Liquidity Constraints in a Monetary Economy*

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Preliminary

Abstract

This paper presents a model where money plays a dual role as medium of exchange and provider of liquidity.

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1 Introduction

Money is the medium used to transfer resources on the spot. Liquidity refers to the availability of a medium to transfer resources over time. This paper is concerned with the relationship between money as a medium of spot trade and a medium of trade over time. We take the point of view that the former role of money is more primitive than the latter. Thus our question is whether money, which already exists in society as a medium of exchange in order to overcome a fundamental impediment to spot trade, can also provide liquidity services. Liquidity is not

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an issue if the market for credit instruments works smoothly. Indeed, if agents could pledge the full value of future returns, they wouldn't have reasons to hold any instrument for liquidity purposes. We consider a world where the fundamental impediment arising in spot trade seeps into the credit market and hinders trade over time, preventing agents from being able to pledge their entire future returns. In such a world money naturally performs two roles, as a provider of exchange and liquidity services.

Our world is inhabited by entrepreneurs and investors. Investors produce an input which entrepreneurs need in order to complete a project paying off in the future. Entrepreneurs and investors meet on an investment market where they contract upon the exchange of the input. Before the project is completed the input may, with some probability, break down. Some - randomly selected- entrepreneurs can, however, transform the input into another good which can be sold on a liquidation market and can serve as additional input for the remaining entrepreneurs. Unlike the market for investment, which is not per se hindered by any friction, the liquidation market is opaque, transactions cannot be observed and traders are anonymous when operating there. It is also a market where buying and selling an input is subject to idiosyncratic production opportunities and needs. In sum, it is a market where money is used as a medium of exchange. The informational friction hindering spot trade on this market affects the investment market. Output produced with the input bought on the liquidation market can always be privately consumed when the liquidation market closes and cannot, therefore, be pledged to investors. This drives a wedge between the full value of an entrepreneur's project - which includes output produced after visiting the liquidation market- and the value that can be pledged to investors -which doesn't include it-. When - because of technological reasons- the pledgeable returns are sufficiently low, entrepreneurs offer some money to investors in an effort to relax their borrowing constraint. In this case, money performs the role of provider of liquidity as well as medium of exchange.

Our paper shares the view of the money search literature (Kiyotaki and Wright (1989) and Lagos and Wright (2005)), which places money as a medium of exchange at the centre of the stage. We adopt the definition of liquidity used by Holmstrom and Tirole (1998) and (2001). Unlike them, we think liquidity issues arise naturally in an economy where money plays the medium of exchange role. We share with Kiyotaki and Moore (2001) the interest in the relationship between money and liquidity, but we reverse their link of causality. For them "evil is the root of all money." We stick to the biblical "money is the root of all evil" (The Bible, 1 Timothy 6:10). We are not the first to think of the business of liquidating assets as, at times, a rough one. Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2007) were there first. Unlike us, they study trade mediated by specialists.

The rest of the paper unfolds as follows. Section 2 presents the model and solves it. Section 3 discusses monetary policy. Section 4 discusses the results. Section 5 concludes. The proofs are in the Appendix.

2 The Model

2.1 The environment

We use a competitive version of the divisible money model developed by Lagos and Wright (2005). Time is discrete and continues for ever. At the start of each period the economy is inhabited by a $[0, 1]$ continuum of homogeneous entrepreneurs and a $[0, 1]$ continuum of homogeneous investors. Each period is divided into three sub-periods: morning, afternoon and evening. Agents derive utility given by $U(y) = y$ from consuming an amount y of final output in any of the three sub-periods. In each sub-period there is one production opportunity. Economic activities in each sub-period are as follows.

Morning. At the beginning of each morning, each investor produces a good. The good is worth zero in the hands of the investor, but entrepreneurs are able to use it as an input to produce a perishable final output. We shall refer to the investors' products as *primary inputs.* During the morning each entrepreneur is randomly matched to one investor. Investors are competitive. An entrepreneur offers a contract to an investor which specifies an amount of the primary input the investor supplies to the entrepreneur and its payment. An investor who has traded with an entrepreneur can meet him during the morning and the afternoon of the current period but not at any other future time. We will be more specific about the terms of the contracts below. Denote by $g(q_1)$ the production technology of primary inputs q_1. The use of the primary inputs is subject to randomness: before production occurs, the primary input q_1 turns out to be intact with probability $\sigma \in (0, 1)$ in which case it can produce $g(q_1)$ units of the final output during the morning, while the input breaks down with probability $1 - \sigma$ in which case it becomes useless for the technology $g(\cdot)$ and produces nothing.

Afternoon. After the day market has closed, another market opens during the afternoon. This is a liquidation market where entrepreneurs can exchange among each other a second input, which is perishable and can be produced using the primary inputs, for further production. We shall refer to the input traded in the liquidation market as *secondary inputs.* The liquidation market is competitive and so agents take the market price, denoted by p, as given. There is an intrinsically worthless good, which is perfectly divisible and storable, called fiat money. We model monetary trade in the afternoon market following the spirit of the monetary search model of Kiyotaki and Wright (1989). There are two main ingredients. First, the trades in the liquidation market are anonymous, and so the trading histories of agents are private knowledge. This implies, among other things, that investors cannot observe the outcome in the liquidation market. Second, entrepreneurs face randomness in their preferences and production possibilities. At the beginning of each afternoon, an entrepreneur is selected to be a seller with probability $1 - \delta \in (0, 1)$. Once a seller, the entrepreneur does not wish to consume any final output in the afternoon but can transform his primary inputs into the secondary inputs, and sell them on the liquidation market. Conversely, an entrepreneur is selected to be a buyer with probability δ. Once a buyer, the entrepreneur does not have the transformation technologies but wishes to consume final output. He can produce the output using the secondary inputs that can be obtained in the liquidation market. We denote by

$h(q_2)$ the production technology of a secondary input q_2. The amount of buyers' final output depends on the amount of both the primary and secondary inputs: if a buyer has only the secondary input q_2 then he can produce $h(q_2)$ units of (perishable) final output; if a buyer has the primary input $q_1 > 0$ still intact, as well as q_2, then he can produce $g(q_1) + h(q_2)$ units during the afternoon. Both of these inputs fully decay at the end of the afternoon.

Evening. During the evening there is another opportunity for production. Agents can produce an output with non-contractible effort. The evening market is walrasian and the output is traded at a per unit price normalized to unity. Fiat money can be traded for the output on this market at a price ϕ per unit.

Preferences and technologies. Agents discount future payoffs at a rate $\beta \in (0, 1)$ across periods, but there is no discounting between the three sub-periods. The marginal costs of all the production are measured in terms of utility, and we normalize all the marginal costs to be one. The transformation technologies are one to one, i.e. one unit of primary inputs can generate one unit of secondary inputs. The production functions $g(\cdot)$ and $h(\cdot)$ are both twice continuously differentiable, strictly increasing and concave in their arguments. They satisfy $g(0) = h(0) = 0$ and the Inada condition, $g'(0) = h'(0) = \infty$.

Money. The assumptions described above, i.e, the random buyer/seller division and the anonymity of transactions, are sufficient to ensure an essential role of money as a medium of exchange in the afternoon market: sellers must receive money for immediate compensation of their products (i.e., secondary inputs). The supply of fiat money is controlled by the government so that $M = \pi M_{-1}$, where M denotes the money stock at a given period and π denotes the gross growth rate of the money supply which we assume to be constant. Subscript -1 (or $+1$) stands for the previous (or next) period. New money is injected, or withdrawn, at the end of each period by lump-sum transfers or taxes at a rate denoted by τ. All agents receive transfers or are taxed equally.

2.2 Efficiency

Given the stochastic production opportunities described above, the planner selects an amount of the primary and secondary inputs, so as to maximize the total expected final output net of the production cost. The planner's problem in each period is described as

$$\max_{q_1,q_2\geq 0} \quad \sigma g(q_1) + \sigma\delta\left(g(q_1)+h(q_2)\right) + (1-\sigma)\delta h(q_2) - q_1 - \delta q_2$$
$$\text{s.t.} \;\; (1-\delta)q_1 \geq \delta q_2.$$

The first term in the objective function represents the total expected output produced during the morning. The amount q_1 is selected before the uncertainty about the use of the primary input is realized. Thus the total cost of morning production is given by q_1 because the primary inputs are generated by a unit mass of investors with unit marginal costs. The second and third terms represent the total output produced during the afternoon - among δ entrepreneurs who are able to produce during the afternoon a proportion σ can use both q_1 and q_2 while the remaining proportion $1-\sigma$ can use only q_2. The amount q_2 is selected after the uncertainty about the afternoon production has been realized. Thus, the total cost that is needed for the afternoon production is given by δq_2. The secondary inputs can be transformed only from the proportion $1-\delta$ of the primary inputs with the one to one technology. Hence, the planner faces the feasibility constraint that the total secondary input δq_2 is no greater than $(1-\delta)q_1$. Notice that there is a fraction $(1-\sigma)\delta$ of the primary inputs q_1 that cannot be transformed into the secondary inputs q_2. Thus, given the second-hand nature of the inputs, the fraction $(1-\sigma)\delta$ of the primary input q_1 can never generate any value in our economy.

The optimality conditions are

$$\sigma(1+\delta)g'(q_1) + (1-\delta)\lambda = 1 \tag{1}$$
$$h'(q_2) = 1+\lambda \tag{2}$$
$$\lambda\left((1-\delta)q_1 - \delta q_2\right) = 0 \tag{3}$$

where $\lambda \geq 0$ represents the multiplier for the feasibility constraint $(1-\delta)q_1 \geq \delta q_2$, and the dash "$'$" stands for the first derivative. (1) equates the social marginal expected benefit of increasing the primary input, which consists of an expected increase in productivity ($= \sigma(1+\delta)g'(\cdot)$) and an expected benefit of having a relaxed constraint ($= (1-\delta)\lambda$), to its social marginal cost ($= 1$). A similar interpretation applies to (2). (3) is the complementary slackness condition. Denote by q_1^*, q_2^*, λ^* the planner's solution.

Lemma 1 *The planner's solution exists, is unique and satisfies:*

1. $q_1^* = g'^{-1}(1/\sigma(1+\delta))$, $q_2^* = h'(1)$, $\lambda^* = 0$ *if and only if* $g'^{-1}(1/\sigma(1+\delta)) \geq \frac{\delta}{1-\delta}h'^{-1}(1)$*;*

2. $q_1^* > g'^{-1}(1/\sigma(1+\delta))$, $q_2^* < h'(1)$, $\lambda^* > 0$ *if and only if* $g'^{-1}(1/\sigma(1+\delta)) < \frac{\delta}{1-\delta}h'^{-1}(1)$*.*

2.3 Steady state monetary equilibrium

In what follows, we construct steady state monetary equilibria where agents of identical type take identical strategies, all real variables are constant over time and money is valued (i.e. $\phi > 0$).

A the start of each period, entrepreneurs are randomly matched to competitive investors, and offer a contract which involves a payment out of future resources in exchange for an amount of primary inputs. There are two important characteristics of the contracts we will describe below. First, in our environments long term contracts are not available because of the random matching process in a large economy: there is no chance for an entrepreneur and an investor who are matched in any given period to meet with each other again in the future. Second, the anonymity in transactions in the liquidation market implies that the final output of entrepreneurs during the afternoon cannot be pledged to outside investors. This is because investors cannot observe the buyer/seller status of entrepreneurs in the liquidation market and the outcome accrues privately to entrepreneurs. Thus, an entrepreneur who enters such a market can always claim without fear of repercussions that he has spent all his money holdings and consumed the entire returns, and holds no resources to pay out to the investor. This monetary nature of trades further implies that investors and entrepreneurs loose track of each other at the end of the afternoon, thereby no financial claims on the evening production, as well as on the afternoon production, can be written.

We assume that the morning output of entrepreneurs is fully pledgeable and that contracts between the entrepreneur and the investor can be made contingent on the morning production. Given the non-pledgeability described above, the payments must happen at the end of the morning, before the liquidation market opens in the afternoon and after the uncertainty about the primary input is realized. A contract between an entrepreneur and an investor specifies the amount q_1 of primary inputs that the entrepreneur buys from the investor and uses for production with technology $g(q_1)$, and its payment - the entrepreneur pays out an amount of output z and a fraction θ^G of his money holdings if his primary input turns out to be viable, and a fraction θ^B of his money holdings if the input turns out to be useless. Formally, z, θ^G, θ^B must satisfy:

$$\begin{aligned} \sigma\left(z+\theta^G\phi m\right)+(1-\sigma)\,\theta^B\phi m &= q_1; && (4)\\ z &\leq g\left(q_1\right), && (5)\\ \theta^i &\in [0,1], \quad i=G,B. && (6) \end{aligned}$$

Condition (4) is the participation constraint of the investor where the L.H.S. represents the expected payment of the entrepreneur and the R.H.S. is the production costs of the investor. Being investors competitive, the entrepreneur makes an offer so that the investor is indifferent between producing or not. The amount ϕm is the entrepreneur's real money holdings at the start of a given period. Condition (5) states that the payment with output cannot exceed the amount produced during the morning, and the conditions (6) state that the payment via money cannot exceed the money holdings of the entrepreneur at the start of the period.

Below, we describe the value function only for the entrepreneurs given that the investors will not carry any money from one period to the next. Although some investors will hold money between the afternoon and the evening, one can assume without loss of generality that they will spend it all in the evening of the same period. While there is no reason for the investors carry money into the future, entrepreneurs will need money in order to buy the secondary input on the liquidation market.

The evening: walrasian market. We work backward and start with the evening market. During the evening, agents trade, consume and produce goods. At the start of any given evening, the expected value of an entrepreneur who holds $\hat{m}$ money and enters the evening market, denoted by $W(\hat{m})$, satisfies

$$\begin{aligned} W(\hat{m}) &= \max_{x,e,m_{+1}\geq 0} x - e + \beta V(m_{+1}) \\ & \quad \text{s.t.} \quad x - e = \phi(\hat{m} - m_{+1}) + \tau \end{aligned}$$

where $V(m_{+1})$ denotes the expected value of entering into the next morning market with holdings m_{+1} of money. The nominal price in the evening market is normalized to 1, and so ϕ represents the relative price of money. Given these prices, the initial money holding $\hat{m}$ and the government tax or transfer τ, the agent chooses an amount of consumption x, effort e and the future money holdings m_{+1}. Note that the initial money holding $\hat{m}$ at the start of a given evening depends on the agent's activities during the morning and afternoon of the same period. If an entrepreneur has started the morning with m money, paid $\theta^i m$ money to the investor, and sold q_2^s (or bought q_2^i) inputs on the liquidation market at a price p, then his initial money holding at the start of the evening is given by $\hat{m} = (1-\theta^i)m + pq_2^s$ (or $\hat{m} = (1-\theta^i)m - pq_2^i$).

Substituting out the term $x - e$ in the value function using the constraint, we obtain the first order condition

$$\beta V'(m_{+1}) = \phi. \tag{7}$$

Observe that m_{+1} is determined independently of $\hat{m}$ (and of m), and hence all entrepreneurs hold the same amount of money at the beginning of any given morning market.

The afternoon: anonymous liquidation market. After the repayment has happened at the end of morning, entrepreneurs either buy a secondary input in the liquidation market and produce and consume final output, or transform their primary input into the secondary input and sell it on the liquidation market during the afternoon. At the start of any given afternoon, the expected value of an entrepreneur who has received an input shock i, holds q_1 primary input and $(1-\theta^i)m$ money, and enters the liquidation market, denoted by $Z^i(q_1,(1-\theta^i)m)$, satisfies

$$\begin{aligned} Z^i(q_1,(1-\theta^i)m) &= \sigma \left\{ \begin{array}{c} \max_{q_2^i \geq 0} \; g(q_1)\xi(i) + h(q_2^i) + W((1-\theta^i)m - pq_2^i) \\ \text{s.t. } pq_2^i \leq (1-\theta^i)m \end{array} \right\} \\ &\quad +(1-\sigma) \left\{ \begin{array}{c} \max_{q_2^s \geq 0} \; -q_2^s + W((1-\theta^i)m + pq_2^s) \\ \text{s.t. } q_2^s \leq q_1 \end{array} \right\} \end{aligned}$$

for $i = G, B$. If the entrepreneur i turns out to be a buyer, which happens with probability δ, then he can buy a secondary input, denoted by q_2^i, up to his money holdings $(1-\theta^i)m$ at the market price p and produce and consume $g(q_1)\xi(i) + h(q_2^i)$ units, where $\xi(i)$ is an indicator function satisfying $\xi(G) = 1$ and $\xi(B) = 0$. He carries $(1-\theta^i)m - pq_2^i$ money to the evening and $W((1-\theta^i)m - pq_2^i)$ is the continuation value specified before. If the entrepreneur turns out to be a seller, which happens with probability $1-\delta$, then he can transform his primary input q_1 into the secondary input, denoted by q_2^s, with unit marginal costs and sell it at the market price p. The seller's continuation value is given by $W((1-\theta^i)m + pq_2^s)$. Given the one to one transformation technology, the seller faces a feasibility constraint $q_2^s \leq q_1$.

The first order conditions are

$$h'(q_2^i) = (\rho^i + \phi)p \tag{8}$$

$$1 + \rho = \phi p \tag{9}$$

for $i = G, B$, where $\rho^i \geq 0$ denotes the multiplier of the buyer i's cash constraint $pq_2^i \leq (1-\theta^i)m$ and $\rho \geq 0$ the multiplier of the seller's feasibility constraint. To derive these conditions, we use the envelope conditions: $\frac{\partial W(\cdot)}{\partial q_2^i} = -\phi p$ for the buyer i and $\frac{\partial W(\cdot)}{\partial q_2^s} = \phi p$ for the seller. Condition (8) states that the buyer i selects the amount of q_2^i so that its marginal product $(= h'(\cdot))$ equals the unit price measured in the real term ($= \phi p$, i.e., one unit of the secondary input is worth p dollars and a dollar is worth ϕ in terms of final output) plus the cost of tightening the budget constraint ($= \rho^i p$). The condition (9) states that the seller transforms its input up to the point where the sum of the marginal production costs ($= 1$) and the cost of tightening the feasibility constraint ($= \rho$) equals the real market price ($= \phi p$). Since the seller's problem is linear, we make a tie-breaking assumption that the seller chooses to produce if indifferent to doing so. Finally, the complementary slackness conditions are

$$\rho^i((1-\theta^i)m - pq_2^i) \quad = \quad 0 \tag{10}$$

$$\rho(q_1 - q_2^s) \quad = \quad 0 \tag{11}$$

for $i = G, B$.

The morning: investment market. At the start of each period, each entrepreneur is randomly matched to an investor. Entrepreneurs offer investors offer the contract, described above, which specifies a payment z, θ_G, θ_B out of their future resources in exchange for an amount of primary inputs q_1. Using the q_1 they produce (and consume) final output during the morning. The repayment happens at the end of the morning. An entrepreneur who holds m money at the start of any given morning has the expected value, denoted by $V(m)$, satisfying

$$\begin{aligned} V(m) \quad = \quad & \max_{q_1, z, \theta_G, \theta_B \geq 0} \sigma \left[g(q_1) - z - \theta_G \phi m + Z^G(q_{1,}, (1-\theta_G)m) \right] \\ & + (1-\sigma) \left[-\theta_B \phi m + Z^B(q_1, (1-\theta_B)m) \right] \end{aligned}$$

subject to the payment constraints (4)-(6). If his primary input q_1 is viable i.e. $i = G$, which happens with probability σ, then the entrepreneur produces and consumes $g(q_1)$ outputs and pays out z outputs and $\theta^G \phi m$ money in real term to the investor. In this case, he carries $(1 - \theta^G)m$ money to the afternoon. If q_1 breaks down i.e. $i = B$, which happens with probability $1 - \sigma$, then the entrepreneur produces and consumes nothing and pays out $\theta^B \phi m$ money in real term to the investor. In this case, he carries $(1 - \theta^B)m$ money to the afternoon. Irrespective of the input shock, the entrepreneur brings the primary input q_1 he has bought from the investor into the next subperiod. $Z^i(\cdot)$, $i = G, B$ are the continuation value described before.

Solving (4) for z and applying this solution to the value function and (5), we can reduce the programme to the following form:

$$
\begin{aligned}
V(m) \quad = \quad & \max_{q_1,\theta_G,\theta_B \geq 0} \sigma\left[g(q_1) + Z^G(q_{1,},(1-\theta_G)m)\right] + (1-\delta)Z^B(q_1,(1-\theta_B)m) - q_1 \\
& \text{s.t.} \qquad q_1 - \phi m\left(\sigma\theta^G + (1-\sigma)\theta^B\right) \leq \sigma g(q_1) \\
& \qquad\qquad \theta^i \in [0,1], \quad i = G, B.
\end{aligned}
$$

Using this expression we obtain the first order conditions:

$$
(1+\delta)\sigma g'(q_1) + (1-\delta)\rho + \mu(\sigma g'(q_1) - 1) \quad = \quad 1 \tag{12}
$$

$$
\mu + \frac{\gamma^i}{\phi m} \quad = \quad \frac{\delta\rho^i}{\phi} + 1 \tag{13}
$$

for $i = G, B$, where $\mu \geq 0$ defines the multiplier of the constraint (5) and $\gamma^i \geq 0$ the multiplier for a constraint $\theta^i \geq 0$ in (6). Note that $\theta^i = 1$ cannot be the solution because of the Inada condition for $h(q_2^i)$, hence the other constraint in (6) $\theta^i \leq 1$ can be ignored. To derive these conditions we use the envelope conditions: $\frac{\partial Z^i(\cdot)}{\partial \theta^i} = -\delta\rho^i m - \phi m$. The L.H.S. of (12) represents the expected marginal benefit of an additional unit of primary inputs q_1 and consists of three parts: the total expected marginal products accruing in the morning and afternoon ($= (1+\delta)\sigma g'(q_1)$); the expected marginal benefit of relaxing the feasibility constraint $q_2 \leq q_1$ as a seller ($= (1-\delta)\rho$); the marginal benefit of relaxing the liquidity constraint (5) ($= \mu(\sigma g'(q_1) - 1)$). The R.H.S. of (12) represents the marginal production cost of q_1 ($= 1$). Using (8) and (9), the conditions (13) can be written as

$$
\mu + \frac{\gamma^i}{\phi m} \quad = \quad \delta\left(\frac{h'(q_2^i)}{1+\rho} - 1\right) + 1
$$

for $i = G, B$. The L.H.S. of this equation represents the marginal benefit of an extra share of monetary payment θ^i, which is the marginal benefit of relaxing the payment constraints (5) and (6). The R.H.S. represents the marginal opportunity costs of increasing θ^i, which is the marginal cost of reducing an extra unit of the money holdings: the entrepreneur loses the net consumption as a buyer during the afternoon, which amounts to $\frac{h'(q_2^i)}{1+\rho} - 1$, and one unit of consumption during the evening.

Finally, the complementary slackness conditions are

$$\mu\left[\sigma g\left(q_1\right)-q_1+\left(\sigma\theta^G+\left(1-\sigma\right)\theta^B\right)\phi m\right] = 0 \tag{14}$$

$$\gamma^i\theta^i = 0 \tag{15}$$

for $i=G,B$ and the envelope condition for m is

$$V'(m)=\phi\left\{\sigma(1-\theta^G)\left(\frac{\delta\rho^G}{\phi}+1\right)+(1-\sigma)(1-\theta^B)\left(\frac{\delta\rho^B}{\phi}+1\right)+\mu(\sigma\theta^G+(1-\sigma)\theta^B)\right\}. \tag{16}$$

Euler equation. We now derive the Euler equation. Plugging (16) into (7) with an updating and rearranging it using (8) and (9), we obtain the Euler equation for money holdings m:

$$\phi=\beta\phi_+\left[\begin{array}{c}\sigma(1-\theta^G)\left\{\delta\left(\frac{h'(q_2^G)}{1+\rho}-1\right)+1\right\}+(1-\sigma)(1-\theta^B)\left\{\delta\left(\frac{h'(q_2^B)}{1+\rho}-1\right)+1\right\}\\ +\mu(\sigma\theta^G+(1-\sigma)\theta^B)\end{array}\right]. \tag{17}$$

In the above equation, the marginal cost of obtaining an extra unit of money today ($=\phi$) equals the discounted value of its expected marginal benefit obtained tomorrow. The marginal value has two components. First, an extra unit of money allows for further production and consumption: the entrepreneur can consume an extra unit during night ($=1$); if the entrepreneur turns out to be a buyer then he can buy an extra amount of secondary inputs equal to $\frac{1}{1+\rho}$. This will generate an additional product given by the marginal product of the secondary inputs ($=h'(\cdot)$) minus the marginal cost needed for such a production $(1+\rho)$. This return of money accrues from its role as a medium of exchange and is captured in the first two terms. Since a fraction θ^i of the money has to be repayed before the production can occur during the afternoon and evening, these terms are multiplied by $\sigma(1-\theta^G)$ if $i=G$ and $(1-\sigma)(1-\theta^B)$ if $i=B$. Second, an extra unit of money reduces the need to pledge output for the payment. This return of money accrues its role as enhancer of liquidity - by liquidity we mean the ease with which future returns can be pledged. It is important to observe that this second role is absent when $\theta^G=\theta^B=0$ and/or $\mu=0$.

Existence, uniqueness and characterization of equilibrium. So far we have described the optimality conditions of individuals, presupposing that all sellers will choose to generate the identical amount of inputs q_2^s. This is because the input shock realized during the morning does not affect the marginal condition of sellers during the afternoon (9). Hereafter, to construct a symmetric equilibrium, we apply a similar reasoning to the buyers' choice, because whether or not a buyer can use a primary input q_1 for the production technology $g(\cdot)$ does not affect its marginal product $h'(\cdot)$ during the afternoon and there is no reason for buyers to spend different amounts of money in the liquidation market. Hence, in what follows, we focus our attention on the case

$$q_2^G = q_2^B \ (\equiv q_2).$$

We now describe the market clearing conditions. These are the final equilibrium requirements in our economy. Market clearing in the morning is guaranteed by bilateral meetings, while market clearing in the afternoon requires

$$\delta q_2 = (1-\delta) q_2^s. \tag{18}$$

Money market clearing implies

$$\frac{\phi_{+1}}{\phi} = \frac{1}{\pi} \tag{19}$$

while market clearing in the evening can be ignored by virtue of Walras Law.

Below we consider policies where $\pi \geq \beta$ and when $\pi = \beta$ (which is the Friedman rule) we only consider the limiting equilibrium as $\pi \to \beta$. Given the symmetry described above, this implies:

Lemma 2 *For $\pi > \beta$, the cash constraint of buyers must be binding and buyers select a share of monetary payment identical across realizations of the input shock i, i.e. $\theta^G = \theta^B$ $(\equiv \theta)$ and $(1-\theta)m = pq_2$.*

Applying the binding cash constraint $(1-\theta)m = pq_2$, $\theta^G = \theta^B$, and (9) to the complementary slackness condition (14), we obtain

$$\mu\left[\sigma g\left(q_1\right) - q_1 + \frac{\theta}{1-\theta}(1+\rho)q_2\right] = 0 \tag{20}$$

Also, note $q_2^G = q_2^B$ and $\theta^G = \theta^B$ imply $\gamma^G = \gamma^B$ ($\equiv \gamma$) by (8) and (13). Applying these results to (8), (9), (13) and (15) we obtain:

$$\begin{aligned} \mu + \frac{\gamma(1-\theta)}{(1+\rho)q_2} &= \delta\left(\frac{h'(q_2)}{1+\rho} - 1\right) + 1; & (21)\\ \gamma\theta &= 0. & (22)\end{aligned}$$

Using these equations and (19), the Euler equation (17) can be simplified to

$$\frac{\pi}{\beta} = \delta\left(\frac{h'(q_2)}{1+\rho} - 1\right) + 1. \tag{23}$$

Definition 1 *A symmetric steady state monetary equilibrium is a set of quantities $q_1, q_2, q_2^s \in (0,\infty)$, prices $p, \phi \in (0,\infty)$ and multipliers $\mu, \gamma, \rho \in [0,\infty)$, and a share of monetary payment $\theta \in [0,1)$ satisfying the first order conditions (9), (12), (21), the Euler equation (23), the market clearing conditions, (18), (19), and the complementary slackness conditions (11), (20), (22).*

Observe that (21)-(23) imply that it is impossible to have the case $\mu = 0$ and $\theta > 0$ i.e., the case in which the liquidity constraint is not binding but a positive amount of money is pledged. This implies that, in our model, the only role of money can play in the morning is to relax the liquidity constraint. Hence, the possible cases are: [1] the liquidity constraint is not binding $\mu = 0$ and no money is pledged $\theta = 0$; [2] the liquidity constraint is binding $\mu > 0$ and no money is pledged $\theta = 0$; [3] the liquidity constraint is binding $\mu > 0$ and a positive amount of money is pledged $\theta > 0$. Below we show either case can emerge.

Our point can be made clear by normalizing the primitive parameters so that the planner's solution satisfies $\lambda^* = 0$, i.e., the feasibility constraint is slack $q_1^* \geq \frac{\delta}{1-\delta}q_2^*$. In what follows, we therefore maintain

Assumption 1

$$g'^{-1}\left(\frac{1}{\sigma(1+\delta)}\right) \geq \frac{\delta}{1-\delta} h'^{-1}(1).$$

Under Assumption 1, the planner's solution solution satisfies: $q_1^* = g'^{-1}(1/\sigma(1+\delta))$; $q_2^* = h'(1)$; $\lambda^* = 0$ (see Lemma 1). The following propositions establish existence and uniqueness of equilibrium and its characterizations.

Proposition 1 *Suppose $\sigma g(q_1^*)/q_1^* \geq 1$. Then, there exists a unique equilibrium in which the liquidity constraint is not binding $\mu = 0$ and no money is pledged $\theta = 0$, and satisfies: $\rho = 0$ and $q_1 = q_1^*$ for all $\pi > \beta$; $q_2 \in (0, q_2^*)$ is strictly decreasing in $\pi > \beta$; $q_2 \to q_2^*$ as $\pi \to \beta$.*

Proposition 2 *Suppose $\sigma g(q_1^*)/q_1^* < 1$. Then, there exists an equilibrium in which the liquidity constraint is binding $\mu > 0$ for all $\pi > \beta$ and satisfies:*

a. $\theta > 0$ and $\rho \geq 0$ for $\pi \in (\beta, \hat{\pi})$; $\theta = 0$ and $\rho \geq 0$ for $\pi \in [\hat{\pi}, \infty)$;

b. $q_1 \in (\hat{q}_1, q_1^*)$ and $q_2 \in (\hat{q}_2, q_2^*)$ are strictly decreasing in $\pi \in (\beta, \hat{\pi})$; $q_1 = \hat{q}_1$ for all $\pi \in [\hat{\pi}, \infty)$ and $q_2 \in (0, \hat{q}_2)$ is strictly decreasing in $\pi \in [\hat{\pi}, \infty)$;

c. $q_1 \to q_1^*$, $q_2 \to q_2^*$, $\theta > 0$, $\rho \to 0$ as $\pi \to \beta$. .

Proposition 1 states that when the technological returns of the primary input q_1 evaluated at the planner's solution, $\sigma g(q_1^*)/q_1^*$, are high enough, entrepreneurs are never liquidity constrained. Equilibrium in this case has a dichotomous nature: the market for investment is insulated from monetary factors.

Instead, when the technological returns of the primary input are not too high, the unobservability which plagues the liquidation market can seep into the investment market inducing a tight bound on the amount of output that can be pledged (Proposition 2). This induces entrepreneurs to pledge some money to relax the liquidity constraint. The investment market and the monetary market now interact and the primary input is affected by inflation as well. As inflation grows, demand for the two inputs decreases. The behavior of entrepreneurs on the investment market changes according to weather the demand for the primary or the secondary input is more elastic to inflation. If the latter is more elastic, then entrepreneurs tend to increase the amount of money they pledge on the investment market when inflation is relatively small. As the amount of the primary input decreases with inflation though, the average returns of the primary input increase because of concavity of the production function. This tends to relax the liquidity constraint and thus tends to make money gradually less useful as provider of liquidity services until eventually - for high enough inflation- money is used only as a provider of exchange services on the liquidation market, where it cannot be dispensed with. At this point the investment market is again insulated from monetary factors. If the demand for the primary input is more elastic to inflation, then entrepreneurs pledge always less money as inflation grows, until eventually they stop altogether at which point money is used only as a medium of exchange.

3 Discussion

Anonymity is crucial in order to make the liquidation market a monetary market. Kocherlakota (1998) has uncovered the crucial role anonymity plays in making money an essential medium of exchange. In our view it is natural to think of liquidation markets as relatively opaque markets where the business of trading assets can become a nuisance. This is also the point of view of a recent literature on trade in the asset market (see Duffie et al. (2005) and Lagos and Rocheteau (2007)). Anonymity in our framework stands for the inability to observe and verify what happened on the liquidation market. Un-observability impairs the possibility to pledge output produced with the assets bought on such an opaque market.

Dropping anonymity would imply two things. First, entrepreneurs could pledge their entire output during the evening, thus making the investment market frictionless. Second, alongside the monetary equilibrium, there would exist another socially superior equilibrium where entrepreneurs use some form of credit to trade on the liquidation market. Money would become - in the jargon of monetary economists- inessential. We thus claim that, in our world, the friction making an economy monetary also impedes the smooth functioning of investment markets.

4 Conclusion

We presented a model where the friction making money a valuable medium of exchange seeps into the investment market and limits the amount of resources that can be pledged. This in turn creates a role for money as a provider of liquidity services as well as exchange services.

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5 Appendix

Proof of Lemma 1

Suppose $\lambda^* = 0$. Then, (1) implies $q_1^* = g'^{-1}(1/\sigma(1+\delta))$ and (2) implies $q_2^* = h'^{-1}(1)$. The complementary slackness condition (3), $(1-\delta)q_1^* \geq \delta q_2^*$, holds if and only if

$$g'^{-1}\left(\frac{1}{\sigma\left(1+\delta\right)}\right) \geq \frac{\delta}{1-\delta} h'^{-1}\left(1\right).$$

Suppose next $\lambda^* > 0$. Then, we must have $(1-\delta)q_1^* = \delta q_2^*$ by (3). Applying this to (1) and (2), we get

$$\sigma\left(1+\delta\right) g'\left(q_1^*\right) + (1-\delta) h'\left(\frac{1-\delta}{\delta} q_1^*\right) = 1,$$

which has a unique solution $q_1^* > g'^{-1}(1/\sigma(1+\delta))$. Applying this solution to the binding constraint $(1-\delta)q_1^* = \delta q_2^*$ and (2), which implies a unique solution $q_2^* < h'^{-1}(1)$, we must have

$$g'^{-1}\left(\frac{1}{\sigma(1+\delta)}\right) < q_1^* = \frac{\delta}{1-\delta} q_2^* < \frac{\delta}{1-\delta} h'^{-1}(1),$$

which is a necessary and sufficient condition for the solution to be $\lambda^* > 0$. ■

Proof of Lemma 2.

Applying $q_2^G = q_2^B$ $(\equiv q_2)$ to (8) we obtain $\rho^G = \rho^B$ $(\equiv \rho^b)$, which further implies $\gamma^G = \gamma^B$ $(\equiv \gamma)$ and $\theta^G = \theta^G$ $(\equiv \theta)$ by (13) and (15). Given these results, the Euler equation (23) can be obtained as explained in the text. Now suppose $\rho^b = 0$. Then, (8) and (9) imply that

$$h'(q_2) = 1 + \rho.$$

This, however, contradicts (23) for $\pi > \beta$, hence if a solution exists for $\pi > \beta$ we must have $\rho^b > 0$ leading to $(1-\theta)m = pq_2$. ■

Proof of Proposition 1 and 2.

Applying the market clearing condition (18) to (15),

$$\rho\left(q_1 - \frac{\delta}{1-\delta} q_2\right) = 0 \tag{24}$$

and solving (12) for $\mu \geq 0$,

$$\mu = \max\left\{0, \frac{(1+\delta)\sigma g'(q_1) - 1 + (1-\delta)\rho}{1 - \sigma g'(q_1)}\right\}. \tag{25}$$

Now the programme is reduced to finding a solution $q_1, q_2, \mu, \gamma, \theta, \rho$ that solves (20)-(25).

Step 1 The case $\mu = 0$, $\theta = 0$, $\rho > 0$ is impossible.

Proof of Step 1. $\mu = 0$ and (25) imply $q_1 > q_1^*$. $\rho > 0$ and (24) imply $q_1 = \frac{\delta}{1-\delta} q_2$. Applying this and a solution $\rho > 0$, which is a solution for $\mu = 0$ in (25), to (23) we get

$$\frac{\pi}{\beta} = \sigma \left[\frac{h'(q_2)}{1 + \frac{1-(1+\delta)\sigma g'\left(\frac{\delta}{1-\delta} q_2\right)}{1-\delta}} \right] + 1.$$

which shows that q_2 is strictly decreasing in $\pi > \beta$ and satisfies $q_2 < q_2^*$ $(\equiv h'^{-1}(1))$ as $\pi \to \beta$. This further implies $q_2 < q_2^*$ and hence $q_1 = \frac{\delta}{1-\delta} q_2 < \frac{\delta}{1-\delta} q_2^*$ for all $\pi > \beta$. However, as Assumption 1 implies $\frac{\delta}{1-\delta} q_2^* \leq q_1^*$ $(\equiv g'^{-1}(1/(1+\delta)\sigma))$, this contradicts $q_1 > q_1^*$. This completes the proof of Step 1.

Since $\sigma g'(q_1^*) = \frac{1}{(1+\delta)} < 1$ by Lemma 1, and g is a concave function with $g(0) = 0$, then $\sigma \frac{g(q_1^*)}{q_1^*} > \sigma g'(q_1^*) = \frac{1}{(1+\delta)}$ and we only need to consider two possibilities: $\sigma \frac{g(q_1^*)}{q_1^*} \geq 1$ or $1 > \sigma \frac{g(q_1^*)}{q_1^*} > \frac{1}{(1+\delta)}$.

Claim 1 *Suppose $\sigma g(q_1^*)/q_1^* \geq 1$. Then, there exists a unique equilibrium in which $\mu = \theta = 0$, $q_1 = q_1^*$ and $q_2 \in (0, q_2^*)$ for all $\pi > \beta$.*

The proof of Claim 1 consists of the following steps.

Step 1 If $\sigma g(q_1^*)/q_1^* \geq 1$, then the case $\mu > 0$, $\theta > 0$, $\rho = 0$ is impossible.

Proof of Step 1. $\rho = 0$ and (23) imply q_2 is strictly decreasing in $\pi > \beta$ and satisfies $q_2 \to q_2^*$ as $\pi \to \beta$. Note $\theta > 0$ implies $\gamma = 0$ by (22). Applying $\gamma = 0$ and $\rho = 0$ to (21) and (25) and substitution out $\mu > 0$ from these equations, we get

$$\delta \left(h'(q_2) - 1 \right) + 1 = \frac{(1+\delta)\sigma g'(q_1) - 1}{1 - \sigma g'(q_1)}.$$

This equation shows that $q_1 \in (g'^{-1}(1/\sigma), q_1^*)$ is strictly increasing in $q_2 \in (0, q_2^*)$, which further implies $q_1 < q_1^*$ for all $\pi > \beta$. Note that since $g(\cdot)$ is strictly concave and $g(0) = 0$, it holds that $g(q_1)/q_1 > g'(q_1)$ for all $q_1 \in (0, \infty)$. This implies $g(q_1)/q_1$ is strictly decreasing in $q_1 \in (0, \infty)$ and so we must have

$$\frac{\sigma g(q_1)}{q_1} > \frac{\sigma g(q_1^*)}{q_1^*} \geq 1$$

for all $\pi > \beta$. However, this contradicts $\theta > 0$ because (20) implies

$$\theta = \frac{q_1 - \sigma g(q_1)}{q_1 - \sigma g(q_1) + q_2} < 0$$

when $\sigma g(q_1)/q_1 > 1$ and $q_2 > 0$. This completes the proof of Step 1.

<u>**Step 2**</u> If $\sigma g(q_1^*)/q_1^* \geq 1$, then the case $\mu > 0$, $\theta > 0$, $\rho > 0$ is impossible.

Proof of Step 2. $\rho = 0$ and (23) imply q_2 is strictly decreasing in $\pi > \beta$ and satisfies $q_2 \to q_2^*$ as $\pi \to \beta$. Note $\theta > 0$ implies $\gamma = 0$ by (22). Applying $\gamma = 0$ and $\rho = 0$ to (21) and (25) and substitution out $\mu > 0$ from these equations, we get

$$\delta\left(h'(q_2) - 1\right) + 1 = \frac{(1+\delta)\sigma g'(q_1) - 1}{1 - \sigma g'(q_1)}.$$

This equation shows that $q_1 \in (g'^{-1}(1/\sigma), q_1^*)$ is strictly increasing in $q_2 \in (0, q_2^*)$, which further implies $q_1 < q_1^*$ for all $\pi > \beta$. Note that since $g(\cdot)$ is strictly concave and $g(0) = 0$, it holds that $g(q_1)/q_1 > g'(q_1)$ for all $q_1 \in (0, \infty)$. This implies $g(q_1)/q_1$ is strictly decreasing in $q_1 \in (0, \infty)$ and so we must have

$$\frac{\sigma g(q_1)}{q_1} > \frac{\sigma g(q_1^*)}{q_1^*} \geq 1$$

for all $\pi > \beta$. However, this contradicts $\theta > 0$ because (20) implies

$$\theta = \frac{q_1 - \sigma g(q_1)}{q_1 - \sigma g(q_1) + q_2} < 0$$

when $\sigma g(q_1)/q_1 > 1$ and $q_2 > 0$. This completes the proof of Step 2.

$$1 = \frac{\beta}{1+\pi}\left\{(1-\gamma)\left[\delta h'\left(\frac{1-\delta}{\delta}q_2^s\right) - \delta\left(1+\rho\right)\right] + \gamma\mu + 1\right\}, \tag{26}$$

$$\mu\left[\sigma g\left(q_1\right) - q_1 + \frac{\gamma}{1-\gamma}\frac{1-\delta}{\delta}q_2^s\right] = 0, \tag{27}$$

$$\mu \leq \left[\delta h'\left(\frac{1-\delta}{\delta}q_2^s\right) - \delta\left(1+\rho\right)\right], \quad = \text{ if } \gamma > 0, \tag{28}$$

$$\mu = \max\left\{0, \frac{\left(1+\delta\right)\sigma g'\left(q_1\right) - 1 + \rho}{1 - \sigma g'\left(q_1\right)}\right\}, \tag{29}$$

$$\rho\left(q_1 - q_2^s\right). \tag{30}$$

Assume $\mu > 0, \gamma > 0, \rho > 0$. Now $q_1 = q_2^s$. Combining (28) at equality and (29), we get

$$\rho = h'\left(\frac{1-\delta}{\delta}q_1\right) - 1 + \frac{1-(1+\delta)\,\sigma g'(q_1)}{\delta\left[1-\sigma g'(q_1)\right]},$$

and plugging it into (26) we have

$$H(q_1, 1+\pi) = \frac{\delta\sigma g'(q_1)}{\left[1-\sigma g'(q_1)\right]} - \frac{(1+\pi)}{\beta} = 0.$$

The function $H(q_1, 1+\pi)$ implicitly defines a continuous and decreasing function $q_1(1+\pi)$. When $1+\pi \to \beta$, $q_2^s \to q_2^*$ and $q_1 \to q_1^*$. Therefore $q_1(1+\pi) \leq q_1^*$. By concavity of g, it follows

$$\sigma\frac{g\left(q_1(1+\pi)\right)}{q_1(1+\pi)} \geq \sigma\frac{g\left(q_1^*\right)}{q_1^*},$$

where the RHS is greater or equal to one by assumption. This implies $\gamma \leq 0$ by (**??**), which contradicts $\gamma > 0$.

Assume now $\mu > 0, \gamma = 0, \rho = 0$. By (27), $q_1 = \sigma g(q_1)$. This is possible only if $\sigma\frac{g(q_1^*)}{q_1^*} = 1$, implying $q_1 = q_1^*$. Using it into (29), we get

$$\mu = \frac{(1+\delta)\,\sigma g'(q_1^*) - 1}{1-\sigma g'(q_1^*)} = 0,$$

which contradicts $\mu > 0$. Assume now $\mu > 0, \gamma = 0, \rho > 0$. By (27), $q_1 = \sigma g(q_1)$. This is possible only if $\sigma\frac{g(q_1^*)}{q_1^*} = 1$, implying $q_1 = q_1^*$. By (26), $\rho = \frac{1-\frac{(1+\pi)}{\beta}}{\delta} < 0$, contradicting $\rho > 0$.

Since $\mu = 0, \gamma > 0, \rho \geq 0$ and $\mu = 0, \gamma = 0, \rho > 0$ are not possible, we are left with $\mu = 0, \gamma = 0, \rho = 0$. In this case the equilibrium system reduces to

$$\delta h'\left(\frac{1-\delta}{\delta}q_2^s\right) - \frac{(1+\pi)}{\beta} + (1-\delta) = 0,$$

$$(1+\delta)\,\sigma g'(q_1) = 1.$$

Hence $q_1 = q_1^*$ and $q_2^s(1+\pi)$ decreasing in $1+\pi$ with $q_2^s(\beta) = q_2^*$. Existence is guaranteed by the Intermediate Value Theorem, using continuity and the Inada condition. Uniqueness follows from concavity of h. Observe that $\rho = 0$, since $q_2^s(\beta) = q_2^* = q_1^*$ by Assumption 1, $q_1(1+\pi) = q_1^*$ for all $1+\pi$ and $q_2^s(1+\pi)$ is decreasing in $1+\pi$.

2. Consider now $\sigma\frac{g(q_1^*)}{q_1^*} < 1$. Assume $\mu = 0, \gamma = 0, \rho = 0$. Then $q_1 = q_1^*$. By (27), $\sigma g(q_1^*) - q_1^* \geq 0$, which contradicts $\sigma\frac{g(q_1^*)}{q_1^*} < 1$. Since $\mu = 0, \gamma > 0, \rho \geq 0$ and $\mu = 0, \gamma = 0, \rho > 0$ are not possible, we are left with $\mu > 0, \gamma > 0, \rho \geq 0$ and $\mu > 0, \gamma = 0, \rho \geq 0$.

2.1. Consider $\mu > 0, \gamma > 0, \rho > 0$ first. By (30), $q_1 = q_2^s$. The equilibrium system is

$$\mu = \frac{(1+\delta)\,\sigma g'(q_1) - 1 + \rho}{1 - \sigma g'(q_1)},$$

$$\gamma = \frac{q_1 - \sigma g(q_1)}{q_1 - \sigma g(q_1) + \frac{1-\delta}{\delta} q_1}, \tag{31}$$

$$\rho = h'\left(\frac{1-\delta}{\delta} q_1\right) - 1 + \frac{1 - (1+\delta)\,\sigma g'(q_1)}{\delta\,[1 - \sigma g'(q_1)]}, \tag{32}$$

$$H(q_1, 1+\pi) = \frac{\delta\sigma g'(q_1)}{[1 - \sigma g'(q_1)]} - \frac{(1+\pi)}{\beta} = 0. \tag{33}$$

The function $H(q_1, 1+\pi)$ implicitly defines a continuous function $q_1(1+\pi)$, with

$$q_1'(1+\pi) = \frac{[1 - \sigma g'(q_1)]^2}{\beta\delta\sigma g''(q_1)} < 0. \tag{34}$$

When $1+\pi \to \beta$, $q_1 \to q_1^*$. Hence, by (31), $\gamma(\beta) > 0$. When $1+\pi \to \infty$, $q_1 \to \overline{q}_1$ s.t. $1 = \sigma g'(q_1)$. By concavity of g, $\overline{q}_1 < \widetilde{q}_1$ s.t. $q_1 = \sigma g(q_1)$. Hence, by (31), $\gamma(\infty) < 0$. Therefore by continuity there exists a value $1+\widetilde{\pi} > \beta$ such that for $1+\pi \geq 1+\widetilde{\pi}$, $\gamma = 0$, $q_1 = \widetilde{q}_1$ s.t. $q_1 = \sigma g(q_1)$. Evaluating $H(q_1, 1+\pi)$ at q_1^*

$$H(q_1^*, 1+\pi) = 1 - \frac{(1+\pi)}{\beta} < 0,$$

for all $1+\pi > \beta$. Evaluating $H(q_1, 1+\pi)$ at $\widetilde{q}_1$

$$H(\widetilde{q}_1, 1+\pi) = \frac{\delta\sigma g'(\widetilde{q}_1)}{[1 - \sigma g'(\widetilde{q}_1)]} - \frac{(1+\pi)}{\beta} > 0,$$

for all $1+\widetilde{\pi} > 1+\pi > \beta$. By continuity of $H(q_1, 1+\pi)$ there exist a solution q_1 for $1+\pi \in (\beta, 1+\widetilde{\pi})$, uniqueness is guaranteed by (34).

For $1+\pi \geq 1+\widetilde{\pi}$, $\rho > 0$ if $1+\overline{\pi} > 1+\widetilde{\pi}$, where

$$1+\overline{\pi} = \beta\left[\delta h'\left(\frac{1-\delta}{\delta} q_1\right) - \delta + 1\right],$$

and $\rho = 0$ if $1+\overline{\pi} \leq 1+\widetilde{\pi}$. Notice that the derivative

$$\gamma'(1+\pi) = \frac{\frac{1-\delta}{\delta} q_1'(1+\pi) q_1(1+\pi)\left[\sigma\frac{g(q_1(1+\pi))}{q_1(1+\pi)} - \sigma g'(q_1(1+\pi))\right]}{\left[q_1(1+\pi) - \sigma g(q_1(1+\pi)) + \frac{1-\delta}{\delta} q_1(1+\pi)\right]^2} < 0.$$

Observe that when $1+\pi \to \beta$, $q_1 \to q_1^*$ and by (32), $\rho = 0$ exactly. Hence $\rho > 0$ cannot happen if

$$\frac{\partial \rho}{\partial (1+\pi)} = \frac{1}{\delta}\left[(1-\delta)\, h''\left(\frac{1-\delta}{\delta} q_1\right) - \frac{(1+\delta)\,\sigma g''(q_1)}{\delta\left[1-\sigma g'(q_1)\right]^2}\right] q_1'(1+\pi) \leq 0. \tag{35}$$

2.2. Consider $\mu > 0, \gamma > 0, \rho = 0$ next. The equilibrium system is

$$\mu = \frac{(1+\delta)\,\sigma g'(q_1) - 1}{1-\sigma g'(q_1)},$$

$$\gamma = \frac{q_1 - \sigma g(q_1)}{q_1 - \sigma g(q_1) + \frac{1-\delta}{\delta} q_2^s}, \tag{36}$$

$$F(q_1, q_2^s) = \sigma g'(q_1) - (1-\delta) - \delta h'\left(\frac{1-\delta}{\delta} q_2^s\right)\left[1-\sigma g'(q_1)\right] = 0, \tag{37}$$

$$G(q_2^s, 1+\pi) = \delta h'\left(\frac{1-\delta}{\delta} q_2^s\right) - \frac{(1+\pi)}{\beta} + (1-\delta) = 0. \tag{38}$$

The function $G(q_2^s, 1+\pi)$ implicitly defines a continuous and decreasing function $q_2^s(1+\pi)$. Existence of q_2^s for all $1+\pi \in (\beta, \infty)$ is guaranteed by continuity of $G(q_2^s, 1+\pi)$ and the Inada condition $h'(0) = \infty$ and uniqueness by concavity of h. The function $F(q_1, q_2^s)$ implicitly defines a continuous function $q_1(q_2^s(1+\pi))$ which is decreasing in inflation. When $1+\pi \to \beta$, by (38), $q_2^s \to q_2^*$ and by (37) $q_1 \to q_1^*$. Hence, by (37), $\gamma(\beta) > 0$. When $1+\pi \to \infty$, by (37) $q_1 \to \overline{q}_1$, where $\overline{q}_1$ is s.t. $1 = \sigma g'(q_1)$. By concavity of g, $\overline{q}_1 < \widetilde{q}_1$, where $\widetilde{q}_1$ is s.t. $q_1 = \sigma g(q_1)$. Hence, there exist a finite value $1+\widetilde{\pi}$ s.t. $\gamma(1+\widetilde{\pi}) = 0$. Moreover for $1+\pi \geq 1+\widetilde{\pi}$, $\rho = 0$. This is the case $\mu > 0$, $\gamma = 0, \rho = 0$. Notice that the sign of the derivative

$$\gamma'(1+\pi) = \frac{\frac{dq_1}{dq_2}\frac{q_2}{q_1}\left[1-\sigma g'(q_1)\right] - \left[1 - \sigma \frac{g(q_1)}{q_1}\right]}{\left[q_1 - \sigma g(q_1) + \frac{1-\delta}{\delta} q_2\right]^2} \frac{1-\delta}{\delta} q_1 q_2'(1+\pi),$$

depends in this case on the elasticity $\frac{dq_1}{dq_2}\frac{q_2}{q_1}$.

Depending on parameters, cases which are mixtures of 2.1 and 2.2 may arise. In all cases $\mu > 0, \gamma > 0$ for low inflation and $\mu > 0, \gamma = 0$ for higher inflation rates.