Default Risk and Collateral in the Absence of Commitment*

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Abstract

Default risk is an important concern for lenders and is a main reason they require borrowers to pledge collateral. There are two reasons for this. The first is that collateral provides some incentive for the borrower to not strategically default. The second is that, in the event of default, the lender can liquidate the collateral and salvage some value from the failed credit relationship. This paper provides a model to study properties of allocations that arise when collateral is part of an optimal lending contract that looks much like a repurchase agreement. In particular, a lack of commitment to future actions implies that collateral must be used to alleviate strategic default. Moreover, because collateral is held by lenders during the credit relationship, there is also a potential incentive for lenders to default on returning collateralized assets. Thus, the optimal contract requires the satisfaction of an incentive constraint for the

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lender, in addition to the one that must be satisfied for the borrower. The paper then discusses how the need to satisfy both constraints places certain restrictions on the allocations that arise when collateral is part of an optimal contract. We conclude by comparing the allocation to a world where agents can commit to future actions.

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1 Introduction

Recent struggles in financial markets have drawn attention to the role of secured credit for lending activity. The declining value of collateral in financial markets has emphasized two important roles for collateral. First, the value to the borrower of the asset pledged as collateral has important implications for the incidence of strategic default. Consequently, collateral provides a borrower with incentive to repay his debt. The greater the borrower's valuation of the collateral, the greater the incentive to repay to avoid forfeiting it. Second, the value to the lender of the asset pledged as collateral has important implications for the degree of insurance that collateral can provide a lender in the event of a default. The greater the lender's valuation of the collateral, the greater the level of insurance that collateral provides.

The goal of this paper is to present an environment in which collateral arises endogenously as part of an optimal financial contract to mitigate strategic default of borrowers and provide insurance to lenders. That is to say, the use of collateral is a necessary part of a constrained-efficient solution that overcomes frictions that would otherwise prevent certain transactions from taking place. This view is similar to models with collateral that are studied in Kehoe and Levine (2008), Rampini (2005), and Lacker (2001). However, it contrasts with other models that study the economic impact of collateralized lending such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) where the role for collateral is not endogenous. Notably, our approach demonstrates that collateral plays a significant role in macroeconomic outcomes such as investment and risk sharing.

In the environment proposed here, we study two frictions that can give rise to collateral. This first is that agents cannot commit to future actions. Thus repayment of debt must satisfy incentive constraints in the spirit of Kehoe and Levine (1993). This provides an opportunity for collateral to mitigate strategic default by strengthening borrower incentive constraints. The second is the presence of idiosyncratic risk that leads to (exogenous) default by some borrowers at the optimum. This additional risk generates an insurance role for collateral in credit arrangements.

Our model is tied to economic fundamentals in that it imposes no institutional assumptions that facilitate trade. In particular, there is no public record-keeping of agent histories (or reports of histories), no repeated relationships between a borrower and a lender, and no enforcement technology by which collateral can be seized. The lack of record-keeping and repeated interactions implies that collateral can uniquely serve to overcome a lack of commitment by borrowers. Collateral strengthens the incentive constraints pertaining to the repayment of debt. Such constraints cannot be strengthened in this model by intertemporal punishments and/or rewards as is done in the typical literature on dynamic risk-sharing with private information, or via the exclusion from future financial contracting as is typical in the literature on limited commitment.[1]

The lack of an enforcement technology by which collateral can be seized implies that collateralized lending must work in the following way. An asset held by the borrower is transferred to the lender to serve as collateral with the expectation that the borrower buys it back in a later period. From this perspective, collateral in our model is a *repurchase agreement* which occurs in many real-world financial contracts

[1] Green (1987) and Kocherakota (1996) provide examples of dynamic risk-sharing arrangements. Kocherlakota (1996) is closest to our model in that his model has two types of risk-averse agents who both lack commitment. See Kehoe and Levine (1993) for an example of permanent punishment via the exclusion from future financial contracts. Alternatively, Krueger and Uhlig (2006) construct a model in which agents cannot be excluded from future financial contracts. In comparison to Kehoe and Levine (1993), Aiyagari and Williamson (2000) study risk-sharing arrangements in a monetary economy. Finally, Reed and Waller (2006) study the role of money for risk-sharing when intertemporal punishments and/or rewards are not possible.

(see Figure 1).

In almost all previous work on collateral, it is assumed that collateral can be seized in the event of a default.[2] Our point here is that, while it may be sufficient to assume some enforcement technology for this purpose, it is not necessary for collateral to emerge as part of an optimum. This is especially relevant if enforcement is costly as is often the case in actual bankruptcy proceedings.[3]

The second complication in our model is the presence of idiosyncratic risk. Borrowers have access to a productive but risky investment technology that may be unsuccessful. Thus, even if collateral can adequately overcome incentives to strategically default, collateral can mitigate the risk to lenders if a borrower cannot repay. In our framework, collateral does not directly provide any utility to a lender but can be liquidated in exchange for money. This is an important feature of the environment. If the lender directly values the collateral, the front end of the repurchase agreement will not resemble debt but act more like a bilateral exchange of goods. This requires that debts be repaid with fiat money, which has value to lenders because lenders can use the proceeds to acquire goods at a later date. In the event that a borrower defaults, a lender can keep the collateral and use it in a secondary market to acquire money.

Our environment contributes to the monetary theory literature that examines the interaction of money and collateralized lending. In particular, Shi (1996) and Mills (2004, 2006) model collateral as repurchase agreements in the absence of commitment. Ferraris and Watanabe (2008) assume an enforcement technology that permits the seizure of collateral in the event of default. Their model allows for the possible resale of collateral in the event of default, leaving open the possibility that collateral can provide insurance for the lender. Because there is no idiosyncratic risk that leads to default in any of these models, however, the impact of the insurance role is not studied.

Due to the lack of commitment on both sides of the financial transaction, the use of collateral generates an additional incentive constraint on the optimal financial contract

[2] Examples include Rampini (2005), Rampini and Viswanathan (2008), Lustig (2007), Kehoe and Levine (2008), Lacker (2001), and Krishnamurthy (2003).

[3] See Bliss (2003).

beyond what is standard in the literature. This incentive constraint applies to the lender and simply requires that the value of collateral to the lender not exceed the value of returning the collateral to the borrower. In our model, where the value of collateral to both the lender and the borrower is known with certainty, this incentive constraint, combined with the standard incentive constraint for the borrower, has an implication that the optimal contract in the absence of commitment leads to full insurance for the lender.

Our result contrasts with other models of collateral such as Lacker (2001), Manove, Padilla and Pagano (2001), Rampini (2005), and Kehoe and Levine (2008) that study the incentive role of collateral, but not on the insurance role. In those models, it is important that the collateral be less valuable to the lender than to the borrower so that some transactions can look like debt transactions as opposed to bilateral trade. While this is typically fine for models in which durable goods such as houses and cars serve as collateral and provide more value to the borrower than the lender, it is not able to explain secured lending involving financial assets such as government bonds. One value to our environment where collateral has resale value to acquire fiat money is that the optimum does not require such an assumption about the value of collateral to the lender and is more able to discuss financial assets that serve as collateral.

In our model, default by lenders will not happen at the optimum, but incentives to default will impact the optimal allocations. However, there are some examples in repo markets where some default by the lenders is occasionally observed.[4] This suggests that a lack of commitment by lenders in a repurchase agreement is not simply a theoretical abstraction.

In order to get a sense of how the lack of commitment and collateral affect allocations, we compare certain characteristics of the optimal contract in the absence of commitment to those that arise in a setting in which there is no commitment problem. When agents can commit to future actions, strategic default is not a concern,

[4]See Jordan and Jordan (1997) and Garbade (2006). Default in this market really means that creditors may choose to intentionally fail to return collateralized assets at the scheduled time in repurchase agreements. The penalties for doing so are usually just that the borrower does not have to pay the interest on the loan.

but exogenous default remains. In this setting, the optimal contract looks more like a standard risk-sharing arrangement between the two types of agents. Not surprisingly, the absence of incentive constraints suggests that a richer set of allocations is feasible. In contrast to the environment without commitment, both agents receive some insurance against the idiosyncratic risk. Moreover, as there are fewer constraints on the allocation of collateralized assets, investment in the risky technology increases so that expected income is higher.

The paper is organized as follows. Section 2 describes the environment. Section 3 describes the optimal contract when there is no commitment and shows some implications of such a contract. Section 4 describes the optimal contract when agents can commit to future actions and some properties of optimal allocations are presented. A comparison of the two optimal contracts is provided in Section 5. Section 6 concludes.

2 Environment

2.1 Preferences, Endowments and Technologies

The model is a pure exchange endowment economy of two-period-lived overlapping generations with two goods at each date, *good* α and *good* β. The economy starts at date $t = 1$. There is a $[0, 1]$ continuum of each of two types of agents, called *Type* A agents and *Type* B agents, born at every date. These two types are distinguished by their endowments, preferences and access to investment technologies.

Each type A agent is endowed with x units of good α when young and nothing when old. Each type B agent is endowed with y units of good β when young and nothing when old.

Let $a^t_{zt'}$ denote consumption of good $z \in \{\alpha, \beta\}$ at date t' by a type A agent of generation-t. The utility of a type A agent is $u^A(a^t_{\alpha t}, a^t_{\beta t})$ where $u^A : \Re^2_+ \to \Re$. Note that a type A agent wishes to consume both good α and good β when young. A type A agent does not wish to consume either good when old. We refer to type A agents as impatient agents. The function u^A is strictly increasing and concave in each argument, is C^1 and has first derivatives such that $u_z^A(0) = \infty$ and $u_z^A(\infty) = 0$ for each argument $z \in \{\alpha, \beta\}$.

Let $b_{zt'}^t$ denote consumption of good $z \in \{\alpha, \beta\}$ at date t' by a type B agent of generation-t. The utility of a type B agent born at date t is $u^B(b_{\alpha,t+1}^t, b_{\beta t}^t)$ where $u^B : \Re_+^2 \to \Re$. A type B agent wishes to consume good β when young and good α when old. We refer to type B agents as patient agents. The function u^B is strictly increasing and concave in each argument, is C^1, and has first derivatives such that $u_z^B(0) = \infty$ and $u_z^B(\infty) = 0$ for each argument $z \in \{\alpha, \beta\}$.

In addition to the endowments of goods α and β, there are also two agent-specific storage technologies involving good α. Type B agents have access to a perfect storage technology that transfers good α across stages within a period. Each unit stored remains available at stage 2. In contrast to type B individuals, type A agents can allocate good α to productive, yet risky, investment technologies. With probability η, the investment is unproductive and does not yield any output. In contrast, with probability $(1 - \eta)$, the investment technology generates R units of α. On average, the investment technology generates higher returns than B's storage technology. That is, $R > \frac{1}{1-\eta}$.

At date $t = 1$, there is a [0,1] continuum of initial old type B agents who are each endowed with M divisible units of fiat money.

2.2 Timing of Events within a Period

The sequence of events within a period is as follows. Each period t is divided into four stages. In the first stage, each generation-t type A agent is matched with a generation-t type B agent. Two activities can take place at this meeting. The first is a potential transfer of some of the good β from the B to the A agent. The second is an allocation of good α into the two agent-specific technologies. An amount, $\sigma \leq x$, can be transferred from the A to the B to be allocated to the (safe) storage technology. The remaining $x - \sigma$ is then invested in the A's risky technology. We assume that good β perishes at the end of this stage, so this is the only opportunity for A to acquire and consume some of good β. Both agents consume their allocation of good β before the end of the first stage.

At the beginning of the second stage, the return on the risky investment is realized. Then each generation-t type A is matched with a generation-$t-1$ type B. In a fraction η of meetings the A does not have any good α to offer and both types leave without it. In the remaining fraction $1-\eta$ of meetings the A agent has $R(1-\sigma)$ of good α to offer. In those meetings, good α is allocated between the two agents.

In the third stage, each generation-t type A is reunited with the generation-t type B agent he met in stage 1. This meeting presents an opportunity for the agents to allocate the σ units of good α between them. Any amount of good α that goes to A is consumed by that agent while the amount of good α that goes to B is carried into the fourth stage.

At the fourth stage of a period there is an aggregate meeting between all generation-t type B agents and all generation-$t-1$ type B agents. This is the final chance for old B's to acquire good α. The aggregate nature of the fourth stage implies that resources can be pulled together and redistributed (as in a market). Finally, we assume that good α perishes at the end of this stage so that it cannot be transferred to the next period.

2.3 Discussion of the Environment

It is useful to briefly comment on some of the important elements of the environment. Recall that our goal is to provide a model where collateral arises as part of an optimal contract to alleviate strategic default from borrowers, and to provide insurance to lenders in the event of any type of default.

Type A and type B agents differ with respect to their preferences and endowments, so both types would benefit from trade. The timing of trading opportunities, however, requires that a young type A individual obtain some of B's endowment before he has something of value to offer in return. Thus, for trade to take place, some type of credit must be extended to the young type A agent.

Because there is no commitment to future actions, a young type A agent must find it incentive compatible to repay the loan. To get around this problem, the type A agent's endowment of good α can be used as collateral. As we present in the next section, the collateral arrangement will work much like a repurchase agreement, such that the collateral is transferred from the type A agent to the type B agent in the first stage of the period. Because the good has value to the type A agent, there is an incentive for him to repay his debt at the third stage of the period in order to reacquire the collateral. The first and third stages, therefore, will correspond to the front and back ends, respectively, of a repurchase agreement.

In our model, the type A agents are sufficiently similar such that an optimal contract can be designed where there is no strategic default by type A agents at the third stage. The presence of some intrinsic risk in the environment, in the form of the risky investment project available to the type A agents, will mean that some type A agents exogenously default, and their type B lenders keep the pledged collateral.

This collateral, therefore, must have value to the type B agents if it is to provide insurance to lenders. In the model, the type B agents do not wish to consume good α when young. This is an important feature of the model because otherwise, there would not be a reason to engage in a credit relationship; agents could simply barter the goods at the first stage. Instead, type B agents wish to consume good α when old. In order for them to do so, they need fiat money to purchase the good at the second stage of old age (recall that good α is not storable beyond each period). It is here that young type A agents sell some of their endowment of good α for fiat money, which they can then use to repay their debts in the next stage.

If a young type A agent must default at the third stage, there is an opportunity in the fourth stage of the period for type B agents stuck with collateral to liquidate that collateral for fiat money. The type B agent can then use the money to buy good α when old. This opportunity at the fourth stage has an interpretation of a resale market for collateral. Thus, the collateral has some liquidation value that enables a lender to acquire goods at a future date.

3 Optimal Allocations in the Absence of Commitment

We begin by studying financial arrangements in the absence of commitment. In addition to a lack of commitment, there is no public record of agents' trading histories. These frictions combine with the specific sequence of events to generate a transactions role for money.[5]

Goods are allocated among four types of agents in this environment: type A and type B agents who have positive money balances and those who have zero money balances.[6]

Notation is needed to represent the amount of good α with which agents leave each stage. Let $z_\alpha^\tau(M)$ denote the amount of good α taken from stage $\tau \in \{2, 3, 4\}$ by agent $z \in \{A, B\}$ when there is monetary exchange. The notation $z_\alpha^\tau(0)$ represents similar allocations for the case when there is not any monetary exchange. Ex-ante steady state social welfare is:

$$\begin{aligned} U &= (1-\eta)u^A[a_\alpha^2(M) + a_\alpha^3(M), a_\beta] + \eta u^A[a_\alpha^3(0), a_\beta] \\ &\quad +(1-\eta)u^B[b_\alpha^2(M) + b_\alpha^4(0), b_\beta] + \eta u^B[b_\alpha^4(M), b_\beta]. \end{aligned} \tag{1}$$

Equation (1) represents every possible consumption opportunity for both types of agents.

Agents play the following game for each date t. In what follows we ignore certain outcomes that we anticipate will not occur in equilibrium. At the first stage of the

[5] See for example, Kocherlatoka (1998) for a general discussion, and Mills (2004) in the particular context of these types of models.

[6] There are actually a few more cases to consider, but can be ruled out in equilibrium. The details are available from the authors.

period, the mechanism suggests that each generation-t type B agent matched with a generation-t type A agent participate in trade by offering a_β units of good β to the type A agent. In return, each type A agent is suggested to promise to repay and transfers an amount, $\sigma \leq x$ of good α. Each agent in the meeting simultaneously chooses whether to participate in exchange or not. Assuming that both individuals agree to trade, type B's consumption of good β is equal to $b_\beta = y - a_\beta$. In addition, the type A individual invests the remaining amount of good α in his risky investment technology, $x - \sigma$. If either agent does not agree with the mechanism's suggestion, trade does not occur and both agents will leave stage 1 with autarky.

At the second stage of date t, each generation-t type A is matched with a generation-$t-1$ type B agent. Type A agents at this stage have either agreed or disagreed to trade in the first stage. Type A agents are further distinguished by whether their investment projects have been successful or not. Generation-$t-1$ type B agents are distinguished by their money balances which reflect their trading activity in the previous period. Some generation-$t-1$ type B agents received money at the third stage of date $t-1$ while some generation-$t-1$ type B's obtained money during the fourth stage of date $t-1$. Because the optimal contract will involve trade, and because of the symmetry in the model, the money balances of type B agents will be the same regardless of which stage at date $t-1$ they acquired money. We denote their money balances by M. Further, we can anticipate that no type B agent will have zero money balances.

The mechanism suggests that type A agents who did not receive any of good β in the first stage not trade. For a type A agent who did receive some good β in the first stage and who had a positive realization of his investment project, the mechanism suggests that he trade $b_\alpha^2(M)$ units of good α if his trading partner has M units of money, and nothing if his trading partner has zero units of money. For a type B agent with M units of money in a meeting with a type A agent who received some good β and who had a positive realization of his investment project, the mechanism suggests that she trade her money balances. If both agree, the trade is carried out. Otherwise, the agents leave the meeting with autarky.

At the third stage of date t, the generation-t type A and type B agents who were matched in the first stage are reunited. Type A agents are distinguished by their monetary balances acquired via trade in the second stage. Type A agents either have zero or M units of money. Type B agents are distinguished by whether or not they have σ units of good α. The mechanism suggests that a type A agent offer their money balances to the type B agent. It suggests that a type B agent offer $a_\alpha^3(M) = \sigma$ in return. If a type A agent does not have money balances, then the mechanism suggests that a type B agent offer $a_\alpha^3(0) = 0$ of good α. As before, if both agents in a meeting agree, the trade takes place. Otherwise, the agents leave in autarky.

Finally, at the fourth stage of date t, all generation-$t-1$ and generation-t type B agents are together in a meeting. The mechanism suggests that generation-t agents with σ units of good α offer them up in exchange for money. It also suggests that the generation-$t-1$ agents who have money balances offer to exchange all of their money for some of good α. The total amount of money that is offered by generation-$t-1$ agents is then evenly distributed to the generation-t agents who agreed to trade. Likewise, the total amount of good α offered by generation-t agents is evenly distributed to the generation-$t-1$ agents. Those who choose not to participate in exchange leave the meeting with autarky. The timing of events is summarized in Figure 2.

The relevant constraints include a number of feasibility and incentive constraints imposed by the timing of events and the lack of full information and commitment.

In the first stage, there are two relevant constraints. The first is that the allocation of good α devoted to both technologies sums to x so that

$$\sigma \leq x. \tag{2}$$

The second has to do with the allocation of good β between the two generation-t agents:

$$a_\beta + b_\beta \leq y. \tag{3}$$

There are also participation constraints for each type of agent. They reflect that the expected value of participating in trade must be at least as good as autarky. These are

$$(1-\eta)u^A[a_\alpha^2(M)+a_\alpha^3(M),a_\beta]+\eta u^A[a_\alpha^3(0),a_\beta] \geq \eta u^A[0,0]+(1-\eta)u^A[Rx,0] \quad (4)$$

for type A agents and

$$(1-\eta)u^B[b_\alpha^2(M)+b_\alpha^4(0),b_\beta]+\eta u^B[b_\alpha^4(M),b_\beta] \geq u^B[0,y] \quad (5)$$

for type B agents.

The second stage feasibility constraint states that agents cannot leave with more of good α than is available at that stage. For the $1-\eta$ meetings in which there is a positive return on the risky technology, the available amount of good α is just the realized return from the investment technology, $R(x-\sigma)$. We anticipate that type B agents participate in trade so that all type B agents enter the second stage with money balances M. The feasibility constraint is then

$$a_\alpha^2(M)+b_\alpha^2(M) \leq R(x-\sigma). \quad (6)$$

For the η meetings in which there is zero return on the risky technology, no goods can be exchanged. This means that the type A agent does not acquire money balances to take with him into the third stage and type B agents will leave the second stage with their money balances.

There are also participation constraints here as well. The first pertains to type A agents. A type A agent agrees to trade if

$$u^A[a_\alpha^2(M)+a_\alpha^3(M),a_\beta] \geq u^A[R(x-\sigma),a_\beta]$$

or simply

$$a_\alpha^2(M)+a_\alpha^3(M) \geq R(x-\sigma). \quad (7)$$

The left-hand side pertains to the fact that if the type A agent agrees to trade, he keeps $a_\alpha^2(M)$ of good α and can use money to acquire $a_\alpha^3(M)$ of good α in the third stage. If he does not agree to trade, then he keeps all of good α available at the second stage, but will have no money balances to acquire any of good α at the third stage ($a_\alpha^3(0) = 0$).

For a type B agent, the decision to participate in trade is a comparison of using her money to acquire good α at the second stage, versus using her money to acquire good α at the fourth stage, which can be expressed as

$$u^B[b_\alpha^2(M) + b_\alpha^4(0), b_\beta] \geq u^B[b_\alpha^4(M), b_\beta]$$

which, because the mechanism suggests $b_\alpha^4(0) = 0$ simplifies to

$$b_\alpha^4(M) \leq b_\alpha^2(M). \tag{8}$$

Note that $b_\alpha^4(M)$ is the amount of good α that a type B agent can expect to consume if she enters the fourth stage with money. This would be the case either if she did not have a trading opportunity in the second stage (because she was matched with a type A agent who suffered a negative shock), or she chooses not to participate in a trade when there is an opportunity.

For the third stage, note that the mechanism suggests

$$a_\alpha^3(0) = 0 \tag{9}$$

and

$$a_\alpha^3(M) = \sigma. \tag{10}$$

Both suggestions are feasible and satisfy participation constraints. The fact that $a_\alpha^3(0) = 0$ in (9) is a product of the fact that there is no commitment and no public record of past transactions. That is to say, if $a_\alpha^3(0) > 0$, a type B agent would have to sacrifice some good α for nothing in return. Thus, $a_\alpha^3(0) = 0$ is the only allocation that would satisfy participation constraints. Constraint (9) represents the insurance role of collateral as a type B agent who is reunited with a type A agent who has no money can leave the third stage with σ units of good α to trade at the fourth stage for money. This is obviously preferred to giving any good α to the type A agent without receiving something in return.

Likewise, constraint (10) satisfies participation constraints. The type A agent trivially wants to consume any amount of good α over nothing, and the type B agent is willing to trade σ for money in order to consume some of good α when old. He is willing to trade now instead of at the fourth stage, because waiting to trade at the fourth stage does not lead to more money and, therefore, more consumption when old than does trading at the third stage.

In the fourth stage, all of the remaining good α gets transferred from the generation-t B's to the generation-$t-1$ B's. Recall that all type B's of each generation meet at this stage. The amount of good α available is that which remains after the generation-t B's transferred some to generation-t A's at the third stage. A fraction, η, of the generation-t B agents were in a third stage meeting with an A agent that had a bad shock and bring σ units of good α to the fourth stage. Similarly, a fraction $(1-\eta)$ of generation-t B agents were in a third stage meeting with an A agent that had a good shock and bring zero units of good α to the fourth stage.

The available good α is distributed to generation-$t-1$ type B agents with positive money balances (a fraction η of all type B agents). Recall that the mechanism suggests

$$b_{\alpha}^{4}(0) = 0 \tag{11}$$

for generation-$t-1$ type B agents who enter the fourth stage without money. As was the case with (9), the lack of commitment and public record of transactions eliminates the possibility that generation-t type B agents give away some of good α to generation-$t-1$ type B agents at this stage. Thus, the feasibility constraint on the fourth stage is

$$\eta b^4_\alpha(M) \leq \eta\sigma$$

or simply

$$b^4_\alpha(M) \leq \sigma. \tag{12}$$

Participation constraints for those with money and those with good α are trivially satisfied. For the generation-t agents, they wish to trade some of good α when young for money used to acquire some of good α when old. For the generation-$t-1$ agents, this is their last opportunity to acquire some of good α for consumption.

Finally, we can now express the following definition.

Definition 1 *The* ***optimal allocation in the absence of commitment*** *is a list* $\{\sigma, a_\beta, a^2_\alpha(M), a^3_\alpha(0), a^3_\alpha(M), b_\beta, b^2_\alpha(M), b^4_\alpha(0), b^4_\alpha(M)\} \in \Re^+$ *that maximizes (1) subject to (2) – (12).*

In order to solve this optimization problem, we set up a Kuhn-Tucker Lagrangian. We plug constraints (9)-(11) into (1) and note that (2) is already incorporated in the other constraints. Thus, we have the following Lagrangian

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & (1-\eta)u^A[a^2_\alpha(M)+\sigma, a_\beta] + \eta u^A[0, a_\beta] \\
& +(1-\eta)u^B[b^2_\alpha(M), b_\beta] + \eta u^B[b^4_\alpha(M), b_\beta] \\
& +\lambda_1[y - a_\beta - b_\beta] \\
& +\lambda_2[R(x-\sigma) - a^2_\alpha(M) - b^2_\alpha(M)] \\
& +\lambda_3[\sigma - b^4_\alpha(M)] \\
& +\gamma_1[a^2_\alpha(M) + a^3_\alpha(M) - R(x-\sigma)] \\
& +\gamma_2[b^2_\alpha(M) - b^4_\alpha(M)] \\
& +\gamma_3\{(1-\eta)u^A[a^2_\alpha(M)+a^3_\alpha(M), a_\beta] + \eta u^A[a^3_\alpha(0), a_\beta] - \eta u^A[0,0] - (1-\eta)u^A[Rx, 0]\} \\
& +\gamma_4\{(1-\eta)u^B[b^2_\alpha(M)+b^4_\alpha(0), b_\beta] + \eta u^B[b^4_\alpha(M), b_\beta] - u^B[0,y]\}
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the multipliers associated with the remaining feasibility constraints, and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the multipliers associated with the remaining participation constraints. We can anticipate that λ_1, λ_2, and λ_3 will all be greater than zero and the associated constraints will bind. Otherwise, resources would be wasted.

The main result comes directly from the characterization of the agents' participation constraints.

Proposition 1 $\sigma = b_\alpha^2(M)$.

Proof. First, note that (12) holds with equality so that $\sigma = b_\alpha^4(M)$. Thus, (8) becomes

$$\sigma \leq b_\alpha^2(M). \tag{13}$$

Because (6) holds at equality we have

$$a_\alpha^2(M) = R(x - \sigma) - b_\alpha^2(M).$$

(7) reduces to

$$a_\alpha^3(M) \geq b_\alpha^2(M). \tag{14}$$

Since the mechanism suggests that $a_\alpha^3(M) = \sigma$, we have

$$\sigma \geq b_\alpha^2(M). \tag{15}$$

Thus, constraints (13) and (15) generate the result. ■

Constraint (15) says that the value of collateral must be at least as large as what it will cost the borrower to obtain it. This is the incentive role of collateral. Here, a type A agent gives up $b_\alpha^2(M)$ to acquire money that he can use later to repay his debt and receive his collateral. Thus, the value of collateral must be at least as large as $b_\alpha^2(M)$.

Constraint (13) says the value of collateral to the lender must be at least as large as the value of reneging on the agreement and selling σ units of good α at the fourth stage.

Taken together, this pins down the amount of good α a type B agent acquires at the second stage when matched with a type A agent with a successful investment project. This leads to the following corollary.

Corollary 1 $b_{\alpha}^{2}(M) = b_{\alpha}^{4}(M)$.

The implication is that when collateral is part of the optimal contract in this environment, the lenders, type B agents, receive full insurance over the risky project. Because $a_{\alpha}^{3}(0) = 0$, type A agents do not receive insurance against the intrinsic risk.

4 Optimal Allocations with Full Commitment

In this section we characterize certain properties of optimal allocations under the assumption that agents can commit to trades. Under such an assumption, neither money nor collateral will be needed to conduct transactions. This serves as a benchmark with which to compare the properties of optimal allocations when commitment is not possible. If agents can commit, there is no need to keep track of incentive constraints. Not surprisingly, fewer constraints leads to potentially better outcomes. These outcomes resemble more risk-sharing among the type A and type B agents, and more flexible portfolio allocations into the two technologies.

There are two decisions that must be made. As in the previous section, the first is how to allocate good α between the A agents' risky investment technology and B agents' safe storage technology. The second is how to allocate both types of goods among the agents given the feasibility constraints imposed by the sequence and make-up of meetings. Goods are allocated among four types of agents: type A agents who have a positive investment realization, type A agents that do not, type B agents who are matched in the third stage with a type A agent who had a positive investment realization, and type B agents who are matched in the third stage with a type A agent who did not have a positive investment realization.

Optimal allocations are those that maximize ex-ante expected steady state utility for type A and B agents. For steady-state consumption levels of good α, we denote $a_\alpha(1)$ and $b_\alpha(1)$ as the consumption of good α when in a third-stage match with a positive investment realization for the type A agent for A and B agents respectively. We denote $a_\alpha(0)$ and $b_\alpha(0)$ as the consumption of good α when in a third-stage match without a positive investment realization for the type A agent for A and B agents respectively. We denote steady-state consumption levels of good β simply as a_β and b_β because these consumption levels are determined before any uncertainty is revealed. Ex-ante expected steady state social welfare is

$$\begin{aligned} U &= (1-\eta)u^A[a_\alpha(1), a_\beta] + \eta u^A[a_\alpha(0), a_\beta] \\ &\quad + (1-\eta)u^B[b_\alpha(1), b_\beta] + \eta u^B[b_\alpha(0), b_\beta]. \end{aligned} \tag{16}$$

Figure 3 summarizes the events that take place when there is commitment.

In contrast to an economy without commitment, the relevant constraints only involve feasibility constraints on trades at each stage. The first is that the allocation of good α among the two technologies sums to x so that

$$\sigma \leq x. \tag{17}$$

The second has to do with the allocation of good β between the two generation-t agents:

$$a_\beta + b_\beta \leq y. \tag{18}$$

The second stage feasibility constraint states that agents cannot leave with more of good α than is available at that stage. For the $1-\eta$ meetings in which there is a positive return on the risky technology, the available amount of good α is just the realized return from the investment technology, $R(x-\sigma)$. The feasibility constraint is then

$$a_\alpha^2(1) + b_\alpha^2(1) \le R(x - \sigma). \tag{19}$$

For the η meetings in which there is zero return on the risky technology, no goods can be exchanged (i.e., $a_\alpha^2(0) = b_\alpha^2(0) = 0$).

The feasibility constraints for the third stage are similar to those in the second stage. In this case, the amount of good α available is σ, the amount generation-t A's gave to generation-t B's in the first stage. Thus, the amount of good α transferred to type A agents must satisfy

$$a_\alpha^3(0) \le \sigma \tag{20}$$

for meetings in which the type A agent had a bad realization in the second stage, and

$$a_\alpha^3(1) \le \sigma \tag{21}$$

for meetings in which the A agent had a good realization in the second stage.

In the fourth stage, all of the remaining good α gets transferred from the generation-t B's to the generation-$t-1$ B's. Recall that all type B's of each generation meet at this stage. The amount of good α available is that which remains after the generation-t B's transferred some to generation-t A's at the third stage. A fraction, η, of the generation-t B agents were in a third stage meeting with an A agent that had a bad shock and bring $(\sigma - a_\alpha^3(0))$ units of good α to the fourth stage. Similarly, a fraction $(1-\eta)$ of generation-t B agents were in a third stage meeting with an A agent that had a good shock and bring $(\sigma - a_\alpha^3(1))$ units of good α to the fourth stage.

The available good α is distributed to two types of generation-$t-1$ B's: those who were matched in the second stage with a generation-t type A with a positive return on the investment technology (a fraction $1-\eta$) and those who were matched with an A agent that did not (a fraction η). Thus, the feasibility constraint on the fourth stage is

$$\eta b_\alpha^4(0) + (1-\eta) b_\alpha^4(1) \le \eta(\sigma - a_\alpha^3(0)) + (1-\eta)(\sigma - a_\alpha^3(1))$$

which can be rewritten as

$$\eta b_{\alpha}^{4}(0)+(1-\eta)b_{\alpha}^{4}(1)+\eta a_{\alpha}^{3}(0)+(1-\eta)a_{\alpha}^{3}(1)\leq\sigma. \tag{22}$$

Finally, we have the consumption levels of good α for generation-t type A agents and generation-$t-1$ agents denoted

$$a_{\alpha}(0)=a_{\alpha}^{3}(0) \tag{23}$$

$$a_{\alpha}(1)=a_{\alpha}^{2}(1)+a_{\alpha}^{3}(1) \tag{24}$$

$$b_{\alpha}(0)=b_{\alpha}^{4}(0) \tag{25}$$

$$b_{\alpha}(1)=b_{\alpha}^{2}(1)+b_{\alpha}^{4}(1). \tag{26}$$

Note that when generation-t A agents and generation-$t-1$ B agents are in a second stage meeting in which there is a positive realization of the risky investment technology, they each get two opportunities to acquire good α for consumption: the second and third stages for the A agent and the second and fourth stages for the B agent. If these same agents are in a second stage meeting in which the realization from the technology is zero, however, they each only have one opportunity to acquire good α: the third stage for the A agent and the fourth stage for the B agent.

From this we can form the following definition.

Definition 2 *The **optimal allocation when there is full commitment**, is a list* $\{\sigma, a_{\beta}, a_{\alpha}^{2}(1), a_{\alpha}^{3}(0), a_{\alpha}^{3}(1), b_{\beta}, b_{\alpha}^{2}(1), b_{\alpha}^{4}(0), b_{\alpha}^{4}(1)\}\in\Re^{+}$ *that maximizes (16) subject to (17) – (26).*

In order to solve the optimization problem, we set up a Kuhn-Tucker Lagrangian. We plug constraints (23) – (26) into (16) and note that (17) is already incorporated in the other constraints. Thus, we have the following Lagrangian.

$$
\begin{aligned}
\pounds \;=\; & (1-\eta)u^A[a_\alpha^2(1)+a_\alpha^3(1),a_\beta]+\eta u^A[a_\alpha^3(0),a_\beta] \\
& +(1-\eta)u^B[b_\alpha^2(1)+b_\alpha^4(1),b_\beta]+\eta u^B[b_\alpha^4(0),b_\beta] \\
& +\lambda_1\,[y-a_\beta-b_\beta] \\
& +\lambda_2\left[R(x-\sigma)-a_\alpha^2(1)-b_\alpha^2(1)\right] \\
& +\lambda_3[\sigma-\eta b_\alpha^4(0)-(1-\eta)b_\alpha^4(1)-\eta a_\alpha^3(0)-(1-\eta)a_\alpha^3(1)] \\
& +\gamma_1[\sigma-a_\alpha^3(0)] \\
& +\gamma_2[\sigma-a_\alpha^3(1)]
\end{aligned}
$$

where $\lambda_1,\lambda_2,\lambda_3,\gamma_1,$ and γ_2 are the multipliers associated with the remaining constraints. We have different notation for the multipliers because we can anticipate that $\lambda_1,\lambda_2,$ and λ_3 will all be greater than zero and the associated constraints will bind. Otherwise, resources would be wasted. It is less obvious whether γ_1, and γ_2, are greater than or equal to zero because any remaining good α could be allocated to generation-$t-1$ B agents in the fourth stage. The first order conditions, anticipating that $\lambda_1,\lambda_2,$ and λ_3 are strictly positive then are as follows:

σ :

$$-\lambda_2 R+\lambda_3+\gamma_1+\gamma_2\leq 0 \tag{27}$$

$$\sigma\geq 0 \text{ with c.s.}$$

a_β :

$$(1-\eta)u_\beta^A[a_\alpha^2(1)+a_\alpha^3(1),a_\beta]+\eta u_\beta^A[a_\alpha^3(0),a_\beta]-\lambda_1\leq 0 \tag{28}$$

$$a_\beta\geq 0 \text{ with c.s.}$$

b_β :

$$(1-\eta)u_\beta^B[b_\alpha^2(1)+b_\alpha^4(1),b_\beta]+\eta u_\beta^B[b_\alpha^4(0),b_\beta]-\lambda_1\leq 0 \tag{29}$$

$$b_\beta \geq 0 \text{ with c.s.}$$

$a_\alpha^2(1)$:

$$(1-\eta)u_\alpha^A[a_\alpha^2(1)+a_\alpha^3(1),a_\beta] \leq \lambda_2 \tag{30}$$

$$a_\alpha^2(1) \geq 0 \text{ with c.s.}$$

$a_\alpha^3(1)$:

$$(1-\eta)u_\alpha^A[a_\alpha^2(1)+a_\alpha^3(1),a_\beta] \leq (1-\eta)\lambda_3+\gamma_2 \tag{31}$$

$$a_\alpha^3(1) \geq 0 \text{ with c.s.}$$

$a_\alpha^3(0)$:

$$\eta u_\alpha^A[a_\alpha^3(0),a_\beta] \leq \eta\lambda_3+\gamma_1 \tag{32}$$

$$a_\alpha^3(0) \geq 0 \text{ with c.s.}$$

$b_\alpha^2(1)$:

$$(1-\eta)u_\alpha^B[b_\alpha^2(1)+b_\alpha^4(1),b_\beta] \leq \lambda_2 \tag{33}$$

$$b_\alpha^2(1) \geq 0 \text{ with c.s.}$$

$b_\alpha^4(1)$:

$$(1-\eta)u_\alpha^B[b_\alpha^2(1)+b_\alpha^4(1),b_\beta] \leq (1-\eta)\lambda_3 \tag{34}$$

$$b_\alpha^4(1) \geq 0 \text{ with c.s.}$$

$b_\alpha^4(0)$:

$$\eta u_\alpha^B[b_\alpha^4(0),b_\beta] \leq \eta\lambda_3 \tag{35}$$

$$b_\alpha^4(0) \geq 0 \text{ with c.s.}$$

λ_1 :

$$y = a_\beta + b_\beta \tag{36}$$

$$\lambda_1 > 0$$

λ_2 :

$$R(x-\sigma) = a_\alpha^2(1) + b_\alpha^2(1) \tag{37}$$

$$\lambda_2 > 0$$

λ_3 :

$$\sigma = \eta b_\alpha^4(0) + (1-\eta) b_\alpha^4(1) + \eta a_\alpha^3(0) + (1-\eta) a_\alpha^3(1) \tag{38}$$

$$\lambda_3 > 0$$

γ_1 :

$$\sigma - a_\alpha^3(0) \geq 0 \tag{39}$$

$$\gamma_1 \geq 0 \text{ with c.s.}$$

γ_2 :

$$\sigma - a_\alpha^3(1) \geq 0 \tag{40}$$

$$\gamma_2 \geq 0 \text{ with c.s.}$$

We now can characterize some of the properties of optimal allocations with the following series of propositions.

Proposition 2 *The list* $\{\sigma, a_\beta, a_\alpha^3(0), b_\beta, b_\alpha^4(0)\} \in \Re_{++}^5$ *at an optimal allocation. Moreover,* $a_\alpha(1) = a_\alpha^2(1) + a_\alpha^3(1) > 0$ *and* $b_\alpha(1) = b_\alpha^2(1) + b_\alpha^4(1) > 0$.

Proof. Inada conditions require $a_\beta, a_\alpha(1), a_\alpha^3(0), b_\beta, b_\alpha(1)$, and $b_\alpha^4(0)$ to all be positive. It then follows from $a_\alpha^3(0) > 0$ and (39) that $\sigma > 0$. ■

It is immediate from Proposition 2 that at an optimal allocation with commitment both type A and type B agents consume some of good α regardless of the realization of the risky technology at stage 2. Thus, both types of agents receive some insurance against the bad outcome.

Using the results of Proposition 2 we can simplify a number of first-order conditions.

First, we rewrite (27):

$$\lambda_3 + \gamma_1 + \gamma_2 = \lambda_2 R \tag{41}$$

$$\sigma > 0.$$

Next, we can combine (28) and (29):

$$\begin{aligned} & (1-\eta)u_\beta^A[a_\alpha^2(1) + a_\alpha^3(1), a_\beta] + \eta u_\beta^A[a_\alpha^3(0), a_\beta] \\ = \; & (1-\eta)u_\beta^B[b_\alpha^2(1) + b_\alpha^4(1), b_\beta] + \eta u_\beta^B[b_\alpha^4(0), b_\beta] \end{aligned} \tag{42}$$

$$a_\beta > 0 \text{ and } b_\beta > 0.$$

Equations (30) and (31) yield:

$$(1-\eta)u_\alpha^A[a_\alpha^2(1) + a_\alpha^3(1), a_\beta] = \min\{\lambda_2, (1-\eta)\lambda_3 + \gamma_2\} \tag{43}$$

$$\begin{aligned} a_\alpha^2(1) &> 0 \text{ if } \min\{\lambda_2, (1-\eta)\lambda_3 + \gamma_2\} = \lambda_2 \\ &= 0 \text{ otherwise} \\ a_\alpha^3(1) &> 0 \text{ if } \min\{\lambda_2, (1-\eta)\lambda_3 + \gamma_2\} = (1-\eta)\lambda_3 + \gamma_2 \\ &= 0 \text{ otherwise} \end{aligned}$$

Finally, we can combine (33) and (34):

$$(1-\eta)u_\alpha^B[b_\alpha^2(1) + b_\alpha^4(1), b_\beta] = \min\{\lambda_2, (1-\eta)\lambda_3\} \tag{44}$$

$$\begin{aligned} b_\alpha^2(1) &> 0 \text{ if } \min\{\lambda_2, (1-\eta)\lambda_3\} = \lambda_2 \\ &= 0 \text{ otherwise} \\ b_\alpha^4(1) &> 0 \text{ if } \min\{\lambda_2, (1-\eta)\lambda_3\} = (1-\eta)\lambda_3 \\ &= 0 \text{ otherwise} \end{aligned}$$

From these simplified conditions, we can present the next proposition.

Proposition 3 *At an optimal allocation, $\gamma_2 = 0$ and constraint (40) is nonbinding.*

Proof. The proof is by contradiction. Suppose that $\gamma_2 > 0$. Then condition (40) requires $a^3_\alpha(1) = \sigma$ which is positive by Proposition 2. Then from $a^3_\alpha(1) > 0$ and condition (43) we get

$$(1-\eta)\lambda_3 + \gamma_2 \leq \lambda_2 \tag{45}$$

which means that

$$(1-\eta)\lambda_3 < \lambda_2 \tag{46}$$

and that by condition (44) $b^4_\alpha(1) > 0$. Given that by Proposition 2 $b^4_\alpha(0) > 0$ as well, and $a^3_\alpha(1) = \sigma$, condition (39) must be nonbinding and $\gamma_1 = 0$. Otherwise, there would be no good α to distribute among the type B agents at stage four and condition (38) would not be satisfied.

Using $\gamma_1 = 0$ and (45) we can manipulate (41) to get

$$\lambda_2 \leq \frac{\eta}{R-1}\lambda_3.$$

Now from (46) we have

$$(1-\eta)\lambda_3 < \frac{\eta}{(R-1)}\lambda_3$$

which simplifies to

$$R(1-\eta) < 1.$$

But it is assumed that

$$R > \frac{1}{(1-\eta)}$$

or

$$R(1-\eta) > 1.$$

Thus, we get a contradiction and $\gamma_2 = 0$. ■

The next result comes directly from Proposition 3.

Proposition 4 *An optimal allocation has the following properties:*

$$a_\alpha^2(1) \;>\; 0 \textit{ if and only if } b_\alpha^2(1) > 0$$
$$a_\alpha^3(1) \;>\; 0 \textit{ if and only if } b_\alpha^4(1) > 0.$$

Proposition 4 states that if type A agents consume good α in the good state at stage two, then so do type B agents, and vice versa. Likewise, if type A agents receive good α in stage three in the good state, then type B agents receive good α in the good state at stage four.

The next proposition further characterizes the optimal allocation.

Proposition 5 *An optimal allocation has $a_\alpha^2(1) > 0$ and $b_\alpha^2(1) > 0$.*

Proof. Suppose not. Next, by Proposition 4, $a_\alpha^2(1) = b_\alpha^2(1) = 0$. Proposition 2 then requires $a_\alpha^3(1) > 0$ and $b_\alpha^4(1) > 0$. From both (43) and (44), this implies that $\lambda_3(1-\eta) < \lambda_2$. From condition (37), $\sigma = x$ and there is no investment in the risky technology. As a result, the shock to the investment in good α should be irrelevant at the optimal allocation so that $a_\alpha^3(0) = a_\alpha^3(1)$ and $b_\alpha^4(0) = b_\alpha^4(1)$.

Now from Proposition 3 we know that $a_\alpha^3(1) < x$. Using $\lambda_3(1-\eta) < \lambda_2$ we can manipulate (41) to

$$\gamma_1 > \lambda_3[(1-\eta)R - 1]$$

which, because $R > \frac{1}{(1-\eta)}$ means that $\gamma_1 > 0$. This and condition (39) imply that $a_\alpha^3(0) = x > a_\alpha^3(1)$ and we get a contradiction. ■

The intuition from Proposition 5 is simple. The assumption that $R > \frac{1}{(1-\eta)}$ guarantees that agents prefer some positive investment in the risky technology. This means there will be some favorable outcomes in stage two and that the young Type A agents and the old Type B agents will share that favorable return at the second stage.

It remains to determine whether $a_\alpha^3(1) > 0$ and $b_\alpha^4(1) > 0$ are also part of an optimal allocation or $a_\alpha^3(1) = b_\alpha^4(1) = 0$ are. It turns out that we cannot rule out either case and so the next series of propositions describes properties of an optimal allocation under both cases.

Proposition 6 *An optimal allocation with* $a^3_\alpha(1) > 0$ *and* $b^4_\alpha(1) > 0$ *has the following properties:*

$$\lambda_2 = (1-\eta)\lambda_3$$

$$\gamma_1 > 0$$

$$a^3_\alpha(0) = \sigma$$

$$a^2_\alpha(1) + a^3_\alpha(1) > a^3_\alpha(0)$$

$$b^2_\alpha(1) + b^4_\alpha(1) = b^4_\alpha(0)$$

and not all ratios of marginal rates of substitution are equal.

Proof. If $a^3_\alpha(1) > 0$ and $b^4_\alpha(1) > 0$ then from (43), (44), and Proposition 5 require $\lambda_2 = (1-\eta)\lambda_3$. This and condition (41) combine to yield

$$R(1-\eta) = 1 + \frac{\gamma_1}{\lambda_3}$$

which, because $R(1-\eta) > 1$, implies that $\gamma_1 > 0$. From (39) it is obvious that $a^3_\alpha(0) = \sigma$. Now from Proposition 2 and (32) we have

$$u^A_\alpha[a^3_\alpha(0), a_\beta] = \lambda_3 + \frac{\gamma_1}{\eta}.$$

Using (41) this becomes

$$u^A_\alpha[a^3_\alpha(0), a_\beta] = \lambda_2 R + \frac{(1-\eta)}{\eta}\gamma_1 > \lambda_2.$$

Thus, from (43) $a^2_\alpha(1) + a^3_\alpha(1) > a^3_\alpha(0)$. Finally, using Proposition 2 and comparing (35) and (44) it is obvious that $b^2_\alpha(1) + b^4_\alpha(1) = b^4_\alpha(0)$. ■

The key result from Proposition 6 is that the stage three resource constraint binds. Another observation is that, in the bad state, the type A agent receives his collateral back. Type A agents receive only partial insurance while type B agents receive full insurance. Regardless, in such a scenario, consumption smoothing among the types and the goods is not complete.

Proposition 7 *An optimal allocation with* $a_\alpha^3(1) = b_\alpha^4(1) = 0$ *and* $\gamma_1 > 0$ *has the following properties:*

$$\lambda_2 < (1-\eta)\lambda_3$$

$$a_\alpha^3(0) = \sigma$$

$$b_\alpha^4(0) = \frac{1-\eta}{\eta}\sigma$$

$$a_\alpha^2(1) > a_\alpha^3(0)$$

$$b_\alpha^2(1) > b_\alpha^4(0)$$

and not all of the ratios of marginal rates of substitution are equal.

Proof. If $a_\alpha^3(1) = b_\alpha^4(1) = 0$ then from (43), (44), and Proposition 5 require $\lambda_2 < (1-\eta)\lambda_3$. Given that $\gamma_1 > 0$ it is obvious from (39) that $a_\alpha^3(0) = \sigma$. Thus, from (38) we have $b_\alpha^4(0) = \frac{1-\eta}{\eta}\sigma$.

By a similar argument to the proof of Proposition 6 we get $a_\alpha^2(1) > a_\alpha^3(0)$. Given $\lambda_2 < (1-\eta)\lambda_3$ it is obvious that $\lambda_2 < \lambda_3$ and by conditions (35) and (44) we get $b_\alpha^2(1) > b_\alpha^4(0)$. ■

As with the previous case, the stage three resource constraint binds for type A agents that had a negative realization of the investment technology. In this case, however, both type A and type B agents receive only partial insurance. However, there is still not complete consumption smoothing across agents and states.

Proposition 8 *An optimal allocation with* $a_\alpha^3(1) = b_\alpha^4(1) = 0$ *and* $\gamma_1 = 0$ *has the following properties:*

$$\lambda_2 < (1-\eta)\lambda_3$$

$$a_\alpha^2(1) > a_\alpha^3(0)$$

$$b_\alpha^2(1) > b_\alpha^4(0)$$

and all the ratios of marginal rates of substitution are equal.

Proof. If $a_\alpha^3(1) = b_\alpha^4(1) = 0$ then from (43), (44), and Proposition 5 require $\lambda_2 < (1-\eta)\lambda_3$. Given that $\gamma_1 = 0$ it is obvious from (39) that $a_\alpha^3(0) < \sigma$. Proving $a_\alpha^2(1) > a_\alpha^3(0)$ and $b_\alpha^2(1) > b_\alpha^4(0)$ follows a similar approach to that in the proof of Proposition 7. ■

In this case the stage three resource constraint for type A agents with bad investment outcomes does not bind. Thus, the ratios of MRS are equalized across types, goods and states. This is the first best allocation of goods. There is only partial insurance for both types of agents.

5 Comparison of the Optimal Contracts

In this section, we compare the allocations that arise optimally in the absence of commitment with those that arise optimally when there is full commitment. Figure 4 summarizes the main differences.

As the table makes clear, the optimal contract without commitment is much more restrictive on the allocations. This is because the incentive constraints for both type A and type B agents restrict the amount of good α that goes to the type B agents to the amount of collateral that is pledged. Thus, when collateral is part of the optimal contract, the generation-$t-1$ type B agent receives a certain amount of good α for consumption regardless of the result of their borrower's return on the investment project. Moreover, the type A agent does not receive any insurance over the risky investment technology because it would require their type B lender to commit to giving away some of good α at the third stage.

When there is full commitment, it is now possible for type B agents to give away some of good α to the type A agents in the third stage. This leads to partial insurance for the type A agents. As a result, the constrained-efficient allocation distorts risk-sharing away at the expense of the type A agents to the potential benefit of the type B agents.

A second important difference is the fact that the portfolio decision of the agents between the investment technology and the storage technology is less constrained in the world without commitment. Again, because of commitment, more transfers across agents are implementable, and the amount invested in the storage technology can efficiently be allocated across agents, leading to a better distribution of good α between the investment technology and the storage technology.

We can summarize the cost of a lack of commitment in the following proposition.

Proposition 9 *The constrained-efficient outcome that arises when collateral is part of an optimal contract distorts both risk-sharing and portfolio allocations relative to the efficient outcome that arises when there is full commitment.*

6 Conclusion

The goal of this paper is to provide an environment in which collateral arises endogenously as part of an optimal contract. In our model collateral serves two roles. The first role, an incentive role, is meant to deal with the typical principal-agent problem of moral hazard in financial contracts. Collateral must have sufficient value to the borrower such that he does not want to forfeit it by failing to repay a loan. Collateral also has an important insurance role. The ability to liquidate collateralized assets protects lenders from all sources of default risk, not just those due to strategic default. Collateral must have sufficient value to the lender such that, in the event of default, it can be liquidated by the lender at a later date.

By studying these two roles, the model demonstrates the importance of the value of collateral to both borrowers and lenders. Thus, it provides some insight into recent struggles in financial markets. In particular, the value of collateralized assets declined dramatically in late 2007 and early 2008. This loss in value meant that some types of securities were no longer effective at providing adequate insurance for creditors in addition to providing less incentive for borrowers to repay loans. To study these issues, our environment could be extended to add some riskiness to the value of collateral. Collateral itself could be risky by assuming that lenders' storage technology is risky. Alternatively, there could be some settlement shock to repayment of loans and/or access to the secondary market at the fourth stage (as in Freeman (1996)). In any case, the value of insurance provided would be lower, perhaps leading to much larger amounts of collateral required for credit to be extended. Moreover, investment and expected income would be much lower.

Recall that the model placed no institutional assumptions on the ability to enforce or commit to transactions. Thus, the model could be extended to think about the impact of additional institutions that improve upon the constrained-efficient allocation without commitment. For example, recent growth in tri-party repos has emerged as one way to provide risk protection to both lenders and borrowers. In particular, agents that provide tri-party repo services will hold onto securities pledged as collateral, protecting the borrower from a lender's incentive to fail to return the collateral. Such financial services could emerge as an optimal mechanism in the model, but would likely require some more institutional assumptions.

Finally, the model could be decentralized to introduce pricing and monetary policy analysis. The standard view of inflation is that it affects the distribution of income in favor of borrowers at the expense of lenders. Given that the constrained-optimal allocation in our model provides little insurance to borrowers, an inflationary monetary policy may undo some of that distortion. Moreover, monetary policy could also have an impact on the value of collateral in our model through its impact on incentive constraints. This would also lead to changes in the amount of investment and risk-sharing in credit markets.

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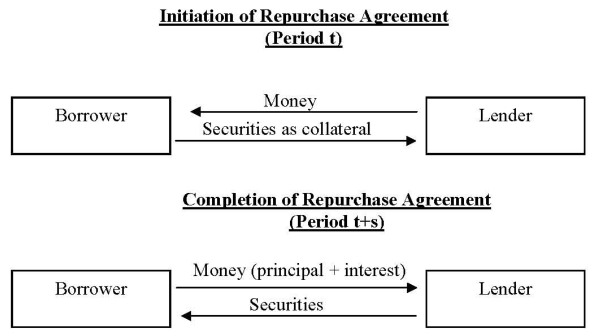

Figure 1: Simple Repurchase Agreement

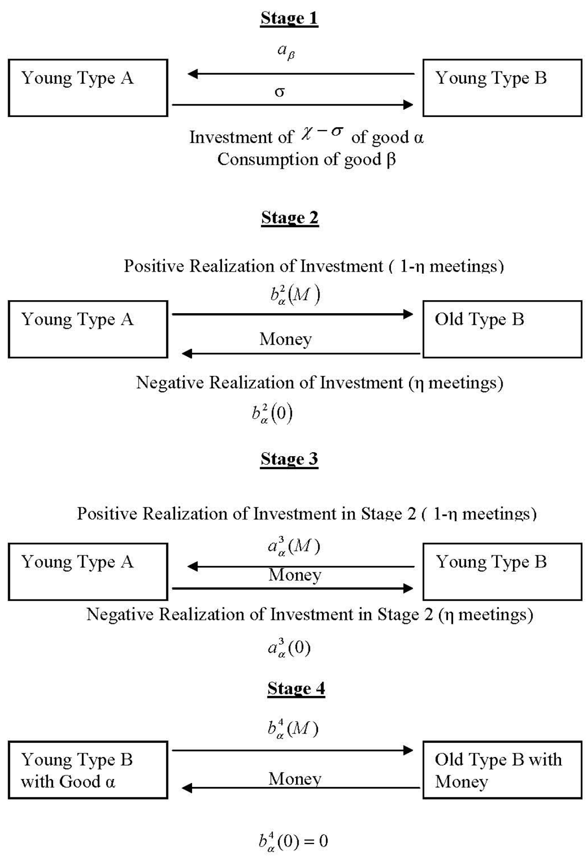

Figure 2: Timing of Events without Commitment

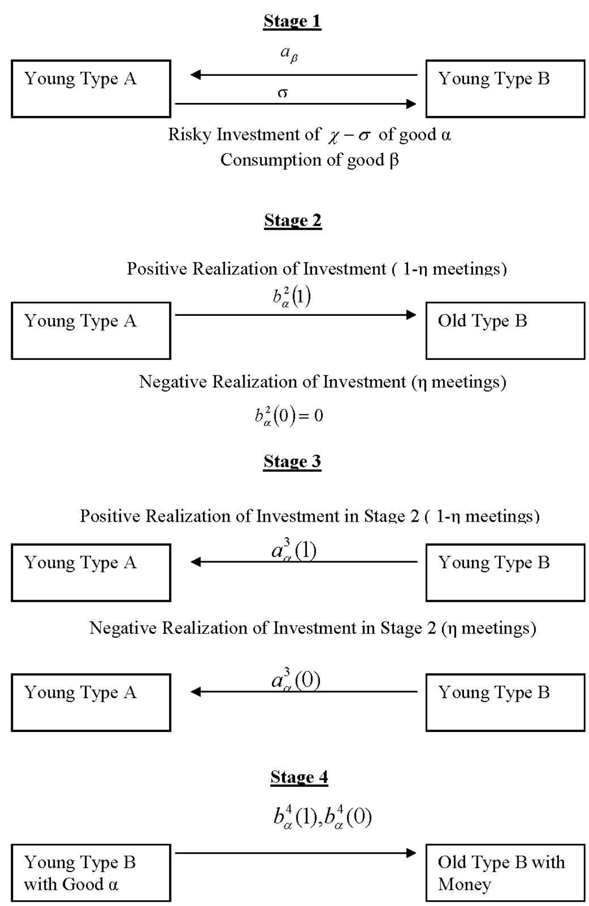

Figure 3: Timing of Events with Commitment

	No Commitment	Commitment
Stage 2	$a^2_\alpha(M) = R(x - \sigma) - \sigma$ $b^2_\alpha(M) = \sigma$	$a^2_\alpha(1) > 0$ $b^2_\alpha(1) > 0$
Stage 3	$a^3_\alpha(0) = 0$ $a^3_\alpha(M) = \sigma$	$a^3_\alpha(0) > 0$ $a^3_\alpha(1) < \sigma$
Stage 4	$b^4_\alpha(0) = 0,$ $b^4_\alpha(M) = \sigma$	$b^4_\alpha(1) \geq 0$ $b^4_\alpha(0) > 0$
Type A Insurance Type B Insurance	No Full	Some Some or full

Figure 4: Comparison of Allocations in Two Environments