to prove this lemma. We first show that Algorithm 2 computes w_E, w_S, α_S, and α_E for all (m, c, k). Second, we prove that Algorithm 2 always delivers some well-defined (α_S, α_E) as outcome. This is a nontrivial step. Because we need to show that w_E as the fixed point of T always exist, and α_S is well defined when Procedure 3 assigns p from equation (11) to it.

After proving the first part of the lemma, we verify in the third step of the proof that α_S satisfies the requirement in Definition 1. Eventually, we prove that in each step of Algorithm 2, w_E is constructed as an equilibrium post-entry value, and the corresponding w_S gives the equilibrium post-survival payoff. Along the way, we also show that (α_S, α_E) is an natural equilibrium strategy.

First, note that the set of all indexing market structures is $\mathbb{M} \equiv \{m \in \mathbb{Z}^{\check{k}}; 1 \le |m| \le \check{m}\}$, which is also the set of all payoff-relevant market structures. Consider any (m, k) pair such that $m \in \mathbb{M}$ and $m_k > 0$, $w_E(m, \cdot, k)$ and $w_S(m, \cdot, k)$ are computed in the step with indexing market structure $(0, \ldots, 0, m_k, m_{k+1}, \ldots, m_{\check{k}})$. For any (m, k) pair such that $m \in \mathbb{M}$, $\alpha_S(m, \cdot, k)$ is computed in the step with indexing market structure $(0, \ldots, 0, 1, m_{k+1}, \ldots, m_{\check{k}})$. For any m such that $m \in \mathbb{M}$ and $m_1 > 0$, $\alpha_E(m, k)$ is computed in the step with indexing market structure m. Therefore, $w_E, w_S, \alpha_S, \alpha_E$ for all payoff-relevant (m, c, k) are computed in Algorithm 2.

Second, because all $\alpha_S, \alpha_E, w_E, w_S$ required to compute the fixed point of T in each step have been either initialized or determined in previous steps, T is always a well-defined contraction mapping with a unique fixed point. Then, w_E is always uniquely determined, as well as w_S and α_E. It remains to show that α_S is also always well-defined, in particular when it is determined from equation (11). Note that for any (m, c, k) such that $k = \underline{k}(m) \equiv \min\{j; m_j > 0\}$, when computing $w_E(m, c, k)$ using T, we always use the initialized value $\alpha_S(m, c, k^+) = 1$ for all $k^+ \ge k$, which leads to the condition that

$$w_E(m, c, k) = \max\{0, w_S(m, c, k)\}.$$

This implies that when $w_E(m, c, k) = 0$, $w_S(m, c, k) \le 0$. Recall that tn Procedure 3, when determining p using equation (11) in step i, it is indeed the case that $w_E(m, c, k^i) = 0$ and $w_S(m, c, k^i) \le 0$. Then, (i) if $w_S(m^i, c, k^i) = w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) > 0$, then the right hand side of equation (11) changes continuously from $w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) > 0$ to $w_S(m, c, k) \le 0$ when p changes from 0 to 1. This means that there exists at least one $p \in [0, 1)$ to satisfy equation (11); (ii) if $w_S(m^i, c, k^i) \le 0$, 0 can be assigned if no p is found to satisfy equation (11). Therefore, we conclude that α_S is always well-defined (although it can take multiple values if multiple p solve equation (11)).

Next, we show that α_S satisfies the requirement in Definition 1 by proving $w_E(m, c, k^1) \geq w_E(m, c, k^2)$ for all m, c and $k^1 \geq k^2$. To this end, for any computed w_E, define a functional space $\mathcal{G}^N$ containing all functions $g_E : \mathbb{M} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \left[0, \frac{\beta\pi(\iota_{\check{k}}, \check{c}, \check{k})}{1-\beta}\right]$ such that $g_E \leq w_E$, and $g_E(m, c, k^1) \geq g_E(m, c, k^2)$ for all (m, c) and $k^1 = k^2 + 1$, with equality holds only when $g_E(m, c, k^1) = g_E(m, c, k^2) = 0$. We aim to prove $T : \mathcal{G}^N \to \mathcal{G}^N$.

Let g_S denote the analogous post-exit value computed by equation (2) using g_E. Under Assumptions 1, for all $g_E \in \mathcal{G}^N$, $g_S(m, c, k^1) > g_S(m, c, k^2)$ for all m, c and $k^1 = k^2 + 1$. Consider the following cases when $(Tg_E)(m, c, k^1)$ is being computed in Algorithm 2, noting that by the OLS ordering of the algorithm, at this moment, $\alpha_S(m, c, k^1)$ remains at its initial value 1 and $\alpha_S(m, c, k^2)$ has been determined in previous computation by Procedure 3.

1. If $\alpha_S(m, c, k^2) = 1$, since both type-k^1, k^2 firms survive with probability one, they expect same post-exit market structure, denoted by M_S.

$$(Tg_E)(m, c, k^1) = \mathbb{E}\left[g_S(M_S, c, k^1) \,\middle|\, M_E = m\right] > \mathbb{E}\left[g_S(M_S, c, k^2) \,\middle|\, M_E = m\right] = (Tg_E)(m, c, k^2).$$

2. If $0 < \alpha_S(m, c, k^2) < 1$, then $\alpha_S(m, c, k^2) = p$ with $p \in [0, 1)$ solving

$$\sum_{j=0}^{m_{k^2}-1} p^{m_{k^2}-1-j}(1-p)^j \binom{m_{k^2}-1}{j} g_S(m - j\iota_{k^2} - \sum_{i=1}^{k^2-1} m_i\iota_i, c, k^2) = 0.$$

   The right hand side is nothing but $(Tg_E)(m, c, k^2)$. Therefore, $(Tg_E)(m, c, k^2) = 0$. Since $g_S \in \mathcal{G}^N$, we have

$$(Tg_E)(m, c, k^1) = \max\left\{0, \sum_{j=0}^{m_{k^2}-1} p^{m_{k^2}-1-j}(1-p)^j \binom{m_{k^2}-1}{j} g_S(m - j\iota_{k^2} - \sum_{i=1}^{k^2-1} m_i\iota_i, c, k^1)\right\} > 0.$$

3. If $\alpha_S(m, c, k^2) = 0$, then $g_E(m, c, k^2) \leq w_E(m, c, k^2) = 0$ for $g_E \in \mathcal{G}^N$. Since $w_E(m, c, k^2) = (T^\infty g_E)(m, c, k^2)$ and T is a monotone operator, $0 = w_E(m, c, k^2) \geq (Tg_E)(m, c, k^2)$ for all $g_E \in \mathcal{G}^N$. Thus, $(Tg_E)(m, c, k^1) \geq 0 \geq (Tg_E)(m, c, k^2)$.

By point-wise comparison, we conclude that $T : \mathcal{G}^N \to \mathcal{G}^N$, hence $w_E(m, c, k^1) \geq w_E(m, c, k^2)$ for all m, c and $k^1 = k^2 + 1$. The proof also verifies that $(Tg_E)(m, c, k^1) > 0$ whenever $\alpha_S(m, c, k^2) > 0$. Since T is a monotone operator, it means that $w_E(m, c, k^1) = (T^\infty g_E)(m, c, k^1) > 0$. Given that in Algorithm 2 α_S is set to be 1 if and only if $w_E(m, c, k^1) > 0$, $\alpha_S(m, c, k^1) = 1$ whenever $\alpha_S(m, c, k^2) > 0$. So α_S satisfies the requirement in Definition 1.

Finally, we prove that (α_S, α_E) forms an Markov-perfect equilibrium. To this end, we first show that w_E constructed by Algorithm 2 is the post-entry payoff under strategy (α_S, α_E). Then, we show that given w_E as payoff, (α_S, α_E) satisfies one-shot deviation proofness.

We begin with showing that w_E is the post-entry payoff under (α_S, α_E) in the first step of Algorithm 2, where $m^1 = \check{m}\iota_{\check{k}}$. In this step, $\mathbb{H}_S^1 = \{(m^1, c, \check{k})|c \in [\hat{c}, \check{c}]\}$. When computing $w_E(H_S^1)$, we use $\alpha_S(H_S^1) = 1$ for all H_1^S, $\alpha_E(\cdot) = 0$, and $w_E(\cdot) = 0$. According to equation (1), if w_E is the post-entry payoff under strategy (α_S, α_E), then it satisfies

$$w_E(H_S^1) = \alpha_S(H_S^1)\mathbb{E}\left[w_S(M_S, c, \check{k}) \,\middle|\, M_E = m^1, C = c, K = \check{k}\right].$$

The construction of α_S in Procedure 3 implies that $\alpha_S(m^1, c, \check{k}) = 1$ if and only if $w_E(m^1, c, \check{k}) > 0$, and $\alpha_S(m^1, c, \check{k}) < 1$ if and only if $w_E(m^1, c, \check{k}) = 0$. Also, note that $\mathbb{E}\left[w_S(M_S, c, \check{k}) \,\middle|\, M_E = m^1, C = c, K = \check{k}\right] = w_S(m^1, c, \check{k})$ if $\alpha_S(m^1, c, \check{k}) = 1$. Then, the above condition for $w_E(H_S^1)$ under the constructed α_S is equivalent to

$$w_E(H_S^1) = \max\{0, I(w_E(H_S^1) > 0)w_S(H_S^1)\} = \max\{0, w_S(H_S^1)\}.$$

By setting $\alpha_S(H_S^1) = 1$, the right hand side of T is identical to this condition. Therefore, setting $\alpha_S(H_S^1) = 1$ is computationally equivalent to using $\alpha_S(H_S^1)$ determined by Procedure 3, i.e., both give the same $w_E(H_S^1)$. Also, under $\alpha_E(\cdot) = 0$, no firm will further enter. This means that $w_E(H_S^1)$ computed as the fixed point of T is the post-entry payoff under strategy (α_S, α_E).

Consequently, for all c, $w_S(m^1, c, \check{k})$ computed by equation (2) using $w_E(m^1, \cdot, \check{k})$ is the post-survival payoff under strategy (α_S, α_E).

Then suppose that the $1, \ldots, i-1$-step of Algorithm 2 have computed the $w_E(m, c, k)$ and $w_S(m, c, k)$ for all $(m, k) \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j} \times \{\underline{k}(m^j)\}$ and all c as the payoffs under (α_S, α_E). Then, Procedure 3 in the first $i-1$-step computes the following part of (α_S, α_E) for all c,

- $\alpha_S(m, c, k)$ for all $(m, k) \in \{(m, k); (m, k) \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j} \times \{\underline{k}(m^j)\}, m - n\iota_{k^i} \neq m^i, \forall n \in \mathbb{N}\}$.
- $\alpha_E(m, c)$ for all $m \in \{m; m \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j}$ with $\underline{k}(m^j) = 1\}$.

Recall that $\underline{k}(m) \equiv \min\{j; m_j > 0\}$. Now, in the i-th step of the algorithm, $H_S^i \in \{(m, c, k); m \in \mathbb{M}_{m^i}, c \in [\hat{c}, \check{c}], k = k^i\}$. To make sure that $w_E(H_S^i)$ and $w_S(H_S^i)$ take their values under (α_S, α_E), we need to use in the construction of T the strategy $\alpha_S(m, \cdot, k)$ for all $m \in \mathbb{M}_{m^i}$ and k such that $m_k > 0$, $\alpha_E(n' + j\iota_1, \cdot)$ for all $j \in \mathbb{N}$ and all possible n', and $w_E(H_S^{i,\prime})$ for (m', k') such that $m' \notin \mathbb{M}_{m^i}$ and $k' \neq k^i$, conditional on type-k^i firms having positive payoff.

We check if the required values are in place.

1. From the argument for step-1 computation, the initialized value $\alpha_S(m, c, k^i) = 1$ leads to the same condition for $w_E(m, c, k^i)$ as the $\alpha_S(m, c, k^i)$ computed by Procedure 3. So, although $\alpha_S(m, c, k^i)$ has not been obtained, we can set it to 1.

2. For any (m, k^+) such that $m \in \mathbb{M}_{m^i}$ and $k^+ > k^i$, as we have shown, $\alpha_S(m, c, k^+) = 1$ conditional $w_E(m, c, k^i) > 0$, which is the same as the initialized value. For any (m, k^-) such that $m \in \mathbb{M}_{m^i}$ and $k^- < k^i$, note that $m \neq m^i$ because $m^i_{k^-} = 0$. By the definition of $\mathbb{M}_{m^i}$, for all $m \in \mathbb{M}_{m^i} \backslash \{m^i\}$, $m \succ m^i$ (so there is some $j < i$-step such that its indexing market structure $m^j = m$) and $m - b\iota^i_k \neq m^i, \forall b \in \mathbb{N}$. Therefore, $\alpha_S(m, c, k^-)$ for all $k^- < k^i$ have been computed.

3. Since according to α_S, all firms with type equal or better than k^i survive, which, together with non-regressive type evolution, implies that $n' \succeq m^i$ and $n' + b\iota_1 \succ m^i$ for all n' and all $b \in \mathbb{N}$. Therefore, for $|n' + b\iota_1| \leq \check{m}$, there is some $j < i$-th step with indexing market structure $m^j = n' + b\iota_1$. So, these $\alpha_E(n' + b\iota_1)$'s values have been computed in the j-th step by Procedure 3. For any n' such that $|n' + \iota_1| > \check{m}$, we use the initialized value $\alpha_E(n' + \iota_1) = 0$.

4. Based on the above argument, for any (m', k') following the transition governed by (α_S, α_E), $m' \succeq m^i$ and $k' \geq k^i$. If $m' \notin \mathbb{M}_{m^i}$ and $k' \neq k^i$, define $\underline{m'}(k') = (0, \ldots, 0, m'_{k'}, \ldots, m'_{\check{k}})$, the market structure which has exactly the same number of type-k' or better firms as m' does, but no type-$k'-1$ or worse firm. Then, $\underline{m'}(k') \succ m^i$, which means that there is some $j < i$-th step such that its indexing market structure $m^j = (0, \ldots, 0, m'_{k'}, \ldots, m'_{\check{k}})$. Since $m' \in \mathbb{M}_{m^j}$, $w_E(m', \cdot, k')$ is then computed in the j-th step. So, all necessary w_E's values have been computed.

Since all the required values of w_E, α_S, α_E have been obtained in earlier steps, $w_E(H^i_S)$ is computed as the payoff under (α_S, α_E), so as $w_S(H^i_S)$.

Then, we verify that (α_S, α_E) is an equilibrium strategy corresponding to w_E, w_S. To this end, we show that $\alpha_S(m, c, k)$ satisfies (4) for all (m, c, k), if all other firms follow α_S as well. For any (m, c, k), consider the following cases

1. If $w_E(m, c, k) > 0$, the algorithm sets $\alpha_S(m, c, k) = 1$. The right-hand-side of (4) is

$$\arg\max_{a \in [0,1]} a w_E(m, c, k) = \{1\} \ni \alpha_S(m, c, k).$$

2. If $w_E(m, c, k) = 0$, then the algorithm sets $\alpha_S \in [0, 1)$. Since any α_S computed by Algorithm 2 satisfies the requirement in Definition 1, it is implied that $\alpha_S(m, c, k^-) = 0$ for all $k^- < k$. Hence, $w_E(m, c, k) = w_E(m - \sum_{i=1}^{k-1} m_i \iota_i, c, k) = \max\{0, w_S(m - \sum_{i=1}^{k-1} m_i \iota_i, c, k)\}$ and $w_S(m - \sum_{i=1}^{k-1} m_i \iota_i, c, k) \leq 0$. We look at three subcases,

   (a) If $w_S(m - (m_k - 1)\iota_k - \sum_{i=1}^{k-1} m_i \iota_i, c, k) > 0$, the algorithm sets $\alpha_S(m, c, k) = p \in [0, 1)$ to satisfy

$$\sum_{j=0}^{m_k - 1} p^{m_k - 1 - j} (1 - p)^j \binom{m_k - 1}{j} w_S(m - j\iota_k - \sum_{i=1}^{k-1} m_i \iota_i, c, k) = 0,$$

The right-hand-side of (4)

$$\arg\max_{a\in[0,1]} a \sum_{j=0}^{m_k-1} p^{m_k-1-j}(1-p)^j \binom{m_k-1}{j} w_S(m-j\iota_k-\sum_{i=1}^{k-1} m_i\iota_i, c, k) = [0,1] \ni \alpha_S(m,c,k).$$

(b) If $w_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i\iota_i, c, k) > 0$ and $\alpha_S(m,c,k) \in [0,1)$ solves the same polynomial as above, same result holds for $\alpha_S(m,c,k)$.

(c) If $w_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i\iota_i, c, k) \le 0$ and $\alpha_S(m,c,k) = 0$. All other type-k firms will exit from the market, so the right-hand-side of (4) is

$$\arg\max_{a\in[0,1]} a w_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i\iota_i, c, k) = \{0\} \ni \alpha_S(m,c,k).$$

For any (m,c,k) such that $m_k = 1$, consider the following cases

1. If $w_E(m,c,k) > 0$, then the algorithm sets $\alpha_S(m,c,k) = 1$. The right-hand-side of (4) is $\arg\max_{a\in[0,1]} a w_S(m,c,k) = \{1\} \ni \alpha_S(m,c,k)$.

2. If $w_E(m,c,k) = 0$, then α_S can not be 1. From the same argument as above, $w_E(m,c,k) = w_E(m-\sum_{i=1}^{k-1} m_i\iota_i, c, k) = \max\{0, w_S(m-\sum_{i=1}^{k-1} m_i\iota_i, c, k)\}$. So $w_S(m-\sum_{i=1}^{k-1} m_i\iota_i, c, k) \le 0$ and the right-hand-side of (4) is $\arg\max_{a\in[0,1]} a w_S(m-\sum_{i=1}^{k-1} m_i\iota_i, c, k) = \{0\} \ni \alpha_S(m,c,k)$.

Therefore, α_S satisfies (4). To show that α_E satisfies (3), first note that $\alpha_E(m,c)$ is determined in the step with indexing market structure m, while $w_E(m+b\iota_1,c,1)$ is computed in step with indexing market structure $m+b\iota_1$, which is (weakly) lexicographically superior than m. Therefore, $w_E(m+b\iota_1,c,1)$ has been determined as a post-entry payoff under α_S. Then, if all potential entrants are using α_E, according to (1), post-entry payoff is $w_E(m+J\iota_1,c,1)$ where J is the largest possible number such that $w_E(m+J\iota_1,c,1)-\varphi > 0$. Therefore, α_E satisfies (3). So, (α_S, α_E) is the equilibrium strategy.

This completes the proof for Lemma 3. □

With Lemma 3 in hand, we proceed to prove Propositions 2 and 3.

Proof for Proposition 2. To prove this proposition, we again establish a lemma first.

Lemma 4. *If v_E is the post-entry payoff in a payoff-monotone natural Markov-perfect equilibrium, it necessarily satisfies that $v_E(m,c,k) > 0$ if and only if $\mathbb{E}[v_S(M_S,c,k)|M_E = m] > 0$, or*

$$v_E(m,c,k) = \max\{0, \mathbb{E}[v_S(M_S,c,k)|M_E = m]\},$$

where the expectation is computed given all equilibrium values $a_S(m,c,k^-)$ for all $k^- < k$, a tentative rule $a_S(m,c,k) = 1$, and $a_S(m,c,k^+) = 1$ for all $k^+ > k$.

Proof. In any symmetric equilibrium, $v_E(m, c, k) > 0$ only if all firms with type k survive. In any natural equilibrium, this also implies that all firms with type k^+ survive as well. Therefore, the "only if" part is true.

The "if" part is true because (i) if $\mathbb{E}[v_S(M_S, c, k)|M_E = m] > 0$ and $a_S(m, c, k-1) > 0$, then it must be the case that in natural equilibrium $v_E(m, c, k-1) > 0$. Also, according to Definition 1, $a_S(m, c, k) = 1$ and $a_S(m, c, k^+) = 1$. Then, $v_E(m, c, k) \geq v_E(m, c, k-1) > 0$; (ii) if $\mathbb{E}[v_S(M_S, c, k)|M_E = m] > 0$ and $a_S(m, c, k-1) < 0$, then in a natural equilibrium $a_S(m, c, k^-) = 0$ for all $k^- < k$, and $\mathbb{E}[v_S(M_S, c, k)|M_E = m] = v_S(m - \sum_{i=1}^{k-1} m_i \iota_i, c, k) > 0$. Recall that we have shown in the proof for Proposition 1 that in the duopoly model, Lemma 1 ensures that $v_E(2\iota_k, c, k) > 0$ if and only if $v_S(2\iota_k, c, k) > 0$. Applying an analogous reasoning, we know that in a payoff-monotone equilibrium, $v_E(m, c, k) > 0$ if $v_S(m - \sum_{i=1}^{k-1} m_i \iota_i, c, k) > 0$. □

Lemma 4 gives a necessary condition for the post-entry payoff in a payoff-monotone natural Markov-perfect equilibrium. For $v_E(m^1, c, \check{k})$, where $m^1 = \check{m}\iota_{\check{k}}$ is the indexing market structure in the first step of Algorithm 2, this condition can be written as

$$v_E(m^1, c, \check{k}) = \max\{0, v_S(m^1, c, \check{k})\}.$$

In the first step of Algorithm 2, $w_E(m^1, c, \check{k})$ is uniquely computed by the contraction mapping generated by the above condition. Thus, it is the only payoff function satisfying the necessary condition for a payoff-monotone equilibrium. Providing that such equilibrium exists, its post-entry payoff $v_E(m^1, c, \check{k})$ has a unique value $w_E(m^1, c, \check{k})$ for all c. Also, $v_S(m^1, c, \check{k}) = w_S(m^1, c, \check{k})$ for all c.

In any succeeding step i of Algorithm 2, with α_S either properly initialized or computed in the previous steps as its equilibrium value (this is shown in Lemma 3), $w_E(m, c, k^i)$ is computed as the unique payoff *under* (α_S, α_E) that satisfies such necessary condition for all c and all $m \in \mathbb{M}_{m^i}$.

Moreover, the (α_S, α_E) constructed in Procedure 3 is also the unique equilibrium strategy given that the w_E, w_S computed in previous steps are unique equilibrium payoffs v_E, v_S. α_E's uniqueness trivially follows its construction. The uniqueness of α_S is due to the monotonicity of (the previously computed part of) w_S: When using (11) to compute the mixing probability p, because $w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) \geq w_S(m - (m_{k^i} - 2)\iota_{k^i}, c, k^i) \geq \ldots \geq w_S(m, c, k^i)$, the right hand side (11) changes continuously and *monotonically* from $w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) > 0$ to $w_S(m, c, k^i) \leq 0$ when p changes from 0 to 1. Therefore, there is only one $p \in [0, 1)$ that satisfies (11). So, α_S is single valued.

Therefore, if there exists a payoff-monotone equilibrium, (α_S, α_E) forms the unique equilibrium and w_E and w_S are the unique equilibrium payoffs. The equilibrium is subsequently unique. □

Proof for Proposition 3. First, note that from the definition of a renegotiation-proof natural Markov-perfect equilibrium, all firms with a same type survive for sure *if and only if* joint continuation gives them positive post-survival payoff. This implies that (i) any such equilibrium's post-entry equilibrium payoff must satisfy the condition in Lemma 4; (ii) if any natural Markov-perfect equilibrium's post-entry payoff satisfies the condition in Lemma 4, such equilibrium is renegotiation-proof.

Since we have shown in Lemma 3 that Algorithm 2 always gives some (α_S, α_E) to form a natural Markov-perfect equilibrium. We have also shown in the proof for Proposition 2 that w_E satisfies the necessary condition in Lemma 4. Therefore, (α_S, α_E) forms a renegotiation-proof natural Markov-perfect equilibrium. $\square$

C An Example of Multiple Equilibria

We construct a three-firm two-type example where the equilibrium payoff is not weakly decreasing in the number of same-type competitors ($\check{m} = 3$ (by setting $\pi(n, c, k) < 0$ for any (n, c, k) if n has more than 3 firms) and $\check{k} = 2$).

Consider the following sequence of c_t: $c_1 = 1, c_2 = 1e^{-6}, c_t = 5$, for all $t \geq 3$. The number of consumers drops to nearly zero in the second period but is boosted to a high level in the third period, and stays high afterwards. We set $\beta = 0.5, \kappa(\mathcal{L}) = \kappa(\mathcal{H}) = 4, \varphi = 1$, and $\Pi_{\mathcal{LH}} = \Pi_{\mathcal{LL}} = 0.5$. We specify π as $\pi(n, c, k) = c\overline{\pi}(n, k) - \kappa$. Some parts of the per-consumer producer surplus $\overline{\pi}$ are summarized in the following table:

$\overline{\pi}(\cdot, \mathcal{L})/\overline{\pi}(\cdot, \mathcal{H})$	$1\iota_{\mathcal{H}}$	$2\iota_{\mathcal{H}}$	$3\iota_{\mathcal{H}}$
$+0\iota_{\mathcal{L}}$	/102	/100	/1
$+1\iota_{\mathcal{L}}$	99/101	0.89/1.24	
$+2\iota_{\mathcal{L}}$	1.23/1.25		

One feature of this surplus structure is that a duopoly market promises much higher per consumer surplus than a triopoly market does. The duopoly-triopoly surplus difference overwhelmingly dominates the $\mathcal{L}, \mathcal{H}$-type difference in surplus. Since from period 3 onwards, the model is essentially an infinitely repeated game, we can use backward induction to compute the equilibrium payoffs[13]. Unlike the results stated in Lemma 1, v_S is not always monotonic in the number of firms,

$$v_S(\iota_{\mathcal{H}}, c_1, \mathcal{H}) = -1.1821, \quad v_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 246, \quad v_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) = -1.5$$

Not surprisingly, the low triopoly surplus implies a low payoff if continuing as one of three type-$\mathcal{H}$ firms. What is counter-intuitive is that the payoff to continuing as a duopolist is

[13] We provide the details of such computation in an online appendix.

better than the payoff to continuing as a monopolist under c_1. This is because in period 2, under a low c_2, a duopoly firm and a monopoly firm make similar flow profit and similar large losses. However, duopoly firms can, by jointly remaining active, preempt any further entrants and enjoy a high duopoly profit after demand increases to a high level in period 3. The future duopoly payoff compensates the loss in period 2, and make $v_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H})$ positive. In contrast, a monopoly market attracts two entrants for sure in period 2, which results in a triopoly market from period 3 onwards. Because demand increases to a high level in period 3, none of these firms will exit and, given the per consumer surplus structure, they will all earn a substantially lower payoff than duopoly firms, which can not compensate the loss in period 2. Consequently, $v_S(\iota_{\mathcal{H}}, c_1, \mathcal{H})$ is negative.

Given the computed non-monotone equilibrium payoff, (a_S, a_E) with $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 1$ is still an natural equilibrium. However, if one duopoly firm chooses to exit with probability 1, the rival firm receives -1.1821 if continuing alone and hence will choose to exit with probability 1 as well. Similarly, if one firm chooses to survive with probability $\frac{-1.1821}{-1.1821-246} = 4.782e^{-3}$, the other firm is indifferent between exiting and survival. Therefore, two other natural equilibria with $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ and $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 4.782e^{-3}$ exist.

Note that when both firms choose $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ or $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 4.782e^{-3}$, they receive zero payoffs. By "renegotiating" on jointly choosing $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 1$, they can strictly improve their equilibrium payoffs. Henceforth, we only restrict attention to equilibria in which there is no room for this type of one-shot joint improvement.

Unfortunately, this type of equilibria may not be unique. Since joint continuation and continuing as monopolist both render payoffs negative, in a one-shot renegotiation-proof equilibrium, a triopoly firm either chooses $a_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ or set $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = p$, where p solves

$$p^2\bar{v}(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) + \binom{2}{1}p(1-p)\bar{v}(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) + (1-p)^2\bar{v}(\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0.$$

This quadratic equation has two roots, $p_1 = 0.0024$ and $p_2 = 0.997$, both between 0 and 1. Hence, there are three one-shot renegotiation-proof equilibria with $a_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H})$ equal to 0, p_1 and p_2, respectively.

Online Appendix for
Simple Markov-Perfect Industry Dynamics

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Contents

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I Details of Proofs

I.1 Details for the Proof of Lemma 1

Proof. We prove that, in any natural Markov-Perfect Equilibrium, v_E satisfies the following properties for all k.

Property 1. For a given $k \in \mathbb{K}$, and for all x such that $0 \leq x \leq \check{k}$, if $v_E(\iota_k + \iota_x, \cdot, k)$ is weakly decreasing in x, we say that v_E satisfies Property 1 for k.

Property 2. For a given $k \in \mathbb{K}$, and for all x such that $0 \leq x \leq \check{k}$, if $v_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x, we say that v_E satisfies Property 2 for k.

Property 3. For a given $k \in \mathbb{K}$, and for all x such that $1 \leq x \leq k-1$, if $v_E(\iota_k + \iota_x, \cdot, x) \leq v_E(\iota_{k-1} + \iota_x, \cdot, x)$, we say that v_E satisfies Property 3 for k.

Property 4. For a given $k \in \mathbb{K}$, if $v_E(\iota_k, \cdot, k) \geq v_E(\iota_{k-1}, \cdot, k-1)$, we say that v_E satisfies Property 4 for k.

First, we prove these properties for $k = \check{k}$. Then, for any $k < \check{k}$, suppose that those properties hold good for $k+1, k+2, \ldots, \check{k}$, we prove that they also hold for k. In this way, we prove that these properties hold for all $k \in \mathbb{K}$. Eventually, we proof Lemma 1 using Property 1.

Now suppose (a_S, a_E) forms a natural equilibrium with equilibrium payoff v_S, v_E. Define $\mathcal{F}$ to be the space of all functions

$$f_E : \left\{ (n_1, \ldots, n_{\check{k}}) : \sum_{i=1}^{\check{k}} n_i \leq 2 \right\} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \left[0, \frac{\beta \pi(\iota_{\check{k}}, \check{c}, \check{k})}{1-\beta} \right],$$

and $T^a : \mathcal{F} \to \mathcal{F}$ with

For any $k^1, k^2 \in \{0, 1, \ldots, \check{k}\}$, if $k^1 \geq k^2$,

$$\begin{aligned}
(T^a f_E)(\iota_{k^1} + \iota_{k^2}, c, k^1) &= \max \left\{ 0, a_S(\iota_{k^1} + \iota_{k^2}, c, k^2) \sum_{i=k^1}^{\check{k}} \sum_{j=k^2}^{\check{k}} \Pi_{k^1,i} \Pi_{k^2,j} \tilde{f}_E(\iota_i + \iota_j, c, i) \right. \\
&\quad \left. + (1 - a_S(\iota_{k^1} + \iota_{k^2}, c, k^2)) \sum_{i=k^1}^{\check{k}} \Pi_{k^1,i} \tilde{f}_E(\iota_i, c, i) \right\} \\
&= \max \big\{ 0, a_S(\iota_{k^1} + \iota_{k^2}, c, k^2) \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K^{2\prime}}, c, K^{1\prime}) | K^2 = k^2] | K^1 = k^1] \\
&\quad + (1 - a_S(\iota_{k^1} + \iota_{k^2}, c, k^2)) \mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}}, c, K^{1\prime}) | K^1 = k^1] \big\}. \qquad (1)
\end{aligned}$$

If $k^1 < k^2$,

$$\begin{aligned}(T^a f_E)(\iota_{k^1}+\iota_{k^2},c,k^1) &= \max\left\{0, \sum_{i=k^1}^{\check{k}}\sum_{j=k^2}^{\check{k}} \Pi_{k^1,i}\Pi_{k^2,j}\tilde{f}_E(\iota_i+\iota_j,c,i)\right\} \\ &= \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1'}}+\iota_{K^{2'}},c,K^{1'})|K^2=k^2]|K^1=k^1]\right\}. (2)\end{aligned}$$

With $\tilde{f}_E(\iota_{k^1}+\iota_{k^2},c,k^1)$ denoting the post-type-transition payoff associate with post-entry payoff f_E when current demand is c, the firm of interest has progressed to type-k^1, and its rival has progressed to type-k^2.

$$\begin{aligned}\tilde{f}_E(\iota_{k^1}+\iota_{k^2},c,k^1) \equiv \beta\mathbb{E}[\pi(\iota_{k^1}+\iota_{k^2},C',k^1) + a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C')f_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C',i) \\ + [1-a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C')]\, f_E(\iota_{k^1}+\iota_{k^2},C',i)|C=c]\end{aligned}$$

where, for definiteness, $a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,c) \equiv 0$ if $k^1, k^2 > 0$.

The space $\mathcal{F}$ is a Banach space (complete with the supremum norm). T^a satisfies Blackwell's sufficient properties for a contraction mapping. The equilibrium payoff v_E is the unique fixed point of T^a. We prove that v_E satisfies Properties 1–4 for all k by showing that the fixed point of T^a lies in the space in which all functions satisfy these properties. To this end, we gradually narrow down the space that this fixed point is in, to eventually reach the space of all functions that satisfy Properties 1–4 for all k.

First, denote a subspace of $\mathcal{F}$ in which any functions f_E satisfy $f_E \geq v_E$ as $\mathcal{F}^0$. Because $\mathcal{F}^0$ is also a non-empty Banach space and $v_E \in \mathcal{F}^0$, $T^a : \mathcal{F}^0 \to \mathcal{F}^0$. We henceforth focus on $\mathcal{F}^0$.

Before proceeding to verify the properties, we introduce a Lemma which we will use repeatedly. Recall that firm types' evolution has the first-order stochastic dominance property, as stated in Assumption 3. The following Lemma exploits this property

Lemma 1. *X, Y are random variables with densities F and G respectively. If X first-order stochastically dominates Y, then $\mathbb{E}[h(X)] \geq \mathbb{E}[h(Y)]$ for all weakly increasing function h.*

Prove Property 2 for $k = \check{k}$. We take two steps to prove Property 2 for $k = \check{k}$.

(i). Consider $\mathcal{F}^1_{\check{k}}$, a subspace of $\mathcal{F}^0$ in which any function f_E satisfies $(T^a f_E)(2\iota_{\check{k}},\cdot,\check{k}) = v_E(2\iota_{\check{k}},\cdot,\check{k})$ and $f_E(2\iota_{\check{k}},\cdot,\check{k}) \leq f_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ for $0 \leq x \leq \check{k}$. This is also a complete metric space. Since at least the function f^*_E which satisfies $f^*_E(2\iota_{\check{k}},\cdot,\check{k}) = v_E(2\iota_{\check{k}},\cdot,\check{k})$ and $f^*_E(2\iota_{\check{k}},\cdot,\check{k}) = f^*_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ for $0 \leq x \leq \check{k}$ is in $\mathcal{F}^1_{\check{k}}$, this subspace is nonempty.

We aim to prove that $T^a : \mathcal{F}^1_{\check{k}} \to \mathcal{F}^1_{\check{k}}$. Since v_E is the unique fixed point of T^a, this result ensures that $v_E \in \mathcal{F}^1_{\check{k}}$ and thus $v_E(2\iota_{\check{k}}, \cdot, \check{k}) \leq v_E(\iota_{\check{k}}, \cdot, \check{k})$.

In a symmetric equilibrium, any rival's exit implies that both firms expect non-positive payoffs from continuing. Therefore, firms earn positive expected payoffs only when both firms continue with probability 1. So

$$v_E(2\iota_{\check{k}}, c, \check{k}) \leq \max\{0, \beta\mathbb{E}[\pi(2\iota_{\check{k}}, C', \check{k}) + v_E(2\iota_{\check{k}}, C', \check{k})|C = c]\} = \max\{0, \tilde{f}_E(2\iota_{\check{k}}, c, \check{k})\}.$$

Note that for all $f_E \in \mathcal{F}^1_{\check{k}}$, we have for any (a_S, a_E), $\tilde{f}_E(\iota_{\check{k}} + \iota_x, \cdot, \check{k}) \geq \tilde{f}_E(2\iota_{\check{k}}, \cdot, \check{k})$ for $0 \leq x \leq \check{k}$. Therefore, $\mathbb{E}[\tilde{f}_E(\iota_{\check{k}} + \iota_{X'}, \cdot, \check{k})|X = x] \geq \tilde{f}_E(2\iota_{\check{k}}, \cdot, \check{k})$, and $\tilde{f}_E(\iota_{\check{k}}, \cdot, \check{k}) \geq \tilde{f}_E(2\iota_{\check{k}}, \cdot, \check{k})$. Then, for any $0 \leq x \leq \check{k}$, we have

$$\begin{aligned}
& (T^a f_E)(\iota_{\check{k}} + \iota_x, c, \check{k}) \\
= \; & \max\left\{0, a_S(\iota_{\check{k}} + \iota_x, c, x)\mathbb{E}[\tilde{f}_E(\iota_{\check{k}} + \iota_{X'}, c, \check{k})|X = x] + (1 - a_S(\iota_{\check{k}} + \iota_x, c, x))\tilde{f}_E(\iota_{\check{k}}, c, \check{k})\right\} \\
\geq \; & \max\{0, \tilde{f}_E(2\iota_{\check{k}}, c, \check{k})\} \\
\geq \; & v_E(2\iota_{\check{k}}, c, \check{k}) \\
= \; & (T^a f_E)(2\iota_{\check{k}}, c, \check{k})
\end{aligned}$$

Therefore, $T^a : \mathcal{F}^1_{\check{k}} \to \mathcal{F}^1_{\check{k}}$ and $v_E(2\iota_{\check{k}}, \cdot, \check{k}) \leq v_E(\iota_{\check{k}} + \iota_x, \cdot, \check{k})$ for $0 \leq x \leq \check{k}$. This further ensures that $v_S(2\iota_{\check{k}}, \cdot, \check{k}) \leq v_S(\iota_{\check{k}}, \cdot, \check{k})$ for any a_S, a_E. Then, an analogous argument to the one on the simultaneous-move survival game in Section **??** leads to

$$v_E(2\iota_{\check{k}}, c, \check{k}) = \max\left\{0, \beta\mathbb{E}\left[\pi(2\iota_{\check{k}}, C', \check{k}) + v_E\left(2\iota_{\check{k}}, C', \check{k}\right) \,\middle|\, C = c\right]\right\}. \tag{3}$$

The right-hand-side of (3) defines a contraction mapping with a unique fixed point $v_E(2\iota_{\check{k}}, \cdot, \check{k})$. So

$$(T^a f_E)(2\iota_{\check{k}}, c, \check{k}) = \max\{0, \tilde{f}_E(2\iota_{\check{k}}, c, \check{k})\}.$$

(ii). We move on to a subspace of $\mathcal{F}^1_{\check{k}}$, which we denote by $\mathcal{F}^2_{\check{k}}$. In this subspace, any function f_E satisfies that $f_E(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x for $0 \leq x \leq \check{k}$. We will further show that $T^a : \mathcal{F}^2_{\check{k}} \to \mathcal{F}^2_{\check{k}}$. Note that for $f_E \in \mathcal{F}^2_{\check{k}}$, $\tilde{f}_E(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x as well. For any k^1, k^2 such that $1 \leq k^1 \leq k^2 \leq \check{k}$, we use $K^{1'}, K^{2'}$ to denote the random variables for the types succeeding $K^1 = k^1, K^2 = k^2$ respectively. $K^{2'}$ stochastically dominates $K^{1'}$. So, according to Lemma 1, $\tilde{f}_E$ shares the monotonicity property as f_E in $\mathcal{F}^2_{\check{k}}$, $\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{1\prime}},\cdot,K^{1\prime})|K^1=k^1] \leq \mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{2\prime}},\cdot,K^{2\prime})|K^2=k^2]$. Therefore,

$$\begin{aligned}
(T^a f_E)(\iota_{\check{k}}+\iota_{k^1},c,k^1) &= \max\left\{0,\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{1\prime}},c,K^{1\prime})|K^1=k^1]\right\} \\
&\leq \max\left\{0,\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{2\prime}},c,K^{2\prime})|K^2=k^2]\right\} \\
&= (T^a f_E)(\iota_{\check{k}}+\iota_{k^2},c,k^2)
\end{aligned}$$

This result guarantees that $T^a:\mathcal{F}^2_{\check{k}}\to\mathcal{F}^2_{\check{k}}$. Therefore, $v_E(\iota_{\check{k}}+\iota_x,\cdot,x)$ is weakly increasing in x and Property 2 is satisfied for $k=\check{k}$. Because in natural equilibrium,

$$v_S(\iota_{\check{k}}+\iota_x,c,x)=\beta\mathbb{E}\left[\pi(\iota_{\check{k}}+\iota_{X'},C',X')+v_E(\iota_{\check{k}}+\iota_{X'},C',X')\,|\,C=c,X=x\right].$$

together with Assumption 1, it is also true that $v_S(\iota_{\check{k}}+\iota_x,\cdot,x)$ is weakly increasing in x, and so is the survival rule $a_S(\iota_{\check{k}}+\iota_x,\cdot,x)$.

Prove Property 1 for $k=\check{k}$. Next, we focus on a subspace of $\mathcal{F}^2_{\check{k}}$, denoted by $\mathcal{F}^3_{\check{k}}$. In this subspace any function f_E must satisfy that $f_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ is weakly decreasing in x for any $0\leq x\leq\check{k}$. Note that for $f_E\in\mathcal{F}^3_{\check{k}}$, $\tilde{f}_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ is weakly decreasing in x as well. Therefore, for any $0\leq k^1\leq k^2\leq\check{k}$, $\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{2\prime}},\cdot,\check{k})|K^2=k^2]\leq\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{1\prime}},\cdot,\check{k})|K^1=k^1]$. Then, using the monotonicity of survival rule that we derived above,

$$\begin{aligned}
& (T^a f_E)(\iota_{\check{k}}+\iota_{k^2},c,\check{k}) \\
= \;& \max\left\{0,a_S(\iota_{\check{k}}+\iota_{k^2},c,k^2)\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{2\prime}},c,\check{k})|K^2=k^2]+(1-a_S(\iota_{\check{k}}+\iota_{k^2},c,k^2))\tilde{f}_E(\iota_{\check{k}},c,\check{k})\right\} \\
\leq \;& \max\left\{0,a_S(\iota_{\check{k}}+\iota_{k^1},c,k^1)\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{2\prime}},c,\check{k})|K^2=k^2]+(1-a_S(\iota_{\check{k}}+\iota_{k^1},c,k^1))\tilde{f}_E(\iota_{\check{k}},c,\check{k})\right\} \\
\leq \;& \max\left\{0,a_S(\iota_{\check{k}}+\iota_{k^1},c,k^1)\mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{K^{1\prime}},c,\check{k})|K^1=k^1]+(1-a_S(\iota_{\check{k}}+\iota_{k^1},c,k^1))\tilde{f}_E(\iota_{\check{k}},c,\check{k})\right\} \\
= \;& (T^a f_E)(\iota_{\check{k}}+\iota_{k^1},c,\check{k})
\end{aligned}$$

Therefore, $T^a:\mathcal{F}^3_{\check{k}}\to\mathcal{F}^3_{\check{k}}$ and $v_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ is weakly decreasing in x, so Property 1 is satisfied for $k=\check{k}$.

Prove Property 3 for $k=\check{k}$. Before proving this property, we need to prove Properties 2 and 1 for $k=\check{k}-1$. We skip the details because later we will demonstrate how to do so for any k. Now suppose that we have verified these two properties for $k=\check{k}-1$, then $v_E\in\mathcal{F}^3_{\check{k}-1}$ where $\mathcal{F}^3_{\check{k}-1}$ is defined analogously to $\mathcal{F}^3_{\check{k}}$. Hence, for $0\leq x\leq\check{k}$, $v_E(\iota_{\check{k}-1}+\iota_x,\cdot,\check{k}-1)$ is weakly decreasing in x and $v_E(\iota_{\check{k}-1}+\iota_x,\cdot,x)$ is weakly increasing in x. Then, denote the subspace of $\mathcal{F}^3_{\check{k}-1}$ in which any f_E functions such that $f_E(\iota_{\check{k}}+\iota_x,\cdot,x) \le f_E(\iota_{\check{k}-1}+\iota_x,\cdot,x)$ for all x such that $0 \le x \le \check{k}-1$ by $\mathcal{F}^4_{\check{k}-1}$.

Then, $\tilde{f}_E(\iota_{\check{k}}+\iota_x,\cdot,x) \le \tilde{f}_E(\iota_{\check{k}-1}+\iota_x,\cdot,x)$ holds for $f_E \in \mathcal{F}^4_{\check{k}-1}$, by Lemma 1, we have

$$\begin{aligned}(T^a f_E)(\iota_{\check{k}}+\iota_x,c,x) &= \max\left\{0, \mathbb{E}[\tilde{f}_E(\iota_{\check{k}}+\iota_{X'},c,X')|X=x]\right\} \\ &\le \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K'}+\iota_{X'},c,X')|X=x]|K=\check{k}-1]\right\} \\ &= (T^a f_E)(\iota_{\check{k}-1}+\iota_x,c,x)\end{aligned}$$

Therefore, $T^a : \mathcal{F}^4_{\check{k}-1} \to \mathcal{F}^4_{\check{k}-1}$ and $v_E(\iota_{\check{k}}+\iota_x,\cdot,x) \le v_E(\iota_{\check{k}-1}+\iota_x,\cdot,x)$ for all x such that $0 \le x \le \check{k}-1$. In particular, this result ensures that $a_E(\iota_{\check{k}}+\iota_1,\cdot) \le a_E(\iota_{\check{k}-1}+\iota_1,\cdot)$.

Prove Property 4 for $k = \check{k}$. Define a subspace of $\mathcal{F}^4_{\check{k}-1}$ in which any function f_E satisfies $f_E(\iota_{\check{k}-1},\cdot,\check{k}-1) \le f_E(\iota_{\check{k}},\cdot,\check{k})$, as $\mathcal{F}^5_{\check{k}-1}$. Because $a_E(\iota_{\check{k}}+\iota_1,\cdot) \le a_E(\iota_{\check{k}-1}+\iota_1,\cdot)$, we have $\tilde{f}_E(\iota_{\check{k}-1},\cdot,\check{k}-1) \le \tilde{f}_E(\iota_{\check{k}},\cdot,\check{k})$ and then $(T^a f_E)(\iota_{\check{k}-1},\cdot,\check{k}-1) \le (T^a f_E)(\iota_{\check{k}},\cdot,\check{k})$. Therefore, $T^a : \mathcal{F}^5_{\check{k}-1} \to \mathcal{F}^5_{\check{k}-1}$ and hence $v_E(\iota_{\check{k}},\cdot,\check{k}) \ge v_E(\iota_{\check{k}-1},\cdot,\check{k}-1)$.

Now suppose that for any $k \le \check{k}$, we have established the following results

Result 1. v_E satisfies Property 1 for $k+1, k+2, \ldots, \check{k}$. That is, $v_E(\iota_{k+1}+\iota_x,\cdot,x), v_E(\iota_{k+2}+\iota_x,\cdot,x), \ldots, v_E(\iota_{\check{k}}+\iota_x,\cdot,x)$ are all weakly increasing in x, $0 \le x \le \check{k}$.

Result 2. v_E satisfies Property 2 for $k+1, k+2, \ldots, \check{k}$. That is, $v_E(\iota_x+\iota_{k+1},\cdot,k+1), v_E(\iota_x+\iota_{k+2},\cdot,k+2), \ldots, v_E(\iota_x+\iota_{\check{k}},\cdot,\check{k})$ are all weakly decreasing in x, $0 \le x \le \check{k}$.

Result 3. v_E satisfies Property 3 for $k+1, k+2, \ldots, \check{k}$. That is, $v_E(\iota_{k+1}+\iota_x,\cdot,x) \ge v_E(\iota_{k+2}+\iota_x,\cdot,x) \ge \ldots \ge v_E(\iota_{\check{k}}+\iota_x,\cdot,x)$ for all $1 \le x \le k$.

Result 4. v_E satisfies Property 4 for $k+1, k+2, \ldots, \check{k}$. That is, $v_E(\iota_{k+1},\cdot,k+1) \le v_E(\iota_{k+2},\cdot,k+2) \le \ldots \le v_E(\iota_{\check{k}},\cdot,\check{k})$.

We then need to prove that v_E satisfies Properties 1-4 for k.

Prove Property 2 for k. We follow three steps to achieve this end.

(i). First, consider $\mathcal{F}^1_k$, the subspace of $\mathcal{F}^5_{k+1}$ in which any function f satisfies that $(T^a f_E)(2\iota_k,\cdot,k) = v_E(2\iota_k,\cdot,k)$, $f_E(\iota_x+\iota_k,\cdot,k)$ is weakly decreasing in x, $k \le x \le \check{k}$, and $f_E(2\iota_k,\cdot,k) \le f_E(\iota_k+\iota_x,\cdot,k)$, for all $0 \le x \le k$. Note that at least a function f^*_E with $f^*_E(2\iota_k,\cdot,k) = v_E(2\iota_k,\cdot,k) = f^*_E(\iota_k+\iota_x,\cdot,k)$ for all x is in $\mathcal{F}^1_k$, so $\mathcal{F}^1_k$ is nonempty. For any $f_E \in \mathcal{F}^1_k$, $\tilde{f}_E$ shares the properties with f_E and hence also has the properties stated in Results

1–4. To prove that $(T^a f_E)(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $k \leq x \leq \check{k}$, consider the following cases for any k^1, k^2 such that $k \leq k^1 < k^2 \leq \check{k}$.

(a) If $k < k^1 < k^2$, according to Lemma 1, $\mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, \cdot, K')|K = k]|K^1 = k^1] \geq \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{2\prime}} + \iota_{K'}, \cdot, K')|K = k]|K^2 = k^2]$. Thus, from equation (2), $(T^a f_E)(\iota_{k^2} + \iota_k, c, k) \leq (T^a f_E)(\iota_{k^1} + \iota_k, c, k)$.

(b) If $k = k^1 < k^2$, first observe that

$$\begin{aligned} (T^a f_E)(\iota_{k^2} + \iota_k, c, k) &= \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{2\prime}} + \iota_{K'}, c, K')|K = k]|K^2 = k^2]\right\} \\ &\leq \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\} \end{aligned}$$

Then, because (i) for all $f_E \in \mathcal{F}^1_k$, $\tilde{f}_E(2\iota_k, \cdot, k) \leq \tilde{f}_E(\iota_k, \cdot, k)$ and $\tilde{f}_E(2\iota_k, \cdot, k) \leq \tilde{f}_E(\iota_k + \iota_1, \cdot, k)$, and (ii) Result 2, we obtain that $\mathbb{E}[\tilde{f}_E(\iota_k + \iota_{K'}, \cdot, K')|K = k] \leq \mathbb{E}[\tilde{f}_E(\iota_{K'}, \cdot, K')|K = k]$, and further $\mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, \cdot, K')|K = k]|K^1 = k] \leq \mathbb{E}[\tilde{f}_E(\iota_{K'}, \cdot, K')|K = k]$. Then,

$$\begin{aligned} (T^a f_E)(\iota_{k^2} + \iota_k, c, k) &\leq \max\{0, a_S(2\iota_k, c, k)\mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k^1 = k] \\ &\quad + (1 - a_S(2\iota_k, c, k))\mathbb{E}[\tilde{f}_E(\iota_{K'}, c, K')|K = k]\} \\ &= (T^a f_E)(\iota_{k^1} + \iota_k, c, k) \end{aligned}$$

To prove that $(T^a f_E)(2\iota_k, \cdot, k) \leq (T^a f_E)(\iota_k + \iota_x, \cdot, k)$, for all $0 \leq x \leq k$, note that,

$$\begin{aligned} (T^a f_E)(2\iota_k, c, k) &= v_E(2\iota_k, c, k) \\ &\leq \max\{0, v_S(2\iota_k, c, k)\} \\ &\leq \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\} \\ &\leq \max\{0, a_S(\iota_k + \iota_x, c, x)\mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{X'} + \iota_{K'}, c, K')|K = k]|X = x] \\ &\quad + (1 - a_S(\iota_k + \iota_x, c, x))\mathbb{E}[\tilde{f}_E(\iota_{K'}, c, K')|K = k]\} \\ &= (T^a f_E)(\iota_k + \iota_x, c, k) \end{aligned}$$

The first inequality is due to equilibrium symmetric; for two type-k firm, either firm's equilibrium payoff is bounded by payoff from joint continuation. The second inequality is because $f_E \in \mathcal{F}^0$ so $f_E \geq v_E$. These results show that $T^a : \mathcal{F}^1_k \to \mathcal{F}^1_k$. An familiar argument on the simultaneous-move survival game leads to

$$v_E(2\iota_k, c, k) = \max\{0, v_S(2\iota_k, c, k)\}. \tag{4}$$

The right-hand-side of (4) defines a contraction mapping with a unique fixed point $v(2\iota_k, \cdot, k)$,

$$(T^a f_E)(2\iota_k, c, k) = \max\left\{0, \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\}.$$

(ii). We move on to a subspace of $\mathcal{F}_k^1$, which we denote by $\mathcal{F}_k^2$. Any function f_E in this subspace satisfy that $f_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing with x. Note that for $f_E \in \mathcal{F}_k^2$, $\tilde{f}_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x as well. Combine it with Result 1, and then we have that $\tilde{f}_E(\iota_d + \iota_x, \cdot, x)$ is weakly increasing in x for all d such that $k \le d \le \check{k}$. Therefore, $\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_x, \cdot, x)|K = k]$ is weakly increasing in x. For any k^1, k^2 such that $1 \le k^1 \le k^2 \le \check{k}$, according to Lemma 1,

$$[\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{K^{1\prime}}, \cdot, K^{1'})|K = k]|K^1 = k^1] \le [\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{K^{2\prime}}, \cdot, K^{2'})|K = k]|K^2 = k^2]. \tag{5}$$

We then consider the following cases.

(a) For $k^1 \le k^2 \le k$, from equation (2), we can observe that equation (5) leads to $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_{\check{k}} + \iota_{k^2}, c, k^2)$.

(b) For $k < k^1 \le k^2 \le \check{k}$, from Result 3, we have $a_S(\iota_k + \iota_{k^1}, \cdot, k) \ge a_S(\iota_k + \iota_{k^2}, \cdot, k)$. Also, from Result 2 and Lemma 1,

$$[\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{K^{i\prime}}, \cdot, K^{i'})|K = k]|K^i = k^i] \le \mathbb{E}[\tilde{f}_E(\iota_{K^{i\prime}}, \cdot, K^{i'})|K^i = k^i], i = 1, 2.$$

From Result 4 and Lemma 1,

$$\mathbb{E}[\tilde{f}_E(\iota_{K^{2\prime}}, \cdot, K^{2'})|K^2 = k^2] \le \mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}}, \cdot, K^{1'})|K^1 = k^1].$$

Using equation (1), we can show $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_k + \iota_{k^2}, c, k^2)$ by exploiting the inequalities.

(c) For $k^1 \le k \le k^2 \le \check{k}$, similarly we can show $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_k + \iota_{k^2}, c, k^2)$ with the above results.

This result guarantees that $T^a : \mathcal{F}_k^2 \to \mathcal{F}_k^3$. Therefore, any equilibrium payoff must satisfy that $v_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x for all $1 \le x \le \check{k}$, which leads to the same monotonicity for v_S and the equilibrium survival rule.

Prove Property 1 for k. Next, we focus on a subspace of $\mathcal{F}_k^3$, denoted by $\mathcal{F}_k^3$. In this subspace, any function f_E satisfies that $f_E(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $0 \le x \le k$. Note that for $f_E \in \mathcal{F}_k^3$, $\tilde{f}_E(\iota_x + \iota_k, \cdot, k)$ is also weakly decreasing in x, $0 \le x \le k$. Combine it with Result 2, and then we have that $\tilde{f}_E(\iota_x + \iota_d, \cdot, d)$ is weakly decreasing in x, $0 \le x \le \check{k}$ and for all d such that $k \le d \le \check{k}$. Therefore, $\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_x, \cdot, K')|K = k]$ is weakly decreasing in x. For any k^1, k^2 such that $0 \le k^1 \le k^2 \le \check{k}$, Lemma 1 implies that

$$[\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{K^{1\prime}}, \cdot, K')|K = k]|K^1 = k^1] \ge [\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{K^{2\prime}}, \cdot, K')|K = k]|K^2 = k^2].$$

Also, we have $a_S(\iota_k + \iota_{k^1}, \cdot, k^1) \le a_S(\iota_k + \iota_{k^2}, \cdot, k^2)$. Therefore, it must be true that $(T^a f_E)(\iota_k + \iota_{k^2}, c, k) \le (T^a f_E)(\iota_k + \iota_{k^1}, c, k)$. So $T^a : \mathcal{F}_k^3 \to \mathcal{F}_k^3$ and the equilibrium payoff $v_E(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $0 \le x \le \check{k}$.

Prove Property 3 for k. Next, we further look into a subspace of $\mathcal{F}_k^3$, denoted by $\mathcal{F}_k^4$, in which any function f_E satisfies that $f_E(\iota_k + \iota_x, \cdot, x) \le f_E(\iota_{k+1} + \iota_x, \cdot, x)$ for all $x < k$. Note that Result 2 and Property 1 ensure that $f_E(\iota_k + \iota_x, \cdot, x) \le f_E(\iota_{k+1} + \iota_x, \cdot, x)$ for all $x \ge k$, so for any $f_E \in \mathcal{F}_k^3$, $\tilde{f}_E(\iota_{k+1} + \iota_x, \cdot, x) \le \tilde{f}_E(\iota_k + \iota_x, \cdot, x)$ for all x. Combine it with Result 3, and then we have $\mathbb{E}[\tilde{f}_E(\iota_{k^1} + \iota_{X'}, \cdot, X')|X = x]$ is weakly decreasing in k^1 for $k \le k^1 \le \check{k}$. According to Lemma 1,

$$\mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K^{1\prime}} + \iota_{X'}, \cdot, X')|X = x]|K^1 = k^1] \le \mathbb{E}[\mathbb{E}[\tilde{f}_E(\iota_{K'} + \iota_{X'}, \cdot, X')|X = x]|K = k].$$

Then using equation (2) we can show that $(T^a f_E)(\iota_{k^1} + \iota_x, c, x) \le (T^a f_E)(\iota_k + \iota_x, c, x)$ for any $x < k$. Therefore, $T^a : \mathcal{F}_k^4 \to \mathcal{F}_k^4$ and $v_E(\iota_{k^1} + \iota_x, c, x) \le v_E(\iota_k + \iota_x, c, x)$ for all x and all k^1 such that $k \le k^1 \le \check{k}$. In particular, this result ensures that $a_E(\iota_{k+1} + \iota_1, \cdot) \le a_E(\iota_k + \iota_1, \cdot)$.

Prove Property 4 for k. Finally, define a subspace of $\mathcal{F}_k^4$, in which any function f_E satisfies $f_E(\iota_k, \cdot, k) \le f_E(\iota_{k+1}, \cdot, k+1)$, as $\mathcal{F}_k^5$. Because $a_E(\iota_{k+1} + \iota_1, \cdot) \le a_E(\iota_k + \iota_1, \cdot)$, we have $\tilde{f}_E(\iota_k, \cdot, k) \le \tilde{f}_E(\iota_{k+1}, \cdot, k+1)$ as well. Combine it with Result 4, and then we have that $\tilde{f}_E(\iota_{k^1}, \cdot, k^1)$ is weakly increasing in k^1 for $k \le k^1 \le \check{k}$. Using Lemma 1, we have $(T^a f_E)(\iota_k, c, k) \le (T^a f_E)(\iota_{k+1}, c, k+1)$ so $T^a : \mathcal{F}_k^5 \to \mathcal{F}_k^5$ and Property 4 is verified for k.

This completes the verification for the sufficient properties for any arbitrary k. Since (a_S, a_E) is also arbitrarily chosen, any natural equilibrium payoff function v_E must satisfy Properties 1, 2, and 4. Then we can prove Lemma 1.

Prove Lemma 1. For any strategy (a_S, a_E), as a special case of Property 1, $v_E(2\iota_k, c, k) \leq v_E(\iota_k, c, k)$ for any $k \leq \check{k}$. To prove $v_S(2\iota_k, c, k) \leq v_S(\iota_k, c, k)$, note that

$$v_S(2\iota_k, c, k) = \mathbb{E}[\mathbb{E}[\tilde{v}_E^a(\iota_{K'} + \iota_{K^{1\prime}}, c, K')|K = k]|K^1 = k]$$

$$v_S(\iota_k, c, k) = \mathbb{E}[\tilde{v}_E^a(\iota_{K'}, c, K')|K = k],$$

where $\tilde{v}_E^a$ is defined analogously as $\tilde{f}_S$. For any $k^1, k^2 \geq k$, Property 1 ensures that $v_E(\iota_{k^1} + \iota_{k^2}, c, k^1) \leq v_E(\iota_{k^1} + \iota_1, c, k^1) \leq v_E(\iota_{k^1}, c, k^1)$, and $\tilde{v}_E^a(\iota_{k^1} + \iota_{k^2}, c, k^1) \leq \tilde{v}_E^a(\iota_{k^1} + \iota_1, c, k^1) \leq \tilde{v}_E^a(\iota_{k^1}, c, k^1)$. Therefore $\mathbb{E}[\mathbb{E}[v_S(\iota_{K'} + \iota_{K^{1\prime}}, c, K')|K = k]|K^1 = k] \leq \mathbb{E}[v_S(\iota_{K'}, c, K')|K]$. $\square$

I.2 Details for the Proof of Proposition 1

Proof. We provide here the details to prove $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$. Define $\mathcal{F}$ to be the space of all functions

$$f_E : \left\{ (n_1, \ldots, n_{\check{k}}) : \sum_{i=1}^{\check{k}} n_i \leq 2 \right\} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \left[0, \frac{\beta\pi(\iota_{\check{k}}, \check{c}, \check{k})}{1-\beta}\right],$$

and $T^\alpha : \mathcal{F} \to \mathcal{F}$ with

$$(T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^1) = \left\{ \begin{array}{ll} (T_{k^1,k^2} f)(C), f(C) \equiv g(\iota_{k^1} + \iota_{k^2}, c, k^2) & \text{if } k^1 \leq k^2 \\ (T_{k^1} f)(C, k^2), f(C, k^2) \equiv g(\iota_{k^1} + \iota_{k^2}, c, k^2) & \text{if } k^1 > k^2 \end{array} \right\}.$$

Thus, T^α is exactly assembled by T_{k^1,k^2} and T_{k^1} in Algorithm 1, and w_E computed by Algorithm 1 is the unique fixed point of T^α. Now consider a subspace of $\mathcal{F}$, which we denote as $\mathcal{F}^N$. In this space, any function f_E satisfies that $f_E \leq w_E$, $f_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq f_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$ for all $k^1 > k^2$.

We aim to prove $T^\alpha : \mathcal{F}^N \to \mathcal{F}^N$. For all $f_E \in \mathcal{F}^N$, $\tilde{f}_E \in \mathcal{F}^N$ as well. Consider the following cases

(i). If Algorithm 1 computes $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) = 1$, then Algorithm 1 also prescribes $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$. Substitute these survival rules into Equation (1) and use Lemma 1 again, we obtain that $(T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^1) \geq (T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^2)$.

(ii). If Algorithm 1 computes $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$, then it must be the case that $w_E(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$. For any $f_E \in \mathcal{F}$, $f_E(\iota_{k^1} + \iota_{k^2}, c, k^2) \leq w_E(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$. Since $w_E(\iota_{k^1} + \iota_{k^2}, c, k^2) = (T^{\alpha\infty} f_E)(\iota_{k^1} + \iota_{k^2}, c, k^2)$ and T^α is a monotone operator, $(T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^2) \leq w_E(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$. Thus, $(T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^1) \geq 0 \geq (T^\alpha f_E)(\iota_{k^1} + \iota_{k^2}, c, k^2)$.

By point-wise comparison, we conclude that $T^\alpha : \mathcal{F}^N \to \mathcal{F}^N$ hence $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$ for all $k^1 > k^2$. This means that whenever $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2) > 0$, $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) > 0$ as well. $\square$

II Computational Techniques

II.1 Computing A Firm's Beliefs about Next Period's State

The computation of the expectation in (1) requires the distribution of M_S conditional on M_E, given that the firm of interest survives and that all other firms use the common strategy a_S. Denote the density (with respect to the appropriate dominating measure) of this distribution with $p_{m_S}(\cdot|M_E = m_E)$. Decompose $m_S = \sum_i m_S^i \equiv \sum_i m_{S,i}\iota_i$ and $m_E = \sum_i m_{E,i}\iota_i$, with $m_{S,i}$ the number of firms who have type-i in the current period and continue to next period, and $m_{E,i}$ the number of firms who are active when the current period's continuation decisions are made. Then, because $M_S^i, \ldots, M_S^{\check{k}}$ are independent conditional on M_E and given that the firm of interest survives, $p_{m_S}(\cdot|M_E = m_E)$ is the convolution of the corresponding conditional densities $p_{m_S^i}(\cdot|M_E = m_E)$ of m_S^i; $i = 1, \ldots, \check{k}$. Denote $\tilde{m}_i \equiv m_{E,i} - I(i = k)$ and $\tilde{m}_{S,i} \equiv m_{S,i} - I(i = k)$. Note that $\tilde{m}_i$ is the number of type-i firms active when continuation decisions are made in the current period, excluding the firm of interest; and $\tilde{m}_{S,i}$ is the number of firms who have type-i in the current period and continue to next period, excluding the firm of interest. Then, for $m_{S,i}$ such that $0 \leq m_{S,i} \leq m_{E,i}$, we have that

$$p_{m_S^i}(\cdot|M_E = m_E) = \binom{\tilde{m}_i}{\tilde{m}_{S,i}} a_S(m_E, c, i)^{\tilde{m}_{S,i}} (1 - a_S(m_E, c, i))^{\tilde{m}_i - \tilde{m}_{S,i}}$$

Computing the expectation in (2) requires the distribution of (N', M', C', K') conditional on M_S, C, K, given that all potential entrants use the common strategy a_E. Denote the density of this distribution with $p(\cdot|M_S, C, K)$. Conditional on (N', C'), M' is independent of (K', M_S, C, K); conditional on (K', M_S, K), N' is independent of C'; conditional on C, C' is independent of (K', M_S, C, K); and conditional on K, K' is independent of H_S. Consequently,

$$\begin{aligned} p(n', m', c', k'|M_S = m_S, C = c, K = k) =& p_M(m'|N' = n', C' = c') \times p_N(n'|K' = k', M_S = m_S, K = k) \\ & \times q(c'|C = c) \times \Pi_{k'k}. \end{aligned}$$

Here, $p_M(\cdot|N', C')$ is the density of next period's post-entry market structure M', conditional on next period's pre-entry market structure N' and demand state C'. And, $p_N(\cdot|K', M_S, K)$

is the density of next period's pre-entry market structure N' conditional on M_S, given that the firm of interest survives with productivity type K'.

First, note that

$$p_M\left(m'|N'=n',C'=c'\right)=\begin{cases}1-a_E\left(m'+\iota_1,c'\right) & \text{if } M'=n';\\ (1-a_E(m'+(m_1'-n_1'+1)\iota_1,c')) & \text{if } m_1'>n_1'\\ \quad\times\prod_{j=1}^{m_1'-n_1'}a_E(n'+j\iota_1,c') & \text{ and } m_2'=n_2',\ldots,m_{\check{k}}'=n_{\check{k}}';\\ 0 & \text{otherwise.}\end{cases}$$

Next, consider $p_N\left(\cdot|K',M_S,K\right)$. Decompose $n'=\sum_i n^i$, with n^i the contribution to next period's pre-entry market structure by the $m_{S,i}$ firms who are of type-i in the previous period and choose to continue. Then, because $N^1,\ldots,N^{\check{k}}$ are independent conditional on M_S and given that the firm of interest survives with productivity type K', $p_N\left(\cdot|K',M_S,K\right)$ is the convolution of the corresponding conditional densities $p_{N^i}\left(\cdot|K',M_S,K\right)$ of N^i; $i=1,\ldots,\check{k}$. Denote $\tilde{n}^i(k,k')\equiv n^i-I(i=k)\iota_{k'}$. This is the contribution of the $m_{S,i}$ firms excluding the firm of interest, to next period's pre-entry market structure. Then, for m_S such that $\tilde{m}_{S,i}\geq 0$, we have that

$$p_{N^i}\left(n^k|K'=k',M_S=m_S,K=k\right)=\prod_{i'=i}^{\check{k}}\binom{\sum_{m=i'}^{\check{k}}\tilde{n}_m^i(k,k')}{\tilde{N}_{i'}^i(k,k')}\Pi_{i'i}^{\tilde{n}_{i'}^i(k,k')}$$

if $\tilde{N}_m^i=0$ for all $m<i$, $\tilde{n}_m^i\geq 0$ for all $m\geq i$, and $\sum_{i'}\tilde{n}_{i'}^i\leq\tilde{m}_i$; and zero otherwise.

II.2 Constructing the Type Transition Matrices in Matlab

II.2.1 The Problem

Given any finite $\check{m}$ and $\check{k}$ and a $\check{k}\times\check{k}$ transition matrix Π, or the triple $(\check{k},\check{m},\Pi)$, we need to compute all the transition matrices for $1,2,\ldots,\check{m}$-firm market structures, conditioning on all realized exits and one surviving firm's type transition. Since any single firm's type transition is characterized by Π, the non-trivial part of this problem is computing all the transition matrices for $1,2,\ldots,\check{m}-1$-firm market structures. W.L.O.G., we discuss how to construct $\check{m}$ such matrices for the triple $(\check{k},\check{m}+1,\Pi)$. For every ordering of all possible market structures with m firms, $m\in\{1,\ldots,\check{m}\}$, there is a representation of transition matrix corresponding to that ordering. We henceforth focus on the transition matrices for OL-ordered market structures. For any m, we denote the transition matrix as Π^m.

II.2.2 The Dimensionality

For the triple $(\check{k}, \check{m}+1, \Pi)$, we know that if there are m surviving firms, the OL-ordered sequence of all possible market structures has $\frac{(m+\check{k}-1)!}{m!(\check{k}-1)!}$ elements. Therefore, Π^m's dimension is $\frac{(m+\check{k}-1)!}{m!(\check{k}-1)!} \times \frac{(m+\check{k}-1)!}{m!(\check{k}-1)!}$.

II.2.3 Recursive Construction of Π^m

We recursively construct Π^m using Π^{m-1} and Π^1, for all $2 \leq i \leq \check{m}$. Note that $\Pi^1 = \Pi$. To describe the construction, we very often use examples. We use *Italic* to distinguish the discussion on general case and *the discussion on an example.*

The link between an element in Π^m and the elements in Π^{m-1} and Π^1 is explained below.

(i). The (a, b) element in Π^m corresponds to a transition probability from an initial m-firm market which has index a in the m-firm OL sequence to a destined m-firm market with index b.

(ii). Suppose that i is the index for the highest type in the initial market a. Taking out one type-i firm from the a leaves an initial $m-1$-firm market structure. Suppose that this market structure has index c in the $m-1$-firm OL sequence.

(iii). Next, suppose that the type-i firm transits to one of the possible types j in the destined market b. This transition has probability $\Pi_{i,j}$. [1]

(iv). Excluding this type-j firm from the destined market leaves a destined $m-1$-firm market. Suppose that this market has index d in the $m-1$-firm OL sequence.

(v). The transition between the initial and the destined $m-1$-firm markets is characterized by $\Pi^{m-1}_{c,d}$.

(vi). The transition from the initial m-firm market to the destined one then has the probability

$$\Pi^m_{a,b} = \sum_{j:b_j>0} \Pi_{i,j} \Pi^{m-1}_{c,d}.$$

[1] Note that we also consider the impossible regression in the types here and throughout this notes. So, we consider all j's including those are lower than i. In such cases, $\Pi_{i,j} = 0$.

Example *Suppose that $\check{k} = 3, \check{m} = 2$. In slightly abused notations, we denote the types as L, M, H. Π^2 is a 6×6 matrix. Now, take its $(2,3)$ element as an example to demonstrate the above procedure.*

(i). This $(2,3)$ element corresponds to the transition from the market HM to HL.

(ii). Taking out the firm with the highest type H from the initial market HM leaves an initial 1-firm market M, which has index 2 in the 1-firm OL sequence.

(iii). Suppose that the H firm transits to H in the destined market. This transition has probability $\Pi_{1,1}$.

(iv). Excluding the H firm from the destined market leaves a destined 1-firm market L, which has index 3 in the 1-firm OL sequence.

(v). The transition between the initial and the destined 1-firm market M and L is characterized by $\Pi^1_{2,3}$.

(vi). Note that H can also transit to L (with 0 probability), the transition from the initial market HM to the destined HL then has the probability

$$\Pi^2_{2,3} = \Pi_{1,1}\Pi^1_{2,3} + \Pi_{1,3}\Pi^1_{2,1}.$$

In short-hand notations, we rewrite the equation $\Pi^m_{ab} = \sum_{j:b_j>0} \Pi_{ij}\Pi^{m-1}_{cd}$ using indices only: $(a,b) := \sum_{j:b_j>0}(i,j) \times (c,d)$, with the understanding that (a,b) always indexes the element in Π^m, (i,j) in Π, and (c,d) in Π^{m-1}. We connect these indices to the objects that they index.

(i). i indexes the highest type in the market a. Therefore, for each given a, i is unique. This implies that in each row of Π^m, all entries share the same i.

(ii). All j's indicate all possible types in market b. Therefore, in each column of Π^m, all entries share the same j's. For any (a,b) entry, there are at most $\check{k}$ possible values of j.

(iii). c indexes the $m-1$ market structure resulted by subtracting a type-i firm from the market a. Therefore, for each (a,b), c is unique and in each row of Π^m, all entries share the same c.

(iv). All d's index all possible $m-1$ market structures resulted by subtracting a type-j firm from the market b. Therefore, in each column of Π^m, all entries share the same d's. For any (a, b) entry, there are at most $\check{k}$ possible values of d.

Henceforth, we call i the first index, all j's the second indices, c the third, and d's the fourth. One may have already developed some intuition that there are regularity patterns in these indices, which can be used to vectorize the calculation of Π^m. Next, we make the regularity pattern visible to intellectual eyes by an example.

Example *As an example, we write* Π^2 *for* $\check{k} = 3, \check{m} = 2$ *using the indices representation. Again, bear in mind that the first two indices index the element in* Π *while the last two index the element in* Π^1.

	HH	HM	HL	MM	ML	LL
HH	$(1,1)\times(1,1)$	$(1,1)\times(1,2)$ $+(1,2)\times(1,1)$	$(1,1)\times(1,3)$ $+(1,3)\times(1,1)$	$(1,2)\times(1,2)$	$(1,2)\times(1,3)$ $+(1,3)\times(1,2)$	$(1,3)\times(1,3)$
HM	$(1,1)\times(2,1)$	$(1,1)\times(2,2)$ $+(1,2)\times(2,1)$	$(1,1)\times(2,3)$ $+(1,3)\times(2,1)$	$(1,2)\times(2,2)$	$(1,2)\times(2,3)$ $+(1,3)\times(2,2)$	$(1,3)\times(2,3)$
HL	$(1,1)\times(3,1)$	$(1,1)\times(3,2)$ $+(1,2)\times(3,1)$	$(1,1)\times(3,3)$ $+(1,3)\times(3,1)$	$(1,2)\times(3,2)$	$(1,2)\times(3,3)$ $+(1,3)\times(3,2)$	$(1,3)\times(3,3)$
MM	$(2,1)\times(2,1)$	$(2,1)\times(2,2)$ $+(2,2)\times(2,1)$	$(2,1)\times(2,3)$ $+(2,3)\times(2,1)$	$(2,2)\times(2,2)$	$(2,2)\times(2,3)$ $+(2,3)\times(2,2)$	$(2,3)\times(2,3)$
ML	$(2,1)\times(3,1)$	$(2,1)\times(3,2)$ $+(2,2)\times(3,1)$	$(2,1)\times(3,3)$ $+(2,3)\times(3,1)$	$(2,2)\times(3,2)$	$(2,2)\times(3,3)$ $+(2,3)\times(3,2)$	$(2,3)\times(3,3)$
LL	$(3,1)\times(3,1)$	$(3,1)\times(3,2)$ $+(3,2)\times(3,1)$	$(3,1)\times(3,3)$ $+(3,3)\times(3,1)$	$(3,2)\times(3,2)$	$(3,2)\times(3,3)$ $+(3,3)\times(3,2)$	$(3,3)\times(3,3)$

To illustrate the regularity in the above matrix, the first trick is to introduce an auxiliary "impossible destined market structure", which possesses index $\frac{(m+\check{k}-1)!}{m!(\check{k}-1)!}+1$ in the OL sequence of m-firm market structures. Its impossibility means that no m-firm market structure can transit to it. *For instance, when $m = 1$, this market structure has index 4. To accommodate such impossible destined market structure, we can expand Π^1 by a fourth column of zeros, so* $\Pi^1_{i,4} = 0, i = 1, 2, 3.$

Then, we can rewrite the above matrix as

	HH	HM	HL	MM	ML	LL
HH	$(1,1)\times(1,1)$ $+(1,2)\times(1,4)$ $+(1,3)\times(1,4)$	$(1,1)\times(1,2)$ $+(1,2)\times(1,1)$ $+(1,3)\times(1,4)$	$(1,1)\times(1,3)$ $+(1,2)\times(1,4)$ $+(1,3)\times(1,1)$	$(1,1)\times(1,4)$ $+(1,2)\times(1,2)$ $+(1,3)\times(1,4)$	$(1,1)\times(1,4)$ $+(1,2)\times(1,3)$ $+(1,3)\times(1,2)$	$(1,1)\times(1,4)$ $+(1,2)\times(1,4)$ $+(1,3)\times(1,3)$
HM	$(1,1)\times(2,1)$ $+(1,2)\times(2,4)$ $+(1,3)\times(2,4)$	$(1,1)\times(2,2)$ $+(1,2)\times(2,1)$ $+(1,3)\times(2,4)$	$(1,1)\times(2,3)$ $+(1,2)\times(2,4)$ $+(1,3)\times(2,1)$	$(1,1)\times(2,4)$ $+(1,2)\times(2,2)$ $+(1,3)\times(2,4)$	$(1,1)\times(2,4)$ $+(1,2)\times(2,3)$ $+(1,3)\times(2,2)$	$(1,1)\times(2,4)$ $+(1,2)\times(2,4)$ $+(1,3)\times(2,3)$
HL	$(1,1)\times(3,1)$ $+(1,2)\times(3,4)$ $+(1,3)\times(3,4)$	$(1,1)\times(3,2)$ $+(1,2)\times(3,1)$ $+(1,3)\times(3,4)$	$(1,1)\times(3,3)$ $+(1,2)\times(3,4)$ $+(1,3)\times(3,1)$	$(1,1)\times(3,4)$ $+(1,2)\times(3,2)$ $+(1,3)\times(3,4)$	$(1,1)\times(3,4)$ $+(1,2)\times(3,3)$ $+(1,3)\times(3,2)$	$(1,1)\times(3,4)$ $+(1,2)\times(3,4)$ $+(1,3)\times(3,3)$
MM	$(2,1)\times(2,1)$ $+(2,2)\times(2,4)$ $+(2,3)\times(3,4)$	$(2,1)\times(2,2)$ $+(2,2)\times(2,1)$ $+(2,3)\times(3,4)$	$(2,1)\times(2,3)$ $+(2,2)\times(2,4)$ $+(2,3)\times(2,1)$	$(2,1)\times(2,4)$ $+(2,2)\times(2,2)$ $+(2,3)\times(3,4)$	$(2,1)\times(2,4)$ $+(2,2)\times(2,3)$ $+(2,3)\times(2,2)$	$(2,1)\times(2,4)$ $+(2,2)\times(2,4)$ $+(2,3)\times(2,3)$
ML	$(2,1)\times(3,1)$ $+(2,2)\times(3,4)$ $+(2,3)\times(3,4)$	$(2,1)\times(3,2)$ $+(2,2)\times(3,1)$ $+(2,3)\times(3,4)$	$(2,1)\times(3,3)$ $+(2,2)\times(3,4)$ $+(2,3)\times(3,1)$	$(2,1)\times(3,4)$ $+(2,2)\times(3,2)$ $+(2,3)\times(3,4)$	$(2,1)\times(3,4)$ $+(2,2)\times(3,3)$ $+(2,3)\times(3,2)$	$(2,1)\times(3,4)$ $+(2,2)\times(3,4)$ $+(2,3)\times(3,3)$
LL	$(3,1)\times(3,1)$ $+(3,2)\times(3,4)$ $+(3,3)\times(3,4)$	$(3,1)\times(3,2)$ $+(3,2)\times(3,1)$ $+(3,3)\times(3,4)$	$(3,1)\times(3,3)$ $+(3,2)\times(3,4)$ $+(3,3)\times(3,1)$	$(3,1)\times(3,4)$ $+(3,2)\times(3,2)$ $+(3,3)\times(3,4)$	$(3,1)\times(3,4)$ $+(3,2)\times(3,3)$ $+(3,3)\times(3,2)$	$(3,1)\times(3,4)$ $+(3,2)\times(3,4)$ $+(3,3)\times(3,3)$

The second trick towards detecting the regularity is to partition the matrix into $\check{k}$ row-blocks. In each row-block, all rows correspond to initial market structures that share the same highest type. *In the above matrix, row 1-3 correspond to initial market structures whose highest type is H, row 4-5 M, and row 6 L.* Because of OL, the index of the row-block is also the index for the highest type. *So, in the above matrix, the first block connects to type 1, the second block to type 2, and the third block to type 3.*

The first row-block of any Π^m matrix contains all initial market structures that has one type-$\check{k}$ firm and $m-1$ other firms with any type. There are $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!}$ such market structures. So the length of the first block is just $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!}$. The second block contains all initial market structures that has one type-$\check{k}-1$ firm and $m-1$ other firms with type no better than $\check{k}-1$. There are $\frac{(m-1+\check{k}-2)!}{(m-1)!(\check{k}-2)!}$ such market structures. So is the length of the second block. In total, there are $\check{k}$ such blocks. The t-th block has the length $\frac{(m-1+\check{k}-t)!}{(m-1)!(\check{k}-t)!}$.

Now each entry of the matrix has four columns of indices. All columns have the same length $\check{k}$ (*3 in this example*). Next, we present the regularity on these columns.

(i). Recall that the first index represents the highest type in the initial market structure. Also recall that in each row-block, all initial market structures share the same highest type. Therefore, the column of first indices in each entry has a single value, which is simply the index of the row-block that this entry is in. Therefore, it remains unchanged for every row in a same block.

(ii). Recall that the second indices represent all the possible types in the destined market structure. After the introduction of the impossible market structure, the column of the second indices in each entry is simply $(1, \ldots, \check{k})$.

(iii). Recall that the third index represents the $m-1$ market structure resulted by subtracting a highest type firm from the initial market structure. Therefore, the column of the third indices has a single value and remains unchanged for every entry in a same row. In each row, this value equals the index of the $m-1$ market structure resulted by subtracting a highest type from the initial market structure. *In the current example, in row 1, the 1-firm market structure resulted by subtracting H from HH is H, which has index number 1 in 1-firm OL sequence. So, in the first row, the third index is 1. In row 2, the 1-firm market structure resulted by subtracting H from HM is M. So, in the first row, the third index is 2.* Observing the following facts

(a) Within each block, this index increases by 1 each row.

(b) The last row in each block corresponds to the most inferior market structure in the OL sense. Hence, this index in the last row of each block must equal to the length of the OL sequence of the $m-1$-firm market structures. *In the current example, the length is 3, which is the value of the third index in row 3, 5, 6.*

(c) The t-th block has the length $\frac{(m-1+\check{k}-t)!}{(m-1)!(\check{k}-t)!}$.

We can conclude that in the t-th row-block, the third index grows from $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!} - \frac{(m-1+\check{k}-t)!}{(m-1)!(\check{k}-t)!} + 1$ to $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!}$ row by row.

(iv). Recall that the fourth indices represents the $m-1$ market structure resulted by subtracting a highest type firm from the destined market structure. Therefore, the column of the fourth indices remains unchanged for every entry in a same column. The regularity pattern of this column is more subtle than any of the above columns. We further explore it. *We write down this column in the above example*

HH	*HM*	*HL*	*MM*	*ML*	*LL*
1	*2*	*3*	*4*	*4*	*4*
4	*1*	*4*	*2*	*3*	*4*
4	*4*	*1*	*4*	*2*	*3*

This matrix of the fourth indices can be engineered from the following 0-1 matrix.

HH	*HM*	*HL*	*MM*	*ML*	*LL*
1	*1*	*1*	*0*	*0*	*0*
0	*1*	*0*	*1*	*0*	*0*
0	*0*	*1*	*0*	*1*	*1*

We can transform the 1's in each row of the above matrix into the ordinals of 1's (the first 1 stays 1, the second 1 is transformed to 2, the third to 3) and the 0's into 4 to go back to the matrix of the fourth indices. This transformation is unique and can always be done for any matrix of the fourth indices. Hence we focus on constructing the later matrix, which we simply call the indexing matrix.

Since the fourth indices are related to the destined market structures, we construct the indexing matrix by exploring Π^m from its column dimension. Now, we introduce the third trick. We partition the Π^m matrix into $\check{k}$ column-blocks. Analogously to the row-blocks, in each column-block, all columns correspond to destined market structures that share the same highest type. *In the example matrix, column 1-3 correspond to destined market structures whose highest type is H, column 4-5 M, and column 6 L. Again, the index of the column-block is also the index for the highest type. So, in the* *above matrix, the first block connects to type 1, the second block to type 2, and the third block to type 3.* Observing the following facts

(a) The indexing matrix for $m = 1$ is a $\check{k} \times \check{k}$ identity matrix.

(b) The indexing matrix has $\check{k}$ rows. Its (e, f) element indicates if the destined market structure f has a type-e firm. If it does, then the (e, f) element of the indexing matrix is 1. Otherwise it is 0. *In the above example, the* $(1, 1)$ *element of the indexing matrix is 1, because the market structure* HH *contains a type-H firm. The* $(2, 1)$ *element is 0, because the market structure* HH *does not contain a type-M firm.*

(c) The indexing matrix can also be partitioned into $\check{k}$ column-blocks.

(d) In its t-th column-block, since the highest type in the destined market structure is t, the first $t-1$ rows of the indexing matrix in this block are all 0's and the t-th row is full of 1's. *In the above example, the first row is full of 1's in block 1 and full of 0's in block 2.*

(e) In its t-th column-block, since the $m-1$-firm market structures resulted by subtracting the highest type firm from the destined market structure are the $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!} - \frac{(m-1+\check{k}-t)!}{(m-1)!(\check{k}-t)!} + 1$ to $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!}$ destined market structures in Π^{m-1}, from the $t+1$-th row onward, the indexing matrix is identical to the $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!} - \frac{(m-1+\check{k}-t)!}{(m-1)!(\check{k}-t)!} + 1$ to $\frac{(m-1+\check{k}-1)!}{(m-1)!(\check{k}-1)!}$ columns of the indexing matrix corresponding to $m-1$. *In column-block 1 in the above example, the* 1*-firm market structures are* $H, M,$ *and* L*, which are the 1,2, and 3 destined market structures of* Π^1*. Hence, from the second row onwards in block 1, the indexing matrix is identical to the 1,2, and 3 columns of the indexing matrix for* $m = 1$*, which is a* 3×3 *identity matrix. In column-block 2 in the above example, the* 1*-firm market structures are* M *and* L*, which are the 2, and 3 destined market structures of* Π^1*. Hence, the third row in block 2 of the indexing matrix is identical to the third row and the 2 and 3 columns of the indexing matrix for* $m = 1$.

With all the regularity patterns pointed out as above, we create a Matlab function to generate all the transition matrices.

II.2.4 The typetransition.m Function

The matlab function typetransition.m takes the triple $(\check{k}, \check{m}, \Pi)$ as input, and produces a $\frac{(\check{m}+\check{k}-1)!}{\check{m}!(\check{k}-1)!} \times (\frac{(\check{m}+\check{k}-1)!}{\check{m}!(\check{k}-1)!}+1) \times \check{m}$ array, in which each page contains a transition matrix and the page number m indicates the number of firms. On each page, the first $\frac{(m+\check{k}-1)!}{m!(\check{k}-1)!}$ rows and the first $\frac{(m+\check{k}-1)!}{m!(\check{k}-1)!}$ columns form the transition matrix for the m-firm market.

This function has several layers of loops. The most outside loop runs from $m = 2$ to $m = \check{m}$. Within this loop, for each given m, the indexing matrix is first created and than transformed to the matrix of the fourth indices. Then, we use the above mentioned regularity patterns to construct the other three columns of indices and compute the transition matrix Π^m row-by-row.

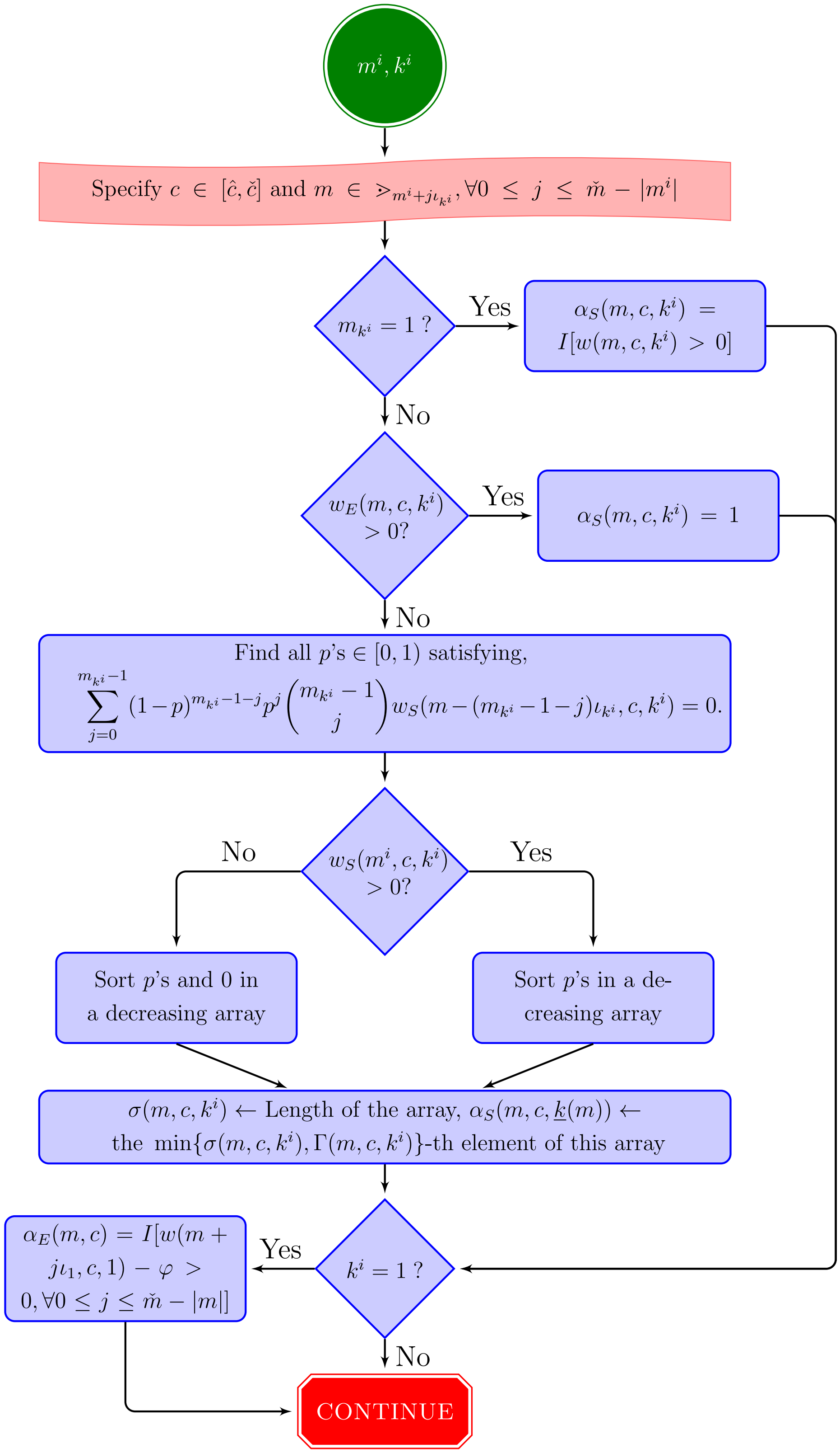
Last, a few words on the computational speed. When $\check{k} = \check{m} = 7$, the transition matrix is computed within 3 seconds. When $\check{k} = \check{m} = 8$, around 60 seconds. When $\check{k} = \check{m} = 9$, a normal PC runs out of memory.

II.3 Computing All Renegotiation-proof Natural Markov-Perfect Equilibria

In this appendix, we first show that when C is discrete, we can compute all renegotiation-proof natural Markov-perfect equilibria. Then, we discuss how to modify Algorithm 2 to compute all such equilibria.

We have seen in Section **??** that the multiplicity of renegotiation-proof equilibria comes from the multiple mixing probabilities that can solve (11). Therefore, to compute any single equilibrium using Algorithm 2, we always need to select the probability corresponding to this equilibrium. To this end, we introduce a flexible selection mechanism which enables us to do so.

A selection rule of such mechanism is summarized by $\Gamma : \mathbb{Z}_\star^{\check{k}} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \{1, \ldots, \check{m}\}$. It works as follows. Suppose that a renegotiation-proof Markov-perfect equilibrium exists, and $a_S(m, c, k)$, as a mixing probability, can take $\sigma(m, c, k)$ values. Sort all these possible values in a (weakly) descending sequence. Then, we use Γ to uniquely pin down $a_S(m, c, k)$ by setting $a_S(m, c, k) = \min\{\sigma(m, c, k), \Gamma(m, c, k)\}$-th possible value in this sequence. To give an example of Γ, if for any (m, c, k), $\Gamma(m, c, k) = 1$. Then, we always pick the first one in the sequence or the largest probability as the survival rule. With a pre-specified Γ, we can modify Procedure 3 to include this mechanism and compute a renegotiation-proof Markov-perfect equilibrium.



Procedure 3: Calculation of Candidate Entry/Survival Rule for the General Model, Non-Monotone Payoffs

Because the number of possible mixing probabilities is bounded by the number of roots of the polynomial in equation (11), which is in turn bounded by the polynomial's order. In the general model, the highest order of any polynomial in equation (11) is $\check{m}$. Thus, from the definition of Γ, it is clear that if C is a discrete variable, the number of distinct Γ mappings is finite. Therefore, we can compute all renegotiation-proof natural Markov-perfect equilibria for the general model by implementing Algorithm 2 repeatedly for all possible Γ's. Although this procedure can be completely parallelized, it is still computationally cumbersome for large $\check{m}, \check{k}$ and large number of possible realizations of C.

Practically, we can reduce the computational burden by avoiding running the algorithm for "redundant" Γ's. For some $(m, c, k) \in \mathcal{S}$ such that $\sigma(m, c, k) < \check{m}$, suppose that under a selection rule Γ, $\Gamma(m, c, k) = \sigma(m, c, k)$. Then any $\tilde{\Gamma}$ with $\tilde{\Gamma}(m, c, k) > \sigma(m, c, k)$ and $\Gamma(n, d, g) = \tilde{\Gamma}(n, d, g)$, for all $(n, d, g) \neq (m, c, k)$ selects the same Markov-perfect equilibrium as Γ. Therefore, all such $\tilde{\Gamma}$ (there are $\check{m} - \sigma(m, c, k)$ of them) are redundant, provided that we have run the algorithm for Γ. This suggests that to find all the renegotiation-proof natural equilibria in a computationally efficient way, we should run the algorithm with no pre-specified Γ but "branch" the algorithm once multiplicity arises. To be more specific, after starting the algorithm, once we reach a (m, c, k) such that $\sigma(m, c, k) > 1$, we create $\sigma(m, c, k)$ branches with $\alpha_S(m, c, k)$ set differently. Different branches then can be computed in parallel. The same branching exercise is done for each parallel session when a new state with multiple choices emerges.

III Computational Details

III.1 Pencil-and-Paper Computation behind Figure 3

One can compute the example graphed in Figure 3 using numerical method, such as value function iteration. Alternatively, this example can be computed exactly using only pencil and paper. This appendix contains the details. Begin with characterizing a duopolist's payoff function which satisfies

$$v_E(2, c) = \begin{cases} 0 & \text{if } Cc \leq \underline{c}_2, \\ \beta \frac{(1-\lambda)\left(\frac{c}{2}\pi(2) - \kappa\right) + \lambda \widetilde{v}(2)}{1 - \beta(1-\lambda)} & \text{if } c > \underline{c}_2, \end{cases}$$

where

$$\underline{c}_2 \equiv \begin{cases} \hat{c} & \text{if } v_E(2,\hat{c}) > 0, \\ \check{c} & \text{if } v_E(2,\check{c}) < 0, \\ \max\{c | v_E(2,c) = 0\} & \text{otherwise,} \end{cases}$$

and

$$\widetilde{v}(2) = \frac{1}{2}\left(\frac{\hat{c}+\check{c}}{2}\right)\pi(2) - \kappa + \int_{\underline{c}_2}^{\check{c}} \frac{v_E(2,c)}{(\check{c}-\hat{c})} dC.$$

We want to calculate the continuation and entry thresholds. If $\hat{c}\pi(2)/2 > \kappa$, firms always earn positive payoff no matter which C is drawn, then $v_E(2,\hat{c}) > 0$ and we settle with the corner solution $\underline{c}_2 = \hat{c}$. If $\hat{c}\pi(2)/2 < \kappa$, we can normalize $[\hat{c},\check{c}]$ to $[0,1]$ with the transformations

$$\begin{aligned} C' &\equiv \frac{C-\hat{c}}{\check{c}-\hat{c}}, \\ \pi'(2) &\equiv \pi(2)\left(\check{c}-\hat{c}\right), \\ \kappa' &\equiv \kappa - \frac{\hat{c}}{2}\pi(2). \end{aligned}$$

We then proceed by first considering two corner solutions.

If $v_E(2,0) > 0$, no duopolist will ever exit the market and they expect to earn average profit from perpetual operation if demand state switches. So $\widetilde{v}(2)$ in this case is

$$\widetilde{v}(2) = \frac{1}{1-\beta}\left(\frac{\pi'(2)}{4} - \kappa'\right).$$

Then $v_E(2,0) > 0$ is

$$v_E(2,0) = \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(0 \times \frac{\pi'(2)}{4} - \kappa'\right) + \frac{\beta\lambda}{\left(1-\beta(1-\lambda)\right)(1-\beta)}\left(\frac{\pi'(2)}{4} - \kappa'\right) > 0$$

Simplify the above expression we produce the necessary and sufficient condition for the corner solution $\underline{c}_2 = 0$ as

$$\gamma - \gamma\beta + \gamma\lambda\beta - \lambda < 0,$$

with $\gamma \equiv \frac{4\kappa'}{\pi'(2)}$.

If $v_E(2,1) < 0$, it is not possible for both duopolist to always choose continuation, so

$$v_E(2,1) = \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(\frac{\pi'(2)}{2} - \kappa'\right) + \frac{\beta\lambda}{1-\beta(1-\lambda)}\left(\frac{\pi'(2)}{4} - \kappa'\right) < 0.$$

Simplify this expression we obtain the necessary and sufficient condition for the corner solution $\underline{c}_2 = 1$ as

$$\gamma > 2 - \lambda.$$

Now we proceed to calculate the interior solution.

Substituting the lower branch of $v_E(2,c)$ into the expression for $\widetilde{v}(2)$ produces

$$\begin{aligned} \widetilde{v}(2) &= \frac{\pi'(2)}{4} - \kappa + \int_{\underline{c}'_2}^{1} v_E(2,C)dC \\ &= \frac{\pi'(2)}{4} - \kappa' + \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(\frac{\pi'(2)}{4}(1-(\underline{c}'_2)^2) - \kappa'(1-\underline{c}'_2)\right) + \frac{\beta\lambda\widetilde{v}(2)}{1-\beta(1-\lambda)}(1-\underline{c}'_2). \end{aligned} \tag{6}$$

In addition $\underline{c}'_2$ by definition must satisfy

$$(1-\lambda)\left(\frac{\underline{c}'_2}{2}\pi'(2) - \kappa\right) + \lambda\widetilde{v}(2) = 0,$$

so

$$\widetilde{v}(2) = \left(\kappa' - \frac{\underline{c}'_2}{2}\pi'(2)\right)\frac{1-\lambda}{\lambda}. \tag{7}$$

Combine (6) and (7) and then rearrange to obtain

$$\begin{aligned} &\left(\frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{4}\right)(\underline{c}'_2)^2 + \left(\frac{\pi'(2)}{2}\frac{1-\lambda}{\lambda} - \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{2}\right)\underline{c}'_2 \\ &\qquad + \left(\frac{\pi'(2)}{4} - \kappa' + \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{4} - \frac{1-\lambda}{\lambda}\kappa'\right) = 0. \end{aligned} \tag{8}$$

(8) is a quadratic equation in $\underline{c}'_2$ of the form $ax^2 + bx + c = 0$ with

$$\begin{aligned} a &= \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{2}, \\ b &= \frac{\pi'(2)}{2}\frac{1-\lambda}{\lambda} - \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{4}, \\ c &= \frac{\pi'(2)}{4} - \kappa' + \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(2)}{4} - \frac{1-\lambda}{\lambda}\kappa'. \end{aligned}$$

Use $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to calculate the roots for (8) as

$$\underline{c}'^1_2 = \frac{1}{\lambda\beta}\left(\beta - 1 + \sqrt{\frac{(1-\lambda-\beta+\gamma\lambda\beta)(1-\beta(1-\lambda))}{1-\lambda}}\right), \text{ and} \tag{9}$$

$$\underline{c}'^2_2 = \frac{1}{\lambda\beta}\left(\beta - 1 - \sqrt{\frac{(1-\lambda-\beta+\gamma\lambda\beta)(1-\beta(1-\lambda))}{1-\lambda}}\right) \tag{10}$$

Because $\beta < 1$, $\underline{c}_2'^2 \notin [0,1]$. The analysis above tell us that the necessary condition for an interior solution to exit is $2-\lambda > \gamma > \frac{\lambda}{1-\beta(1-\lambda)}$. We use this to check that $\underline{c}_2'^1$ is in the admissible set $(0,1)$.

Apparently $\underline{c}_2'^1$ is increasing in γ. Substitute $\gamma = \frac{\lambda}{1-\beta(1-\lambda)}$ into (9) to calculate the lower bound for $\underline{c}_2'^1$ as

$$\underline{c}_2'^1 = \frac{1}{\lambda\beta}\left(\beta - 1 + \sqrt{(1-\beta)^2}\right) = 0.$$

Substitute $\gamma = 2-\lambda$ into (9) to calculate the upper bound for $\underline{c}_2'^1$ as

$$\underline{c}_2'^1 = \frac{1}{\lambda\beta}\left(\beta - 1 + \sqrt{(1-\beta(1-\lambda))^2}\right) = 1.$$

Therefore, when $2-\lambda > \gamma > \frac{\lambda}{1-\beta(1-\lambda)}$, the interior solution for $\underline{c}_2'$ is given by (9). Finally, we restore $\underline{c}_2$ by using $\underline{c}_2 = \underline{c}_2'(\check{c}-\hat{c})+\hat{c}$.

Before we turn to the analysis for a monopolist, we calculate $\overline{c}_2$ which is defined as

$$\overline{c}_2 \equiv \begin{cases} \hat{c} & \text{if } v_E(2,\hat{c}) > \varphi, \\ \check{c} & \text{if } v_E(2,\check{c}) < \varphi, \\ \max\{c | v_E(2,c) - \varphi = 0\} & \text{otherwise,} \end{cases}$$

After obtaining $\underline{c}_2$, using (7), we can calculate $\widetilde{v}(2)$. Then by setting $v_E(2,\overline{c}_2) = \varphi$, we calculate $\overline{c}_2$ as

$$\overline{c}_2 = \frac{2}{\pi(2)}\left(\frac{\varphi(1-\beta(1-\lambda)) - \lambda\beta\widetilde{v}(2)}{\beta(1-\lambda)} + \kappa\right). \tag{11}$$

With this in hand, we can define a monopolist's payoff

$$v_E(1,c) = \begin{cases} 0 & \text{if } c \le \underline{c}_1, \\ \beta\frac{(1-\lambda)(c\pi(1)-\kappa)+\lambda\widetilde{v}(1)}{1-\beta(1-\lambda)} & \text{if } c > \underline{c}_1, \end{cases}$$

where

$$\underline{c}_1 \equiv \begin{cases} \hat{c} & \text{if } v_E(1,\hat{c}) > 0, \\ \check{c} & \text{if } v_E(1,\check{c}) < 0, \\ \max\{c | v_E(1,c) = 0\} & \text{otherwise,} \end{cases}$$

and

$$\widetilde{v}(1) = \left(\frac{\hat{c}+\check{c}}{2}\right)\pi(1) - \kappa + \int_{\overline{c}_2}^{\check{c}} \frac{v_E(2,C)}{(\check{c}-\hat{c})}dC + \int_{\underline{c}_1}^{\overline{c}_2} \frac{v_E(1,C)}{(\check{c}-\hat{c})}dC.$$

If $\hat{c}\pi(1) > \kappa$, firm always earns positive payoff no matter which C is drawn, then $v_E(1, \hat{c}) > 0$ and we settle with the corner solution $\underline{c}_1 = \hat{c}$. If $\hat{c}\pi(1) < \kappa$, we can normalize $[\hat{c}, \check{c}]$ to $[0, 1]$ with the transformations

$$\begin{aligned}
C' &\equiv \frac{C - \hat{c}}{\check{c} - \hat{c}}, \\
\pi'(1) &\equiv \pi(1)\left(\check{c} - \hat{c}\right), \\
\kappa' &\equiv \kappa - \hat{c}\pi(1).
\end{aligned}$$

Because

$$\int_{\overline{c}_2'}^{1} v_E(2, C')dC'$$

is readily computable by using the definition of $v_E(2, C)$. We hereafter denote its value by ν.

Similarly to the duopolist case, we then proceed by first considering two corner solutions.

If $v_E(1, 0) > 0$, no monopolist will exit the market and we can calculate $\widetilde{v}(1)$ in this case as

$$\widetilde{v}(1) = \frac{1}{2}\pi'(1) - \kappa' + \nu + \frac{(\overline{c}_2')^2}{2}\frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\pi'(1) - \frac{\beta(1-\lambda)\overline{c}_2'}{1-\beta(1-\lambda)}\kappa' + \frac{\beta\lambda\overline{c}_2'}{1-\beta(1-\lambda)}\widetilde{v}(1).$$

This gives

$$\widetilde{v}(1) = \frac{\left(\pi'(1)/2 - \kappa' + \nu\right)\left(1 - \beta(1-\lambda)\right) + \beta(1-\lambda)\left((\overline{c}_2')^2\pi'(1)/2 - \overline{c}_2'\kappa'\right)}{1 - \beta + \lambda\beta(1 - \overline{c}_2')}.$$

Then $v_E(1, 0) > 0$ is

$$\begin{aligned}
v_E(1, 0) =& \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(-\kappa'\right) + \frac{\beta\lambda}{\left(1-\beta(1-\lambda)\right)} \\
&\times \frac{\left(\pi'(1)/2 - \kappa' + \nu\right)\left(1 - \beta(1-\lambda)\right) + \beta(1-\lambda)\left((\overline{c}_2')^2\pi'(1)/2 - \overline{c}_2'\kappa'\right)}{1 - \beta + \lambda\beta(1 - \overline{c}_2')}.
\end{aligned}$$

Simplify the above expression we produce the necessary and sufficient condition for the corner solution $\underline{c}_1 = 0$ as

$$\gamma' < \frac{(\overline{c}_2')^2(1-\lambda)\beta}{\left(1-\beta(1-\lambda)\right)} + 1,$$

with $\gamma' \equiv \frac{2}{\pi'(1)}\left(\frac{\kappa'}{\lambda} - \nu\right)$.

If $v_E(1, 1) < 0$, a monopolist will never remain in the market and entry will never happen, so

$$v_E(1, 1) = \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(\pi'(1) - \kappa'\right) + \frac{\beta\lambda}{1-\beta(1-\lambda)}\left(\frac{\pi'(1)}{2} - \kappa'\right) < 0.$$

Simplify this expression we obtain the necessary and sufficient condition for the corner solution $\underline{c}_1 = 1$ as

$$\frac{2-\lambda}{2}\pi'(1) < \kappa'.$$

Because $\nu = 0$ in this case, we can add $\lambda\nu$ to the the right hand side without changing the result. Divide both sides by $\pi'(1)\lambda/2$

$$\gamma' > \frac{2-\lambda}{\lambda}.$$

Now we proceed to calculate the interior solution.

Substituting the lower branch of $v_E(1, c)$ into the expression for $\widetilde{v}(1)$ produces

$$\begin{aligned} \widetilde{v}(1) &= \frac{\pi'(1)}{2} - \kappa' + \nu + \int_{\underline{c}'_1}^{\overline{c}'_2} v_E(1, C)dC \\ &= \frac{\pi'(1)}{2} - \kappa' + \nu + \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\left(\frac{\pi'(1)}{2}((\overline{c}'_2)^2 - (\underline{c}'_1)^2) - \kappa'(\overline{c}'_2 - \underline{c}'_1)\right) \\ &\quad + \frac{\beta\lambda\widetilde{v}(1)(\overline{c}'_2 - \underline{c}'_1)}{1-\beta(1-\lambda)}. \end{aligned} \tag{12}$$

In addition $\underline{c}'_1$ by definition must satisfy

$$(1-\lambda)\left(\underline{c}'_1\pi'(1) - \kappa'\right) + \lambda\widetilde{v}(1) = 0,$$

so

$$\widetilde{v}(1) = \left(\kappa' - \underline{c}'_1\pi'(1)\right)\frac{1-\lambda}{\lambda}. \tag{13}$$

Combine (12) and (13) and then rearrange to obtain

$$\begin{aligned} &\left(\frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}\frac{\pi'(1)}{2}\right)(\underline{c}'_1)^2 + \left(\pi'(1)\frac{1-\lambda}{\lambda} - \frac{\beta(1-\lambda)\overline{c}'_2}{1-\beta(1-\lambda)}\pi'(1)\right)\underline{c}'_1 \\ &\quad + \left(\frac{\pi'(1)}{2} - \kappa' + \nu + \frac{\beta(1-\lambda)(\overline{c}'_2)^2}{1-\beta(1-\lambda)}\frac{\pi'(1)}{2} - \frac{1-\lambda}{\lambda}\kappa'\right) = 0. \end{aligned} \tag{14}$$

(14) is a quadratic equation in $\underline{c}'_1$ of the form $ax^2 + bx + c = 0$. Use $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to calculate the roots for (14) as

$$\underline{c}'^1_1 = (\overline{c}'_2 - 1) + \frac{\beta-1}{\lambda\beta} + \sqrt{\frac{(\xi^2 - 2\eta\xi\overline{c}'_2 - \eta) + \eta\gamma'}{\eta^2}}, \text{ and} \tag{15}$$

$$\underline{c}'^2_1 = (\overline{c}'_2 - 1) + \frac{\beta-1}{\lambda\beta} + \sqrt{\frac{(\xi^2 - 2\eta\xi\overline{c}'_2 - \eta) + \eta\gamma'}{\eta^2}} \tag{16}$$

with

$$\begin{array}{rcl} \xi & = & \frac{1-\lambda}{\lambda}, \\ \eta & = & \frac{\beta(1-\lambda)}{1-\beta(1-\lambda)}. \end{array}$$

Because $\beta < 1$ and $\overline{c}_2' < 1$, $\underline{c}_1'^2 \notin [0,1]$. The analysis above tell us that the necessary condition for an interior solution to exit is $\frac{2-\lambda}{\lambda} > \gamma' > \frac{(\overline{c}_2')^2(1-\lambda)}{(1-\beta(1-\lambda))\lambda} + 1$. We use this to check that $\underline{c}_1'^1$ is in the admissible set $(0,1)$.

Apparently $\underline{c}_1'^1$ is increasing in γ'. Substitute $\gamma' = \frac{(\overline{c}_2')^2(1-\lambda)}{(1-\beta(1-\lambda))\lambda} + 1$ to calculate the lower bound for $\underline{c}_1'^1$.

$$\underline{c}_1'^1 = \overline{c}_2' - 1 + \frac{\beta-1}{\lambda\beta} + \sqrt{\frac{\xi^2 - 2\eta\xi\overline{c}_2' - \eta + \eta + \eta^2(\overline{c}_2')^2}{\eta^2}} = 0.$$

which can be simplified to

$$\underline{c}_1'^1 = \overline{c}_2' - \frac{\xi}{\eta} + \sqrt{\frac{(\xi - \eta\overline{c}_2')^2}{\eta^2}} = 0.$$

Substitute $\gamma = \frac{2-\lambda}{\lambda}$ into (15) to calculate the upper bound for $\underline{c}_1'^1$.

$$\underline{c}_1'^1 = (\underline{c}_2' - 1) + \frac{1}{\lambda\beta}(\beta - 1) + \sqrt{\frac{(1-\beta(1-\lambda) - 2\lambda\beta\overline{c}_2' + 2\lambda\beta)(1-\beta(1-\lambda))}{\lambda^2\beta^2}}.$$

Remember when monopolist exit the market for sure, no duopolist will choose to survive. So $\overline{c}_2' = 1$ and above equation simplifies to

$$\underline{c}_1'^1 = \frac{1}{\lambda\beta}(\beta - 1) + \sqrt{\frac{(1-\beta(1-\lambda))^2}{\lambda^2\beta^2}} = 1.$$

Therefore, when $\frac{2-\lambda}{\lambda} > \gamma' > \frac{(\overline{c}_2')^2(1-\lambda)}{(1-\beta(1-\lambda))\lambda} + 1$, the interior solution for $\underline{c}_1'$ is given by (15). Finally, we restore $\underline{c}_1$ by using $\underline{c}_1 = \underline{c}_1'(\check{c} - \hat{c}) + \hat{c}$.

Last thing is to calculate $\overline{c}_1$, which is defined as

$$\overline{c}_1 \equiv \begin{cases} \hat{c} & \text{if } v_E(1,\hat{c}) > \varphi, \\ \check{c} & \text{if } v_E(1,\check{c}) < \varphi, \\ \max\{c | v_E(1,c) - \varphi = 0\} & \text{otherwise,} \end{cases}$$

After obtaining $\underline{c}_1$, using (13), we can calculate $\widetilde{v}(1)$. Then by setting $v_E(1,\overline{c}_1) = \varphi$, we calculate $\overline{c}_1$ as

$$\overline{c}_1 = \frac{1}{\pi(1)}\left(\frac{\varphi(1-\beta(1-\lambda)) - \lambda\beta\widetilde{v}(1)}{\beta(1-\lambda)} + \kappa\right). \tag{17}$$

Because firms' payoffs are linear functions in C, knowing $\underline{c}_1, \overline{c}_1, \underline{c}_2, \overline{c}_2$ is sufficient for determining the payoffs for duopolist and monopolist. The calculation is completed.

III.2 Computing the Example of Multiple Equilibria in Section 4.2

Note that this model is deterministic. We compute some important equilibrium payoffs which account for equilibrium multiplicity as below

(i). Since $c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) > \kappa$, if three type-$H$ firms are active in the second period, they can always recover fixed cost and make positive profit by remaining active from the third period onwards. Moreover, if less type-H firms are active in the second period, they receive higher profit from the third period onward. Therefore, for $t \geq 2$, $a_S(3\iota_{\mathcal{H}}, c_t, \mathcal{H})$ is a dominant strategy and

$$v_E(3\iota_{\mathcal{H}}, c_t, \mathcal{H}) = \frac{\beta(c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) - \kappa)}{1-\beta} = 1.$$

(ii). Since $c_3\overline{\pi}_{\mathcal{L}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) > \kappa$, $c_3\overline{\pi}_{\mathcal{H}}(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}) > \kappa$ and $c_3\overline{\pi}_{\mathcal{L}}(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}) > \kappa$, for the same reason, for $t \geq 2$,

$$\begin{aligned}
v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_t, \mathcal{L}) &= v_S(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_t, \mathcal{L}) \\
&= \frac{\beta(1-\Pi_{\mathcal{LH}})}{1-\beta(1-\Pi_{\mathcal{LH}})}(c_3\overline{\pi}_{\mathcal{L}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) - \kappa) \\
&\quad + \left(\frac{\beta}{1-\beta} - \frac{\beta(1-\Pi_{\mathcal{LH}})}{1-\beta(1-\Pi_{\mathcal{LH}})}\right)(c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) - \kappa) \\
&= 0.8167 \\
v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_t, \mathcal{H}) &= v_S(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_t, \mathcal{H}) \\
&= \frac{\beta(1-\Pi_{\mathcal{LH}})}{1-\beta(1-\Pi_{\mathcal{LH}})}(c_3\overline{\pi}_{\mathcal{H}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) - \kappa) \\
&\quad + \left(\frac{\beta}{1-\beta} - \frac{\beta(1-\Pi_{\mathcal{LH}})}{1-\beta(1-\Pi_{\mathcal{LH}})}\right)(c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) - \kappa) \\
&= 1.4 \\
v_E(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_t, \mathcal{L}) &= v_S(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_t, \mathcal{L}) \\
&= \frac{1}{1-\beta(1-\Pi_{\mathcal{LH}})^2}((1-\Pi_{\mathcal{LH}})^2(c_3\overline{\pi}_{\mathcal{L}}(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}})) \\
&\quad +(1-\Pi_{\mathcal{LH}})\Pi_{\mathcal{LH}}(c_3\overline{\pi}_{\mathcal{H}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) + v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_{t+1}, \mathcal{H})) \\
&\quad +(1-\Pi_{\mathcal{LH}})\Pi_{\mathcal{LH}}(c_3\overline{\pi}_{\mathcal{L}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) + v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_{t+1}, \mathcal{L})) \\
&\quad +\Pi_{\mathcal{LH}}^2(c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) + v_E(3\iota_{\mathcal{H}}, c_{t+1}, \mathcal{L})) - \kappa) \\
&= 1.2881
\end{aligned}$$

(iii). Since $v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_2, \mathcal{L}) < \varphi$ and $v_E(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_2, \mathcal{L}) > \varphi$, we have that $a_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_2, \mathcal{L}) = 0$ and $a_E(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_2, \mathcal{L}) = 1$. This means that in the second period, two type-L firms will enter a market occupied by a type-H monopoly, while no firm will enter a market occupied by two type-H duopoly. Since demand stays constant from the third period on, the market structures at the end of the second period will never be changed. Therefore, for $t \geq 2$,

$$\begin{aligned}
v_E(2\iota_{\mathcal{H}}, c_t, \mathcal{H}) &= v_S(2\iota_{\mathcal{H}}, c_t, \mathcal{H}) \\
&= \frac{\beta(c_3\overline{\pi}_{\mathcal{H}}(2\iota_{\mathcal{H}}) - \kappa)}{1-\beta} \\
&= 496 \\
v_E(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_t, \mathcal{H}) &= v_S(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_t, \mathcal{H}) \\
&= \frac{1}{1-\beta(1-\Pi_{\mathcal{LH}})^2}((1-\Pi_{\mathcal{LH}})^2(c_3\overline{\pi}_{\mathcal{H}}(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}})) \\
&\quad +2(1-\Pi_{\mathcal{LH}})\Pi_{\mathcal{LH}}(c_3\overline{\pi}_{\mathcal{H}}(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}) + v_E(2\iota_{\mathcal{H}} + \iota_{\mathcal{L}}, c_{t+1}, \mathcal{H})) \\
&\quad +\Pi_{\mathcal{LH}}^2(c_3\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) + v_E(3\iota_{\mathcal{H}}, c_{t+1}, \mathcal{L})) - \kappa) \\
&= 1.6357
\end{aligned}$$

(iv). For a type-H monopolist who is active in the first period, the payoff to continuation is

$$v_S(\iota_{\mathcal{H}}, c_1, \mathcal{H}) = \beta\left((c_2\overline{\pi}_{\mathcal{H}}(\iota_{\mathcal{H}}) - \kappa) + v_E(\iota_{\mathcal{H}} + 2\iota_{\mathcal{L}}, c_2, \mathcal{H})\right) = -1.1821.$$

For a type-H duopolist who is active in the first period together with another type-H rival, the payoff to continuation, given the rival also continues, is

$$v_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = \beta\left((c_2\overline{\pi}_{\mathcal{H}}(2\iota_{\mathcal{H}}) - \kappa) + v_E(2\iota_{\mathcal{H}}, c_2, \mathcal{H})\right) = 246.$$

For a type-H triopolist who is active in the first period together with another two type-H rivals, the payoff to continuation, given the rivals also continue, is

$$v_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) = \beta\left((c_2\overline{\pi}_{\mathcal{H}}(3\iota_{\mathcal{H}}) - \kappa) + v_E(3\iota_{\mathcal{H}}, c_2, \mathcal{H})\right) = -1.5.$$

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