UNOBSERVABLE HETEROGENEITY IN DIRECTED SEARCH

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ABSTRACT. This paper provides a directed search model designed to explain the residual part of wage variation left over after the impact of education and other observable worker characteristics has been removed. Workers have private information about their characteristics at the time they apply for jobs. Firms can observe these characteristics once workers apply, and hire the worker with the characteristic that they like. The paper focuses on the case in which firms aren't able to condition their wage offers on these characteristics. The paper shows how to extend directed search arguments to deal with arbitrary distributions of worker and firm types. The paper then illustrates how data on the relationship between exit wage and unemployment duration can be used to identify the unobserved distributions of worker and firm types. The model also has testable predictions. For example, certain easily checked properties of the offer distribution of wages imply that workers who are hired by the highest wage firms should also be the workers who have the shortest unemployment duration. This is in strict contrast to the usual directed search story in which high wages are always accompanied by higher probability of unemployment.

1. INTRODUCTION

One shortcoming of most directed search models compared to models with purely random matching is the fact that they are based on strong symmetry assumptions. Even papers that are explicitly designed to allow for differences among traders (for example Shimer (2005) and Shi (2002)) restrict themselves to distributions of wages and types with finite support. At a minimum, this makes it difficult for them to match with econometric data that tends to involve continuous distributions. Furthermore, the empirical content of these models applies to the relationship between observables. For example, workers with more education will receive higher wages. Apart from differences in these observables, all traders are assumed to be the same, leaving the models mute about the large part of the variation in wages that is apparently unrelated to observables.

The purpose of this paper is to provide an alternative version of the directed search model that can be used to explain this residual variation. In the model, this variation is due to unobservable worker characteristics, and to the (equally unobservable) value of these characteristics to firms. The empirical content of the model comes from the relationship it establishes between the offer distribution of wages and the relationship between wages at which workers leave unemployment and their average search duration.[1] Addison, Centeno, and Portugal (2004) present evidence to suggest that exit wages and unemployment duration are negatively correlated. In a more standard directed search terminology, this means that employment probability seems to be positively correlated with wage. The evidence is not strong, but it is striking that it provides no support at all for the classic prediction of directed search - that workers who apply at high wage firms will have a lower probability of employment.

It is shown in the model below that any systematic tendency in this data is tied to the properties of the wage offer distribution. The relationship between exit wage and unemployment duration is driven by two considerations. The first is completely intuitive - higher wage firms will tend to hire workers whose quality is higher, and these workers will tend to be more likely to find jobs no matter where they apply. This is confounded by the possibility that higher quality workers will tend to apply where there are a lot of other high quality workers. This is where the directed search model plays a role since it ties down the application strategy for workers of different qualities. The characterization of the equilibrium application strategy provided below makes it possible to provide a readily checked property of the offer distribution that will determine the relationship between exit wage and duration. Assuming that the wage offer distribution is has the usual skewed (log normal) shape, it will support an inverse relationship between exit wage and duration provided the density doesn't decline too rapidly to the right of its peak.

A second phenomena that the model in this paper can be used to address is duration dependence (Machin and Manning (1999) or Addison, Centeno, and Portugal (2004)). Workers who have been unemployed for a long time tend to wait longer for a job offer than workers who

[1]The model in the paper is static, so the actual relationship is between the probability of employment and the wage at which a worker is hired. In the steady state of such a model, the expected duration of unemployment is just the reciprocal of the probability of employment.

are newly unemployed. Directed search models typically assume that all workers are the same or differ only in ways that are observable to an outsider. Workers apply to high wage firms only because their equilibrium mixed strategies require them to do so with some positive probability. When workers' application behavior is driven by an underlying characteristic, workers who adopt a risky application strategy in one period will tend to persist in this behavior. Workers who have been unemployed for a long time are more likely to be workers whose types support a risky application strategy. Duration dependence is then simply a consequence of the workers underlying characteristic and does not reflect any kind of discouragement effect. Again, the model below captures this phenomenon in a simple way.

Finally, the model below illustrates how data on the wage distribution and the relationship between exit wage and unemployment duration can be used to attribute wage variation to either variation in the unobservable characteristics of workers, or to variation in firms valuation of these characteristics.

Most of the paper is concerned with the case where there is a continuum of workers and firms. However, the basic logic of directed search involves mixed application strategies where workers apply with higher probability at higher wage firms. It isn't at all clear what mixed application strategies mean when there are a continuum of different firms to choose from all offering different wages. To get some insight into this process the paper starts with the analysis of the equilibrium of the application sub-game that occurs after a finite number of firms have posted their wages for a finite number of workers. By taking limits of the equilibrium payoffs as the number of firms and workers grow large, this approach defines payoffs in a continuum model in which workers and firms best reply to a distribution of wage offers, and a *reservation wage* application strategy for workers. The payoff functions defined by these limits define a large game that captures the logic of directed search when there is a continuum of different traders on each side of the market. The construction of this large game is one of the central contributions of the paper. The equilibrium arguments are based on an adaptation of the argument in Peters and Severinov (1997), and resemble the mixed equilibrium that were characterized in Shi (2002) and Shimer (2005), albeit under much different assumptions. There is a unique symmetric equilibrium in which workers randomize over the wages at which they submit applications. In any finite directed search game, this application strategy is conceptually straightforward, but complex since it involves a potentially large number of different application probabilities for each of the different wages being offered in equilibrium. The paper shows the sense in which this application strategy converges to a simple reservation wage rule similar to the one in Shimer (2005) as the number of traders becomes large.

After a brief digression about the relationship between wage and employment probability in the standard directed search model, the paper begins with a description of search equilibrium with a finite number of firms and workers. This section is an attempt to motivate the reservation wage strategy that workers use in the large game. The paper then presents a basic set of limit theorems that are used to define the payoff functions in the large search game. The equilibrium in the large search game is characterized in Section 4, and the relationship between wage offer distributions and search duration is analyzed. The final section concludes. Detailed proofs of the limit theorems are contained in an appendix.

2. Directed Search

Most directed search models compute payoffs by assuming that a single deviator offers a wage that differs from the wage offered by all the others (for example, Burdett, Shi, and Wright (2001)). The symmetry among the non-deviating firms is what makes the calculation possible. This approach makes it tough to deal with even the simplest wage distributions. To illustrate how directed search can be modified to deal with differences among firms, a brief digression to explain a theorem in Peters (2000) helps to illustrate what is different in the model here.

That paper begins with a finite model in which firms offer wages and workers choose application strategies that maximize their expected wage. There are many different firms who play a Nash equilibrium in wages, understanding the way that the Nash equilibrium in workers' application strategies depends on the wages they set. The theorem in the paper shows that as the number of firms and workers participating in the market gets large, there is a payoff μ such that any firm who offers a wage w while all other firms play their Nash equilibrium strategies receives payoff that converges to

$$(1 - e^{-k})(P - w) \tag{1}$$

where k satisfies

$$\frac{1 - e^{-k}}{k} w = \mu \tag{2}$$

and P is the profit the firm earns when it hires a worker (which is exogenous and differs across firms).

Equation (2) is the simple part of the theorem. It puts a specific form on the relationship between the matching probability and the wage at which a worker applies. The deeper part of the theorem is (1) which relates the probability with which firms fill vacancies to the probability with which workers match. The important point is that the k that appears in (1) is the same as the one in (2). This means that when the number of workers and firms is large, every wage that is offered in every Nash equilibrium must be close to the wage that maximizes (1) for some firm type constrained by (2). Then if we know the types of the firms, we could find the equilibrium distribution of wages by finding the distribution of solutions to this problem for a given μ, then adjusting μ until the average value of k associated with wages across the distribution is equal to the actual ratio of workers to firms in the market.

Conversely, given a distribution of wages, one could work backwards and find the distribution of firm types that would support the observed distribution as an equilibrium distribution.[2] To find the preferences of these firms it is only necessary to have information about the relationship between wages and unemployment probability, which can be estimated from the duration data. In the model above, wage and employment probability must vary in a way that traces out a worker's indifference curve. Firms preferences at each wage are then found by finding types whose indifference curves are tangent to this worker indifference curve. In particular, this argument makes it clear that wage and employment probability must be inversely related.

Employment probability isn't observable, and can only be guessed by looking at duration. Assuming some steady state dynamic version of the directed search model in which workers independently carry out their application randomizations in every period, two predictions emerge. First, workers who leave unemployment quickly will tend to be those who apply to low wage firms. All workers are more likely to apply to high wage firms, so those who exit after long duration will tend to leave at high wages. So duration and exit wage should be directly related. Second, a worker who has been unemployed for a long time will use the same application strategy as a worker who is newly unemployed. So there should be no duration dependence.

[2]This structural procedure is generally impossible in models with a finite set of types, since most wage distributions cannot be explained with a finite set of types.

It should be apparent that both these outcomes result largely from the fact that workers with the same observable characteristics are assumed to be identical. The model that follows is designed primarily to relax this assumption.

3. Fundamentals

A labor market consists of sets M and N of firms and workers respectively. To begin, assume these sets are finite and consist of m firms and n workers with $n = \tau m$. Each worker has a characteristic y contained in a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^+$. These characteristics are observable to firms once workers apply, but initially, they are private information. When M and N are finite, it will be assumed that each worker's characteristic is independently drawn from a distribution F. The distribution F is assumed to be differentiable and monotonically increasing and to satisfy the property that $\frac{F'(y)}{1-F(y)}$ is uniformly bounded.[3] It is assumed impossible for firms to reward this characteristic directly. A worker's payoff is simply the wage he receives. Workers are risk neutral.

Firms differ, but their characteristics are common knowledge.[4] Each firm j has a single job that it wants to fill. It chooses the wage that it wishes to pay the worker who fills this job. Each firm's wage is chosen from a compact interval $W \subset \mathbb{R}^+$. Payoffs for firms depend on the wage they offer and on the characteristic of the worker they hire. The payoff for firm j is $v_j : W \times Y \to \mathbb{R}$. It is assumed that v_j is jointly continuous, that the family $v_j(\cdot, y)$ is an equi-continuous family of functions from Y into $\mathbb{R}$, and that the derivative of v with respect to y is bounded for all w.

The Bayesian game that determines wages and matches starts with firms simultaneously choosing their wages. After observing the wage offers each worker applies to one and only one firm. Once applications are made to the firms, each firm chooses to hire the worker who applies to it who has the highest characteristic. Since all workers are in some sense equally well qualified for the jobs that firms offer, we assume that the firm does not have the option of refusing to hire once it observes the characteristics of the workers who apply.

[3]One necessary condition for this to hold is that the density of F at the highest wage in the support of the distribution must be zero. This rules out, for example, a uniform distribution, or a distribution in which all workers have an identical characteristic that is commonly know.

[4]It would make little difference if these characteristics were private to firms, since all workers and other firms care about is the firm's wage.

4. Equilibrium of the Worker Application Sub-game

To motivate the argument that follows, we start by considering where workers of different types apply once they see the wage offers of firms. This section begins to develop the concept of a reservation wage application strategy that is the basic building block for analysis of the large game.

A strategy for worker i in the application sub-game is a function $\pi^i : W^N \times Y \rightarrow S^{m-1}$, where $S^{m-1} = \left\{\pi \in \mathbb{R}_+^m : \sum_{i=1}^m \pi_i = 1\right\}$.[5] This section analyzes symmetric equilibria in which every worker uses an application strategy that is a common function of his or her type. The idea that is fundamental to directed search is that these application strategies depend on the array of wages on offer. For the purposes of characterizing the equilibrium in the application sub-game associated with a fixed set of wages, the notation that captures this will be suppressed and we write $\pi_j(y)$ to be the probability with which each worker whose type is y applies to firm j.

Since firms always hire the worker with the highest type who applies, worker i will match with firm j in equilibrium so long as every other worker in the market either has a lower type than he does, or applies to some other firm. So the probability that a worker is hired if he applies to firm j is given by

$$\left[1 - \int_y^{\overline{y}} \pi_j\left(y'\right) dF\left(y'\right)\right]^{n-1}$$

The payoff to the worker is equal to this probability multiplied by the wage that the firm offers. It will simplify the argument in this section to assume that wages are ordered in such a way that $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_m$.

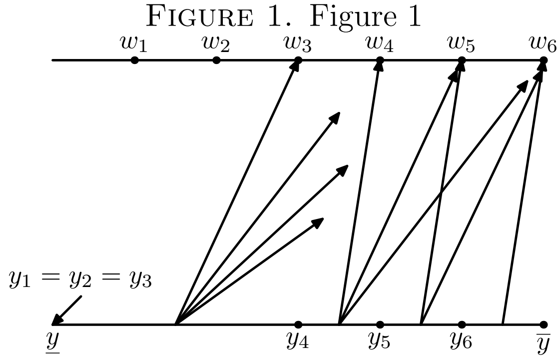
The unique (symmetric) equilibrium for the application sub-game is given by the following Lemma.

Lemma 4.1. *For any array of wages $w_1, \ldots w_m$ offered by firms for which $w_1 > 0$, there is an array $\{y_K, \ldots, y_m\}$ with $K \geqslant 1$ and a set $\{\pi_j^k\}_{k \geqslant K; j \geqslant k}$ of probabilities satisfying $\pi_j^k > 0$ and $\sum_{j=k}^m \pi_j^k = 1$ for each k and such that the strategy*

$$\pi_j(y) = \begin{cases} \pi_j^k & \text{if } j \geqslant k; y \in [y_k, y_{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

[5] We ignore the possibility that a worker might not apply to any firm since that is a strictly dominated strategy given the assumptions about payoffs.

FIGURE 1. Figure 1



is almost everywhere a unique continuation equilibrium application strategy. The probabilities π^i_j *satisfy*

$$\left(\frac{\pi^i_j}{\pi^i_i}\right)^{n-1} = \frac{w_i}{w_j} \tag{3}$$

for each $j > i$.

Furthermore, the numbers $\{y_k\}$ *and* $\{\pi^k_j\}$ *depend continuously on the wages offered by firms.*

The proof is included in the appendix. The theorem is hard to state because there are many different probabilities that have to be described. However, the important content of the theorem is displayed in Figure 1.

The lower line in the figure represents the set of possible types that workers might have, from $\underline{y}$ to $\overline{y}$. The upper line represents the array of wages on offer. There are six firms in this example, each offers a distinct wage. The theorem says that the set of types can be partitioned into $m - K + 1$ different subsets. In the picture there are four such subsets with cutoff points given by $y_4, \ldots y_6$. The interpretation of the interval is that workers whose types are in $(y_j, y_{j+1}]$ all choose firm j as the firm with the lowest index to which they will apply. In other words, there is a lowest wage to which a worker of type y_j will apply. Following Shimer (2005) we refer to this wage as worker j's *reservation wage.* The leftmost arrows emanating from each point on the horizontal axis point to this reservation wage. The second important part of the theorem is illustrated by the fact that there are shorter arrow pointing at higher wages as well. Specifically, this means that the worker of type y_j will apply with positive probability at every wage above his or her reservation wage.

The complexity of the statement comes from the fact that the probability with which the worker of type y_j applies to each of the wages above his or her reservation wage is different. Nonetheless, the theorem provides considerable regularity since it shows both that all workers whose types are in the interval $(y_j, y_{j+1}]$ have the same reservation wage, and apply to each wage above their reservation wage with the same probability.

Workers in the interval $(y_4, y_5]$ in the picture above, won't apply to any firm whose wage is below w_4. Workers whose types are in the interval $(y_5, y_6]$ won't apply at any wage below w_5. So these reservation wages are non-decreasing functions of type.

All types of workers except for the highest randomize when they apply. The second part of the theorem suggests how this randomization works out when the labor market is very large. Equation (3) requires that workers apply with the highest probability to wages near their reservation wage. The higher a wage offer is relative to a worker's reservation wage, the lower the probability with which the worker will apply. Notice that this is very different from what happens in the standard directed search story in which higher wages always imply higher application probabilities. In the standard story, workers have to be indifferent about applying at all different wages. Application probabilities have to be higher at high wage firms so that the probability of being hired falls as the wage rises. Here indifference also drives the application strategy, but the probability of being hired at the higher wage firm falls for a worker of any given type because he faces stiffer competition at the higher wage firms, not because there are more applicants.

Notice, however, that (3) suggests that differences in application probabilities is a 'small numbers' phenomena. If n is very large, then π_i^i and π_j^i have to be almost equal equal to one another in order for (3) to hold, no matter what the difference between w_i and w_j is. If n is very large, these differences in application probabilities should tend to disappear leading to a situation in which workers appear to be applying to all wages above their reservation wage with the same probability. We report a formal result of this kind below.

Finally, it is worthwhile to give some intuition about why the equilibrium works out the way it does. For example, an alternative that might seem reasonable is that workers and firms match assortatively with the highest quality workers applying to the highest wage firms, with lower quality workers applying exclusively at lower wages. To see why this won't work, consider the lowest worker type y_j who applies at wage w_j. If workers application strategies involve sorting, worker y_j will be hired for sure if he applies at any wage below w_j and will be hired at the wage w_j if no higher type workers apply. Any worker who has type $y_k < y_j$ is less likely than worker y_j to be hired at a firm offering a wage $w' < w_j$, but has exactly the same chance as worker y_j of being hired at wage w_j. So workers with types below y_j would strictly prefer to apply at wage w_j than at any lower wage. To prevent this, there must be some chance that workers with types below y_j also apply at wage w_j in equilibrium.

All the outcomes depend on the number of workers and firms. To avoid adding notation, everything is indexed by n and it is assumed that the ratio of the number of workers to firms is constant and equal to τ. Fix an array of wages. Let $\{\pi_1(\cdot), \ldots, \pi_m(\cdot)\}$ be the continuation equilibrium associated with this array of wages. Observe that by Lemma 4.1, each function $\pi_j(y)$ is a step function with jumps at the points y_j. Define the step function

$$\omega_n(y) = \min\{w_j : \pi_j(y) > 0\}$$

As above, this notation suppresses the fact that the function depends on the array of wages on offer. This is the lowest wage to which a worker of type y applies with positive probability in the continuation equilibrium. Alternatively, it is worker y's reservation wage. This function is a step function whose 'steps' occur at the critical points y_j identified by Lemma 4.1. The limit from the left of $\omega_n(y_j)$ at y_j is w_{j-1}, while the right limit of $\omega_n(y_j) = w_j$. Denote its 'inverse' function by

$$y_n^*(w) = \sup\{y' : \omega_n(y') \leq w\}$$

In words, the inverse function gives the highest type who chooses a firm offering wage w with strictly positive probability. Despite the fact that the notation suppresses this, bear in mind that the functions y_n^* and ω_n both depend on the array of wages on offer.

4.1. **Wages.** Consider the firm who offers the wage w_j (i.e., the j^{th} lowest wage). The probability with which a worker drawn randomly both comes to firm j and has a type at least y is

$$\int_y^{y_n^*(w_j)} \pi_j(y')dF(y')$$

Let $\tilde{j}(y) = \{j' : \omega_n(y) = w_{j'}\}$ be the index of the lowest wage to which a worker of type y applies. Using this the integral above can be written

$$\sum_{j'=\tilde{j}(y)}^{j} \pi_j^{j'} \left[F(y_{j'+1}) - \max\left[F(y_{j'}), F(y)\right]\right]$$

if $\omega_n(y) \leq w_j$. The integral is zero otherwise. Then from firm j's point of view, it looks exactly as if n worker types are being independently drawn from the probability distribution

$$\phi_j(y) \equiv 1 - \pi_j^{\tilde{j}(y)}[F(y_{\tilde{j}(y)+1}) - F(y)] - \sum_{j'=\tilde{j}(y)+1}^{j} \pi_j^{j'}[F(y_{j'+1}) - F(y_{j'})]$$

The distribution function $\phi_j(y)$ has an atom of size

$$1 - \sum_{j'=1}^{j} \pi_j^{j'}[F(y_{j'+1}) - F(y_{j'})]$$

at $\underline{y}$.

The firm will always hire the worker who has the highest type. The probability distribution for the type hired by the firm is then the probability distribution of the highest order statistic from this distribution. This gives the expected payoff for firm j as

$$\int_{\underline{y}}^{\overline{y}} v_j(w_j, y) d\phi_j^n(y) \tag{4}$$

By Lemma 4.1, the distribution function $\phi^n(y)$ is continuous at each point y in firm j's wage. So $\phi^n(\cdot)$ varies continuously in the weak topology with firm j's wage. As the family of functions $v_j(\cdot, y)$ is equicontinuous, the integral is a continuous function of the wage w_j that the firm offers. The existence of a mixed strategy equilibrium in firms' wages then follows from standard theorems.

5. Equilibrium in a Large Game

Despite the conceptual simplicity of firms' payoff implied by (4), it is difficult to provide much in the way of characterization of the Nash equilibrium of the firms' part of the game. It is tempting to jump to a continuum of workers and firms to see if this helps the characterization. A significant complication in this regard arises from the fact that in a large game, payoffs should be defined for every feasible action against every possible distribution of the actions of the others. For firms this is at least conceptually straightforward since wage distributions are well understood. For workers, the application 'strategy' is not a well defined object in the continuum. The distribution of such things is then a moot point.

To get around this difficulty, this section provides a theorem that shows that limit payoffs of all traders depend on their own actions, on the distribution of wages, and on a single reservation wage function for workers. This result suggests a natural definition of equilibrium for the continuum game.

Let G be a distribution of wages. Suppose that firms' preferences can be parameterized by elements in some interval $\mathcal{H}$ of the real line. Let H be a distribution function on $\mathcal{H}$. Firms preferences can then be written $v_h(w,y)$ where $h \in \mathcal{H}$. To approximate the distribution of wages, let G_n be a sequence of step functions that converges weakly to G. Let $\{w_1^n, \ldots, w_m^n\}$ be the finite array of wages whose distribution is G_n.

Now fix a firm type h and a wage w to be offered by that firm. Suppose that in this approximation, firm h has the $j^{n^{th}}$ highest wage. For any distribution function, G_n, let $G_n^-(w)$ be the left limit of G_n, so that $(1 - G_n^-(w))m$ is the number of firms whose wage offer is *at least* w. Recall that the non-decreasing function $\omega_n : Y \to W$ represents the lowest wage at which worker of type y will apply, and that $y_n(w) = \sup_y \omega_n(y) \le w$ is the highest worker type who will apply to a firm who offers wage w.

Theorem 5.1. *Let G be a distribution of wages, w an arbitrary wage offered by a firm of type h, and w^-, the largest wage in the support of G that is less than or equal to w. Let G_n be a sequence of distributions that converges weakly to G. Let j_n be the corresponding sequence of indices of firm h's wage (i.e., such that w is the j_n^{th} lowest wage in the distribution associated with G_n). There is a non-decreasing right continuous function $\omega(y)$ and a non-decreasing right continuous function $y^*(w)$ (both of which depend on G) such that*

$$\lim_{n\to\infty}\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y)dF(y)\right]^{n-1} = \frac{w^-}{w} e^{-\int_y^{y^*(w^-)} k(y')dF(y')} \tag{5}$$

and

$$\lim_{n\to\infty}\int_{\underline{y}}^{\overline{y}} v_h(w,y)d\phi_{j_n}^n(y) =$$

$$\frac{w^-}{w}\int_{\underline{y}}^{y^*(w^-)} k(y)v_j(w,y)e^{-\int_y^{y^*(w^-)} k(y')dF(y')}F'(y)dy + v_h(w,y^*(w))\left(1 - \frac{w^-}{w}\right) \tag{6}$$

where

$$k(y) = \frac{\tau}{1 - G^-(\omega(y))}$$

Furthermore $y^*(w) = \sup\{y : \omega(y) \le w\}$.

The proof of the theorem is, again, included in the appendix. A number of more descriptive comments are in order here.

First, the formulas above differ depending on whether the wage offer a firm makes, or the wage at which a worker applies, are in the support of the existing distribution G of wages. If they are, then w and w^- are the same. The probability with which a worker is hired when he or she applies to a firm offering wage w is then given by the simpler formula

$$e^{-\int_y^{y^*(w)} k(y')dF(y')}$$

while the profit for the firm who offers a wage in the support of G is

$$\int_{\underline{y}}^{y^*(w)} k(y)v_j(w,y)e^{-\int_y^{y^*(w)} k(y')dF(y')}F'(y)dy$$

At first glance, these formulas seem independent of the distribution G. Recall however, that $k(y) = \frac{\tau}{1-G^-(\omega(y))}$. The function $\omega(y)$ is the common reservation wage function used by all workers, and $y^*(w)$ is it's 'inverse'. So all traders payoffs are determined by their own actions (offer or apply at wage w), the distribution of wages G and the reservation wage function ω.

To see how these formulas work when a firm offers a wage outside the support of the existing distribution, it might help to consider the case where G has all firms offering the same wage, say w_0. Then $1 - G^-(w) = 1$ for every wage less than or equal to w_0. Since the reservation wage function will have its range in the support of the distribution G, $\omega(y) = w_0$ for all y. Then $k(y) = \tau$ for all y. By the definition of the inverse function $y^*(w_0) = \overline{y}$.

With these preliminaries, a worker of type y who applies to a firm offering the wage w_0 is hired with probability

$$e^{-\int_y^{y^*(w)} k(y')dF(y')} = e^{-\int_y^{\overline{y}} \tau dF(y')} = e^{-\tau(1-F(y))}$$

Recall that this formula is slightly different from the formula in the usual story because workers types differ. The worker is hired if no worker with a better type shows up. In the usual story, the worker is hired if he is the winner of a lottery in which applicants are chosen with equal probability.

The payoff to a firm who offers w_0 in the support of the distribution is

$$\int_{\underline{y}}^{\overline{y}} \tau v_j(w_0,y)e^{-\tau(1-F(y)}F'(y)dy$$

Now suppose that a single deviating firm offers a wage $w_1 > w_0$. Then by (5), the probability with which a worker is hired when he applies to this deviating firm is

$$\frac{w_0}{w_1}e^{-\tau(1-F(y))}$$

This is just the analog of (2) in the sense that it says that the distribution G supports a 'market' payoff for each type equal to $w_0 e^{-\tau(1-F(y))}$ which will be unchanged by the deviation.

The deviating firm's payoff is given by

$$\frac{w_0}{w_1}\int_{\underline{y}}^{\overline{y}} \tau v_j(w_1,y)e^{-\tau(1-F(y))}F'(y)dy + v_h(w_1,\overline{y})\left(1-\frac{w_0}{w_1}\right)$$

Equilibrium can now be calculated by finding a wage w_0 for which this latter expression is never larger than the payoff the firm receives when it offers w_0.

What is most important about the theorem is that it gives limit payoffs that, apart from exogenous stuff, depend only on firm and worker choices (the wage offered or applied to), the distribution of wages G, and a single reservation wage rule $\omega(y)$. These formulas provide a natural definition of equilibrium in the large game. First, if every firm chooses a wage that is a best reply to the distribution G and the function ω, then the associated distribution of best replies should be equal to G. Secondly, each worker type should find that his or her expected payoff is the same when they apply to any wage at or above their reservation wage $\omega(y)$, and this payoff should be at least as large as it is when they apply at any wage below their reservation wage.

This is the notion of equilibrium that is adopted in the rest of the paper. Before proceeding, it is useful to clarify somewhat the relationship between this approach and the right approach, which is to compute the limits of sub-game perfect Nash equilibria as the number of workers and firms grow. A brief digression on this follows.

Let H be a distribution of firms' types and $H^n$the distribution of firms' types in a finite approximation consisting of n workers and m firms, where $\frac{n}{m} = \tau$. Imagine constructing the sequence of approximations so that H^n converges weakly to H. Since firms' first stage payoff functions are continuous when there are finitely many workers and firms according to Lemma 4.1, an equilibrium for the entire game must exist in which firms use mixed strategies to set their wages in the first stage. Suppose fortuitously that every finite approximation has an equilibrium in which each firm uses a pure strategy in equilibrium when setting its wage. This pure strategy equilibrium supports a distribution G_n of wages. This distribution has a weak limit G as n becomes large. Then the formulas in Theorem 5.1 give the limit values of payoffs to workers of each type facing the equilibrium distribution of wages. The Theorem also provides the limit value of the payoff for any wage w played against the equilibrium wages of all the other firms. If G is not an equilibrium distribution in the limit game (in the sense described above, that G is a best reply to itself), then some type of firm does strictly better in the limit game than he could do by playing the limit of his equilibrium pure strategy. Then convergence means that he will do strictly better in the finite approximations as well when n is large enough. So the weak limits of (pure strategy) equilibrium wage distributions from large finite games must be equilibrium distributions in the limit game. This is the sense in which we can use the payoff functions from the limit game to approximate what happens in large finite approximations.

The converse isn't necessarily true. There is no guarantee that equilibria from the limit game are necessarily close to pure strategy equilibria in any finite approximation. This isn't too big a problem, since if there are such limits, the equilibrium for the large game will find them. On the other hand, the argument above isn't obviously true for sequences of mixed equilibria. So finding equilibrium in the large game may not uncover all the limits of equilibria from large finite games.

Finally, one limit calculation helps to illustrate why the probability functions disappear in the limit. To make the argument a little easier, assume that G is differentiable. The following argument calculates the limit value of the probability with which a worker of type y applies to some firm whose wage is less than or equal to w. The point is to show that this probability will be equal to the ratio of the measure of firms who offer wages between worker y's reservation wage and w, to the measure of firms who offer wages above y's reservation wage. The interpretation is that when there are many workers and firms, y behaves almost as if he is applying with equal probability to every firm who offers a wage above his reservation wage.

The formal calculation of this probability in the game with n workers is

$$\sum_{j'=j:w_j=\omega_n(y)}^{j:w_j=w} \pi_{j'}^n(y) =$$

$$\frac{m}{n-1} \sum_{j'=j:w_j=\omega_n(y)}^{j:w_j=w} (n-1)\pi_{j'}^n(y) \frac{G_n(w_{j'+1}) - G_n(w_{j'})}{w_{j'+1} - w_{j'}} (w_{j'+1} - w_{j'}) =$$

Observe that in this formula $m(G_n(w_{j'+1}) - G_n(w_{j'})) = 1$ by the construction of the approximation. This can be written as

$$\frac{m}{n-1} \int_{\omega_n(y)}^{w} (n-1)\pi_w^n(y) G_n'(w) dw$$

making the obvious substitutions. By the bounded convergence theorem and the differentiability of G, this converges to

$$\frac{1}{\tau}\int_{\omega(y)}^{w} k(y)G'(w)dw$$

Substituting for $k(y)$ gives

$$\int_{\omega(y)}^{w} \frac{G'(w)}{1-G(\omega(y))}dw$$

This gives the desired result.

6. Properties of Equilibrium

This section simply forges ahead with the idea that the distribution G and reservation wage function ω must satisfy the two conditions of equilibrium described above. Fix G. Since this case hasn't been addressed in the literature, assume G is differentiable. The second condition for equilibrium says that workers should receive the same payoff by applying at every firm whose wage is above their reservation wage. Formally

$$we^{-\int_{y}^{y^*(w)} k(y')dF(y')} = \text{constant} \tag{7}$$

for each $w \geq \omega(y)$. The total derivative of the function with respect to wage should then be zero. That is

$$we^{-\int_{y}^{y^*(w)} k(y')dF(y')}k(y^*(w))F'(y^*(w))\frac{dy^*(w)}{dw} = e^{-\int_{y}^{y^*(w)} k(y')dF(y')}$$

giving

$$w\frac{\tau}{1-G(w)}F'(y^*(w)) = \frac{1}{dy^*(w)/dw} \tag{8}$$

Since $y^*(w)$ is the inverse function of $\omega(y)$ at each point where this derivative exists,

$$\omega(y)\frac{\tau}{1-G(\omega(y))}F'(y) = \frac{d\omega(y)}{dy} \tag{9}$$

The boundary condition is simply that $\omega(\overline{y}) = \overline{w}_G$ where $\overline{w}_G$ is the highest wage in the support of G. The solution to this equation gives the equilibrium reservation wage strategy ω for workers in response to any differentiable distribution G of wage offers by firms.

One implication of (9) is that $\omega(y)$ is strictly increasing. Then $y^*(\omega(y)) = y$, and from (5), a worker of type y who applies to a firm offering wage $\omega(y)$ is offered a job with probability 1. Workers with higher types than y simply won't apply to a firm who offers a wage

$\omega(y)$. This means that the symmetric reservation wage function $\omega(y)$ also describes workers' payoff in the continuation equilibrium associated with a distribution G. The constant of the right hand side of (7) is equal to $\omega(y)$. It follows from the same equation that the probability with which a worker of type y trades with a firm offering wage w is simply equal to $\frac{\omega(y)}{w}$. Then we can rewrite (7) as

$$we^{-\int_y^{y^*(w)} k(y')dF(y')} = \omega(y) \tag{10}$$

The equilibrium distribution G will vary with the distribution of firms' and workers' types which are unobservable. In particular, any distribution G can be supported for an appropriate choice for the distribution of firm types. This is another way of saying that the theory itself does not impose any restrictions on the distribution of wages apart from those implied by assumptions about the distribution of firms' types.

The only other observables are the labour market experiences of workers, the wages that they are hired at and their unemployment experience. Their matching probability is not observed, but their duration at the time they find jobs presumably is observable. Duration is just the inverse of the matching probability. The theory does impose restrictions on the relationship between the wage distribution and duration.

The probability that a worker of type y is hired by *some* firm, using the reasoning above, is given by

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} e^{-\int_y^{y^*(w)} k(y')dF(y')} \frac{G'(w)}{1 - G(\omega(y))} dw$$

Since the expected wage is constant for a worker of type y at every wage above $\omega(y)$, this can be written as

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} \frac{\omega(y)}{w} \frac{G'(w)}{1 - G(\omega(y))} dw$$

The expected duration of unemployment for a worker of type y is $\frac{1}{Q(y)}$. Of concern below is how this matching probability varies with the worker's type. Since higher types have higher reservation wages, the way this function varies with type depends on the way the function

$$\psi(w) = \int_w^{\overline{w}_G} \frac{w}{w'} \frac{G'(w')}{1 - G(w)} dw'$$

in wage w. This function represents the expectation of the ratio of any wage to wages in the distribution G that are higher than that wage. This function isn't particularly simple, but its' properties are readily checked numerically using only data from the wage distribution. Of particular interest are the situations in which this function is monotonic. If $\psi'(w) > 0$, for example, this means that the ex ante probability with which a worker of type y trades is an increasing function of y because of the monotonicity of $\omega(y)$.

Again, y is unobservable. What is observable is the actual duration of workers hired at different wages. From (6), the probability that a worker hired by the firm who offers wage w has a type less than or equal to y_0 is given by

$$\frac{\int_{\underline{y}}^{y_0} k(y) e^{-\int_y^{y^*(w)} k(y')dF(y')} F'(y)dy}{\int_{\underline{y}}^{y^*(w)} k(y) e^{-\int_y^{y^*(w)} k(y')dF(y')} F'(y)dy} =$$

$$\frac{\int_{\underline{y}}^{y_0} \frac{\tau}{1-G(\omega(y))} \frac{\omega(y)}{w} F'(y)dy}{\int_{\underline{y}}^{y^*(w)} \frac{\tau}{1-G(\omega(y))} \frac{\omega(y)}{w} F'(y)dy}$$

Note that this probability is conditional on some worker being hired by the firm, which explains the denominator. The equality follows by substituting for $k(y)$ and using (10). Substituting (9) gives an even simpler formulation

$$\frac{\int_{\underline{y}}^{y_0} \omega'(y)dy}{w - \omega(\underline{y})} \tag{11}$$

This expression is readily seen to be declining in w. The interpretation is that an increase in the wage moves the distribution function for the type hired by the firm to one that first order stochastically dominates the original distribution.

Now use this distribution to take expectations of $Q(y)$ to get the following:

Proposition 6.1. *If $\psi(w)$ is monotonically increasing (decreasing) then the expected duration of unemployment for a worker hired by a firm is an decreasing (increasing) function of the wage offered by the firm.*

The unusual part of this proposition is the first part. When $\psi(w)$ is increasing, workers who are hired at high wage firms will tend on average to have spent less time searching for jobs than workers who are hired by low wage firms. This is quite unlike standard directed search where high wages and long duration must go together. This prediction is not a particularly strong test of the model, since the function $\psi(w)$ may not be monotonic. Notice however, that it is a testable consequence of the model that does not rely on any knowledge about the distributions of the unobservables.

The expected duration for a worker hired by a firm offering a wage w is given by the reciprocal of

$$\int_{\underline{y}}^{y^*(w)} \frac{\omega'(y)}{w - \omega(\underline{y})} \psi(\omega(y)) dy$$

using the expression for the density of the type of worker hired by the firm that was derived above. It is apparent from this expression that when $\psi(w)$ is non-monotonic, then there will be no systematic relationship between the wage at which a worker is hired and his probability of matching measured as his expected duration. Even in this dimension, the result is quite different from the standard directed search model where wage and employment probability must be inversely related.

Let $\Phi(w)$ be the expected duration of unemployment for workers hired at wage w, which could be estimated from existing data. This function, along with the actual wage distribution constitute the observables in this problem. Fix a set of types, say $[0, 1]$. Using the last expression, the functional equation

$$\int_{\underline{y}}^{y^*(w)} \frac{\omega'(y)}{w - \omega(\underline{y})} \psi(\omega(y)) dy = \frac{1}{\Phi(w)}$$

can be solved to recover the function $\omega(y)$. The distribution of worker types is then recovered by solving (9). The distribution of firm types must then be chosen to support the observed distribution G when firms best reply to G and the symmetric strategy $\omega(\cdot)$ used by workers. This makes it possible to recover the productivity of the unobserved distribution of types.

The point of this last argument is simply to show how the model can be used to decompose the wage variation into variation in workers' and firms' types. We leave the analysis of this for future work.

7. Conclusion

This paper illustrates how a directed search model can be used to account for the residual part of wage variation. Part of this involves adjusting the directed search model to allow for rich variation in the types of workers and firms. This improves on existing models that use extensive symmetry assumptions that force the models to behave in counter-factual ways. In the variant proposed here, rich distributions of firm and worker characteristics can be incorporated.

The directed search model does impose some structure on the data. Surprisingly it restricts the relationship between the wage distribution and the function relating unemployment duration and exit wage. Some wage distributions (the uniform being an example) have the property that workers who leave unemployment at high wages must also have shorter unemployment duration. This prediction is distinctly different from standard directed search models where unemployment duration and wage must be positively related.

The driving force in the model presented here is the equilibrium of the workers' application sub-game. Contrary to what one might expect, low quality workers do not restrict their applications to low wage firms. On the contrary, low quality workers make applications at all kinds of different wages. The higher the unobservable quality of the worker, the more discriminating the worker is in the wages at which he applies. It is this property that breaks the strong relationship between wage and unemployment probability. Higher quality workers are more likely, everything else constant, to be hired by firms. High quality workers also apply to higher wage firms on average. In this sense high wages and short duration should be related. This relationship is not unambiguous however. As a workers quality rises, he is more likely to be hired at any given firm, but he will also restrict his applications to firms whose wages are higher. This by itself reduces the probability of employment because high wage firms have bigger queues - the usual directed search story.

Finally, the paper illustrates how observable data on wages and duration can be used to recover the unobserved distributions of firms' and workers' types.

8. Appendix

8.1. **Proof of Lemma 4.1.**

Proof. The proof is inductive.

Evidently a worker with the highest type will be hired with probability one where ever he applies, so every equilibrium strategy must have the highest type worker apply to one of the firms who offer the highest wage. If $w_{m-1} = w_m$ set $y_m = 1$ and $\pi_m^m = 1$. In this case observe that a worker of type y_m is just indifferent between applying to firm m and $m-1$.

Otherwise, fix an open interval $(y_m, \overline{y})$. The expected payoff to worker i with a type in this interval who applies to firm m is

$$\left[1 - \int_{y}^{\overline{y}} \pi_m(y')\, dF(y')\right]^{n-1} w_m$$

The expected payoff to applying to any firm j whose wage is $w_j < w_m$ is

$$\left[1-\int_{y}^{\overline{y}} \pi_j(y')dF\left(y'\right)\right]^{n-1} w_j$$

Now observe that for y_m close enough to $\overline{y}$, workers will strictly prefer applying to firm m than applying to firm j, even if all the workers whose types are higher apply to firm m with probability 1. In other words, for workers whose type is close enough to $\overline{y}$, applying to one of the firms whose wage is highest strictly dominates any other choice. Thus there is some interval near $\overline{y}$ such that workers whose types are in this interval apply to firm m with probability 1 in every Bayesian equilibrium. The lowest type for which this is true is the type y_m such that

$$\left[1-\int_{y_m}^{\overline{y}} dF\left(y'\right)\right]^{n-1} w_m = w_{m-1}$$

or the type y that satisfies,

$$[F\left(y\right)]^{n-1} w_m = w_{m-1} \tag{12}$$

Then $\pi_m^i(y) = 1 \equiv \pi_m^m$ for every i and for every $y \in (y_m, \overline{y}]\}$ must be true in every Bayesian equilibrium of this sub-game.

Note that y_m is a continuous function of w_m and w_{m-1} and that $y_m \to 1$ as $w_{m-1} \to w_m$. Since π_m^m is constant, it is trivially a continuous function of w_m and w_{m-1}. Furthermore, note that a worker of type y_m gets the same payoff from every firm whose index is greater than or equal to $m-1$.

Now suppose that we have defined cutoff valuations $\{y_{k+1}, \ldots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k+1, \ldots m$ and $j' \geqslant k'$, satisfying $\sum_{j' \geqslant k'} \pi_{j'}^{k'} = 1$ for each k'. Suppose that these satisfy the following conditions:

- (C.1) - $\pi_{j'}(y) = \pi_{j'}^{k'}$ for each $y \in (y_{k'}, y_{k'+1})$ and $\pi_{j'} = 0$ otherwise, in every symmetric Bayesian equilibrium;
- (C.2) - a worker of type $y_{k'}$ where $y_{k'} \in \{y_{k+1}, \ldots, y_m\}$, gets the same payoff from every firm whose index is at least $k'-1$;
- (C.3) - each of these numbers is a continuous function of wages $w_k, \ldots w_m$.

If $y_{k+1} = \underline{y}$, then we have shown that the Bayesian continuation equilibrium for this sub-game is almost everywhere uniquely defined (the exceptions are the cutoff values y_k). So suppose $y_{k+1} > \underline{y}$. We now show that properties (C.1) to (C.3) can be extended to some interval $[y_k, y_{k+1})$ which will be non-degenerate provided $w_k < w_{k-1}$.

If $w_k = w_{k+1}$, or $w_{k-1} = w_k$, set $y_k = y_{k+1}$, $\pi_k^k = 0$ and $\pi_j^k = \pi_j^{k+1}$ for each $j > k$. It is straightforward that valuations $\{y_k, \dots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k, \dots m$ satisfy conditions (C.1) to (C.3) of the induction hypothesis.

Otherwise either $w_{k-1} < w_k < w_{k+1}$ or $k = 1$. Each of these cases can be analyzed the same way. In the former case, observe that in this construction, worker types larger than y_{k+1} will never apply to firm k. Thus for y close enough to y_{k+1} applying to *any* firm with wage rate below w_k will be strictly dominated by applying to firm k no matter what workers with types in the interval (y, y_{k+1}) choose to do. In the case where $k = 1$ firm k is already the lowest wage firm. In either case, we conclude that there is an interval of types (y_k, y_{k+1}), with y_k possibly equal to $\underline{y}$, such that workers with types in this interval will apply with positive probability only to firms with wages at least w_k in every Bayesian equilibrium.

By the induction hypothesis, a worker of type y_{k+1} will receive the same payoff from each firm $k+1$ through m. This payoff is given by

$$\left[1 - \sum_{i=1}^{j} \pi_{k+j}^{k+i}[F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j}$$

when this worker applies to firm $k+j$. By the induction hypothesis, this payoff is equal to w_k for each $j \geqslant 1$. Notice that this payoff is independent of what workers whose types are in the interval (y_k, y_{k+1}) choose to do. A worker i of type $y \in (y_k, y_{k+1})$ who applies to firm $k+j$ receives payoff

(13)
$$\left[1 - \int_{y}^{y_{k+1}} \pi_{k+j}(y) dF(y) - \sum_{i=1}^{j} \pi_{k+j}^{k+i}[F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j}$$

while the same worker who applies to firm k gets

$$\left[1 - \int_{y}^{y_{k+1}} \pi_k(y) dF(y)\right]^{n-1} w_k \tag{14}$$

The function described in (14) is non-decreasing in y and has a limit from the left at y_{k+1} equal to w_k. Since applying to firms whose wages are lower than w_k is a strictly dominated strategy of a worker of type y close enough to y_{k+1}, it must be the case that for every i, $\int_y^{y_{k+1}} \pi_{k+j}^i(y) dy$ is strictly positive for some j. Then from (13) and (14), $\int_y^{y_{k+1}} \pi_{k+j}(y) dy$ must be strictly positive for all j.

The payoff must be the same at firm k and $k+j$ for each $j > 0$ and for every $y \in (y_k, y_{k+1})$. This requires that (13) and (14) must be equal identically in y. Differentiating this identity repeatedly gives

$$\left(\frac{\pi_{k+j}^{k}(y)}{\pi_{k}^{k}(y)}\right)^{n-1}=\frac{w_k}{w_{k+j}} \tag{15}$$

implying that π_{k+j}^{k} are constant.

They can all be determined from the condition

$$\sum_{j=0}^{m-k}\pi_{k+j}^{k}=1 \tag{16}$$

Notice that by the induction hypothesis, the limits from the left of (13) and (14) at y_{k+1} must both be equal to w_k. Thus (15) and (16) are also sufficient for identity of the payoffs.

Having found the value for π_k^k we can determine the lower bound y_k. Since workers with higher types and higher investments only apply to firms whose wages are at least w_k, this worker is sure to be hired if he applies to the $k-1^{st}$ firm, assuming that there is one. On the other hand, since he is lowest type who applies to the k^{th} firm, he will be hired by the k^{th} firm only if no other worker with a higher type applies. Then define y_k as follows: if $k=1$, then $y_k=y_1=\underline{y}$; otherwise if

$$\left[1-\pi_k^k\left(F(y_{k+1})-F(y)\right)\right]^{n-1}w_k=w_{k-1} \tag{17}$$

has a solution that exceeds $\underline{y}$, set y_k equal to this solution; otherwise set $y_k=\underline{y}$.

This argument extends conditions (C.1) and (C.2) by construction. Property (C.3) is readily verified using, for example, the maximum theorem since $w_{k+j}>0$ by assumption. □

8.2. **A preliminary Result.**

Lemma 8.1. *For any sequence G_n there is a sub-sequence such that $\omega_n(y)$ converges weakly to a right continuous non-decreasing function $\omega(y)$. Define $y_n^*(w)=\sup\{y:\omega_n(y)\le w\}$. The sequence $y_n^*(\cdot)$ converges weakly to a right continuous non-decreasing function $y^*(\cdot)$.*

Proof. By construction each $\omega_n(y)$ is right continuous and non-decreasing, and for each n $\int_{\underline{w}}^{\overline{w}}dw_n(y)\le\overline{w}_G-\underline{w}_G$ where $\overline{w}_G$ and $\underline{w}_G$ are the maximum and minimum points in the support of G respectively. Hence by Helly Compactness Theorem, $w_n(y)$ has a subsequence that converges weakly to an non-decreasing right continuous function. Similarly, $y_n^*(\cdot)$ is non-decreasing and right continuous, and so has a weak limit $y^*(\cdot)$. □

8.3. **Proof of Lemma 8.2.**

Lemma 8.2. *For any wage w, let w^- be the largest wage in the support of G that is less than or equal to w. Let j^n be the index of w in the n^{th} approximation to G. Let $\omega(y)$ be a limit of the sequence $\omega_n(y)$ as defined in Lemma 8.1. Then* $\lim_{n\to\infty} \pi^n_{j^n}(y)(n-1) = \frac{\tau}{1-G^-(\omega(y))}$ *and*

$$\lim_{n\to\infty}\left(1-\int_y^{y^*(w)} \pi^n_{j^n}(y')dF(y')\right)^{n-1} =$$

$$\frac{w^-}{w}e^{-\int_y^{y^*(w^-)}\frac{\tau}{1-G^-(\omega(y'))}dF(y')}$$

Proof. Let $y < \overline{y}$. The expected payoff to any type is strictly positive in equilibrium. Let $\overline{v}(y)$ be any point-wise limit for this payoff as n goes to infinity. For any finite n, there is a firm m who offers the highest wage in the support of the distribution G_n. Call this highest wage w_m. This highest wage is no higher than $\overline{w}$.

By Lemma 4.1, every worker type y applies to this firm with strictly positive probability. Then from (13), the payoff to a worker of type y who applies to firm m is bounded above by

$$(1-\pi^n_m(y)(1-F(y)))^{n-1}\,\overline{w} \tag{18}$$

This is an upper bound for two reasons. Firm m will offer a wage that is no higher than $\overline{w}$, and workers whose types are higher than y will apply to firm m with probability at least as high as a worker of type y.

We first show that $\pi^n_m(y)(n-1)$ is uniformly (in y) bounded above by an integrable function, then work down to show the result for firms with indices lower than m. Suppose to the contrary, it has a limit that exceeds some constant $b(y)$. Then the upper bound given by (18) can be no larger than

$$\lim_{n\to\infty}(1-(\frac{b(y)}{n-1})(F(\overline{y})-F(y)))^{n-1}\overline{w}$$

$$= \exp\left\{\frac{\log\left(1-\frac{1}{n-1}b(y)(1-F(y))\right)}{\frac{1}{n-1}}\right\}\overline{w}$$

$$= e^{-b(y)(F(\overline{y})-F(y))}\overline{w}$$

where the last result follows by L'Hopital's rule. Since the upper bound on the worker's payoff must be at least $\overline{v}(y)$, then $\lim_{n\to\infty}\pi^n_m(y)(n-1)$ must have an upper bound $b(y)$ that satisfies $e^{-b(y)(F(\overline{y})-F(y))}\overline{w} \geq \overline{v}(y)$, for all y, or $b(y)(F(\overline{y}) - F(y)) \leq -\log\left(\frac{\overline{v}(y)}{\overline{w}}\right)$. Now observe that

$$\int_y^{\overline{y}} b(y')dF(y') \leq \int_y^{\overline{y}} b(y')(1-F(y'))\frac{F'(y')}{1-F(y')}dy' \leq -\log\left(\frac{\overline{v}(y)}{\overline{w}}\right)\cdot B\cdot(\overline{y}-y)$$

where B is the uniform bound on the ratio $\frac{F'(y')}{1-F(y')}$. Thus the bound $b(y)$ is an F-integrable function of y that uniformly bounds $\pi_m^n(y)(n-1)$ for all n large enough. Define $\overline{k}(y) = \lim_{n\to\infty} \pi_m^n(y)(n-1)$.

Now we extend this uniform upper bound to firms with indices below m. From (15)

$$\pi_j^n(y)(n-1) = \left(\frac{w_m}{w}\right)^{\frac{1}{n-1}} \pi_m^n(y)(n-1) \tag{19}$$

for each j such that $\pi_j^n(y) > 0$. From the previous result, the right hand side of this equation is uniformly bounded by the F-integrable function $\frac{w_m}{w}b(y)$, so the left hand side is also uniformly bounded. Furthermore, taking limits in (19) with respect to n gives

$$\lim_{n\to\infty} \pi_{j^n}^n(y)(n-1) = \overline{k}(y)$$

Recall that $\omega_n(y)$ is the lowest wage to which a worker of type y applies with positive probability in the continuation equilibrium with n workers. From (19)

$$\sum_{j:w_j\geq\omega_n(y)} \pi_j^n(y)(n-1) = \pi_m^n(y)(n-1) \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} \tag{20}$$

The sum on the left hand side of this last equation is $n-1$ since the application probabilities sum to one. On the right hand side, observe that

$$\sum_{j:w_j\geq\omega_n(y)} 1 \leq \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} \leq \left(\frac{w_m}{w_{j_n^*}}\right)^{\frac{1}{n-1}} \sum_{j:w_j\geq\omega_n(y)} 1$$

Dividing this by m gives

$$(1-G_n^-(\omega_n(y))) \leq \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} /m \leq \left(\frac{w_m}{w_{j_n^*}}\right)^{\frac{1}{n-1}} (1-G_n^-(\omega_n(y)))$$

This implies that

$$\lim_{n\to\infty} \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} /m = 1 - G^-(\omega(y))$$

where $\omega(y)$. Then from (20)
(21)

$$\lim_{n\to\infty} \pi^n_{j^n}(y)(n-1) = \lim_{n\to\infty} \frac{n-1}{1-G^-_n(\omega_n(y))\cdot m} = \frac{\tau}{1-G^-(\omega(y))} = k(y)$$

which gives the first result in the Lemma.

For the second result, recall that j^n is the index of wage w in the n^{th} approximation to G. The wage w_{j^n-1} is the highest wage in the n^{th} approximation that is less than or equal to w . Now observe that

$$\lim_{n\to\infty}\left(1-\int_y^{y^*_n(w)} \pi^n_{j^n}(y')dF(y')\right)^{n-1} =$$

(22)

$$\lim_{n\to\infty}\left(1-\int_y^{y^*_n(w_{j^n-1})} \pi^n_{j^n}(y')dF(y') - \int_{y^*_n(w_{j^n-1})}^{y^*_n(w)} \pi^n_{j^n}(y')dF(y')\right)^{n-1}$$

By the definition of $y^*_n(w_{j^n-1})$, a worker of this type who applies to the firm offering wage w will be hired with probability

$$\left(1-\int_{y^*_n(w_{j^n-1})}^{y^*_n(w)} \pi^n_{j^n}(y')dF(y')\right)^{n-1}$$

He will be hired for sure if he applies to the firm offering w_{j^n-1}. So

$$\int_{y^*_n(w_{j^n-1})}^{y^*_n(w)} \pi^n_{j^n}(y')dF(y') = 1-\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}}$$

Substitute this into (22) above to get

$$\lim_{n\to\infty}\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}} - \int_y^{y^*_n(w_{j^n-1})} \pi^n_{j^n}(y')dF(y')\right)^{n-1} =$$

$$\lim_{n\to\infty}\exp\left\{(n-1)\log\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}} - \frac{1}{n-1}\int_y^{y^*_n(w_{j^n-1})} \pi^n_{j^n}(y')(n-1)dF(y')\right)\right\}$$

Since the exponential function is continuous, the limit can moved inside the first bracket. So we compute

$$\text{(23)}\quad \lim_{n\to\infty}\frac{\log\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}} - \frac{1}{n-1}\left\{\int_y^{y^*_n(w_{j^n-1})} \pi^n_{j^n}(y')(n-1)dF(y')\right\}\right)}{\frac{1}{n-1}}$$

which can be written as

$$\lim_{x\to 0}\frac{\log\left(\left(\frac{w(x)}{w}\right)^{x} - x\left\{\int_y^{y^*_n(w(x))} \pi^n_{j^n}(y')(n-1)dF(y')\right\}\right)}{x}$$

By construction, w_{j^n-1} converges to w^-, the largest wage in the support of G that is less than or equal to w. The integral $\int_y^{y_n^*(w_{j^n-1})} \pi_{j^n}^n(y')(n-1)dF(y')$ converges by the bounded convergence theorem and (21) to

$$e^{-\int_y^{y^*(w^-)} k(y')dF(y')}$$

So the integral should be thought of as a function of x whose derivative at 0 is zero. The function $w(x)$ is simply the sequence w_{j^n-1}, which converges, so should again be thought of as function whose derivative at zero is zero. Note that with this definition, $w(0) = w^-$. Then apply L'Hopital's rule to get the limit of (23) as

$$\frac{w^-}{w} e^{-\int_y^{y^*(w^-)} k(y')dF(y')}$$

□

8.4. **Proof of Theorem 5.1.**

Proof. The proof of Theorem 5.1 now follows from Lemmas 8.2 and 8.1. A firm of type h who offers wage w has profit

$$\int_{\underline{y}}^{\overline{y}} v_h(w,y)d\phi_{j_n}^n(y)$$

where j_n is the index of the wage w in the distribution G_n associated with the n^{th} approximation. Substituting for ϕ gives

$$\int_{\underline{y}}^{y_n^*(w)} v_h(w,y)d\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n =]$$

$$\int_{\underline{y}}^{y_n^*(w_{j^n-1})} v_h(w,y)d\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n$$

$$\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} v_h(w,y)d\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n =$$

$$\int_{\underline{y}}^{y_n^*(w_{j^n-1})} v_h(w,y)n\pi_{j_n}^n(y)\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^{n-1} F'(y)dy+$$

$$v_h(w,y_n^*(w)) - v_h(w,y_n^*(w_{j^n-1}))\left[1-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n +$$

$$\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} \left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n v_y(w,y)dy$$

That last two terms in this expression are derived by integrating by parts. Now observe that a worker of type $y_n^*(w_{j^n-1})$ is just indifferent between applying at the wage w_{j^n-1} and being hired for sure, or applying at wage w and being hired with probability

$$\left[1-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)}\pi_{j_n}^n(y')dF(y')\right]^{n-1}$$

So substitute $\frac{w_{j^n-1}}{w}$ for this probability in the second term, and take limits to get

$$\int_{\underline{y}}^{y^*(w^-)} v_h(w,y)k(y)\frac{w^-}{w}e^{-\int_y^{y^*(w^-)}k(y')dF(y')}+$$

$$v_h(w,y^*(w))\left(1-\frac{w^-}{w}\right)$$

The first term follows from the bounded convergence theorem and Lemma 8.2. The second term follows from the substitution made above, and from the fact that $y_n^*(w)-y_n^*(w_{j^n-1})$ converges to zero with n (if not, the probability of being hired at wage w for traders between $y_n^*(w)$ and $y_n^*(w_{j^n-1})$ goes to zero. This also reduces the last term to zero because the derivative of v with respect to y is bounded (and the term multiplying it is less than 1).

The last part of the argument is to show that

$$y^*(w)=\sup\{y:\omega(y)\leq w\}$$

Suppose the contrary that for some w, $y^*(w)>\sup\{y:\omega_n(y)\leq w\}=y_n^*(w)$ for all large n. Observe that for each n, $\omega_n(y_n^*(w))\geq w$. Furthermore, note that a worker of type $y_n^*(w)$ has a type that is at least as high as any other worker who applies at wage w. So such a worker is hired for sure at wage w. Let $y_0=\lim_{n\to\infty}\sup\{y:\omega_n(y)\leq w\}<y^*(w)$.

At the other extreme, if $y^*(w)$ is not a continuity point of ω, then since the latter function is right continuous and non-decreasing, there is a point $y_0<y_1<y^*(w)$ at which ω is continuous (and $\omega(y_1)\leq w$). For large n, it must be that $\omega_n(y_1)>w$ since otherwise $y_n^*(w)$ would be at least as large as y_1. Yet since y_1 is a continuity point of ω and ω_n converges weakly to ω, $\omega_n(y_1)\to\omega(y_1)$.

Then using Lemma 8.2, the payoff to a worker of type $y_n^*(w)$ who applies at the wage $\omega_n(y)$ is converging to

$$we^{-\int_{y_0}^{y_1}k(y')dF(y')}<w$$

This contradicts the property that workers should receive the same expected payoff by applying to all wages that are at least as large as their reservation wage.

A similar argument establishes a contradiction when $y_0 = \lim_{n\to\infty} \sup\{y : \omega_n(y) \le w\} > y^*(w)$.

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