Equilibrium Unemployment in a Generalized Search Model*

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February 25, 2007

Abstract

We present a generalization of the standard Diamond-Mortensen-Pissarides search model of unemployment in which there is both an intensive and extensive margin of employment creation (corresponding to hiring by existing firms and entry by new firms). In our model, each firm can invest to gain access to a production technology with diminishing returns to labor and then post vacancies in order to recruit workers. Entry by new firms corresponds to the extensive margin of employment creation, while job creation by existing firms captures the intensive margin. As in the baseline Diamond-Mortensen-Pissarides search model and theories of the firm developed by Stole and Zwiebel (1996a,b) and Wolinsky (2000), wages are determined by continuous bargaining between the firm and its employees. A steady-state equilibrium corresponds to a level of unemployment, labor market tightness, firm size distribution and wage distribution. We characterize the steady-state equilibrium and show how it can be computed. We then analyze the out-of-steady-state dynamics of the model, which are represented by a system of partial differential equations. We analyze how the economy responds to productivity shocks. The novel feature of the model is the ability to differentiate the response of intensive and extensive margins of employment to shocks. Our preliminary calibrations suggest that the two margins work in opposite directions: upon the arrival of a positive productivity shock, existing firms gradually reduce their employment while newly entering firms increase theirs. However, this pattern does not appear to increase the responsiveness of unemployment and of labor market tightness to productivity shocks at business cycle frequencies. In future work, we will investigate both the behavior of the intensive and extensive margins of employment creation and the responsiveness of unemployment to business cycle shocks more systematically.

*First version: July 23, 2006. This paper reports on ongoing work. Comments are welcome.

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1 Introduction

The search and matching model developed by, among others, Diamond (1982a,b), Mortensen (1982), Pissarides (1985), and Mortensen and Pissarides (1994), has become the leading framework for the analysis of equilibrium unemployment. The baseline Diamond-Mortensen-Pissarides (DMP) model has three building blocks: (i) a matching function, which incorporates the frictions involved in labor market search; (ii) a production structure where each firm can employ a single worker, and (iii) a protocol for bargaining between firms and workers for wage determination. These features lead to a very tractable model, with a range of comparative statics. Nevertheless, despite its success in many dimensions, this framework does not distinguish between intensive and extensive margins of employment creation and destruction (that is, employment decisions by existing firms and new firms). Existing evidence suggests that new jobs are created both by existing firms and by the entry of new firms (see for example, Davis, Haltiwanger, and Schuh, 1996), but the cyclical behaviors of job creation and destruction by existing and new firms (and by firm exit) are very different. In this paper, we extend the baseline DMP model to enable an analysis of the intensive and extensive margins of employment. The main difference from the baseline model is that the production technology exhibits diminishing returns to labor.

In the model economy, a large number of firms can invest a fixed amount k to gain access to a production function with decreasing returns. Once active, firms can post vacancies in order to hire workers. Employment relationships come to an end both because of exogenous worker separations from continuing firms and because of firm shutdowns. Wages are determined by bargaining between individual workers and the firm as in the standard DMP model. Since there are more than two players in the bargaining game, we use the bargaining protocol of Shapley (1953) adapted to a dynamic setting. The Shapley value has been previously used in a static model of employment and wage determination by Stole and Zwiebel (1996a,b) and in a dynamic setting by Wolinsky (2000). It is an attractive wage determination protocol because it generalizes the Nash bargaining solution used in the standard DMP model in a natural way, because it captures the notion that players cannot enter into binding agreements, and because it has appealing microfoundations, especially in our environment in which the firm is "essential" for production.

Our main contribution is therefore to present a relatively tractable model that combines the general equilibrium structure of the baseline DMP model with the model due to Stole and Zwiebel and Wolinsky of wage and employment determination in the absence of binding contracts between firms and workers.

In Section 2, we start with a model in which each firm can employ a countable number of workers. An equilibrium in this economy determines not only unemployment and vacancy rates, but also the size distribution of firms and an endogenous wage distribution. The economy with a countable number of workers is intuitive and thus a good starting point for our analysis. Nevertheless, characterizing the equilibrium of this economy is made difficult by integer issues. For this reason, we work with the limiting economy in which the number of workers a firm can employ is a continuous variable (and in this case, we use the limiting bargaining solution of Aumann and Shapley (1974), also used in Stole and Zwiebel (1996a) and Wolinsky (2000)). This limiting economy, which is presented in Section 3, is our benchmark model. The steady-state equilibrium in this economy can be represented by a set of differential equations, which can be solved explicitly.

While the characterization of the steady-state equilibrium is relatively tractable, the behavior of the economy out of the steady state is more complex, because not only the vacancy and unemployment rates, but the entire size distribution of firms and the wage distribution of workers change along the process of adjustment to the steady state. We present a set of partial differential equations that characterize the transitional dynamics of this economy and we use both the steady-state characterization and these partial differential equations to discuss a number of new questions.

Most importantly, our model suggests that the intensive and extensive margins of employment creation play different roles at different stages of adjustment. Consider, for example, an increase in the productivity of firms starting from a steady-state equilibrium. We show that under a variety of reasonable parameterizations of the model, the new steady-state equilibrium has a tighter labor market and lower unemployment, but also fewer workers per firm. This latter feature seems to be a consequence of the greater outside option of firms and the higher cost of hiring workers, which together induce established firms to reduce their steady-state size.

This comparison between steady states implies that there is a more than compensating increase in the number of firms in the new steady-state equilibrium. Thus between the two steady states (in the "medium run") there is a substantial adjustment in the extensive margin. What is not clear from this steady-state comparison is the time path of the extensive and intensive margins of employment creation. One possibility is that because newly entering firms are unable to grow to their target size instantaneously, existing firms may first increase their employment and then subsequently reduce it towards their new steady-state level. Another possibility is monotonic adjustment in the size of existing firms, which corresponds to the intensive and the extensive margins of employment moving in opposite directions both in the short run and across steady states (existing firms contracting while new firms are expanding). These two scenarios not only differ in terms of the behavior of the extensive and intensive margins of employment creation, but they also have different implications for the responsiveness of unemployment and labor market tightness to business cycle shocks.

Since it is not possible to characterize the dynamics of the equilibrium analytically, we use a simple calibration of our baseline model to investigate the dynamic adjustment of the economy to a positive productivity shock. Our preliminary calibrations show that in response to a positive shock, the intensive and the extensive margins of employment move in opposite directions and the size of existing establishments decline monotonically towards the new steady state. While this pattern of adjustment in the extensive and the intensive margins is interesting in its own right, it does not seem to generate larger responses to productivity shocks in the short run than in the medium run.[1]

Our work is related to various different strands of the search literature. As noted above, we build on and generalize the baseline search-matching model of Diamond (1982b), Pissarides (1984, 1985) and Mortensen and Pissarides (1994), and we also perform quantitative analyses similar to those in Shimer (2005) and Hall (2005).[2] Hawkins (2006a,b) also considers a search economy with multi-worker firms, but assumes directed search and wage posting rather than bargaining.[3] Models of firm-worker bargaining with diminishing returns at the firm level without labor market frictions have been considered by Stole and Zwiebel (1996a,b) and Acemoglu, Antrás, and Helpman (2005). Previous studies incorporating dynamic firm-worker bargaining with diminishing returns include Bertola and Caballero (1994),

[1] Our preliminary calibration results notwithstanding, we believe that the differential behaviors of the intensive and the extensive margins of employment creation in the short and the medium run may still have important implications for the responsiveness of unemployment and wages to productivity shocks over different horizons. A recent influential paper by Shimer (2005) has argued that the baseline DMP model cannot generate quantitatively plausible fluctuations in unemployment because wages are "excessively" responsive to changes in productivity (see also Hall, 2005). Shimer (2005) also showed that the responsiveness of labor market tightness (and wages) to productivity in the short run can be approximated by the long-run elasticities. This conclusion would be altered in this model if the employment level of existing firms increased in response to the arrival of a positive productivity shock, since in the short run in that case, aggregate employment would increase both due to entry of new firms and additional hiring by existing firms. Whether this is the case is a quantitative question; our investigations suggest that the unemployment rate in fact follows a monotonic transition path to the new steady state, and that the elasticity of unemployment with respect to productivity both in the short and long run is of the same order of magnitude as that found in Shimer (2005).

[2] See also Merz (1995), Andolfatto (1996) and Hagedorn and Manovskii (2005) for other quantitative investigations.

[3] See Montgomery (1991), Peters (1991), Moen (1997), Shimer (1996), Acemoglu and Shimer (1999a,b) and Burdett, Shi, and Wright (2001) for directed search models.

Smith (1999), Wolinsky (2000), Bertola and Garibaldi (2001), Cahuc and Wasmer (2001), and Cahuc, Marque, and Wasmer (2005). Of these, the paper most closely related to our work is the important paper by Wolinsky (2000), which also considers the dynamic bargaining problem between workers and firms in the presence of diminishing returns. However, Wolinsky's analysis is essentially partial equilibrium since the arrival of new workers to firms is assumed to be exogenous. Consequently, Wolinsky's model does not endogenize the unemployment rate and cannot be used for equilibrium analysis in the labor market. In addition, none of the other papers fully solve for wage determination and firm size distribution in general equilibrium with forward-looking bargaining.[4]

The rest of the paper is organized as follows. Section 2 presents the baseline model. Section 3 considers the limit economy in which workers size is small. This section provides an explicit-form characterization of the steady-state equilibrium and proves the existence of a state-state equilibrium. It also provides some simple comparative static results. Section 4 analyzes the out-of-steady-state dynamics of the economy of Section 3. Sections 5 and 6 present preliminary calibrations of our basic model, comparing steady-state distributions and also looking at the path of adjustment in response to a positive productivity shock. Section 7 concludes.

2 Motivating Model (Countable Number of Employees)

In this section we introduce the benchmark model in which each firm can hire a countable number of workers (1, 2,...). We will use to this model to illustrate the basic ideas and the structure of the model in an intuitive and familiar environment. In the next section, we will consider the limiting equilibrium in which each firm can employ a continuum of workers. More specifically, the model economy is infinite horizon in continuous time and is populated by a continuum of workers of size $\varepsilon > 0$ each (in this section, there is no loss of generality if the reader prefers to think of the case in which $\varepsilon = 1$). In Section 3 we consider the limiting economy where $\varepsilon \to 0$, which will enable a sharper characterization of both the steady-state and the dynamic equilibria.[5] Thus, the model in this section can be considered as motivation for the continuous employment case.

[4] Bertola and Caballero (1994) solve a search model with multiple workers per firm assuming that wages are determined by Nash bargaining. Nash bargaining is difficult to justify when there is multilateral bargaining, however. Cahuc, Marque, and Wasmer (2005) use a multilateral bargaining rule similar to the Shapley value, but incorrectly impose the envelope condition, which holds for the last worker hired by the firm, throughout rather than fully solving for wages in general equilibrium.

[5] In particular, when $\varepsilon > 0$, integer problems can lead to an economically uninteresting multiplicity of equilibria, and the limit $\varepsilon \to 0$ avoids this.

2.1 Environment

Consider the following continuous time infinite-horizon economy. There is a continuum of mass 1 of identical workers. All workers supply labor inelastically, are risk neutral, and discount the future at the rate r. In particular, the utility of the worker at time t is:

$$U(t) = \int_t^{\infty} \exp(-r(\tau - t))\, c(\tau)\, d\tau,$$

where $c(\tau)$ is consumption at time τ, equal to $b > 0$ if the worker is unemployed, and to his wage at time τ, $w(\tau)$, if employed.

On the other side of the market, there is a large mass of potential firms. All firms are owned by the workers and also act in in risk-neutral manner, maximizing the net present discounted value of profits. Since this has no effect on any aspect of the equilibrium, the exact distribution of shares of firms is not specified. At any instant, each potential firm can pay a fixed cost of k in order to become active (for example, k can be interpreted as the price of the necessary capital equipment). After doing so, the firm has access to the production function, $F : \varepsilon\mathbb{Z}_+ \to \mathbb{R}_+$, which specifies flow rate of output as a function of the mass of employees at that instant, denoted by n. Here $\varepsilon\mathbb{Z}_+$ is the set $\varepsilon\mathbb{Z}_+ \equiv \{0, \varepsilon, 2\varepsilon, 3\varepsilon, ...\}$, so that n takes values from this set.[6] This amounts to assuming that the "size" of each employee is equal to ε; it is useful since we will later take the limit as $\varepsilon \to 0$ and allow the firm to employ any number of workers. For the expressions in this section, there would be no loss in generality in imposing $\varepsilon = 1$, so that the number of employees of a firm takes integer values.

We assume that $F(n)$ is strictly increasing, continuously differentiable and strictly concave and satisfies $F(0) = 0$. Moreover, we assume that it satisfies a weaker version of the standard Inada condition, $\lim_{n\to\infty} F'(n) < b$, where the prime denotes differentiation.

Matching between firms and workers is frictional as in the standard DMP model. An active firm has the option of posting a single vacancy at any time; doing so has a flow cost of γ. If a firm does not post a vacancy, then it cannot meet new workers and thus cannot hire additional workers.[7] The flow of firm-worker meetings at every instant is determined by an aggregate matching function, $M(u, v)$, where u is the mass of unemployed workers, u, and v is the mass of vacancies posted by firms. The function $M(u, v)$ is assumed to be strictly increasing and continuously differentiable in its two arguments and to exhibit constant returns to scale. This

[6] Note that since workers are of fixed mass ε each, the mass of workers employed at a given firm must be an element of the set $\varepsilon\mathbb{Z}_+$ of non-negative integer multiples of ε. That is, for a given ε, the number of workers per firm can take on only countably many values. Recall, however, that there are a continuum of workers and firms.

[7] In equilibrium, the assumptions about bargaining to be specified below will ensure that all meetings between a firm and a worker will lead to a match; we therefore use the words 'meeting' and 'matching' interchangeably.

implies that the flow mass of unemployed workers that meet a given vacancy can be expressed as

$$\begin{aligned} q(\theta) &\equiv \frac{M(u,v)}{v} \\ &\equiv M\left(\theta^{-1},1\right), \end{aligned}$$

where $\theta \equiv v/u$ is the tightness of the labor market, and $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable decreasing function. Since each worker is of mass ε, the flow rate at which a vacancy is contacted by a worker is $q(\theta)/\varepsilon$. This implies that it becomes more difficult for firms to match with workers in a tighter (high θ) labor market. The flow rate of a match for an unemployed worker is $\theta q(\theta)$, which is also continuously differentiable and is assumed to be increasing in θ, so that finding a firm becomes easier for workers in a tighter labor market. We denote the decision of the firm about whether to post a vacancy by $h \in [0,1]$; $h = 1$ denotes that the firm posts a vacancy, $h = 0$ denotes that it does not, and $h \in (0,1)$ denotes a mixed strategy.

Once a match between a firm and a worker is formed, it continues until either it is exogenously destroyed, or until one of the two agents chooses to end the match. Exogenous destruction can take place because of two reasons: first, the firm may shut down, which happens according to a Poisson process with arrival rate δ. If a firm is hit by such a shock, then it is destroyed; the scrapping value is 0, and all workers previously employed at the firm become unemployed. In addition, each employed worker is separated from his current employer and returns to unemployment, according to a Poisson process with arrival rate s. All these stochastic processes are independent. This implies that a firm that employs a mass n of workers will experience a separation from one of its workers according to a Poisson process with arrival rate sn/ε.

Since matching between firms and workers is frictional, firms will only grow slowly and there will be a distribution of firm sizes. For $m \in \varepsilon\mathbb{Z}_+$, let $g(m,t)$ be the mass of firms employing m at time t. The distribution $g(\cdot,\cdot)$ will be the aggregate state variable of the economy, while $n(t)$, the number of workers employed by a firm at time t, will be the individual state variable.

For tractability, in this section we will only consider steady-state equilibria and will suppress time dependence when this causes no confusion. Moreover, we assume that both firms and workers are *anonymous*, so that equilibria must have a symmetric Markovian structure, with strategies depending only on payoff-relevant state variables.[8]

[8] Anonymity is useful in ruling out "unreasonable" equilibria in which agents in the economy use punishment-type strategies against specific agents.

More specifically, let $\mathcal{G}$ be the set of density functions $g : \varepsilon\mathbb{Z}_+ \rightarrow \mathbb{R}_+$, which, by definition, satisfy $\sum_{m=0}^{\infty} g(m) = 1$. Throughout we use "value" and "net present discounted value" interchangeably. An anonymous steady-state allocation can be defined as follows:

Definition 1. *A steady-state allocation is a tuple* $\langle \theta, V^u, g(\cdot), J(\cdot,\cdot), V(\cdot,\cdot), h(\cdot,\cdot), w(\cdot,\cdot), u \rangle$ *such that*

- $\theta \in \mathbb{R}_+$ *is the tightness of the labor market.*
- $V^u \in \mathbb{R}_+$ *is the value of an unemployed worker.*
- $g \in \mathcal{G}$ *is the distribution of firm sizes.*
- $J : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \rightarrow \mathbb{R}_+$ *is the value of a firm with* $m \in \varepsilon\mathbb{Z}_+$ *employees when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $V : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \rightarrow \mathbb{R}_+$ *is the value of a worker employed by a firm of size* $m \in \varepsilon\mathbb{Z}_+$ *when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $h : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \rightarrow [0,1]$ *is the vacancy posting decision of a firm with* $m \in \varepsilon\mathbb{Z}_+$ *employees when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $w : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \rightarrow \mathbb{R}_+$ *be the wage of a worker employed by a firm of size* $m \in \varepsilon\mathbb{Z}_+$ *when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $u \in [0,1]$ *is the unemployment rate.*

Notice that a steady-state allocation also defines and endogenous wage distribution, since g determines the distribution of firm sizes and w determines wages corresponding to different firm sizes.

2.2 Equilibrium Characterization

An anonymous steady-state equilibrium will be such that both active and inactive firms maximize their values (profits) and all workers maximize their utilities. Before we can define this equilibrium we need to specify wage determination, which in turn requires us to be more explicit about the value functions. Throughout, we will simplify notation by writing the value, policy and wage functions only as a function of the number of employees, e.g., $J(m)$ instead of $J(m, g(\cdot))$. Also, denote by Δ the first-difference operator, so that

$\Delta J(m) = [J(m) - J(m-\varepsilon)]/\varepsilon$. Standard arguments immediately imply that in steady state the value function for a firm of size $m \in \mathbb{Z}_+$ satisfies the following recursive equation:

$$\begin{aligned} rJ(m) &= F(m) - mw(m) - \delta J(m) - \frac{ms}{\varepsilon}[J(m) - J(m-\varepsilon)] \\ &\quad + h(m)\left\{-\gamma + \frac{q(\theta)}{\varepsilon}[J(m+\varepsilon) - J(m)]\right\} \end{aligned} \tag{1}$$

or rearranging:

$$(r+\delta)J(m) = F(m) - mw(m) - ms\Delta J(m) + h(m)\{-\gamma + q(\theta)\Delta J(m+\varepsilon)\}. \tag{2}$$

This equation is easy to understand. The left hand side is by definition the flow value of the asset of a firm of size m, $rJ(m)$. The right hand side gives the components of this flow value. A firm of size m produces a flow rate of output equal to $F(m)$ and pays a wage rate of $w(m)$ to each of its m workers. At the flow rate δ, the firm shuts down and loses its value $J(m)$ (and receives the scrap value of zero). At the flow rate sm/ε, one of the employees is separated from the firm, so the firm suffers a loss equal to $J(m) - J(m-\varepsilon)$.[9] Finally, as noted above, $h(m) \in \{0,1\}$ denotes whether a firm of size m opens a vacancy. If it does, $h(m) = 1$, then it incurs a cost of γ and hires a worker at the flow rate $q(\theta)/\varepsilon$, realizing a capital gain equal to $J(m+\varepsilon) - J(m)$. This expression is simplified due to two features. First, we have imposed that upon matching, the firm will hire the worker (which will always be the case in equilibrium). Second, since we are in steady state, there is no further term denoting the change in the asset value of $J(m)$ over time (i.e., there is no term $\dot{J}(m) = 0$).

By a similar reasoning, the value of the worker employed in a firm of size $m \in \mathbb{N}$ is given by

$$\begin{aligned} rV(m) &= w(m) - (\delta+s)[V(m) - V^u] - h(m)\frac{q(\theta)}{\varepsilon}[V(m) - V(m+\varepsilon)] \\ &\quad + \frac{s(m-\varepsilon)}{\varepsilon}[V(m-\varepsilon) - V(m)] \end{aligned} \tag{3}$$

or

$$(r+\delta+s)[V(m) - V^u] = w(m) - rV^u + h(m)q(\theta)\Delta V(m+\varepsilon) - (sm-\varepsilon)\Delta V(m). \tag{4}$$

The worker receives the wage rate $w(m)$, and loses his job either because of firm shutdown or separation (total flow rate of $\delta + s$), in which case he becomes unemployed. At the rate $s(m-\varepsilon)/\varepsilon$, one of the other workers is separated, in which case the worker realizes a capital

[9] The term "loss" here will be justified, since we will see below that in equilibrium $J(\cdot)$ is indeed increasing.

gain equal to $V(m-\varepsilon)-V(m)$.[10] Finally, if the firm chooses to post a vacancy, $h(m)=1$, then at the rate $q(\theta)/\varepsilon$, there will be a further hire and the worker will experience a loss of $V(m)-V(m+\varepsilon)$.

It is also straightforward to see that the value of an unemployed worker will be given by

$$rV^u = b + \theta q(\theta)\frac{\sum_{m=0}^{\infty} g(m)h(m)[V(m+\varepsilon)-V^u]}{\sum_{m=0}^{\infty} g(m)h(m)}. \tag{5}$$

Intuitively, the worker receives a flow of consumption equal to b when unemployed, finds a job at the flow rate $\theta q(\theta)$, and this job comes from a firm that already has m employees, thus giving the worker and value of $V(m+\varepsilon)$, with probability equal to the proportion of vacancies posted by such firms; this is equal to $g(m)\,h(m)/\left[\sum_{n=0}^{\infty} g(n)h(n)\right]$.

Clearly, V^u, as well as the strategies of workers and firms, depend on the distribution $g(\cdot)$. This is in turn given by a simple accounting equation:

$$\left(\delta + \frac{sm}{\varepsilon} + \frac{h(m)\,q(\theta)}{\varepsilon}\right) g(m) = \frac{h(m-\varepsilon)\,q(\theta)}{\varepsilon} g(m-\varepsilon) + \frac{s(m+\varepsilon)}{\varepsilon} g(m+\varepsilon), \tag{6}$$

for given $\theta \in \mathbb{R}_+$ and $g(0) \in [0,1]$. Intuitively, firms of size m leave this state if they shut down (which happens at the flow rate δ), if they lose a worker (which happens at the rate sm/ε), and if they hire a worker (which happens at the rate $q(\theta)/\varepsilon$ as long as they choose $h(m)=1$). Entry into this state either comes from firms of size $m-\varepsilon$ (at the rate $q(\theta)/\varepsilon$ if they post a vacancy, i.e., $h(m-\varepsilon)=1$) or from firms of size $m+\varepsilon$ that lose a worker (at the rate $s(m+\varepsilon)/\varepsilon$).

To complete the description of the environment, we also need to specify wage determination. As in the standard DMP search-matching models, and matched firm-worker pair creates a quasi-rent, since their value together exceeds the sum of their outside options. Wages are then assumed to be determined by some type of bargaining. Given that there is a relationship between one firm and many workers, a natural bargaining concept is the Shapley value, introduced in the seminal work of Shapley (1953). In this context a natural bargaining protocol described by Stole and Zwiebel (1996a) leads to the Shapley value. In particular, Stole and Zwiebel (1996a) show that if the firm bargains sequentially with each worker, until agreement is reached with all, the equilibrium division of rents will correspond to the Shapley value. Intuitively, when a firm contemplates disagreeing with a worker, it realizes that this will increase the bargaining power of the remaining workers because of diminishing marginal return in the production function $F(\cdot)$. Since there is a single essential player (in the sense that workers do not produce any output without the firm) the Shapley value takes a simple form.

[10] Contrary to $J(\cdot)$, $V(\cdot)$ will be decreasing, so that this is indeed a gain for the worker.

First, recall that the Shapley value specifies that in a bargaining game with a finite number of players every player's payoff is the average of her contributions to all coalitions that consist of players ordered below her in all feasible permutations. More explicitly, in a game with $m+1$ players, let $\pi = \{\pi(0), \pi(1), ..., \pi(m)\}$ be a permutation of $0, 1, 2, ..., m$, where player 0 denotes the firm and players $1, 2, ..., m$ are the suppliers, and let $z_\pi^j = \{j' \mid \pi(j) > \pi(j')\}$ be the set of players ordered below j in the permutation π. Let the set of feasible permutations be Π and the value of any coalition from this set of permutations be given by the function $v : \Pi \to \mathbb{R}$. Then the Shapley value of player j is

$$\text{Shapley value}_j = \frac{1}{(m+1)!} \sum_{\pi \in \Pi} \left[v\left(z_\pi^j \cup j\right) - v\left(z_\pi^j\right) \right].$$

In other words, the Shapley value of player j is the average of her contribution to possible coalitions ordered below her according to all possible permutations. The Shapley value equation is symmetric in the sense that all players are treated identically. Applying this to our context would imply the following simple equation in terms of value functions: $J(m) - J(m - \varepsilon) = \varepsilon [V(m) - V^u]$, which implies that the incremental value that the firm receives from employing the worker is equalized to the value that the worker obtains by being employed rather than unemployed; the factor ε on the right side of this equation arises from the fact that the worker is of size ε. This equation assumes symmetry between the firm and the workers in the bargaining protocol. Instead, as in the standard DMP setup, we will use a slight generalization on this equation, which allows differential bargaining powers between workers and firms. In particular, our wage determination equation will be

$$\beta [J(m) - J(m - \varepsilon)] = (1 - \beta)\varepsilon [V(m) - V^u],$$

or

$$\beta \Delta J(m) = (1 - \beta) [V(m) - V^u], \tag{7}$$

for all $m \in \mathbb{N}$, where $\beta \in (0, 1)$ denotes the bargaining power belonging to workers.

Now we can define *a steady-state equilibrium.*

Definition 2. *A tuple $\langle \theta, V^u, g(\cdot), J(\cdot,\cdot), V(\cdot,\cdot), h(\cdot,\cdot), w(\cdot,\cdot) \rangle$ is a steady-state equilibrium if*

- *$J(m)$, $V(m)$ and $w(m)$ satisfy (2), (4) and (7).*
- *$g(m) \in \mathcal{G}$ satisfies (6).*

- *the value of an unemployed worker, V^u, satisfies (5).*
- *there is optimal vacancy posting, i.e.,*

$$h(m) = \begin{cases} 1 & \text{if } -\gamma + q(\theta)\,\Delta J(m+\varepsilon) > 0 \\ 0 & \text{if } -\gamma + q(\theta)\,\Delta J(m+\varepsilon) < 0. \end{cases} \tag{8}$$

- *there is free entry, i.e.,*

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness.} \tag{9}$$

- *$V^u \in \mathbb{R}_+$ is the value of an unemployed worker.*

Notice that the steady-state equilibrium did not specify the unemployment rate u. This is because, as in the standard DMP model, the unemployment rate can be determined after the other endogenous variables. In particular, in steady state, a standard accounting argument implies that the u unemployed workers will be matched and thus hired at the flow rate $\theta q(\theta)$. On the other side, workers lose their job because of separations at the flow rate s and because of firm shutdowns at the flow rate δ. Consequently, the steady-state unemployment rate is given by equating flows into unemployment, $(1-u)(s+\delta)$ with flows out of unemployment, $u\theta q(\theta)$, thus

$$u = \frac{s+\delta}{s+\delta+\theta q(\theta)}. \tag{10}$$

It is straightforward to verify that, as in the standard DMP model, u is a monotonically decreasing function of θ: steady-state unemployment is lower when the labor market is tighter.

We can provide a partial characterization of a steady-state equilibrium. In particular, a particularly neat formula relates wages to the production function and the outside option of workers (that is, the flow value of being unemployed, rV^u). This is the subject of the following lemma.

Lemma 1. *In a steady-state equilibrium, wages and firm-value functions satisfy*

$$\left(m + \frac{1-\beta}{\beta}\varepsilon\right) w(m) - (m-\varepsilon) w(m-\varepsilon) = \varepsilon \Delta F(m) + \frac{1-\beta}{\beta}\varepsilon r V^u + (h(m) - h(m-\varepsilon))\left(\gamma - q\Delta J(m)\right) \tag{11}$$

for $m \geq 2\varepsilon$. In addition,

$$\frac{\varepsilon}{\beta} w(\varepsilon) = \varepsilon \Delta F(\varepsilon) + \frac{1-\beta}{\beta}\varepsilon r V^u + (h(\varepsilon) - h(0))\left(\gamma - q(\theta)\,\Delta J(\varepsilon)\right). \tag{12}$$

Proof. Subtracting the firm's Bellman equation (2) for $m-\varepsilon$ from that for m yields

$$qh(m)\Delta J(m+\varepsilon)-[qh(m-\varepsilon)+\varepsilon(r+\delta)+sm]\Delta J(m)+s(m-\varepsilon)\Delta J(m-\varepsilon)$$
$$=mw(m)-(m-\varepsilon)w(m-\varepsilon)-F(m)+F(m-\varepsilon)-[h(m)-h(m-\varepsilon)]\,\gamma;$$

this equation holds for $m\geq 2\varepsilon$. In the case $m=\varepsilon$ it follows that

$$q\Delta J(2\varepsilon)-(q+\varepsilon(r+\delta)+s\varepsilon)\,\Delta J(\varepsilon)=\varepsilon w(\varepsilon)-F(\varepsilon)+F(0)-[h(\varepsilon)-h(0)]\,\gamma.$$

Also, substituting from the Shapley bargaining equation into the worker's Bellman equation for n, multiplying by $\varepsilon(1-\beta)/\beta$, and rearranging gives that

$$qh(m)\Delta J(m+\varepsilon)-(qh(m)+\varepsilon(r+\delta)+sm)\,\Delta J(m)+s(m-\varepsilon)\Delta J(m-\varepsilon)=\frac{1-\beta}{\beta}\varepsilon\,[rV^u-w(m)]\,.$$

Comparing the previous equations establishes the relationships (11) and (12). □

The wage equation derived in the lemma takes a particularly simple form in the case when there is an m^* such that $h(m)=1$ for $m<m^*$ and $h(m)=0$ for $m\geq m^*$. An equilibrium of this form is referred to as a *threshold equilibrium* (with hiring cutoff m^*). Naturally, only steady-state equilibria of this sort are of interest and in the rest of the paper, by a (steady-state) equilibrium, we always mean a threshold steady-state equilibrium.[11]

The proof of the following corollary is immediate on substituting the form of $h(\cdot)$ in (11).

Corollary 1. *In a steady-state equilibrium with hiring cutoff m^*, for all $m\neq\varepsilon$ and $m\neq m^*$, wages satisfy the following difference equation:*

$$\left(m+\frac{1-\beta}{\beta}\varepsilon\right)w(m)-(m-\varepsilon)w(m-\varepsilon)=\varepsilon\Delta F(m)+\frac{1-\beta}{\beta}\varepsilon rV^u.$$

In addition, for $m=\varepsilon$ and $m=m^*$, we have

$$\frac{\varepsilon}{\beta}w(\varepsilon)=\varepsilon\Delta F(\varepsilon)+\frac{1-\beta}{\beta}\varepsilon rV^u$$
$$\left(m^*+\frac{1-\beta}{\beta}\varepsilon\right)w(m^*)-(m^*-\varepsilon)w(m^*-\varepsilon)=\varepsilon\Delta F(m^*)+\frac{1-\beta}{\beta}\varepsilon rV^u+(q\,(\theta)\,\Delta J(m^*)-\gamma)\,.$$

[11] To see intuitively why threshold equilibria are natural to study, suppose that the equilibrium does not take this form, in the sense that there exist m_1 and $m_2>m_1$, such that $h\,(m)=1$ for some $m>m_2$, but $h\,(m)=0$ for $m\in[m_1,m_2]$. Let m^* be the smallest firm size such that $h\,(m^*+\varepsilon)=0$. It is straightforward to verify that, when we view firm size as a Markov process, all states $m>m^*$ are transient. In particular, let $P^\tau\,(m,[\tilde{m},\tilde{m}'])$ be the probability of reaching a state in the interval $[\tilde{m},\tilde{m}']$ starting from state m in τ steps. Then from the accounting equations in (6), it is evident that $\lim_{\tau\to\infty}P^\tau\,(m,[\tilde{m},\tilde{m}'])=0$ for all $\tilde{m}'\geq\tilde{m}>m^*$ and for all m, since firms always lose workers at positive flow rates and will never grow beyond m^*.

This analysis does not rule out the existence of non-threshold equilibria in which firms employ mixed vacancy-posting strategies, so that there exist $m_2>m_1$ such that $h(m_1)\in[0,1]$ but $h(m_2)=1$. It is hence not a proof that no other equilibria exist, but it is indicative that there is little loss of generality in restricting attention to threshold steady-state equilibria.

The recurrence relation can be solved forward inductively to give an expression for each $w(m)$, $m \in \varepsilon\mathbb{Z}_+$ as a weighted sum of the terms $\Delta F(\mu)$, $\mu \le m$. For the special case in which the bargaining powers of firms and workers are symmetric, i.e., $\beta = 1/2$, this representation takes a particularly useful form, which is recorded in the following corollary:

Corollary 2. *Suppose that $\beta = 1/2$. In an equilibrium with hiring cutoff m^*, then wages are given by*

$$w(m) = \begin{cases} \frac{1}{2} rV^u + \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] & m < m^* \\ \frac{1}{2} rV^u + \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] + \frac{m^*}{m(m+\varepsilon)}\left[q(\theta)\,\Delta J(m^*) - \gamma\right] & m \ge m^*. \end{cases} \tag{13}$$

That is, wages at a firm with m workers are given by an average of the flow value of the worker's outside option, rV^u, and a term that measures the difference between the average labor productivity at the firm with m workers and a weighted average of the marginal product over the whole range of workers from the first up to the mth worker. The presence of this difference term reflects the essence of the Shapley value assumption, which recognizes the endogeneity of the outside option of the firm. In particular, if the firm were to disagree in its bargaining with the mth worker, this would increase the bargaining power of all the other workers and force the firm to pay higher wages to them. These wages would be related to these workers' marginal products after the mth worker has quit, thus, with this reasoning, the marginal product of each worker up to the mth features in the wage equation.

Equation (13) may become more intuitive when we rewrite the second term as follows:

$$\begin{aligned} \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] &= \frac{\varepsilon}{m\,(m+\varepsilon)}\sum_{\mu=0}^{m/\varepsilon}\left[F(m) - F(\mu\varepsilon)\right] \\ &= \frac{1}{m(m+\varepsilon)}\sum_{\mu=0}^{m/\varepsilon}\sum_{\nu=\mu+1}^{m/\varepsilon}\Delta F(\nu\varepsilon) \\ &= \frac{1}{m(m+\varepsilon)}\sum_{\nu=1}^{m/\varepsilon}\nu\Delta F(\nu\varepsilon). \end{aligned}$$

This expression makes it clear that wages paid by a firm employing m workers will depend most heavily on the marginal products that would apply if the firm were to employ slightly fewer than m workers, while marginal products when it employs fewer workers receive less weight in the wage formula. This is intuitive in light of the discussion above; disagreement with the mth worker will force the firm to pay higher wages to remaining workers commensurate with their marginal products and the threat points they would have in that case, which relate most directly to the marginal product the firm would experience if it had just a few less workers.

Taking rV^u and $q(\theta)$, the optimal hiring behavior of firms and equilibrium wage bargains are straightforward to characterize as solutions to a set of linear equations or as a solution to a difference equation (see also Wolinsky, 2000). However, characterizing or even proving the existence of a general equilibrium is difficult. The reason for this can be seen in the equation for wage determination (13). Consider a cutoff equilibrium at m^*, and suppose that, given the values of rV^u and $q(\theta)$, a firm is indifferent between hiring an additional worker when it has $m^* - \varepsilon$ employees (thus $q\Delta(m^*) = \gamma$). Then for appropriately-chosen parameter values, there may also exist a cutoff equilibrium at $m^* - \varepsilon$. Now a small change in rV^u and $q(\theta)$ will induce a discontinuous change in the hiring behavior of firms. Thus generally the worker's value function $V(\cdot)$ does not vary continuously in $(q(\theta), rV^u)$, and therefore, neither does the right side of equation (5). This makes it difficult to establish the existence of a solution to the system of equations that characterize the equilibrium The same issues also lead to a large range of multiplicity of equilibrium for most parameterizations, since whether firms stop hiring at m^* or $m^* - \varepsilon$ can have a first-order effect on the vector $(q(\theta), rV^u)$, making multiple such vectors potentially part of steady-state equilibria. Both of these difficulties arise from the presence of workers with discrete size. Motivated by this reasoning, we next study the limiting economy as worker size ε becomes small.

3 Baseline Model (Continuous Employment)

3.1 Basics

In this section, we consider the results of the model defined by taking the limit of the model introduced in the previous Section as $\varepsilon \to 0$. That is, we transform the number of workers employed by the firm from a discrete variable $m \in \varepsilon\mathbb{Z}_+$ to a continuous variable $n \in \mathbb{R}_+$. This has several technical advantages; in particular, the equations that define the solution are easier to work with. As an added advantage, this setting allows us to transform the difference equations that arise in the discrete-worker setting to differential equations, which can nearly be solved in closed form.

More formally, we shall take the limit as $\varepsilon \to 0$ in equations (2), (4), (6), and (7) from the previous section. This essentially amounts to replacing the discrete first difference operator with a differential operator, while leaving the equations of the model otherwise unchanged. We therefore begin by introducing these close analogs of the basic equations of the model;

the intuition is identical to that provided in the discussion in the previous section, and so discussion of the model setup will be brief here.

As in the previous section, we focus on anonymous steady-state equilibria and further limit attention to threshold equilibria. In particular, this implies that all firms will use the same equilibrium hiring strategy, which takes the form of a cutoff value n^* such that firms with fewer than n^* employees post a vacancy and those with more than n^* do not; i.e., $h(n^*) = 1$ if $n < n^*$ and $h(n^*) = 0$ if $n > n^*$. Finally, for natural reasons, we will focus on solutions in which the firm's value function $J(\cdot)$ is twice continuously differentiable in the number of employees the firm currently has, n (i.e., $J(\cdot) \in C^2(\mathbb{R}_+)$). The Bellman equations and the Shapley value bargaining equations (the differential analogs of (4) and (7) above) then require the wage and the worker's value function be continuously differentiable in the number of employees at the firm (i.e., $V(\cdot) \in C^1(\mathbb{R}_+)$, $w(\cdot) \in C^1(\mathbb{R}_+)$).Below, we will prove the existence of a steady-state equilibrium with these features.

The setup is otherwise identical to that in the previous section.

Recalling equation (1) and taking the limit as $\varepsilon \to 0$ (under the assumption that a differentiable solution $J(\cdot)$ exists) immediately yields

$$(r+\delta)\, J(n) = F(n) - nw(n) + \max\{0, -\gamma + q(\theta)J'(n)\} - snJ'(n). \tag{14}$$

Under the assumption that firms are using a hiring strategy with cutoff at n^*, this equation can be further simplified to

$$(r+\delta)\, J(n) = \begin{cases} F(n) - nw(n) - \gamma + (q(\theta) - sn)J'(n) & n \le n^* \\ F(n) - nw(n) - snJ'(n) & n > n^* \end{cases} \tag{15}$$

Recall that $J(\cdot) \in C^2$. First, the fact that the firm's hiring strategy changes discretely from $h(n) = 1$ for $n < n^*$ to $h(n) = 0$ for $n > n^*$ requires the following boundary condition to be satisfied:

$$-\gamma + q(\theta)J'(n^*) = 0. \tag{16}$$

This condition might be termed a smooth pasting condition, but it will be true whether or not the cutoff n^* that the firm chooses is in fact optimal.[12] It arises from the fact that when its employment reaches n^*, the firm stops paying the flow cost γ of posting a vacancy. Since this boundary condition is true for any cutoff n^*, it is not sufficient to characterize the solution to the differential equation (15). An additional boundary condition comes from a standard smooth pasting argument. Intuitively, notice that the firm is solving an optimal stopping

[12] See Dixit (1993, p. 42) for a discussion of smooth pasting and super-contact conditions in related problems.

problem—at what point to stop posting additional vacancies. Since the cost of posting a vacancy is a constant flow cost, it is intuitive that a 'super-contact' condition on the second derivative of $J(\cdot)$ will be required. Intuitively, the super-contact condition requires that n^* is an optimal stopping point for the firm, in the sense that a small change in n^* should have no impact on the value of the firm. This is equivalent to the second-order condition

$$J''(n^*) = 0. \tag{17}$$

In addition, as in the discrete worker case of the previous section, we have a free entry condition, which takes the form

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness,} \tag{18}$$

which is identical to (9) in the previous section, and requires that in order for there to be positive activity in equilibrium, the value of k needs to be sufficiently low for firms to find it attractive to gain access to the production technology.

To make further progress, we need to use the Shapley bargaining formula, together with the worker's Bellman equation, to solve for the wage function $w(\cdot)$. Similarly to the argument for the firm's Bellman equation, taking the limit as $\varepsilon \to 0$ in the worker's Bellman equation (3) in the previous section yields:

$$(r+s+\delta) V(n) = \begin{cases} w(n) + [q(\theta) - sn] V'(n) + (s+\delta) V^u & n \leq n^* \\ w(n) - snV'(n) + (s+\delta) V^u & n > n^*. \end{cases} \tag{19}$$

By an analogous argument, the Shapley value equation is

$$(1-\beta)[V(n) - V^u] = \beta J'(n). \tag{20}$$

To simplify the resulting differential equations and communicate the basic qualitative features of the model, let us now focus on the case where $\beta = 1/2$, so that the Shapley bargaining equation takes the simpler form

$$V(n) - V^u = J'(n). \tag{21}$$

Moreover, for future use, let us differentiate this equation to observe that

$$V'(n) = J''(n). \tag{22}$$

The equation for rV^u that arises from the Bellman equation for an unemployed worker closes the model. To write this equation, it is necessary to introduce additional notation for the steady-state distribution of firms as a function of the number of workers already employed by the firm; this differs slightly from the case with discrete worker size. As before, we will solve the model only in steady-state, so we do not need to index the distribution of firms by size according to time.

Denote the firm-size distribution by $G(n)$. First, suppose that there exists a steady-state for $G(\cdot)$ such that the density function $g(n)$ is continuous on $(0, n^*)$, together possibly with an atom of mass G^* at n^*. Such a distribution $G(\cdot)$ is a steady-state distribution if $g(\cdot)$ satisfies the steady state accounting equation equating, for each $n \in (0, n^*)$ and each $\varepsilon > 0$ sufficiently small, the flow of firms into and out of the interval $(n - \varepsilon/2, n + \varepsilon/2)$:

$$\left(\frac{q(\theta)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta\right) g(n) = \frac{q(\theta)}{\varepsilon} g(n - \varepsilon) + \frac{s(n + \varepsilon)}{\varepsilon} g(n + \varepsilon) + O(\varepsilon). \tag{23}$$

Taking limits as $\varepsilon \to 0$, it follows that any differentiable solution to this equation must satisfy the following differential equation for $g(n)$:

$$\frac{g'(n)}{g(n)} = \frac{\delta - s}{sn - q(\theta)}. \tag{24}$$

Integrating (24) gives that the general solution (for $q(\theta) - sn \geq 0$) is given by

$$g(n) = A(q(\theta) - sn)^{\frac{\delta - s}{s}}. \tag{25}$$

There are two cases to consider: the case where $n^* \geq q(\theta)/s$ and the one where $n^* < q(\theta)/s$. The importance of this distinction is that when $n = q(\theta)/s$, then ignoring firm death, flows due to hiring and worker separation balance out; for n larger, there is a net loss of firms of size n even disregarding firm death. Thus the firm size distribution is supported on $[0, \min\{n^*, q(\theta)/s\}]$. In the case of interest, in which $n^* < q(\theta)/s$, then even at $n = n^*$, there is a positive "flow" of firms at $n - \varepsilon$ for any $\varepsilon > 0$; this leads to an atom at n^*. To calculate the size of this atom, again equate the flow of firms into and out of n^*, to obtain:

$$(q(\theta) - sn^*)g(n^*) = \delta G^*. \tag{26}$$

Notice that (26) assumes the outflow of firms from the state n^* is given only by firm death. This is a consequence of the feature that $h(n^*) = sn^*/q,(\theta)$ i.e., firms at n^* still post vacancies and are hiring continuously, but only at a rate that allows them to counteract their loss of workers to the separation shock. This hiring is what maintains the atom at G^*. A more intuitive explanation for this is that at n^*, the firm is indifferent between posting a vacancy or not. If it posted a vacancy for sure when at n^*, so that $h(n^*) = 1$, it would quickly hire workers faster than it loses them to separation, since $q(\theta) > sn^*$ by assumption; the firm would then move to having $n > n^*$ workers. However, as soon as this happens, it would then strictly prefer not to post a vacancy, and would therefore lose workers again until its workforce size falls to n^*. Conversely, if the firm does not post a vacancy at n^*, so that $h(n^*) = 0$, then it would quickly drop to $n < n^*$ and begin hiring again. The only possibility is that the firm must use a mixed strategy and post a vacancy with a flow probability of $h(n^*) = sn^*/q(\theta)$; this ensures that the rate at which the firm attracts new workers, $h(n^*)q(\theta)$, equals the rate at which it loses workers, sn^*.[13]

Now combining (25) and (26), we obtain:

$$A(q - sn^*)^{\frac{\delta}{s}} = \delta G^*.$$

To solve for A, note that since $G(\cdot)$ is a probability distribution, we must have

$$G^* + \int_0^{n^*} g(n)\,dn = 1.$$

This implies $A = \delta q(\theta)^{-\delta/s}$, so that the steady-state firm size distribution is given by

$$g(n) = \frac{\delta}{q(\theta)}\left(1 - \frac{sn}{q(\theta)}\right)^{\frac{\delta-s}{s}} = \frac{\delta}{q(\theta)^{\frac{\delta}{s}}}\left(q(\theta) - sn\right)^{\frac{\delta-s}{s}} \qquad \text{if } n \in [0, \min(n^*, \frac{q(\theta)}{s}) \tag{27}$$

with an atom of mass

$$G^* \equiv \left(1 - \frac{sn^*}{q(\theta)}\right)^{\frac{\delta}{s}} = \left(\frac{q(\theta) - sn^*}{q(\theta)}\right)^{\frac{\delta}{s}} \tag{28}$$

at n^* in the case where $n^* < \frac{q(\theta)}{s}$.

Figure 1 shows this distribution for a case in which $sn^* < q(\theta)$ and $\delta < s$, which is the empirically relevant case. The size G^* of the atom at n^* is not shown, although its location n^* is indicated with a vertical line.

Finally, observe that since the stochastic process for a firm's size satisfies an ergodicity condition, the steady state distribution $G(\cdot)$ is unique, so there was no loss of generality in

[13] In the discrete model with workers of size $\varepsilon > 0$ of the previous section, mixed strategies are not required, but an analogous result holds. Firms with n^* workers in this case generally strictly prefer not to post a vacancy, while firms with $n < n^*$ strictly prefer to do so. Thus a firm that has n^* workers will lose a worker with flow probability sn^*/ε, while a firm with $n^* - \varepsilon$ workers will gain an additional worker with probability q/ε and lose one with probability $sn^*/\varepsilon - 1$. Since as $\varepsilon \to 0$, $q(\theta)/\varepsilon - [sn^*/\varepsilon - 1] \to \infty$, one can verify that a firm that has n^* workers will, in the future, have exactly n^* workers for a fraction $1 - sn^*/q(\theta)$ of the time until it is destroyed, and exactly $n^* - \varepsilon$ workers for a fraction $sn^*/q(\theta)$ of this time. That is, the firm will post a vacancy a fraction $sn^*/q(\theta)$ of the time. Averaging over all firms with n^* workers establishes the result.

It is also useful to note the difference between the $\varepsilon > 0$ economy with workers on discrete size and the limit $\varepsilon \to 0$ economy in this respect. While in the economy with workers of discrete size, aggregate dynamics of the firm size distribution are deterministic, each individual firm's size follows a nontrivial stochastic process. In contrast, in the limit economy, the growth process of each firm is deterministic.

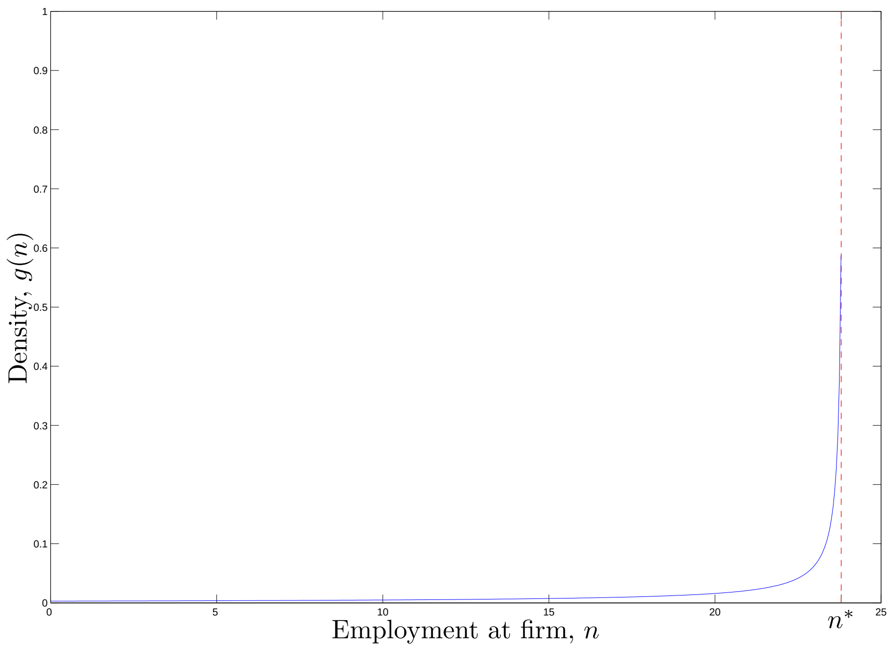

Figure 1: Steady state density function $g(n)$

solving only for a distribution in which $g(\cdot)$ is continuously differentiable on $(0, n^*)$.[14]

We can now write the Bellman equation for an unemployed worker. In the case where $q(\theta) > sn^*$ (i.e., when there is an atom of positive mass of firms at n^*), recall that we have $h(n^*) = \frac{sn^*}{q(\theta)}$; thus we have to incorporates the probability of an unemployed worker being hired by a firm of size n^*. Consequently, the Bellman equation for an unemployed worker is simply

$$rV^u = b + \theta q(\theta) \left[-V^u + \frac{\int_0^{n^*} V(n) g(n)\, dn + \frac{sn^*}{q(\theta)} V(n^*) G^*}{1 - \left(1 - \frac{sn^*}{q(\theta)}\right) G^*} \right] \tag{29}$$

The equations given above characterize almost completely any equilibrium of the model in the class we are considering (those that satisfy the differentiability assumptions and in which firms use symmetric cutoff hiring strategies). The only additional requirement is to check that firms and workers are behaving optimally.

This completes the characterization of an equilibrium, which we record as the following definition, analogous to Definition 2.

[14]For the ergodicity argument it is convenient to think of firm death as a shock that changes the size of a firm to 0, rather than causing entry of a new firm; in this case the uniqueness of the invariant distribution is immediate from Theorem 11.9 of Stokey, Lucas, and Prescott (1989).

Definition 3. *A tuple* $\langle \theta, V^u, g(\cdot), J(\cdot), V(\cdot), h(\cdot), w(\cdot) \rangle$ *is an* anonymous (threshold) steady-state equilibrium *if*

- $J(m)$, $V(m)$, *and* $w(m)$ *satisfy (16), (17), (15), (19) and (20).*
- $g(m) \in \mathcal{G}$ *satisfies (27) and (28).*
- *the value of an unemployed worker,* V^u, *satisfies (29).*
- *there is optimal vacancy posting, i.e.,*

$$h(m) = \begin{cases} 1 & \text{if } -\gamma + q(\theta) J'(m) > 0 \\ 0 & \text{if } -\gamma + q(\theta) J'(m) < 0. \end{cases} \tag{30}$$

- *there is free entry in the sense that equation (18) holds.*

As in the previous section, the steady-state unemployment rate can be determined after the other endogenous variables, and is given by (10), making the unemployment rate, u, once again a monotonically decreasing function of the tightness of the labor market, θ.

3.2 Equilibrium Characterization

To begin analysis of such equilibria, first observe that not surprisingly, a continuous analog of equation (13) holds and gives a simple expression for the wage function $w(\cdot)$. To obtain this equation, substitute from the bargaining equations (20) and (22) into the worker's Bellman equation (19) and rearrange to observe that for $n \in (0, n^*)$,

$$(r + s + \delta)J'(n) - (q(\theta) - sn)J''(n) = w - rV^u. \tag{31}$$

On the other hand, differentiating the firm's Bellman equation gives that for $n \in (0, n^*)$,

$$(r + s + \delta)J'(n) - (q(\theta) - sn)J''(n) = F'(n) - w(n) - nw'(n). \tag{32}$$

Equating (31) and (32) shows that in any twice differentiable solution to the program above, it must be that for $n \in (0, n^*)$,

$$2w(n) + nw'(n) = F'(n) + rV^u. \tag{33}$$

A similar argument shows that this equation also holds for $n > n^*$, though this is less important for our analysis. Integrating (33) by parts gives:

$$w(n) = \frac{1}{2} rV^u + \frac{1}{n}\left[F(n) - \frac{1}{n}\int_0^n F(m)\,dm\right] + \frac{K}{n^2},$$

where K is a constant of integration. Next note that if $K \neq 0$, then as $n \to 0$, the firm's total wage bill satisfies $nw(n) = O(1/n)$. For finite $q(\theta)$, this implies that the firm's value function can be written as

$$J(0) = E \int_0^\infty e^{-(r+\delta)t} \left[F(n(t)) - n(t) \, w(n(t)) \right] dt,$$

where $n(t)$ denotes the number of employees at time t. Evidently, $J(0)$ is equal to $+\infty$ if $K < 0$ and to $-\infty$ if $K > 0$. Therefore, the constant of integration must be equal to zero, i.e., $K = 0$. This argument establishes the following lemma.

Lemma 2. *In a steady-state equilibrium, wages satisfy*

$$w(n) = \frac{1}{2} r V^u + \frac{1}{n} \left[F(n) - \frac{1}{n} \int_0^n F(m) \, dm \right] \qquad (\forall n > 0). \tag{34}$$

This lemma shows that despite the additional general equilibrium interactions, wages in this model take a form identical to those in Stole and Zwiebel (1996a,b) and Wolinsky (2000). The first term in (34) is the contribution of the (flow value of the) outside option of the worker to his wage. The second term is the worker's contribution to the value of the firm, taking into account that if the worker were to quit, this would also influence the wages of other employees of the firm.

This explicit form characterization of wages will be important in further characterization and proving the existence of a steady-state equilibrium. A graphical representation of the dependence of wages on the number of workers employed at the firm is indicated in Figure 2. (This figure shows the wage function arising in the example of a Cobb-Douglas production function used in the calibrations in Section 5.) Also shown are a horizontal line indicating the flow value of the unemployed, rV^u, and the marginal product function, $F'(n)$.

Our next results show that, as depicted in Figure 2, wages and flow profits satisfy convenient boundary conditions.

Lemma 3. *In a steady-state equilibrium, wages are strictly positive, strictly decreasing with firm size, and, satisfy*

$$\lim_{n \to 0^+} w(n) = +\infty \text{ and } \lim_{n \to \infty} w(n) = \frac{1}{2} r V^u.$$

Moreover, the flow profit $\pi(n) = F(n) - nw(n)$ *of the firm is maximized at some* $n \in [0, \infty)$ *and satisfies*

$$\lim_{n \to 0^+} \pi(n) = 0 \text{ and } \lim_{n \to \infty} \pi(n) = -\infty.$$

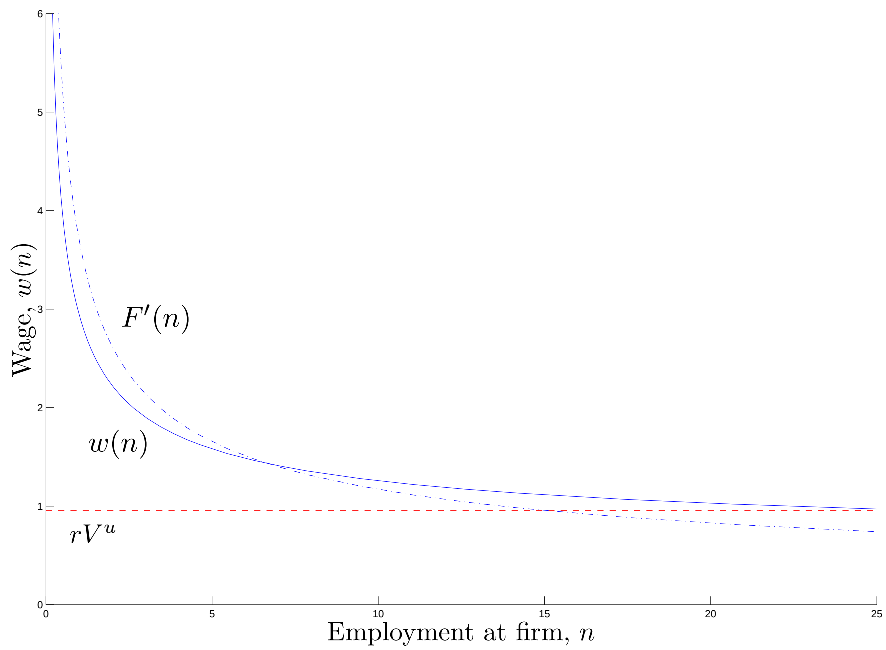

Figure 2: Wage, marginal product, and outside option

Proof. See Appendix. □

The fact that wages at very small firms become very large arises from the Inada condition on the firm's production function, since the marginal product also becomes arbitrarily large as n decreases to 0.

The closed form equation for wages also allows the value functions for firms to be derived in closed form as shown in the following lemma.

Lemma 4. *In a steady-state equilibrium with labor market tightness θ, the firm's value function $J(\cdot)$ satisfies*

$$J(n) = (q(\theta) - sn)^{-\frac{r+\delta}{s}} \left[q(\theta)^{\frac{r+\delta}{s}} k + \int_0^n (q(\theta) - sm)^{\frac{r+\delta}{s}-1} \left(\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right) dm \right] \tag{35}$$

for all $0 < n < n^*$.

Proof. See Appendix. □

While the closed form solution for the firm's value function looks complicated, it has a relatively simple structure and enables us to further characterize the form of the equilibrium.

In particular, differentiating this value function, $J(\cdot)$, with respect to n allows the boundary conditions at n^* also to be written in closed form. More specifically, equation (16) becomes

$$J(n^*) = \frac{1}{r+\delta}\left[-\frac{n^*}{2}rV^u - \frac{sn^*\gamma}{q(\theta)} + \frac{1}{n^*}\int_0^{n^*} F(m)\,dm\right], \tag{36}$$

while the smooth pasting condition (17) becomes

$$\frac{1}{n^*}\left[F(n^*) - \frac{1}{n^*}\int_0^{n^*} F(m)\,dm\right] = \frac{1}{2}rV^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \tag{37}$$

It is interesting to observe that this latter can also be expressed as

$$w(n^*) = rV^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \tag{38}$$

Equation (38) is very intuitive and states that the firm continues to hire until the wage it pays equals the outside option of the worker, rV^u, plus a term that is proportional to the severity of the labor market friction (parameterized by the flow cost of posting an application, γ, divided by the productivity of that posting, $1/q(\theta)$). This wage equation is therefore comparable to the result obtained in a static setting by Stole and Zwiebel (1996a) (see their Corollary 1 on page 396) and generalized to a dynamic setting by Wolinsky (2000). In these previous analyses, since there is no hiring margin (and no frictions), the second term is absent. Consequently, those models always imply "over-hiring" relative to a hypothetical competitive benchmark; firms will hire more than this competitive benchmark in order to reduce the marginal product of workers and thus their bargaining power according to the Shapley bargaining protocol (see Stole and Zwiebel, 1996a). Our analysis shows that this over-hiring result may or may not apply in general equilibrium; when γ is small, it will, but it could fail to do so when γ is large and $q(\theta)$ is relatively small.

Substituting the closed form expressions for $J(n^*)$ given by (36) into the formula for $J(\cdot)$ given by (35) and rearranging gives an expression which will be useful in characterizing equilibria.

$$\begin{aligned} k = J(0) = J(n^*)&\left(\frac{q(\theta)-sn^*}{q(\theta)}\right)^{\frac{r+\delta}{s}} \\ &- q(\theta)^{-\frac{r+\delta}{s}}\int_0^{n^*}(q(\theta)-sm)^{\frac{r+\delta}{s}-1}\left[\gamma + \frac{m}{2}rV^u - \frac{1}{m}\int_0^m F(\nu)\,d\nu\right]dm. \end{aligned} \tag{39}$$

Equation (38), together with the worker's Bellman equation (19) and the closed form equation for wages given by (34), allow the worker's Bellman equation to be expressed more simply also.

Lemma 5. *For $0 < n < n^*$, the worker's value function $V(\cdot)$ satisfies*

$$V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - sm)^{\frac{r+\delta}{s}} w(m)\, dm. \tag{40}$$

Equivalently,

$$V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q(\theta)-sn^*}{q(\theta)-sn} \right)^{\frac{r+\delta+s}{s}}}{2(r+\delta+s)} rV^u + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - sm)^{\frac{r+\delta}{s}} \psi(m)\, dm. \tag{41}$$

where

$$\psi(n) \equiv \frac{1}{n} \left[F(n) - \frac{1}{n} \int_0^n F(m)\, dm \right].$$

Proof. See Appendix. □

These equations now enable us to represent a steady-state equilibrium as the intersection of two curves in $(q(\theta), rV^u)$-space. In particular, suppose we know that in some equilibrium, the rate at which firms meet workers and the flow value of an unemployed worker equal $q(\theta)$ and rV^u respectively. Then firms take as given the path of wages that they will have to pay, $w(\cdot)$, given by (34). This means that their decision as to when to stop hiring, n^*, is given as the solution to an optimal stopping problem; n^* is then determined as the solution to (37), and the value of $J(n^*)$ can be deduced from (36). Denote this value by $n^*(q, rV^u)$. Next, solving the differential equation (35) with initial condition $(n^*(q, rV^u), J(n^*(q, rV^u)))$ allows us to solve for $J(0)$. The resulting expression is given in closed form by the right side of (39). If the free entry condition is satisfied, then the value of $J(0)$ so derived must equal the capital cost of entry k to ensure that (39) is satisfied. This provides one condition for (q, rV^u) to be part of an equilibrium. The remaining condition for (q, rV^u) to be part of an equilibrium is $(q, rV^u, n^*(q, rV^u))$ must satisfy (29), the Bellman equation of an unemployed worker. It remains to check that firm and worker behavior is optimal; for firms this is true by construction of $n^*(q, rV^u)$, while for workers, this follows since according to Lemma 3 and equation (38), the wage at any firm that is represented in equilibrium is strictly greater than rV^u, so that it is always optimal to accept any job offer. We record this conclusion as the following Proposition.

Proposition 1. *Let $q(\theta) > 0$ and $rV^u > 0$ be given. Then there is a steady-state equilibrium allocation with queue length $q(\theta)$ and value of an unemployed worker given by rV^u if and only* *if* (29) *and* (39) *are satisfied with* n^* *defined as the unique solution to* (37) *and* $J(n^*)$ *given by* (36).

Proof. Most of the proof is given in the discussion preceding the statement of the proposition. The uniqueness of $n^*(q, rV^u)$ follows from the proof of Lemma 3, together with (37). $\square$

We are now in a position to prove an existence theorem.

Theorem 1. *An steady-state equilibrium with cutoff hiring strategies exists.*

Proof. See Appendix. $\square$

The proof of the Theorem consists of showing that there exist (q, rV^u) satisfying the hypothesis of Proposition 1. Here, we present a diagrammatic exposition, emphasizing the intuition. The proof of Theorem 1 establishes that an equilibrium with positive activity exists if

$$k < \max_{n>0} \left\{ \frac{1}{n} \int_0^n F(m)\, dm - nb \right\},$$

where the existence of the maximum on the right side follows from Lemma 3. In this case, Figure 3 shows a diagram depicting in (q, rV^u)-space the two curves described in the discussion preceding Proposition 1. The upward-sloping curve is the free-entry condition of firms, equation (39); it is upward-sloping since, all else equal, an increase in rV^u, must be compensated by an increase in q, which makes entry into the labor market more profitable for firms. This is because a higher rV^u translates into higher wages, so that the profit margins of firms decline. Zero profits can only be ensured by leaving vacancies unfilled for shorter durations, thus by an increase in q. The downward-sloping curve is the Bellman equation for unemployed workers. It is downward-sloping since an increase in rV^u on the right side of (29) corresponds to an increase in wages; to keep the flow value of an unemployed worker satisfying this equation, it must be that hiring is more rapid (that is, q is larger), so that when hired, the worker spends less time earning the high wage he receives when his firm is smaller.

Comparative statics of the response of the endogenous variables $q(\theta)$ and rV^u can now be obtained from the diagrammatic representation of the equilibrium. While general conclusions are difficult to draw, the general features of the comparative statics are quite clear. The movements of the free-entry condition, (39), are generally unambiguous. For example, in response to an increase in productivity, it moves upwards. This is because for a given (q, rV^u), increased productivity increases flow profits for all firms, and so increases the implied value of entry, $J(0)$; to keep the free entry condition satisfied, rV^u must increase for each q. Similarly, the free-entry condition, (39) moves downwards in response to an increase in k. Since a

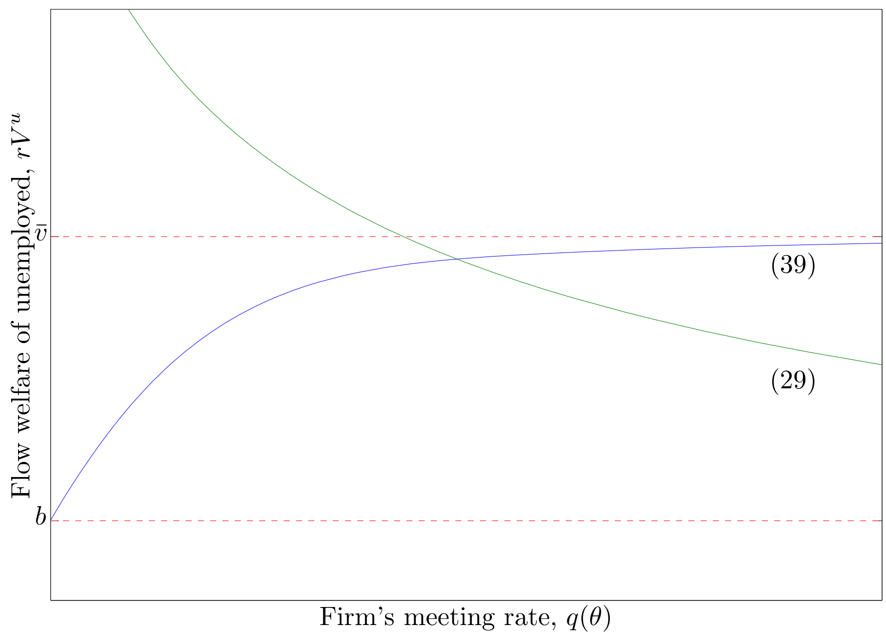

Figure 3: Equilibrium determination

change in k does not affect the other curve, it has unambiguous effects on the steady-state equilibrium in the situation depicted in the diagram in which the worker's Bellman equation is downward-sloping; an increase in k reduces rV^u and increases $q(\theta)$. This also corresponds to a decline in θ and therefore, from (10), to an increase in the steady-state unemployment rate. In all calibrated examples we have investigated, the worker's Bellman equation has indeed been downward-sloping. We therefore conjecture that an increase in the cost of entry unambiguously reduces the tightness of the labor market, θ, and increases steady-state unemployment, u, but we do not at present have a proof of this assertion.

The impact of a productivity shock on equilibrium variables, on the other hand, is ambiguous because productivity shocks have a potentially ambiguous effect on the other curve. This is because the impact of a productivity increase on the optimal employment level of firms, $n^*(q, rV^u)$, is ambiguous. For a given (q, rV^u), the wages paid at a firm with any fixed number n of workers, $w(n)$, increase; however, the increase in n^* means that more workers are employed at larger firms, which, all else equal, pay lower wages. In calibrated examples, the first effect tends to dominate, so that the curve moves upwards. An example where this is the case is shown in Figure 4. The dashed lines indicate the movement of the curves after

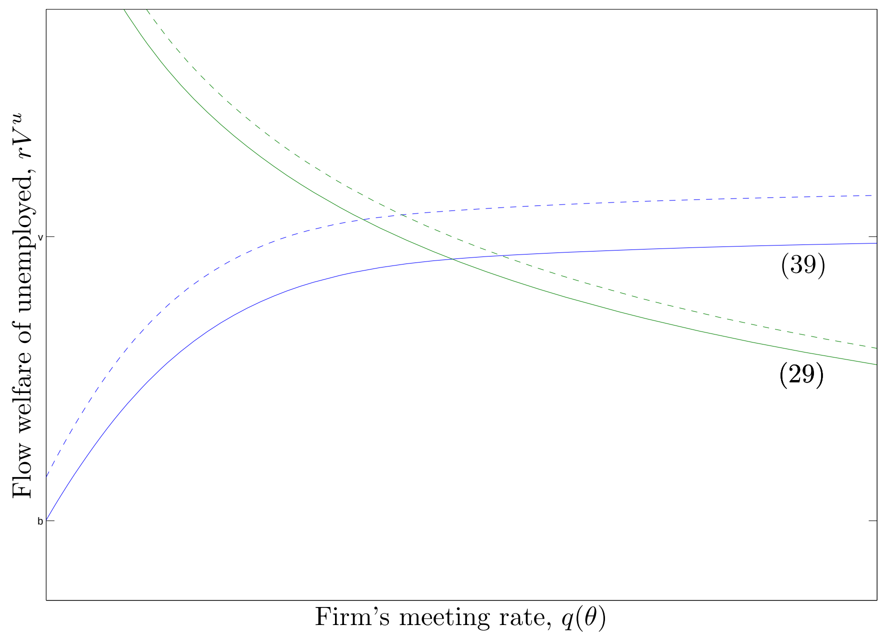

Figure 4: Response to a positive productivity shock

a Hicks-neutral increase in productivity. In this case, the utility of workers increases unambiguously, but the response of the equilibrium job-finding rate for firms, $q(\theta)$, is ambiguous. Nevertheless, in many calibrated examples, including the example shown in Figure 4, $q(\theta)$ decreases in response to the increase in productivity, so that workers' job-finding rate rises and steady-state unemployment falls. Another interesting feature of this example is that n^* also falls in response to the positive productivity shock. This implies that in the new steady state firms are, on average, smaller. Consequently, much of the adjustment to the new steady-state takes place *at the extensive margin*, that is, by the entry of new firms, while existing firms in fact decline in size. This is a pattern we find consistently in the calibrations, and underlines the importance of considering separating the intensive and extensive margins of employment creation.

Since the implications of an increase in productivity are potentially ambiguous, we therefore now investigate the response of unemployment, wages, and the vacancy-to-unemployment ratio to productivity shocks by using a simple calibrated version of the model.

4 Equilibrium Dynamics

In this section, we study the out of steady-state behavior in the baseline model (with continuous employment) introduced in the last section. So far we used the notation $J(n)$ to denote the value of a firm with n employees, $V(n)$ to denote the value of a worker employed in a firm with n employees, $w(n)$ for the wage of a worker in a firm with n employees, $g(n)$ for the density of firms with n employees, and finally $h(n)$ to denote the vacancy posting decision over firm with n employees. We also used $J'(n)$ and $V'(n)$ to denote their derivatives. This notation was made possible because in steady state all these objects did not depend on time. To study the dynamic equilibrium of this economy, we need to condition all of these objects on time, t, as well as on the number of employees, n, for example, $J(n,t)$ for the value of the firm at time t when it employs n workers. Correspondingly, we will denote the partial derivatives of this function by $J_n(n,t)$ and $J_t(n,t)$, and its second partial derivatives by $J_{nn}(n,t)$, etc. The same applies to $V(n,t)$.

A dynamic equilibrium is defined similarly to a steady-state equilibrium, but is naturally more involved, since all objects are time-varying. Before defining such an equilibrium more formally, let us develop the equivalent of the steady-state Bellman equations. A standard derivation gives the firm's Hamilton-Jacobi-Bellman equation, which requires that the value of a firm with n workers at time t satisfy:

$$\begin{aligned}(r+\delta)J(n,t) &= F(n,t) - nw(n,t) - snJ_n(n,t) \\ &\quad + \max_{h\in[0,1]}\{h(-\gamma + q(\theta(t))J_n(n,t))\} + J_t(n,t),\end{aligned} \tag{42}$$

where $\theta(t)$ is labor market tightness at time t, $F(n.t)$ is the output of a firm with n employees at time t, $w(n,t)$ is the wage function at time t, which will again be determined in equilibrium. All the terms have similar interpretation to the Bellman equation (15) above, except that there is also the time derivative of the value function, $J_t(n,t)$ on the right hand side.

Free entry still holds at all dates, so we also need to have

$$J(0,t) \leq k, \tag{43}$$

with equality whenever there is entry.

In addition, the analogs of the smooth pasting and optimal hiring (super-contact) conditions are given by

$$-\gamma + q(\theta(t))\, J_n(n^*(t), t), \tag{44}$$

and

$$J_{nn}\left(n^{*}\left(t\right),t\right)=0, \tag{45}$$

where $n^{*}(t)$ is the "ideal size" of the firm which is now time varying, because productivity and labor market tightness vary over time. Recall that as in the steady-state analysis, $n^{*}(t)$ is such that in the threshold dynamic equilibrium we have $h(n,t)=0$ for $n>n^{*}(t)$ and $h(n,t)=1$ for $n<n^{*}(t)$.

Also, with a similar argument, the Hamilton-Jacobi-Bellman equation for the workers implies that the value function $V(n,t)$ must satisfy

$$(r+\delta+s)V(n,t)=w(n,t)+h(n,t)[q(\theta(t))-sn]V_{n}(n,t)+V_{t}(n,t), \tag{46}$$

which again only differs from the state-state equation (19) because of the time derivative $V_t(n,t)$ on the right hand side.

Wages are again determined by continuous Shapley bargaining, which now implies that

$$V(n,t)-V^{u}(t)=J_{n}(n,t), \tag{47}$$

which is identical to (20). The value of an unemployed worker, which is now time varying, $V^{u}(t)$, In particular, we have

$$rV^{u}(t)=b+\theta(t)\,q(\theta(t))\left[-V^{u}(t)+\frac{\int_{0}^{\infty}V(m,t)g(m,t)h(m,t)\,dn}{\int_{0}^{\infty}g(m,t)h(m,t)\,dm}\right]+V_{t}^{u}(t). \tag{48}$$

To complete the description of the environment, we need to specify the distribution of firm sizes over time, represented by $g(n,t)$. To derive this distribution, let us reason as in the previous section and start with the case in which each worker is of size $\varepsilon>0$. In this case, away from steady state, the rate of change in $g(n,t)$ over time, $g_t(n,t)$, is given by the difference between flows in and flows out of firms into the "state" of having n employees. Therefore, for $n<n^{*}(t)$,

$$g_{t}(n,t)=-\left(\frac{q(\theta(t))}{\varepsilon}+\frac{sn}{\varepsilon}+\delta\right)g(n,t)+\frac{q(\theta(t))}{\varepsilon}g(n-\varepsilon,t)+\frac{s(n+\varepsilon)}{\varepsilon}g(n+\varepsilon,t)+O(\varepsilon).$$

Now taking the limit $\varepsilon\to 0$, we obtain the partial differential equation for $n<n^{*}(t)$:[15]

$$g_{t}(n,t)=-(\delta-s)\,g(n,t)+(sn-q(\theta(t)))\,g_{n}(n,t). \tag{49}$$

In addition, as in the steady state, there may be an atom at $n^{*}(t)$, the employment level beyond which firms do not hire at time t. Assuming that $n_{t}^{*}(t)$ always satisfies $-sn^{*}(t)<$

[15] For $n>n^{*}(t)$, we can derive a similar partial differential equation without the $q(\theta(t))$ terms. Nevertheless, if there are initially no firms above $n^{*}(t)$ and provided that $n_{t}^{*}(t)>-sn^{*}(t)$ at all t, there never will be any such firms, so that this partial differential equation is of limited interest for our focus.

$n_t^*(t) < q(\theta(t)) - sn^*(t)$, then firms with $n^*(t)$ workers will be able to choose a hiring policy $h \in [0,1]$ such that they remain at the cutoff $n^*(t)$ as t varies. Also, assume that there are no firms with $n > n^*(t)$. In this case (which is the relevant one in our calibrations), the change in $G^*(t)$ over time arises from adding those firms near enough to $n^*(t)$ that by hiring with $h = 1$, they reach $n^*(t)$, and subtracting those firms at $n^*(t)$ that die exogenously.

$$G_t^*(t) = [q(\theta(t)) - sn^*(t) - n_t^*(t)]\, g(n^*, t) - \delta G^*(t)\,. \tag{50}$$

Given these equations, we can define a dynamic equilibrium as follows:

Definition 4. *A tuple* $\langle \theta(t), V^u(t), g(n,t), J(n,t), V(n,t), h(n,t), w(n,t) \rangle$ *is an* anonymous (threshold) dynamic equilibrium *if*

- *$J(n,t)$, $V(n,t)$ and $w(n,t)$ satisfy (42), (44), (45), (46) and (47).*
- *$g(n,t) \in \mathcal{G}$ satisfies (49) and (50).*
- *the value of an unemployed worker, $V^u(t)$, satisfies (48).*
- *there is optimal vacancy posting, i.e.,*

$$h(n,t) = \begin{cases} 1 & \text{if } -\gamma + q(\theta(t))\, J_n(n,t) > 0 \\ 0 & \text{if } -\gamma + q(\theta(t))\, J_n(n,t) < 0. \end{cases} \tag{51}$$

- *there is free entry in the sense that equation (43) holds.*

As in the analysis of the steady-state equilibrium, we can differentiate the firm's Bellman equation with respect to n and compare with the worker's Bellman equation, which yields

$$(r+\delta)J_n(n,t) = F_n(n,t) - w(n,t) - nw_n(n,t) + q(\theta(t))\, J_{nn}(n,t) - sJ_n(n,t) - snJ_{nn}(n,t) + J_{nt}(n,t)$$

and

$$(r+\delta+s)J_n(n,t) - (q(\theta(t)) - sn)J_{nn}(n,t) = w(n,t) - rV^u(t) + J_{nt}(n,t) + V_t^u(t)\,.$$

Combining these two equations, we obtain a relatively simple partial differential equation characterizing the behavior of wages:

$$2w(n,t) + nw_n(n,t) = F_n(n,t) + rV^u(t) - V_t^u(t)\,,$$

which can be written in a form similar to the wage equation in stating-state, where the wage a worker in a firm with n employees at time t is given as the sum of to terms again, his outside option and his contribution to the value of the firm:

$$w(n,t) = \frac{1}{2}\left[rV^u(t) - V_t^u(t)\right] + \frac{1}{n}\left[F(n,t) - \frac{1}{n}\int_0^n F(m,t)\,dm\right]. \tag{52}$$

The only difference between this equation and (34), which applied in this steady-state analysis, is the presence of $V_t^u(t)$ in the first term, which represents that the effect of the change in the outside option of the worker on his current wage.

The wage equation (52) enables us to simplify the system of partial differential equations to the following:

$$\begin{aligned}
(r+\delta)J(n,t) &= F(n,t) - nw(n,t) - snJ_n(n,t) \\
&\quad + \max_{h\in[0,1]}\left\{h\left(-\gamma + q(\theta(t))J_n(n,t)\right)\right\} + J_t(n,t) \\
q(\theta(t))J_n(n^*(t),t) &= \gamma \\
J_{nn}(n^*(t),t) &= 0 \\
w(n,t) &= \frac{1}{2}\left[rV^u(t) - V_t^u(t)\right] + \frac{1}{n}\left[F(n,t) - \frac{1}{n}\int_0^n F(m,t)\,dm\right] \\
rV^u(t) &= b + \theta(t)\,q(\theta(t))\frac{\int_0^\infty J_n(m,t)g(m,t)h(m,t)\,dm}{\int_0^\infty g(m,t)h(m,t)\,dm} + V_t^u(t) \\
J(0,t) &\leq k.
\end{aligned}$$

Further elimination of the wage equation from this set of equations, taking the path of the tightness of the labor market $\theta(t)$ as given, and assuming that the production function is time-invariant over the time horizon we are considering (e.g., after a one-time show), we can reduce the system of equations to a hyperbolic first-order partial differential equation in J, J_n, and J_t for $n \in [0, n^*(t)]$:

$$\begin{aligned}
J_t(n,t) &= (r+\delta)J(n,t) - \frac{1}{n}\int_0^n F(m)\,dm + \gamma + (sn-q)J_n(n,t) \\
&\quad + \frac{n}{2}\left[b + \theta(t)\,q(\theta(t))\frac{\int_0^\infty J_n(m,t)g(m,t)h(m,t)\,dm}{\int_0^\infty g(m,t)h(m,t)\,dm}\right],
\end{aligned} \tag{53}$$

with boundary conditions

$$\begin{aligned}
J(0,t) &= k \\
J_n(n^*(t),t) &= \frac{\gamma}{q(\theta(t))} \\
J_{nn}(n^*(t),t) &= 0.
\end{aligned}$$

Therefore, the dynamics in this economy can be represented by a relatively simple system of partial differential equations, given by (53) together with (49) and (50). Unfortunately, this system of partial differential equations does not have a closed-form solution. Nevertheless, we can provide numerical solutions in order to investigate the response of the equilibrium to changes in productivity.

5 Calibration: Comparison of Steady States

In this section, we investigate a calibrated version of the model with continuous employment studied in the last two sections. We first discuss parameter choices in a benchmark calibration of the model, and then investigate the dynamic response of the economy to a positive productivity shock.

5.1 Benchmark Parameters

Although our primary objective is to illustrate the qualitative features of the equilibrium dynamics in this generalized search model, we choose parameters to approximate the labor market equilibrium in U.S. data. For comparability with the benchmark calibration results in this literature, we use the calibrations of Shimer (2005) where possible. Following Shimer, we first normalize the mean wage paid to workers to be 1, and set the flow utility enjoyed by the unemployed, b, to be equal to 0.4. Next, we choose the unit of time to be equal to a quarter, and therefore can again borrow Shimer's calibrations by setting the interest rate r equal to 0.012. We also follow Shimer and Petrongolo and Pissarides (2001) in assuming that the aggregate matching function $M(u, v)$ takes a Cobb-Douglas form,

$$M(u, v) = Zu^{\eta}v^{1-\eta},$$

with $\eta = 0.72$ and Z chosen so that the average job-finding rate for the unemployed, $\theta q(\theta)$, is 1.355. In addition, we take from Shimer's paper (along with Davis, Haltiwanger, and Schuh, 1996) that the gross rate of job destruction, $s + \delta$ should equal 0.10. For the sake of tractability, we set the bargaining power parameter of workers equal to $\beta = 0.5$.

We also need to match basic facts about the firm size distribution. The average employment of U.S. firms (both publicly- and privately-held) is reported by Davis, Haltiwanger, Jarmin and Miranda (2006) as 23.8, and we calibrate so that the mean firm size matches this figure.

For computational reasons, at this stage we are only able to solve the system of partial differential equations derived in the previous section for the case where $\delta = 0$. We therefore focus on steady state comparisons also assuming $\delta = 0$.[16] In this case, the requirement that average employment equal 23.8 takes the form that the firm's hiring policy should take the form $h(n) = 1$ when $n < n^* = 23.8$. The fact that there is no firm death also implies that in steady state all firms will hire up to the threshold n^* and thus have workforces of size equal to n^*.

We set the flow cost of posting a vacancy as $\gamma = 1.2$, slightly larger than the average wage of 1. Choosing the vacancy posting cost much lower than this makes it impossible to find an equilibrium in which average employment equals 23.8, while choosing it significantly higher seems implausible.

We assume a Cobb-Douglas production function, of the form $F(n) = An^{\alpha}$. We only have one condition that the two parameters (A, α) are required to satisfy, which is that $n^* = 23.8$. Another is obtained by requiring that the labor share of output correspond to that observed in the US; we therefore choose $\alpha = 0.5$; this corresponds to $A = 7.65$ and a labor share of 0.637.

5.2 Steady State Comparisons

Figure 5 shows the steady-state value function indicating the value of a firm employing n workers for $0 \leq n \leq n^* = 23.8$. The lower curve (indicated by a solid line and labeled $A = 7.65$) is for the steady state before the increase in productivity.

Several properties of this function are worth noting. First, the calibrated value of a firm with zero employment is equal to 1016.7; this value must equal the cost of entry, k, which seems of plausible value for the cost of upfront investment for firms. This corresponds to 6.81 years of output (ignoring discounting). Second, the calibrated value function is indeed increasing and concave, justifying the assumption that a threshold hiring rule is optimal.

The dashed line, marked with the caption $A = 7.72$, in Figure 5 shows the steady-state second value function for the higher level of productivity. This value function corresponds to an economy identical to the one described above, except that the productivity of workers, A, is higher by 1 percent. As expected, the increase in productivity increases the steady-state value of firms. The general features of the value function are similar to that for $A = 7.65$.

[16] In our steady-state calibrations in an earlier draft of this paper, we chose to set the ratio of separations due to plant closure to those equal to 5, so that $\frac{s}{23.8\delta} = 5$; this roughly matches data for U.S. manufacturing plants from Davis, Haltiwanger, and Schuh (1996).They note in Figure 2.3, page 29, that if one looks at quarterly data, 11.6% of job destruction occurs in plant shutdowns; in annual data, 22.9% of job destruction occurs in plant shutdowns; the reasons for the discrepancy are that closure takes time and that transitory plant-level employment changes are more important in higher frequency data (fn. 9, p. 27). An intermediate value seems appropriate: our calculation implies that one-sixth, or 16.7%, of job destruction occurs because of firm destruction. Although the steady-state comparisons presented here assume $\delta = 0$, the broad features are similar when we allow for $\delta > 0$ along the lines described in this footnote.

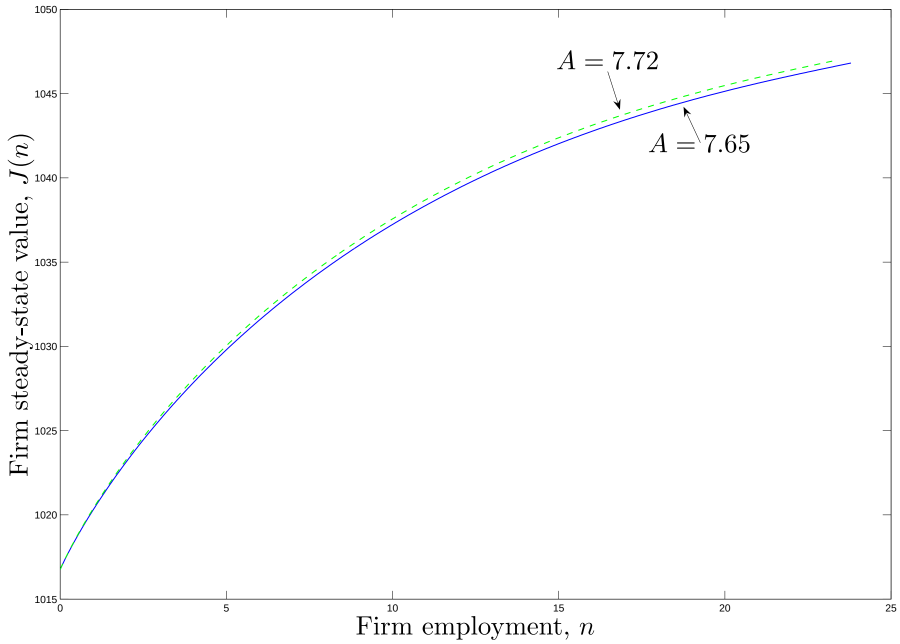

Figure 5: Steady state value functions

The most interesting result from this steady state comparison is that, in response to the increase in productivity, the threshold employment level n^* decreases; the new value is 23.33. The elasticity of n^* with respect to A is -1.95. Compensating this decline in the steady-state firm size is an increase in the number of firms. In particular, the mass of firms in the economy rises from 0.0391 to 0.0399, corresponding to an elasticity of 2.05 with respect to productivity. The intuition for this result seems to be that, with the increase in productivity the labor market becomes tighter (i.e., θ increases), raising both the outside option of workers and the cost of hiring an additional employee for each firm. Both of these effects counteract the increase in productivity for existing firms and reduce their desired level of employment.[17] This implies that the adjustments to the new steady state must involve different responses in the extensive and the intensive margins. Ultimately, existing firms will have to reduce their size, while new firms will enter and hire workers to reach their steady-state employment level. This steady-state comparison does not reveal, however, whether in the short run the intensive and the extensive margins will move in the same or in opposite directions. We will investigate this in the next section.

[17]In general, the effect of an increase in productivity on n^* is ambiguous. Nevertheless, in various different calibrations we have found that an increase in productivity leads to a lower steady-state level n^*.

The final point to note relating to the steady-state calibration is the response of unemployment to the productivity shock. In the original economy, unemployment equals 6.87%, whereas after the increase in productivity, unemployment falls to 6.81%. This corresponds to an elasticity of around -0.86, slightly higher than the elasticity of -0.45 reported for the benchmark Diamond-Mortensen-Pissarides model by Shimer (2005).

6 Equilibrium Dynamics in Response to a Positive Productivity Shock

We now investigate the short-run response of the economy to productivity shocks by a numerically solving the system of partial differential equations given by (49), (50) and (53). In principle, different types of behaviors are possible depending on the response of the intensive margin of employment following the increase in productivity. For example, it is possible for existing firms to increase their hiring in the short run because new entrant firms will remain small for an extended period of time owing to the frictional hiring process. In this case, the intensive margin of employment creation will follow a non-monotonic pattern, the size of existing firms first rising and then falling towards the new steady-state level. With such an adjustment pattern, we would also expect the short-run responses of unemployment and labor market tightness to be greater than in the medium run. However, an alternative adjustment pattern is also possible. Existing firms could reduce their hiring immediately following the positive productivity shock, adjusting monotonically to the new steady state. In this case, both in the short run and in the medium run, the intensive and the extensive margins of employment creation would be moving in opposite directions. While these opposing movements of these two margins are interesting, in this case we expect the short-run and the medium-run responses of labor market tightness to productivity shocks to be similar. In this section, we undertake a preliminary calibration exercise to understand which type of adjustment path is more likely.

The main challenge in characterizing the equilibrium dynamics in response to a productivity shock is to solve the system of partial differential equations given by (49), (50) and (53), or more simply, to characterize the solution to equation (53). The numerical challenges posed by this problem are non-trivial, since even the single partial differential equation (53) features the endogenous variables $\theta(t)$ and $n^*(t)$, so that both the functional form of the partial differential equation and its boundary condition at $n^*(t)$ are determined jointly with these variables.

As a first attempt, we focus on the case where $\delta = 0$, which significantly simplifies the computational burden. In future work, we will relax this restriction. With $\delta = 0$ there are no firm deaths, so we conjecture (and impose) that the response to a positive productivity shock occurs via a burst of entry at time 0. Consequently, the firm size distribution can be represented by the tuple $(g^o, n^o(t), g^y, n^y(t))$, indicating the number of existing firms g^o, the number of entering firms g^y, and the sizes of these firms at each time t, given by $n^o(t)$ and $n^y(t)$ respectively. Next, suppose that the time path for $n^*(t)$ is known. Then it is possible to determine the time path of $(\theta(t), n^o(t), n^y(t))$ from the observation that the hiring behavior of firms satisfies

$$h(n,t) = \begin{cases} 1 & 0 \leq n < n^*(t) \\ \frac{1}{q(\theta(t))} \left[s n^*(t) + n_t^*(t) \right] & n = n^*(t) \\ 0 & n > n^*(t). \end{cases} \tag{54}$$

In addition, by definition, $\theta(t)$ is the ratio of vacancies and unemployment, so that

$$\theta(t) = \frac{g^o h(n^o(t), t) + g^y h(n^y(t), t)}{1 - g^o n^o(t) - g^y n^y(t)}. \tag{55}$$

Finally, we know that

$$n_t^x(t) = q(\theta(t)) h(n^x(t), t) - s n^x(t), \tag{56}$$

where $x \in \{o, y\}$. The previous three equations (54)-(56) can be solved simultaneously numerically to calculate the evolution of $(n^o(t), n^y(t), \theta(t))$. Thus, conditional on a guess for the time path $n^*(t)$, the coefficient functions and boundary conditions of the partial differential equation (53) are therefore known. This problem can then be solved by standard methods; we choose to use the Lax version of the finite differences method.[18] Since the equilibrium conditions imply that the solution to the partial differential equation (53) is over-determined (only one boundary condition is required, and we have three), we can complete the numerical solution method by verifying whether our guess for the time path of $n^*(t)$ was correct by checking whether the other two boundary conditions are satisfied at all times. We choose to impose the boundary condition that $J_n(n^*(t), t) = \gamma / q(\theta(t))$ and check whether the solution satisfies the other boundary conditions.

Unfortunately, to estimate the equilibrium path of $n^*(t)$ precisely turns out to be difficult because of numerical errors. In particular, small changes in the time path of $n^*(t)$ make only small differences to the solution of the differential equation, and these are hard to distinguish from the numerical errors resulting from the discrete approximation to the continuous partial differential equation.[19] Consequently, the results presented in this version of the paper should

[18] For example, Press, Teukolsky, Vetterling, and Flannery (1992, §19.1)

[19] The size of this error can be roughly estimated by applying the finite differences solution method used here to the steady-state solution, which is known in closed form.

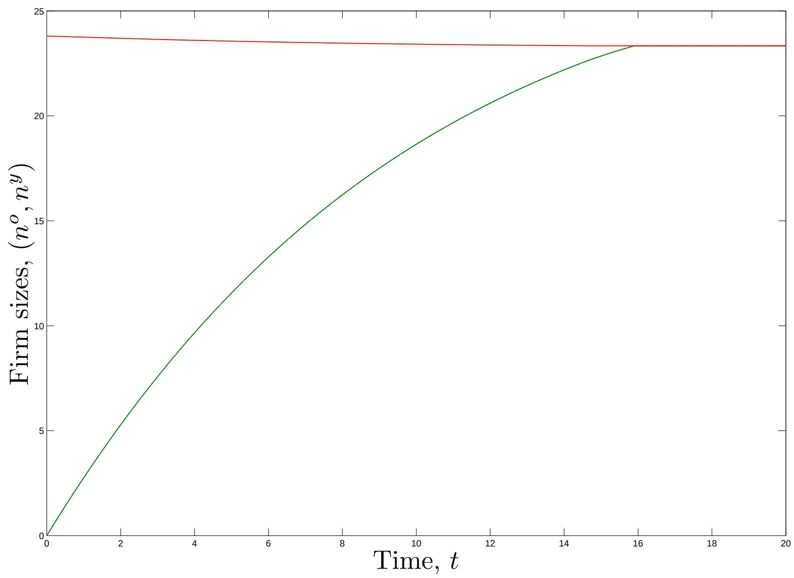

Figure 6: Time path for firm sizes

be viewed as approximate solutions. In future versions, we will use different numerical algorithms to improve the accuracy of our solutions and thus hope to arrive at better estimates of the equilibrium path of the key variables.

With these caveats in mind, Figure 6 shows the time path for the firm size distribution in response to the 1% increase in productivity. Newly entering firms start at zero employment and then increase the size of their workforces, while existing firms shrink slowly from their initial employment level towards the lower firm size of the new steady state. Both of these time paths are monotonic. Adjustment to the new steady state takes around 16 quarters.

Figure 7 shows the associated paths of the hiring policies of newly entering and incumbent firms. It suggests that the extensive and the intensive margins of employment move in opposite directions starting immediately after the shock. In particular, new firms set $h = 1$ until they reach the new steady-state employment level, n^*, while incumbents choose a level of h strictly between 0 and 1 (i.e., makes between posting and not posting vacancies). This enables them to track the behavior of $n^*(t)$ rather than expanding their employment while $n^*(t)$ declines or overshooting this threshold level. Our calibration also shows that there is a nonmonotonicity in this curve near $t = 16$, but we suspect that this is simply an artefact of the solution method and numerical errors. The fact that employment at incumbent firms falls monotonically to the new steady state appears relatively robust. This suggests that the intensive and extensive

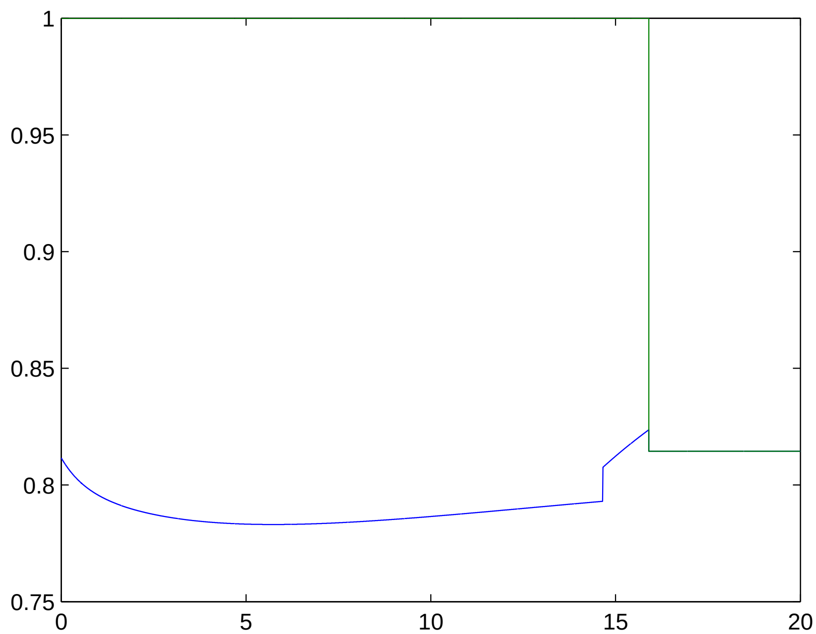

Figure 7: Time path for hiring policies

margins of adjustment to the new steady state both move monotonically.

Figure 8 and Figure 9 show the evolution of the unemployment rate and of the tightness of the labor market. Consistent with the opposite movements of the intensive and the extensive margins, the short-run impact of the productivity shock appear to be no larger than the medium-run impact. Figure 8 indicates that unemployment is higher after a few months than either before the shock or in the new steady state. However, this again seems to be related to numerical errors in the solution method. Variations on the baseline do not yield this result.

It should be emphasized that the numerical solution procedure is fragile, and so the results should be viewed as tentative and preliminary.[20] In further work, we will provide further characterization for the system of partial differential equations and also improve the accuracy of the numerical approximation procedure. We thus hope to obtain more reliable estimates of the intensive and the extensive margins of employment creation and the short-run responses of unemployment and labor market tightness to productivity shocks.

[20] To illustrate the severity of this problem, note that we have been unable to test the maintained assumption that all entry occurs at time zero.

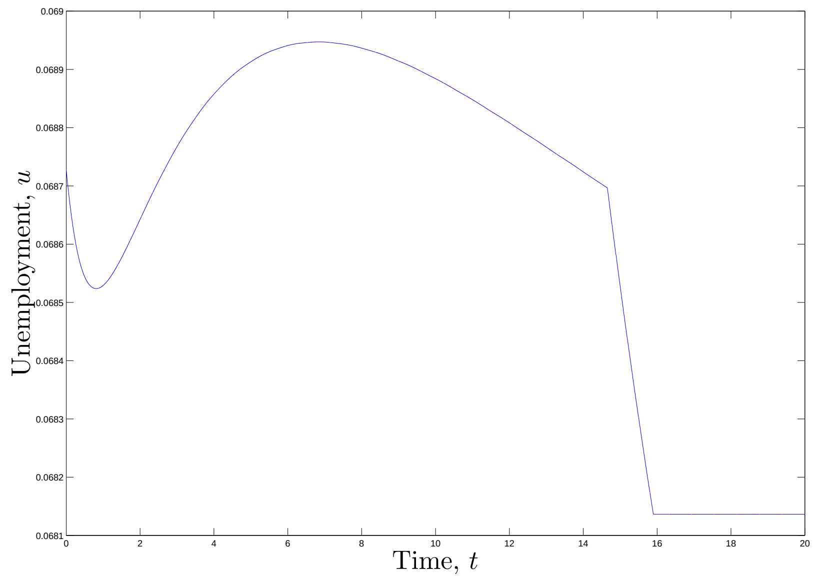

Figure 8: Time path for unemployment

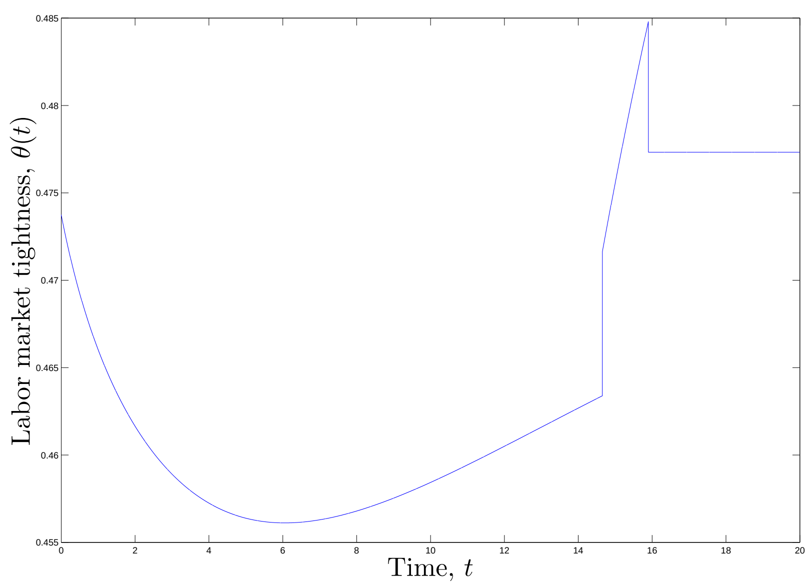

Figure 9: Time path for θ

7 Conclusion

This paper presented a generalization of the standard Dimond-Mortensen-Pissarides (DMP) search model of unemployment, in which there is both an intensive and an extensive margin of employment creation. In our model, each firm can invest to gain access to a production technology with diminishing returns to labor and then post vacancies in order to recruit workers. Entry by new firms corresponds to the extensive margin of employment creation, while job creation by existing firms captures the intensive margin. As in the baseline Diamond-Mortensen-Pissarides search model and theories of the firm developed by Stole and Zwiebel (1996a,b) and Wolinsky (2000), wages are determined by continuous bargaining between the firm and its employees.

We first characterized certain properties of the equilibrium in a baseline model where each firm can hire a countable number of workers. Although this model is a natural generalization of the standard DMP model, the fact that the choice variable of each firm is a discrete variable makes its analysis difficult. We therefore provided a more complete characterization of equilibrium and the proof of existence of a steady-state equilibrium in the limit economy, where the size of each worker becomes infinitesimally small. Another advantage of this limit economy is that the steady-state equilibrium can be characterized by a set of differential equations, which admits closed-form solutions. Consequently, despite the presence of general equilibrium interactions and forward-looking bargaining between the firm and multiple workers, the equilibrium takes a relatively tractable form.

After providing some simple comparative static results, we also undertook a simple calibration of our baseline economy to investigate how unemployment, wages and firm size change as productivity increases, and whether the intensive and the excessive margins of employment move in the same or in opposite directions over different horizons. Steady-state comparisons are relatively straightforward to compute and show results consistent with our characterization and comparative static results. One interesting finding is that an increase in productivity reduces unemployment, but also leads to smaller equilibrium firm sizes. Therefore, comparing steady states, we see intensive and the extensive margins of employment to move in opposite directions. This seems to be because the greater outside option of workers and the higher cost of filling vacancies discourage hiring by existing firms in the new steady state. We also investigated the equilibrium dynamics of our economy, which is represented by a system of partial differential equations. These partial differential equations do not admit closed-form solutions and are also difficult to investigate numerically. Our preliminary solutions show some tentative patterns, whereby even in the short run the intensive and the extensive margins of employment creation move in opposite directions, and the short-run responses of employment and wages to productivity shocks is not appreciably larger than those suggested by steady-state comparisons.

8 Appendix A: Omitted Proofs

Proof of Lemma 3. Define

$$\psi(n) = \frac{1}{n}\left[F(n) - \frac{1}{n}\int_0^n F(m)\,dm\right],$$

then $\psi(n) > 0$, $\psi'(n) < 0$, and $\lim_{n\to 0^+}\psi(n) = +\infty$ and $\lim_{n\to\infty}\psi(n) = 0$. To see these facts, observe first that $n^2\psi(n) = \int_0^n [F(n) - F(m)]\,dm > 0$ since F is strictly increasing. Next, observe that

$$\frac{n^3}{2}\psi'(n) = \int_0^n F(n)\,dn - nF(n) + \frac{n^2}{2}F'(n),$$

whereas from a third-order Taylor expansion for $n \mapsto \int_0^n F(m)\,dm$ at 0 about n, we have that for some $\nu \in (0,n)$,

$$0 = \int_0^n F(n)\,dn - nF(n) + \frac{n^2}{2}F'(n) - \frac{n^3}{6}F''(\nu);$$

since $F''(\nu) < 0$, it follows that $\psi'(n) < 0$ for all $n > 0$. Next, a similar second-order Taylor expansion implies that for all $n > 0$,

$$\psi(n) = \frac{1}{2}F'(\nu)$$

for some $\nu \in (0,n)$; it follows from the Inada condition that $\lim_{n\to 0^+}\psi(n) = +\infty$. Finally, that $\lim_{n\to\infty}\psi(n) = 0$ follows from observing that $0 < \psi(n) < \frac{1}{n}F(n) \to 0$ as $n \to \infty$, together with the squeeze principle.

The results concerning the profit function follow immediately from observing that

$$\pi(n) = \frac{1}{n}\int_0^n F(m)\,dm - \frac{n}{2}rV^u.$$

It follows that $\pi'(n) = \psi(n) - \frac{1}{2}rV^u$, from which it follows immediately that π'' is strictly concave. The remaining claims follow from substituting $n = 0$ in the formula for $\pi(\cdot)$, and from observing that

$$\lim_{n\to\infty}\pi'(n) = \lim_{n\to\infty}\psi(n) - \frac{1}{2}rV^u = -\frac{1}{2}rV^u < 0.$$

□

Proof of Lemma 4. First, substituting from (34) into (15) establishes that for $0 < n < n^*$, the firm's value function $J(\cdot)$ satisfies

$$(q(\theta) - sn)J'(n) = (r+\delta)\,J(n) + \gamma + \frac{n}{2}rV^u - \left[\frac{1}{n}\int_0^n F(m)\,dm\right]. \tag{57}$$

If $q(\theta) - sn^* > 0$, then the function $n \mapsto \frac{r+\delta}{q(\theta)-sn}$ satisfies a Lipschitz condition on $[0, n^*]$; it follows from Picard's theorem and associated results on ordinary differential equations that there is a unique solution to (57) on $(0, n^*)$ in this case. This solution is given by

$$J(n) = (q - sn)^{-\frac{r+\delta}{s}} \left[K + \int_0^n (q - sn)^{\frac{r+\delta}{s} - 1} \left(\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right) dm \right] \tag{58}$$

where K is a constant of integration. It follows from the free entry condition (18) that $K = q^{\frac{r+\delta}{s}} k$. $\square$

Proof of Lemma 5. The proof is analogous to that of Lemma 4. Write $T(n) = V(n) - V^u$. Then the worker's Bellman equation (19) is

$$(r + s + \delta) T(n) = w(n) + (q - sn) T'(n).$$

A boundary condition is given by the fact that at n^*, the worker is paid $w(n^*)$ until the job ends (with flow probability $\delta + s$); this implies that $V(n^*)$ satisfies

$$rV(n^*) = w(n^*) + (s + \delta)[V^u - V(n^*)],$$

or, since $w(n^*) = rV^u + (r + \delta + s)\gamma/q$,

$$T(n^*) = \frac{\gamma}{q}.$$

The usual integration argument shows that the unique closed form solution for $T(\cdot)$ is as given in the statement of the Lemma. $\square$

Proof of Theorem 1. Define two functions $\chi, \omega : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$\chi(q, rV^u) = J(n^*) \left(\frac{q - sn^*}{q} \right)^{\frac{r+\delta}{s}} - \frac{1}{q^{\frac{r+\delta}{s}}} \int_0^{n^*} (q - sm)^{\frac{r+\delta}{s} - 1} \left[\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right] dm \tag{59}$$

$$\omega(q, rV^u) = rV^u - b - \theta q \left[-V^u + \frac{\int_0^{n^*} V(n) g(n)\, dn + \frac{sn^*}{q} V(n^*) G^*}{1 - \left(1 - \frac{sn^*}{q}\right) G^*} \right] \tag{60}$$

$$\begin{aligned} &= rV^u - b - \theta q \int_0^{n^*} \left\{ \frac{\gamma}{q} \left(\frac{q - sn^*}{q - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q-sn^*}{q-sn} \right)^{\frac{r+\delta+s}{s}}}{2(r + \delta + s)} rV^u \right. \\ &\qquad \left. + (q - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q - sm)^{\frac{r+\delta}{s}} \psi(m)\, dm \right\} d\hat{G}(n) \end{aligned} \tag{61}$$

where n^* and $J(n^*)$ are defined as in Proposition 1, according to (36) and (37), and where $\hat{G}(\cdot)$ is the probability measure with continuous density

$$\frac{g(n)}{1-\left(1-\frac{sn^*}{q}\right)G^*}$$

on $[0, n^*]$ and an atom of mass sn^*G^*/q at n^*. Note that $\chi(q, rV^u)$ represents the maximum value of an entrant firm that takes q and rV^u as given and expects to pay wages $w(\cdot)$ as given by (34); $rV^u - \omega(q, rV^u)$ is the value of an unemployed worker in an economy populated by such firms. As observed in the discussion preceding Proposition 1, (q, rV^u) is part of an equilibrium allocation iff

$$k = \chi(q, rV^u) \tag{62}$$

$$0 = \omega(q, rV^u). \tag{63}$$

We will show that such an intersection exists by first observing that (62) defines a continuous 1-manifold in $\mathbb{R}^+ \times \mathbb{R}$, then observing that ω restricted to this manifold defines a continuous function, showing that it takes positive and negative values, and applying the intermediate value theorem.

To see that (62) defines a continuous 1-manifold, first note that $\chi(q, rV^u)$ is nondecreasing in q and nonincreasing in rV^u, with both relationships being strict if there is positive activity (that is, if it is optimal for a firm with zero workers to hire). This follows immediately from the definition of $\chi(q, rV^u)$ as the maximal value of the problem for the firm as described. First, clearly if rV^u increases, then for any q, $\chi(q, rV^u)$ must decrease, since if the firm keeps the same hiring strategy as before, then it would increase the value of its program as $w(n)$ decreases for each n; reoptimizing the hiring strategy can only increase this effect. Second, if q increases to $q' > q$, then the firm could keep the value of its program the same by keeping the same cutoff n^* but posting a vacancy with probability $q/q' < 1$ for each $n < n^*$. Moreover, for each $q > 0$, it's clear that there is a unique $rV^u \in \mathbb{R}$ such that $\chi(q, rV^u) = k$. Finally, since $\chi(\cdot)$ is continuous, it follows that (62) defines a continuous curve in $\mathbb{R}^+ \times \mathbb{R}$. Call this curve C. Since it's clear that $\omega|_C$ is continuous (since ω itself is continuous), an equilibrium will exist iff we can find points $(q_1, v_1), (q_2, v_2) \in C$ such that $\omega(q_1, v_1)$ and $\omega(q_2, v_2)$ differ in sign.

To do this, first define $\bar{v}$ to solve

$$k = \frac{1}{r+\delta} \max_{n>0} \left[\frac{1}{n}\int_0^n F(m)\, dm - \frac{n}{2}v\right].$$

It therefore follows that $\lim_{q\to\infty} \chi(q, \bar{v}) = 0$. Also, for $v > \bar{v}$, $\chi(q, v) < k$ by construction.

Now, if $q \to \infty$, then any firm will instantaneously hire n^*; thus in the limit, $J(n) = J(n^*)$ for all $n \in [0, n^*]$. From the definition of $\omega(\cdot)$, it follows that in this case, $\omega(q, \bar{v}) = \bar{v} - b$.

In the other extreme, let $\hat{q} > 0$ satisfy $\chi(\hat{q}, b) = 0$; such a $\hat{q}$ will exist provided that $\bar{v} > b$. Suppose that this is true. Then it is clear from the definition of $\bar{v}$ that $n^*(\hat{q}, b) > 0$. Also, by definition,

$$\omega(\hat{q}, b) = -\theta q \int_0^{n^*} \left[-V^u + V(n)\, dG(n)\right].$$

If q is finite, then $\theta q > 0$, and $\hat{G}(\cdot)$ is a probability measure that places positive measure on every subset of $[0, n^*]$ of positive Lebesgue measure, while $V(n) - V^u > 0$ for each $n > 0$. Thus the only possibilities are that $q = +\infty$ (which is impossible since $\bar{v} \neq b$), or that $\omega(\hat{(q)}, b) < 0$. Thus if $\bar{v} - b > 0$, then C contains points at which ω takes values of opposite signs, which completes the proof of the existence of an equilibrium via the intermediate value theorem.

If $\bar{v} - b \leq 0$ then it is trivial to prove that there is an equilibrium in which no firm ever enters. $\square$

9 Appendix B: Heuristic Argument For the Smooth Pasting Condition

Here we present a brief heuristic argument for the form of the boundary condition for the optimal $n^*(t)$, the 'super-contact' condition $J_{nn}(n^*(t), t) = 0$.

Consider the case of a firm whose current employment level is n_0, while the target level of employment at which the firm stops hiring is $n^*(t)$. Assume that $n^*(0) = n_0^*$ and that $\frac{dn^*(t)}{dt} = v + O(t)$, with $v > -s$. In this case, the optimum policy for a firm with $n_0 > n_0^*$ workers is not to hire at all and allow its workforce to fall by attrition at rate sn. (The assumption that $v > -s$ assures that the time for the workforce to fall to $n^*(t)$ occurs at some t such that the first-order approximation $n^*(t) \approx n_0^* + vt$ is valid.) Then in this case, we can deduce that since

$$n(t) = n_0 e^{-st}$$

it follows that the time taken for $n(t)$ to fall to $n^*(t)$ is the time τ satisfying that

$$n_0 e^{-s\tau} = n_0^* + v\tau$$

or

$$n_0^* + \delta n = (n_0^* + v\tau) e^{s\tau}$$

where $\delta n = n_0 - n_0^*$. Brute force calculation (optionally using the Lagrange inversion formula for power series) shows that

$$\tau = \frac{\delta n}{v + s n_0^*} - \frac{\delta n^2}{2} \frac{s^2 n_0^* + 2vs}{(v + s n_0^*)^3} + O(\delta n^3).$$

Now, in order to write an expression for $J(n, t)$ for such n, introduce the notation that the flow net profit of a firm, ignoring vacancy posting costs (that is, production less wages) is denoted $\phi(n)$. Then we can write

$$J(n_0, 0) = \int_0^\tau e^{-(r+\delta)t} \phi(n(t))\, dt + e^{-(r+\delta)\tau} J(n^*(\tau), \tau), The$$

while

$$J(n_0^*, 0) = \int_0^\tau e^{-(r+\delta)t} \left[\phi(n^*(t) - \frac{\gamma}{q}(v + s n^*(t)) \right] dt + e^{-(r+\delta)\tau} J(n^*(\tau), \tau).$$

Subtracting gives that

$$\begin{aligned}
J(n_0, 0) - J(n_0^*, 0) &= \int_0^\tau e^{-(r+\delta)t} \left[\phi(n(t)) - \phi(n^*(t)) + \frac{\gamma}{q}(v + s n^*(t)) \right] dt \\
&= \int_0^\tau e^{-(r+\delta)t} \left[\phi'(n) \delta n \left(1 - \frac{t}{\tau} \right) + \frac{\gamma}{q}(v + s(n_0^* + vt)) \right] dt + O(\tau^3) \\
&= \left[\delta n \phi'(n) + \frac{\gamma}{q}(v + s n_0^*) \right] \int_0^\tau e^{-(r+\delta)t}\, dt + \left[\frac{\gamma s v}{q} - \frac{\delta n \phi'(n)}{\tau} \right] \int_0^\tau t e^{-(r+\delta)t}\, dt + O(\tau^3) \\
&= \left[\delta n \phi'(n) + \frac{\gamma}{q}(v + s n_0^*) \right] \left[\tau - \frac{r+\delta}{2} \tau^2 \right] + \left[\frac{\gamma s v}{q} - \frac{\delta n \phi'(n)}{\tau} \right] \left[\frac{\tau^2}{2} - \frac{(r+\delta)\tau^3}{3} \right] + O(\tau^3) \\
&= \frac{\gamma}{q} \delta n - \frac{\delta n^2}{2(v + s n_0^*)} \left[\phi'(n) - \frac{\gamma}{q}(r + \delta + s) \right] + O(\delta n^3),
\end{aligned}$$

where some algebra has been omitted, first substituting in for τ in terms of δn, and then simplifying in the last line. It follows that $J_n(n_0^*, 0) = \frac{\gamma}{q}$, the familiar condition that the firm should stop hiring once the marginal benefit of doing so equals the hiring cost. In addition, $J_{nn}(n_0^*, 0) = 0$ if and only if $\phi'(n) - \frac{\gamma}{q}(r + \delta + s) = 0$.

Next, let us consider whether a firm would like to deviate from this plan. Consider a firm that has a workforce of n_0^* workers at time 0. For n_0^* to be optimal, it must be that the firm wants to set its workforce equal to $n_*(t)$ at all $t \geq 0$ (note that this assumes that $n^*(t)$ never moves fast enough that this is actually feasible for the firm, something that isn't obvious a priori, though it is true in the calibrated examples we consider). The obvious deviation to consider is that the firm increases its hiring by a discrete amount for a small unit of time τ, then allows its workforce to fall by attrition back to the optimal level at some $t > \tau$. Without loss of generality we can assume that the firm increases its hiring from the level required to keep on the path of $n^*(t)$, which is $v + s n^*(t)$, by some level $\delta h > 0$ such that

$-sn^*(t) < v + sn^*(t) + \delta h \leq q$, for a duration $\tau > 0$. (Think of δh as 'large' and τ as 'small,' despite the notation. The inequality restriction on δh implies that hiring this amount is feasible for the firm.) Then, from time τ to time $\tau + \tau' > \tau$, the firm reduces its hiring by δh below the level required had it kept on the path of $n^*(t)$. (This will require that also $-sn^*(t) < v + sn^*(t) - \delta h \leq q$, at least for τ sufficiently small.)

To calculate the benefit to the firm of this strategy, we need first to calculate the path of the variable $\hat{n}(t)$, defined by $\hat{n}(t) = n(t) - n^*(t)$. We know that

$$\hat{n}_t(t) = \begin{cases} \delta h - s\hat{n} & t < \tau \\ -\delta h - s\hat{n} & \tau < t < \tau + \tau' \end{cases}$$

so that

$$\hat{n}(t) = \begin{cases} \frac{\delta h}{s}\left(1 - e^{-st}\right) & t < \tau \\ \frac{\delta h}{s}\left(-1 + (2 - e^{-s\tau})e^{-s(t-\tau)}\right). \end{cases}$$

Since τ' is defined by $\hat{n}(\tau + \tau') = 0$, it follows that

$$e^{s\tau'} = 2 - e^{-s\tau}$$

or, solving up to second order,

$$\tau' = \tau - s\tau^2 + O(\tau^3).$$

Next, taking a first order Taylor expansion of $\phi(\cdot)$ about n_0^*, we can calculate the benefit to the firm from this strategy as equal to

$$\int_0^\tau e^{-(r+\delta)t}\left[\phi'(n_0^*)\hat{n}(t) - \frac{\gamma\delta h}{q}\right] dt + e^{-(r+\delta)\tau}\int_0^{\tau'} e^{-(r+\delta)s}\left[\phi'(n_0^*)\hat{n}(s+\tau) + \frac{\gamma\delta h}{q}\right] ds + O(\tau^3).$$

Substitute for

$$\hat{n}(t) = \begin{cases} \delta h t + O(\tau^2) & 0 < t < \tau \\ \delta h\left(\tau - (t - \tau)\right) + O(\tau^2) & \tau < t < \tau + \tau' \end{cases}$$

and write first-order expansions of the other terms under the integrals to get that the benefit is equal to

$$\begin{aligned} B = &\int_0^\tau (1 - (r+\delta)t)\left[\phi'(n_0^*)\delta h t - \frac{\gamma\delta h}{q}\right] dt \\ &+ (1 - (r+\delta)\tau)\int_0^{\tau'} (1 - (r+\delta)s)\left[\phi'(n_0^*)\delta h\left(\tau - s\right) + \frac{\gamma\delta h}{q}\right] ds + O(\tau^3) \\ = &\left[-\tau\frac{\gamma\delta h}{q} + \frac{1}{2}\phi'(n_0^*)\delta h\tau^2 + \frac{1}{2}(r+\delta)\frac{\gamma\delta h}{q}\tau^2\right] \\ &+ \left[\tau\frac{\gamma\delta h}{q} + \tau^2\delta h\left(\frac{1}{2}\phi'(n_0^*) - \frac{\gamma}{q}\left[s + \frac{3}{2}(r+\delta)\right]\right)\right] + O(\tau^3) \\ = &\,\delta h\tau^2\left[\phi'(n_0^*) - \frac{\gamma}{q}(r+\delta+s)\right] + O(\tau^3). \end{aligned}$$

Since δh can be positive or negative, it is therefore a necessary condition that the term in brackets equal zero. That is, $\phi'(n_0^*) = \frac{\gamma}{q}(r + \delta + s)$, or $J_{nn} = 0$.

References

Acemoglu, D., P. Antrás, and E. Helpman (2005): "Contracts and the Division of Labor," NBER Working Paper 11356, National Bureau of Economic Research, Cambridge, MA.

Acemoglu, D., and R. Shimer (1999a): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107(5), 893–928.

——— (1999b): "Holdups and Efficiency with Search Frictions," *International Economic Review*, 40(4), 827–49.

Andolfatto, D. (1996): "Business Cycles and Labor-Market Search," *American Economic Review*, 86(1), 112–132.

Aumann, R. J., and L. S. Shapley (1974): *Values of Non-Atomic Games*. Princeton University Press, Princeton, NJ.

Bertola, G., and R. J. Caballero (1994): "Cross-Sectional Efficiency and Labor Hoarding in a Matching Model of Unemployment," *Review of Economic Studies*, 61(3), 435–456.

Bertola, G., and P. Garibaldi (2001): "Wages and the Size of Firms in Dynamic Matching Models," *Review of Economic Dyanmics*, 4(2), 335–368.

Burdett, K., S. Shi, and R. Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), 1060–1085.

Cahuc, P., F. Marque, and E. Wasmer (2005): "A Theory of Wages and Labor Demand with Intrafirm Bargaining and Matching Frictions," unpublished.

Cahuc, P., and E. Wasmer (2001): "Labour Market Efficiency, Wages and Employment when Search Frictions Interact with Intrafirm Bargaining," Discussion Paper 2812, Center for Economic Policy Research, London.

Davis, S. J., J. C. Haltiwanger, R. Jarmin, and J. Miranda (2006): "Volatility and Dispersion in Business Growth Rates: Publicly Traded versus Privately Held Firms," to appear in 2006 NBER Macroannual.

Davis, S. J., J. C. Haltiwanger, and S. Schuh (1996): *Job Creation and Destruction*. MIT Press, Cambridge, MA.

Diamond, P. A. (1982a): "Aggregate Demand Management in Search Equilibrium," *Journal of Political Economy*, 90(5), 881–894.

——— (1982b): "Wage Determination and Efficiency in Search Equilibrium," *Review of Economic Studies*, 49(2), 217–27.

Dixit, A. K. (1993): *The Art of Smooth Pasting*, vol. 55 of *Fundamentals of Pure and Applied Economics*. Routledge, Abingdon, Oxon.

Hagedorn, M., and I. Manovskii (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies Revisited," unpublished, University of Pennsylvania.

Hall, R. E. (2005): "Employment Fluctuations with Equilibrium Wage Stickiness," *American Economic Review*, 95(1), 50–65.

Hawkins, W. B. (2006a): "Competitive Search, Efficiency, and Multi-worker Firms," MIT Ph.D. thesis, Chapter 1.

——— (2006b): "Fluctuations and Multi-worker Firms under Competitive Search," MIT Ph.D. thesis, Chapter 2.

Merz, M. (1995): "Search in the labor market and the real business cycle," *Journal of Monetary Economics*, 36(2), 269–300.

Moen, E. R. (1997): "Competitive Search Equilibrium," *Journal of Political Economy*, 105(2), 385–411.

Montgomery, J. D. (1991): "Equilibrium Wage Dispersion and Interindustry Wage Differentials," *Quarterly Journal of Economics*, 106(1), 163–79.

Mortensen, D. T. (1982): "Property Rights and Efficiency in Mating, Racing, and Related Games," *American Economic Review*, 72(5), 968–79.

Mortensen, D. T., and C. A. Pissarides (1994): "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, 61(3), 397–415.

Peters, M. (1991): "Ex Ante Price Offers in Matching Games: Non-steady States," *Econometrica*, 59(5), 1425–54.

Petrongolo, B., and C. A. Pissarides (2001): "Looking into the Black Box: A Survey of the Matching Function," *Journal of Economic Literature*, 39(2), 390–431.

Pissarides, C. A. (1984): "Efficient Job Rejection," *Economic Journal*, 94(Supplement: Conference Papers), 97–108.

——— (1985): "Short-Run Equilibrium Dynamics of Unemployment, Vacancies, and Real Wages," *American Economic Review*, 75(4), 676–690.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery (1992): *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, Cambridge, UK, second edn.

Shapley, L. S. (1953): "A value for n-person games," in *Contributions to the Theory of Games II*, ed. by H. W. Kuhn, and A. W. Tucker, vol. 28 of *Ann. Math. Studies*, pp. 307–317. Princeton University Press, Princeton, NJ.

Shimer, R. (1996): "Contracts in a Frictional Labor Market," Ph.D. thesis, MIT, Chapter 1.

——— (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 95(1), 25–49.

Smith, E. (1999): "Search, Concave Production and Optimal Firm Size," *Review of Economic Dynamics*, 2(2), 456–471.

Stokey, N. L., R. E. Lucas, Jr., and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.

Stole, L. A., and J. Zwiebel (1996a): "Intra-firm Bargaining under Non-binding Contracts," *Review of Economic Studies*, 63(3), 375–410.

——— (1996b): "Organizational Design and Technology Choice under Intrafirm Bargaining," *American Economic Review*, 86(1), 195–222.

Wolinsky, A. (2000): "A Theory of the Firm with Non-Binding Employment Contracts," *Econometrica*, 68(4), 875–910.