Money and Asset Prices with Uninsurable Risks*

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Abstract

We develop a model where the coexistence of money and a higher yielding asset is endogenously obtained when no restriction is placed on the use of either object as a medium of exchange. Due to the presence of uninsurable risks, agents have, in equilibrium, different relative valuations of the asset to money, and hence, the use of money as a means of payment is strictly preferred. This endogenous difference in the *willingness* of agents to use money over the asset implies that money carries a greater liquidity premium than the asset. We obtain that the asset strictly dominates money in terms of the expected rate of return, except at the Friedman rule. Since buyers are risk neutral with respect to the dividend payment of the asset, they hold both the asset and money despite the rate of return dominance of asset to money. We obtain that for low levels of inflation the price of the asset, and therefore its real rate of return, is not affected by the rate of inflation. This is because the asset is not used as a means of payment, only money is, and the price of the asset is given by its fundamental value. But when the inflation rate becomes large enough, real balances are so low that buyers will then start using the asset as a means of payment in addition to money, and here, the asset will carry a liquidity premium as well, although it is smaller than that of money. The price of the asset is then increasing in the inflation rate, and its real rate of return is therefore negatively correlated with the inflation rate.

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1 Introduction

This article is a theoretical study of the impact of monetary policy and inflation on asset prices. There has been a number of such studies, e.g., Danthine and Donaldson (1986), Lucas (1980, 1982, 1984), Svensson (1985), Townsend (1987), Marshall (1992) and Bansal and Coleman (1996), but all these studies consider consumption based asset pricing models *à la* Lucas (1978), in which money is introduced either by putting money in the utility function (MIU), or through a cash-in-advance (CIA) or CIA-like constraint, or a transaction cost function (TC). Others (e.g. Freeman, 1985) have used overlapping-generations (OG) models. Although OG models are deep models of money because they capture the store-of-value property of money, they do not really capture its transaction role, which is the feature that distinctively differentiates money from other assets. However, these models do not explain why money is used as a means of payment rather than other, potentially higher yielding, assets. Hence, when these models study the impact of inflation in asset prices, because they do not allow agents to substitute away from money as a means of payment, they are potentially missing an important ingredient in how assets are priced.

On the other hand, the deep models of money that are search-theoretic monetary models were, until recently, ill-equipped to study asset prices. In fact, in models of the tradition of Kiyotaki and Wright (1991, 1993), Shi (1995) and Trejos and Wright (1995), all trades are decentralized, and it is therefore difficult to price assets. In a recent paper, Lagos and Wright (2005), LW hereafter, have developed a search-theoretic model of money that blends centralized and decentralized trade, opening the door for a great number of application that previous search-theoretic models of money could not handled, among them asset pricing.

However, when different assets can be used as means of payment, it is not clear why fiat money, which is usually return dominated, would circulate. In Lagos and Rocheteau (2007) and Geromichalos et al. (2007) money competes against neoclassical capital and Lucas trees respectively as a means of exchange, and they obtain that as long as the efficient stock of capital or the stock of Lucas trees is large enough to carry out transactions in the decentralized market, a monetary equilibrium exists only at the Friedman rule, in which case it is not essential. Others papers like Aruoba et al. (2007b), Berentsen et al., (2008) or Telyukova and Wright (2008) assume that counterfeiting claims to asset or productive capital or IOUs is costless, whereas money cannot be counterfeited, so money circulates as a means of payment even if it is return dominated. Lagos (2007) considers, in a version of his asset pricing model based on LW, a model with an exogenous proportion of trades where stocks can be used, whereas bonds (or money) can always be used as a means of payment.[1] Lester et al. (2008) consider a model where a fraction of agents can recognize counterfeits, but those who cannot recognize them accept only money as a means of payment. These papers obtain the coexistence of money and higher yielding assets because assets cannot be used in some types of meetings whereas money can be used in all types of meetings. That is, money is more liquid, and therefore money carries a higher liquidity premium. In these papers, agents are homogeneous, but there is an exogenous difference in the liquidity properties of money and other assets due to a difference in agents' *ability* to use money compared to these other assets.

In this paper we develop a model where the coexistence of money and a higher yielding asset is endogenously obtained where no restriction is placed on the use of either object as a medium of exchange. In our paper, due to the presence of uninsurable risks, in equilibrium, agents have different relative valuations of the asset to money, and hence, the use of money as a means of payment is strictly preferred. This endogenous difference in the *willingness* of agents to use money over the asset implies that money carries a greater liquidity premium than the asset.

More specifically, we consider a OG variation of the LW model, with alternating centralized markets (CM) and decentralized markets (DM), where agents of each generation live for three subperiods.[2] Agents can be either buyers and sellers: buyers want to consume in the DM and have a constant disutility of effort in the last CM of their lives, whereas sellers can only produce in the DM and face an uninsurable risk[3] in the last CM of their lives because their disutility of effort is stochastic and negatively correlated with the dividend payment of the asset. This is what creates a wedge between the valuations of the asset of buyers and sellers, with buyers valuing the asset more than sellers. This implies that, although buyers have all the bargaining power in the decentralized market and therefore there is no holdup problem, trade in the decentralized market is inefficient when only the asset can be used as a means of payment.

When money is introduced, we obtain that, conditional on having brought money and asset into the DM,[4] buyers strictly prefer paying sellers using money rather than the asset. This is because buyers value the asset relative to money more than sellers, and therefore buyers prefer to pay using money only, as long as they are not too constrained. If buyers have too little real balances, then they will use the asset as a means of payment as well. Hence, we obtain endogenously that

[1] A similar path has been followed in some non-search theoretic asset pricing models, e.g., Holmström and Tirole (2001) and Kiyotaki and Moore (2005).

[2] See, for instance, Zhu (2007), Rocheteau (2008), and Waller (2008), for a similar OG setup.

[3] We assume that agents cannot commit to honor contracts, and since agents live for three subperiods they cannot be punished should they renege on a risk-sharing contract in the last subperiod of their lives.

[4] and under mild conditions for monetary policy.

money has better liquidity properties than the asset because buyers *choose* to pay using money first. The fact that buyers prefer to pay using money implies that money carries a greater liquidity premium than the asset, and we therefore obtain that the asset strictly dominates money in terms of expected rate of return, except at the Friedman rule. Since buyers are risk neutral with respect to the dividend payment of the asset, they hold both the asset and money despite the rate of return dominance of asset to money.

Regarding the impact of inflation on asset prices when the monetary authority follows a stationary monetary policy in the sense that the inflation rate is kept constant, our model delivers that for low levels of inflation the price of the asset, and therefore its real rate of return, is not affected by the rate of inflation. This is because for low levels of inflation the asset is not used as a means of payment, only money is, and the price of the asset is given by its fundamental value, which is the expected present value of its dividend flow. Hence, when the inflation rate increases, although buyers' real balances decrease, thereby reducing the quantity they can buy from sellers in the DM, buyers will still not be using the asset to buy from sellers. This is because the differentiated relative valuation of the asset to money implies that the opportunity cost of using money in terms of tomorrow's expected effort savings is too high compared to the utility benefits of the extra quantity of goods they can obtain in the DM. In other words, when the inflation rate is low we endogenously obtain a *dichotomy*: money is used as a means of payment in the DM,[5] the asset is used to transfer resources across periods, and the rate of return on money does not impact the real rate of return on the asset. When the inflation rate becomes large enough, real balances are so low that buyers will then use the asset as a means of payment in complement to money. In this case the asset will carry a liquidity premium as well, although it is smaller than that of money. In this case, when the inflation rate increases, because real balances fall, which constrains buyers further in the DM, the liquidity premia on both money and the asset increase. Hence, the price of the asset increases, thereby reducing its real rate of return. We therefore obtain a negative relationship between inflation and the asset's real rate of return.

Our paper is related to papers by Engineer and Shi (1998) and Berentsen and Rocheteau (2003) who show that in a model of decentralized trade where agents' preferences over each other's goods is exogenously asymmetric, barter is inefficient, even if agents can trade every period. This is because the agents' differentiated valuation implies that the surplus of match is generically not maximized. In their environments the introduction of money improves welfare because all agents

[5] Money is also used by old sellers in the CM.

value it symmetrically, which enables agents to maximize the surplus of the match.[6]

Among the search-theoretic models of money dealing with asset prices mentioned above, the most closely linked to our paper are that of Lagos and Rocheteau (2007), Geromichalos et al. (2007) and Lester et al. (2008). However, in these papers agents would like to be able to use the asset, or capital, as a means of payment in the DM, but either they do not have enough of it or they cannot use it because of informational asymmetry. In our model it is the choice of buyers to use money over the asset as a means of payment. This also means that the liquidity premium that money carries over the asset is completely endogenous. Moreover, in this paper whether the price and real rate of return on the asset, or capital in the case of Lagos and Rocheteau, is affected by the inflation rate depends on whether the stock of the asset is large enough to carry out transactions in the DM. In contrast, we obtain that the level of inflation is what determines whether the asset price is affected by the inflation rate. Finally, our paper is also related to the paper by Rocheteau (2008), who obtains very similar results to ours, albeit in an environment with asymmetric information.

The paper is organized as follows. The environment is laid out in the next section. Section 3 considers the model without fiat money, and the model with fiat money is studied in section 4. The implications of the monetary model in a stationary monetary policy environment is considered in section 5, and section 6 concludes.

2 The Environment

Time is discrete and the horizon infinite. In each period two markets are opened sequentially: a Centralized (Walrasian) Market (CM) and a Decentralized Market (DM). In each period the state of the world is one of two possible states $z \in \{1, 2\}$, and for the time being we assume that the state follows an *iid* process such that the probability that the state is 1 is $\pi \in (0, 1)$. The state of the world for a given period is revealed before the CM opens.

We adopt an Overlapping-Generations structure: a unit mass of agents is born at the beginning of each CM, and all of these agents die at the end of the next CM, i.e., agents live for three subperiods. Discounting happens between periods and the common discount factor is $\beta \in (0, 1)$. We say agents are young in the first two subperiods of their lives and old in the last subperiod. Within each generation there are two types of agents: half are called buyers, indexed by b, while

[6] These results are those of Berentsen and Rocheteau (2003). Engineer and Shi's (1998) results are more nuanced, but as Berentsen and Rocheteau show the difference in results come from some special assumptions made by Engineer and Shi.

the other half are called sellers, indexed by s.

In the first subperiod of each date agents consume and produce a perishable consumption good which is traded in the CM. The utility of consuming x units of the good is $U(x)$ for all agents, and we assume that U is strictly increasing, strictly concave and twice continuously differentiable, with $U(0) = 0$ and $\lim_{x\to+\infty} U'(x) = 0$. To produce the good requires effort, and effort is turned into the good one-for-one. The utility cost of supplying effort h for both types while young is independent of the state and is h, as it is for old buyers, whereas the cost of supplying h units of effort for a type-j agent, $j \in \{s, b\}$, when the state of the world is z is $\eta^{j,z}h$. We assume[7] that $\eta^{b,1} = \eta^{b,2} = 1$, and $\eta^{s,1} < 1 < \eta^{s,2}$, with

$$\pi\eta^{s,1} + (1-\pi)\,\eta^{s,2} = 1,$$

so that both types of agents have the same expected disutility of effort.[8]

In the second subperiod of each date trade is decentralized: each buyer is matched at random with a seller. The cost of producing y units of the good for the seller is $c(y)$ while the utility of the buyer in consuming y is $u(y)$. We assume that both u and c are twice continuously differentiable with $u(0) = c(0) = 0$, that $u(y) - c(y)$ is strictly concave, and that $\lim_{y\to0} u'(y) = \lim_{y\to+\infty} c'(y) = +\infty$ and $\lim_{y\to+\infty} u'(y) = \lim_{y\to0} c'(y) = 0$. The terms of trade in the DM are determined by take-it-or-leave-it offers that buyers make to sellers.[9]

In addition to the two perishable consumption goods, there are two other non-perishable objects in the economy. The first is a Lucas tree (Lucas, 1978), which yields a state-dependent dividend consisting of a quantity d^z of consumption goods every period in the CM. It is assumed that $d^1 > d^2$. There is also an intrinsically worthless object called fiat money. Both objects are perfectly divisible and storable and can be traded freely in either markets. We assume that there does not

[7]It is possible to instead assume that the difference across the two groups of agents is in their marginal productivity of labor. The mechanics would be exactly the same, though the formulation would be more cumbersone for computation and presentation purposes.

[8]We could have assumed more generally that $\eta^{s,1} < \eta^{b,1} \leq 1 \leq \eta^{b,2} < \eta^{s,2}$ with $\pi\eta^{j,1} + (1-\pi)\,\eta^{j,2} = 1$ for $j = s, b$, but all that is required for our results to go through is that the seller has a greater variability in his disutility of effort. The assumption that both types of agents have the same expected disutility of effort guarantees that both types of agents have the same intertemporal marginal rate of substitution of effort.

[9]If we were to adopt the generalized Nash bargaining solution we would have to worry about the choice of buyers regarding the part of their portfolio to bring into the DM because of the hold-up problem. See Geromichalos et al. (2007) and Lester et al. (2008) for a discussion of the problem in a similar set-up, and Aruoba et al. (2007a) for a general discussion of the hold-up problem in monetary models where terms of trade are determined through bargaining.

exist a technology keeping track of individuals' trading histories, and that agents cannot commit to honoring private contracts they have agreed to. These two assumptions rule out the use of credit in the DM, for otherwise money cannot be essential.[10] Furthermore, the lack of commitment also implies that agents cannot credibly agree on ex ante risk-sharing contractual arrangements for the last subperiod of their lives.

The Lucas tree is in fixed supply while money is injected, or withdrawn, by the government in the centralized market by lump-sum transfers, or taxes, to the young. Let A denote the fixed stock of the asset and M_t the quantity of money in period t. We consider different policies: we first consider stationary monetary policies where the money stock grows at a constant rate, so that real balances and the inflation rate across time and states. We then consider cases where the money growth rate, and therefore the inflation rate, is state-dependent. A monetary rule is a function F that gives tomorrow's stock of money for each state as a function of today's stock of money and tomorrow's state of the world. Formally, a monetary rule is

$$\begin{array}{rccc} F: & \mathbb{R} & \to & \mathbb{R} \times \mathbb{R} \\ & M & \mapsto & \left(\widehat{M}^1, \widehat{M}^2\right). \end{array}$$

3 The Economy without Fiat Money

Let us first consider the equilibrium for an economy without money. The problem an agent faces in the first subperiod of his life is to choose how much to consume, work and asset to purchase (from the old) in order to maximize his expected discounted lifetime utility. Since all young agents are identical across periods and old agents will always sell all of their assets in the last CM, we have that the demand and supply of asset in the CM are the same in all periods. It is therefore natural to look for an equilibrium where the price of the asset is constant over time and independent of the state. Formally, a type j agent's problem, $j = s, b$, in his first subperiod when the state of the world is z is

$$\begin{aligned} & \underset{(x,h,a)}{Max} \left\{U(x) - h + W^j(a)\right\} \\ \text{s.t. } x &= h - \psi a, \end{aligned} \tag{1}$$

equation (1) where $W^j(a)$ is the value of entering the DM with a units of asset, ψ is the price of the asset, and it is assumed that the asset is traded *ex dividend.* For buyers and sellers we have

[10] See Kocherlakota (1998) and Wallace (2001).

that

$$W^b(a) = u(y) + \beta \mathbb{E}_{\widehat{z}}\left[v^{b,\widehat{z}}(a - q_a)\right], \text{ and} \tag{2}$$

$$W^s(a) = -c(y) + \beta \mathbb{E}_{\widehat{z}}\left[v^{s,\widehat{z}}(a + q_a)\right], \tag{3}$$

where $v^{j,\widehat{z}}(\widehat{a})$ is the value for type j agents of entering the last CM with $\widehat{a}$ units of assets when the state is $\widehat{z}$. Note that as long as the price of the asset is state-independent the value of entering the DM is also independent of the state of the world in that period. Assuming that $h > 0$, we have that an agent will choose to consume x^* such that $U'(x^*) = 1$, and the amount he chooses to work depends on his asset choice: $h(a) = x^* + \psi a$. The first-order condition, which is necessary and sufficient, for the asset holding decision is

$$\psi \geq W_a^j(a), \ = \text{ if } a > 0. \tag{4}$$

Let $a^s(\psi)$ and $a^b(\psi)$ denote the asset holding choices of buyers and sellers for the price ψ, the market clearing condition for the asset is

$$0.5a^s(\psi) + 0.5a^b(\psi) = A \tag{5}$$

Before moving on to the DM, let us consider the last CM of an agent's life. If we denote by $v^{j,\widehat{z}}(\widehat{a})$ the value for a type j agent of entering the last subperiod of his life with $\widehat{a}$ units of assets when the state of the world is $\widehat{z}$, we have that

$$v^{j,\widehat{z}}(\widehat{a}) = \underset{(x,h)}{Max}\left\{U(x) - \eta^{j,\widehat{z}}h\right\} \tag{6}$$

$$\text{s.t. } x = h + \left(\psi + d^{\widehat{z}}\right)\widehat{a}.$$

Assuming interiority of the solution for h, it follows that $x = x_{\widehat{z}}^j$ and $h(\widehat{a}) = x_{\widehat{z}}^j - \left(\psi + d^{\widehat{z}}\right)\widehat{a}$, where $U'(x_{\widehat{z}}^j) = \eta^{j,\widehat{z}}$.[11] Hence, the expected value for a type j agent of entering the last CM of his life with $\widehat{a}$ units of assets can be expressed as

$$v^j(\widehat{a}) = v^j(0) + v_a^j\widehat{a}, \tag{7}$$

where $v^j(0) = \pi(U(x_1^j) - \eta^{j,1}x_1^j) + (1-\pi)\left(U(x_2^j) - \eta^{j,2}x_2^j\right)$ and $v_a^j = \pi\eta^{j,1}\left(\psi + d^1\right) + (1-\pi)\eta^{j,2}\left(\psi + d^2\right)$ is the constant marginal value of the asset to a type j agent. The last expression can be simplified to $v_a^j = \psi + \overline{d}^j$ with $\overline{d}^s \equiv \pi\eta^{s,1}d^1 + (1-\pi)\eta^{s,2}d^2$ and $\overline{d}^b = \overline{d} \equiv \pi d^1 + (1-\pi)d^2$ being the expected value of the dividend for sellers and buyers.

[11] This naturally requires that $x_{\widehat{z}}^* \geq \left(\psi + d^{\widehat{z}}\right)\widehat{a}$, which will be verified (guaranteed) later.

It is worth noting that the marginal value of the asset for buyers is greater than that of sellers, i.e., $v_a^b > v_a^s$. In fact, $\overline{d} > \overline{d}^s$. The intuition for this difference in valuation of the asset is simple: the asset pays a high dividend when sellers have a low disutility of effort, and therefore do not mind too much exerting effort, whereas the asset pays a low dividend in the state when it is the most costly for them to work. In other words the asset is a bad one for sellers for self-insurance against effort cost because the correlation is of the wrong sign. Buyers, contrary to sellers, do not face any variation in their marginal disutility of effort, and therefore the dividend payouts are not correlated with their disutility of effort.

In the DM, when a buyer with quantity of asset a meets a seller with quantity of asset $\widetilde{a}$, the surplus for the buyer is[12]

$$S^b(y;a) = u(y) + \beta \mathbb{E}_{\widehat{z}} \left[v^{b,\widehat{z}}(a - q_a) - v^{b,\widehat{z}}(a) \right],$$

where q_a is the quantity of asset the buyer transfers to the seller in exchange for the quantity y of the good. From (7), the above expression can be rewritten as

$$S^b(y;a) = u(y) - \beta v_a^b q_a.$$

Symmetrically, the surplus for a seller can be written as

$$S^s(y) = -c(y) + \beta v_a^s q_a.$$

We therefore have that when the buyer and seller have asset holdings a and $\widetilde{a}$ respectively, the terms of trades (y, q_a) solve

$$\begin{aligned} & Max_{(y,q_a)}\ S^b(y;a) \\ s.t.\ S^s(y) \quad \geq \quad & 0\ (\lambda)\,,\, q_a \leq a\ \ (\mu_a) \end{aligned} \tag{8}$$

where the terms in brackets denote the Lagrange multipliers for the respective constraints..

Definition 1 *An non-monetary equilibrium is a price for the asset ψ, asset holdings for sellers and buyers a^s and a^b and terms of trade for the DM (y, q_a), such that: (i) given ψ, a^j satisfies (4) for $j \in \{s, b\}$, ψ is such that (5) is satisfied; and (ii) (y, q_a) solves (8).*

[12] Here we are anticipating the result that the terms of trade in the DM do not depend on the seller's portfolio.

3.1 Terms of Trade in the DM

In this section we will derive the terms of trade in the DM, taking as given the asset choices of agents. When matched with a seller, a buyer makes him a take-it-or-leave-it offer, that is the terms of trade solve (6). The first-order conditions[13] are

$$FOC\ (y) \quad : \quad u'(y) = \lambda c'(y); \tag{9}$$

$$FOC\ (q_a) \quad : \quad \beta v_a^b + \mu_a \geq \lambda \beta v_a^s, \ = \ \text{for } q_a > 0. \tag{10}$$

Since the marginal gains from trading the good are infinitely large at $y = 0$, clearly the two agents will trade, provided that the buyer holds a non-empty portfolio. Hence, provided the buyer comes into the DM with some asset, the second FOC holds with equality and can be rewritten as

$$FOC\ (q_a) : \frac{v_a^b}{v_a^s} + \frac{\mu_a}{\beta v_a^s} = \lambda.$$

We then have the following lemma.

Lemma 1 *In an equilibrium without money, when the buyer's asset holding is a, the terms of trade in the DM (y, q_a) are such that:*

$$y = \min\{c^{-1}(\beta v_a^s a), y^*(\epsilon_a)\}, \ and\ q_a = \min\{a, a^*(\epsilon_a)\}, \tag{11}$$

where $\epsilon_a \equiv v_a^b/v_a^s$, $y^*(\epsilon_a)$ solves $\ell(y;\epsilon) = 0$ with $\ell(y;\epsilon) \equiv \epsilon^{-1}[u'(y)/c'(y)] - 1$, and $a^*(\epsilon_a) \equiv c(y^*(\epsilon_a))/\beta v_a^s$.

Proof. In the Appendix. ■

If a buyer is not constrained by his asset holding he will purchase the quantity $y^*(\epsilon_a)$ from the seller he is matched with. This requires that he holds a quantity of asset of at least $a^*(\epsilon_a)$. If a buyer holds less than $a^*(\epsilon_a)$ when he comes into the DM he will be constrained, in which case he gives away his entire asset holding, i.e., $q_a = a$ and receives in exchange $y = c^{-1}(\beta v_a^s a)$, the maximum quantity the seller is willing to produce for $v_a^s a$.

In this environment without money the buyer will never purchase the efficient quantity y^* from the buyer, y^* being such that $u'(y^*) = c'(y^*)$. In fact, a buyer is willing to exchange the asset against goods in the DM up to the point where his marginal gain is equal to his marginal opportunity cost, which is the expected value of effort the marginal unit of asset would have saved him. However, to

[13] The Lagrangian is concave for its Hessian is negative semi-definite.

obtain a quantity y of goods in the DM the buyer has to transfer $q_a = c(y)/\beta v_a^s$ units of asset to the buyer. The marginal cost of obtaining y is therefore

$$v_a^b \frac{c'(y)}{v_a^s},$$

whereas the marginal gain is $u'(y)$. The maximum quantity a buyer is willing to purchase from a seller is therefore $y^*(\epsilon_a)$, and is such that

$$\frac{u'(y^*(\epsilon_a))}{c'(y^*(\epsilon_a))} = \frac{v_a^b}{v_a^s}. \tag{12}$$

Even though we have not solved yet for the price of the asset, it is clear that $v_a^b > v_a^s$. It follows that $y^*(\epsilon_a) < y^*$.

3.2 The Asset Holding Decisions

Now that we have established the terms of trade in the DM, we turn our attention to the choice of asset holdings for the buyers and sellers in the first subperiod of their lives. The first order conditions for buyers and sellers are respectively

$$\begin{aligned} \psi &\geq W_a^b(a), \quad = \text{ if } a^b > 0; \\ \psi &\geq W_a^s(a), \quad = \text{ if } a^s > 0. \end{aligned} \tag{13}$$

When the agent is a seller his asset holdings has no incidence on the terms of trade in the DM, and

$$W^s(a) = -c(y) + \beta \mathbb{E}_{\widehat{z}} \left[v^{s,\widehat{z}}(a + q_a) \right],$$

and therefore the expected marginal value for the asset in the first CM is

$$W_a^s(a) = \beta v_a^s. \tag{14}$$

In this case the marginal value of the asset is simply the discounted expected value of the asset next period. This is because when an agent is a seller in the DM he does not need the asset to carry out transactions there, and therefore the asset does not provide him with any liquidity services. Hence, a seller's valuation is based solely on the dividend payouts in the next CM.

For buyers we have

$$W^b(a) = u(y) + \beta \mathbb{E}_{\widehat{z}} \left[v^{b,\widehat{z}}(a - q_a) \right],$$

and the quantity of asset he has brought with him into the DM matters because he will need some asset to purchase from the seller he has been matched with. We have the following lemma.

Lemma 2 *In a meeting where the buyer's asset holding is a,*

$$W_a^b(a) = \beta v_a^b \left[1 + L(y;\epsilon_a)\right], \tag{15}$$

where $L(y;\epsilon) \equiv \max\{0;\ell(y;\epsilon)\}$ *is the liquidity premium.*

Proof. In the Appendix. ∎

If the value of the buyer's asset holding in the eyes of the seller is at least equal to $a^*(\epsilon_a)$, then the buyer is able to purchase the quantity $y^*(\epsilon_a)$ from the seller, and in that case if the buyer brings some more asset into the DM the terms of trade will not change. Therefore, the expected value of the extra units of asset brought in is simply βv_a^b, i.e., the discounted expected price of the asset next period. If, however, the buyer did not have enough asset initially to purchase the desired quantity $y^*(\epsilon_a)$, then if he increases the value of the portfolio brought into the DM he will increase the quantity of good he can purchase from the seller, and this increases his utility to the tune of $\beta(1/\epsilon_a)\left(u'(y)/c'(y)\right)$. The term $L(y;\epsilon_a)$ captures the liquidity premium that the asset yields *in excess* of the pure financial returns.

Plugging (14) and (15) into the FOCs (13) yield

$$\begin{aligned} \psi &\geq \beta v_a^b \left[1 + \ell(y;\epsilon_a) \cdot 1\left\{a < a^*(\epsilon_a)\right\}\right], &&= \text{ if } a^b > 0; \\ \psi &\geq \beta v_a^s, &&= \text{ if } a^s > 0. \end{aligned} \tag{16}$$

Since $v_a^b > v_a^s$ it follows that $a^b = A$, $a^s = 0$ and $\psi = \beta v_a^b \left[1 + L(y;\epsilon_a)\right]$. Hence, if the stock of asset in the economy is large enough for buyers to be able to purchase the desired quantity of good $y^*(\epsilon_a)$ in the DM, then the asset does not carry any liquidity premium, in which case buyers' marginal valuation of the asset is simply v_a^b. If the stock of asset is not large enough, then the asset carries a liquidity premium $L(y;\epsilon_a) > 0$ and buyers value the asset in excess of v_a^b. In either case buyers value the asset more than sellers, and therefore are willing to pay more for it. We then have the following proposition.

Proposition 1 *There exists a unique steady-state equilibrium without money. There exists* $\widetilde{A}$ *solving*

$$\frac{\beta\left[\beta\overline{d} + (1-\beta)\overline{d}^s\right]}{1-\beta}\widetilde{A} = c(\widetilde{y}^*)$$

such that:

(i) if $A \geq \widetilde{A}$*, then* $\psi = \widetilde{\psi}^* = \beta \overline{d}/(1-\beta)$*,* $y = \widetilde{y}^* \equiv y^*(\widetilde{\epsilon}_a^*)$*, and* $q_a = \widetilde{A}$*, where*

$$\widetilde{\epsilon}_a^* = \frac{\overline{d}}{\beta \overline{d} + (1-\beta)\overline{d}^s};$$

(ii) if $A < \widetilde{A}$*, then* $\psi = \psi(A) > \widetilde{\psi}^*$*,* $y = \widetilde{y}(A) < \widetilde{y}^*$*, and* $q_a = A$*, with* $\psi'(A) < 0$ *and* $\widetilde{y}'(A) > 0$*.*

Proof. In the Appendix. ■

Hence, we obtain that the economy without fiat money is always inefficient because of the difference in valuations of the asset of buyers and sellers. As in Engineer and Shi (1998) and Berentsen and Rocheteau (2003), this inefficiency arising from the difference in valuations of the asset suggest that, as we will indeed show, that the introduction of money in this environment can increase welfare, and therefore make money essential despite being return dominated by the asset.

4 The Economy with Fiat Money

We will now consider the economy with fiat money in addition to the Lucas tree. In this case the problem an agent faces in the first subperiod of his life is to choose how much to consume and work and what portfolio to bring into the DM in order to maximize his expected discounted lifetime utility. It is still true that all young agents are identical across periods and that old agents will always sell all of their assets in the last CM of their lives. Moreover, since we are considering only monetary rules where the money stock growth can depend on next period's state but not on today's state, we are considering equilibria where real balances are the constant in all periods and states, so that, focusing on stationary monetary rules, the price of the asset are both state-independent. We therefore have that a type j agent's problem in his first subperiod in either state of the world is

$$\begin{aligned} & \underset{(x,h,a,m)}{Max} \left\{ U(x) - h + W^j(a,m) \right\} \\ \text{s.t. } x &= h - \psi a - \phi m, \end{aligned} \tag{17}$$

where $W^j(a,m)$ is the value of entering the DM with portfolio (a,m), ψ and ϕ are the prices of the asset and money respectively (it is still assumed that the asset is traded *ex dividend*). Assuming the solution for h is interior, we have that a young agent will choose to consume x^* such that $U'(x^*) = 1$ in each state, and the amount he chooses to work is $h(a,m) = x^* + \psi a + \phi m$. The first-order conditions, which are necessary and sufficient, for the asset and money holding decisions are

$$\psi \geq W_a^j(a,m), \ = \text{ for } a^j > 0, \tag{18}$$

$$\phi \geq W_m^j(a,m), \ = \text{ for } m^j > 0, \tag{19}$$

Let $a^j(\psi,\phi)$ and $m^j(\psi,\phi)$ denote the asset and money choices of type j agents given the prices of asset and money are ψ and ϕ. The market clearing conditions for the asset and money in either state are

$$0.5a^s(\psi,\phi) + 0.5a^b(\psi,\phi) = A, \text{ and} \tag{20}$$

$$0.5m^s(\psi,\phi) + 0.5m^b(\psi,\phi) = M, \tag{21}$$

where M is the stock of money for that period.

Before moving on to the DM we will again first consider the last CM of an agent's life. Denoting by $v^{j,\widehat{z}}(\widehat{a},\widehat{m})$ the value for a type j agent of entering the last subperiod of his life with portfolio $(\widehat{a},\widehat{m})$ when the state of the world is $\widehat{z}$, we have that

$$v^{j,\widehat{z}}(\widehat{a},\widehat{m}) = \underset{(x,h)}{Max}\left\{U(x) - \eta^{j,\widehat{z}}h\right\} \tag{22}$$
$$\text{s.t. } x = h + \left(\psi^{\widehat{z}} + d^{\widehat{z}}\right)\widehat{a} + \phi^{\widehat{z}}\widehat{m}.$$

Assuming interiority of the solution for h, it follows that $x = x^*_{\widehat{z}}$ and $h^{\widehat{z}}(\widehat{a},\widehat{m}) = x^*_{\widehat{z}} - \left(\psi^{\widehat{z}} + d^{\widehat{z}}\right)\widehat{a} - +\phi^{\widehat{z}}\widehat{m}$, where $U'(x^*_{\widehat{z}}) = \eta^{j,\widehat{z}}$.[14] Hence, the expected value for a type j agent of entering the last CM of his life with portfolio $(\widehat{a},\widehat{m})$ can be expressed as

$$v^j(\widehat{a},\widehat{m}) = v^j(0,0) + v_a^j\widehat{a} + v_m^j\widehat{m}, \tag{23}$$

where $v^j(0,0) = \pi\left(U(x_1^*) - \eta^{j,1}x_1^*\right) + (1-\pi)\left(U(x_2^*) - \eta^{j,2}x_2^*\right)$, $v_m^j = \pi\eta^{j,1}\phi^1 + (1-\pi)\eta^{j,2}\phi^2$ is the constant marginal value of the money, and v_a^j is defined as before. This time it is not necessarily the case that buyers value the asset more than sellers. This is because the price of the asset can be state-dependent, and if the price of the asset in state 1 is sufficiently smaller than in state 2 we can obtain that sellers actually value the asset more than buyers.

In the DM, we continue to assume that the buyer makes a take–it-or-leave-it offer to the seller. When a buyer with portfolio $(\widehat{a},\widehat{m})$ meets a seller (we now know his portfolio is irrelevant for the

[14] This naturally requires that $x^*_{\widehat{z}} \geq \left(\psi^{\widehat{z}} + d^{\widehat{z}}\right)\widehat{a} + \phi^{\widehat{z}}\widehat{m}$, which will be verified (guaranteed) later.

terms of trade as long as interiority is guaranteed), the surplus for the buyer is

$$S^b(y; a, m) = u(y) + \beta \mathbb{E}_{\widehat{z}} \left[v^{b,\widehat{z}}(a - q_a, m - q_m) - v^{b,\widehat{z}}(a, m) \right],$$

where q_m is the quantity of money the buyer transfers to the seller. From (26), the above expression can be rewritten as

$$S^b(y; a, m) = u(y) - \beta \left(v_a^b q_a + v_m^b q_m \right).$$

Symmetrically, the surplus for a seller can be written as

$$S^s(y) = -c(y) + \beta \left(v_a^s q_a + v_m^s q_m \right).$$

We therefore have that when the buyer has portfolio (a, m), the terms of trades (y, q_a, q_m) solve

$$\begin{aligned} & Max_{(y,q_a)} \; S^b(y; a, m) \quad (24) \\ s.t. \; S^s(y) \;\; & \geq \;\; 0 \;\; (\lambda), \; q_a \leq a \;\; (\mu_a), \; q_m \leq m \;\; (\mu_m). \end{aligned}$$

where the terms in brackets denote the Lagrange multipliers for the respective constraints.

Definition 2 *A monetary equilibrium is a price for the asset ψ, asset and money holdings for sellers and buyers $\left(a^s, m^s, a^b, m^b\right)$, and terms of trade in the DM (y, q_a, q_m), such that: (i) given ψ and the monetary rule F, $\left(a^j, m^j\right)$ satisfies (18) and (19) for $j \in \{s, b\}$, and (20) and (21) are satisfied; and (ii) (y, q_a, q_m) solves (24).*

4.1 Terms of Trade in the DM and the Essentiality of Money

In this section we will derive the terms of trade in the DM, taking as given the portfolios that agents bring in. The first-order conditions for the buyer's maximization problem are

$$FOC \; (y) \;\; : \;\; u'(y) = \lambda c'(y); \quad (25)$$

$$FOC \; (q_a) \;\; : \;\; \beta v_a^b + \mu_a \geq \lambda \beta v_a^s, \; = \; \text{if } q_a > 0; \text{ and} \quad (26)$$

$$FOC \; (q_m) \;\; : \;\; \beta v_m^b + \mu_m \geq \lambda \beta v_m^s, \; = \; \text{if } q_m > 0. \quad (27)$$

Since the marginal gains from trading the good are infinitely large at $y = 0$, clearly the two agents will trade, provided that the buyer holds a non-empty portfolio. And since the seller needs to be compensated by the buyer for the goods he will provide him with, we therefore have that at least one of the last two FOCs holds with equality.

The last two FOCs can be reexpressed as follows:

$$\begin{aligned} FOC\ (q_a) &: \frac{v_a^b}{v_a^s} + \frac{\mu_a}{\beta v_a^s} \geq \lambda; \text{ and} \\ FOC\ (q_m) &: \frac{v_m^b}{v_m^s} + \frac{\mu_m}{\beta v_m^s} \geq \lambda. \end{aligned}$$

One can see from the above FOCs that the buyer is indifferent between using money or the asset as a means of payment if his *relative* valuation of the asset to money is identical to the seller's *relative* valuation of the asset to money, i.e.,

$$\frac{v_a^b}{v_m^b} = \frac{v_a^s}{v_m^s}.$$

In this case only the total expected value of the portfolio in the next CM matters. Define by $\omega(q_a, q_m) = v_a^s q_a + v_a^s q_m$ the expected value of the transfer (q_a, q_m) for the seller. When the buyer has portfolio (a, m), $\omega(a, m)$ denotes the value of the entire portfolio of the buyer in the eyes of the seller. We then have the following lemma.

Lemma 3 *If the buyer and the seller have the same relative valuation of asset to money, i.e., $\epsilon_a \equiv v_a^b/v_a^s = v_m^b/v_m^s \equiv \epsilon_m$, and the buyer's portfolio is (a, m), the terms of trade (y, q_a, q_m) are such that*

$$y = \min\{c^{-1}(\beta\omega(a, m), y^*(\epsilon_m)\}, \text{ and } \omega(q_a, q_m) = \min\{\omega(a, m), \omega^*(\epsilon_m)\},$$

where $y^*(\epsilon_m)$ solves $\ell(y; \epsilon_m) = 0$, and $\omega^*(\epsilon_m) \equiv c(y^*(\epsilon_m))/\beta$.

Proof. In the Appendix. ∎

As mentioned above, when the buyer and the seller have the same relative valuation of money to asset the buyer is indifferent between using money or the asset as a means of payment, as they have, in a sense, the same "exchange rate," v_a/v_m, of asset to money. When, in addition, the buyer and the seller value each means of payment the same way, i.e., $\epsilon = 1$, we have that, provided the buyer's portfolio value is large enough for him not to be constrained, i.e., $\omega(a, m) \geq \omega^*(1)$, the buyer will purchase the efficient quantity y^* from the seller.

However, when the buyer and the seller have the same relative valuation of asset to money, but $\epsilon \neq 1$, they need not be trading the efficient quantity even when the buyer is unconstrained. In fact, in this case one can have that the buyer and the seller do not value each means of payment the same way since $\epsilon \neq 1$, which implies that when the buyer is unconstrained he will purchase the quantity $y^*(\epsilon)$ from the seller, with $y^*(\epsilon) > y^*$ for $\epsilon < 1$, and $y^*(\epsilon) < y^*$ when $\epsilon > 1$. If we consider the case $\epsilon > 1$, the buyer values money and the asset more than the seller, whereas when $\epsilon = 1$ they both value the two means of payment the same way. Hence, when $\epsilon > 1$ the seller is willing to produce less than when $\epsilon = 1$ for the same transfer, or in other words it is more costly for the buyer to purchase a given quantity, and he will therefore buy less.

When the value of the buyer's portfolio is not large enough to purchase $y^*(\epsilon)$, he is constrained, and uses all of his portfolio in exchange for goods: when $\omega(a, m) < \omega^*(\epsilon)$ the buyer gives his entire portfolio to the seller, and in exchange the latter is willing to produce the quantity $y = c^{-1}(\beta\omega(a, m)) < y^*(\epsilon)$. Note that when $\epsilon < 1$, since in this case $y^*(\epsilon) > y^*$, it is possible that the efficient quantity of goods will be produced even though the buyer is constrained.

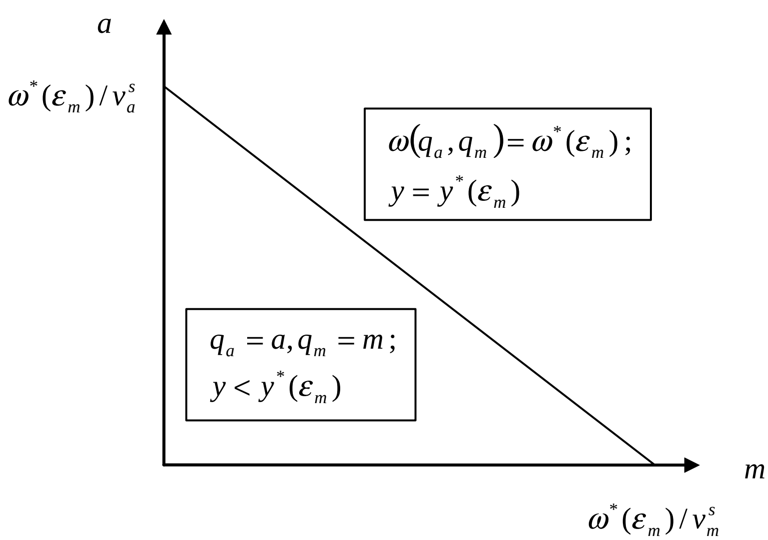

Figure 1

When the buyer and the seller have different relative valuations of asset to money, the buyer has a strict preference for using one of the two as a means of payment. When the buyer values the asset relative to money more than the seller, i.e.,

$$\frac{v_a^b}{v_m^b} > \frac{v_a^s}{v_m^s},$$

the buyer strictly prefers to use money as a means of payment.

Lemma 4 *In a meeting where the buyer values the asset relative to money more than the seller, i.e., $\epsilon_a > \epsilon_m$, and the buyer's portfolio is (a, m), there exist $m^*(\epsilon_m)$ and $m^*(\epsilon_a)$, $m^*(\epsilon_m) > m^*(\epsilon_a)$,* *such that the terms of trade (y, q_a, q_m) are:*

$$(y, q_a, q_m) = \begin{cases} (y^*(\epsilon_m), 0, m^*(\epsilon_m)) & \text{, if } m \geq m^*(\epsilon_m); \\ (y(0,m), 0, m) & \text{, if } m \in [m^*(\epsilon_a), m^*(\epsilon_m)); \\ (y^*(\epsilon_a), q_a(m;\epsilon_a), m) & \text{, if } m < m^*(\epsilon_a) \text{ and } \omega(a,m) \geq \omega^*(\epsilon_a); \\ (y(a,m), a, m) & \text{, if } m < m^*(\epsilon_a) \text{ and } \omega(a,m) < \omega^*(\epsilon_a), \end{cases}$$

where $m^*(\epsilon) = c(y^*(\epsilon)/(\beta v_m^s)$, $y(q_a, q_m) = c^{-1}[\beta(v_m^s q_m + v_a^s q_a)]$, *and*

$$q_a(m;\epsilon_a) = \frac{c(y^*(\epsilon_a))}{\beta v_a^s} - \frac{v_m^s}{v_a^s} m. \tag{28}$$

Proof. In the Appendix. ■

Let us lay out the intuition for the result. First, it is obvious that since the buyer has all the bargaining power he will offer terms of trade to the seller that will make him indifferent between trading or not, i.e., $S^s(y) = 0$. The transfer (q_a, q_m) made by the buyer to the seller to obtain a quantity y of the good satisfies $v_a^s q_a + v_m^s q_m = c(y)/\beta$. Using this, the surplus to the buyer in obtaining the quantity y is

$$S^b(y; a, m) = u(y) - c(y) \times \frac{v_a^b q_a + v_m^b q_m}{v_a^s q_a + v_m^s q_m}.$$

Hence, if we abstract from the value that the buyer holds in his portfolio for each asset, it follows that the buyer prefers to use money rather than the asset to pay the seller: if the buyer uses only money, i.e., $q_a = 0$ and $q_m > 0$, from the above expression we obtain that the surplus in obtaining y is

$$u(y) - c(y) \times \frac{v_m^b}{v_m^s},$$

whereas if he uses only the asset the surplus is

$$u(y) - c(y) \times \frac{v_a^b}{v_a^s}.$$

However, we have that buyers' valuation of the asset relative to money exceeds that of sellers, i.e.,

$$\frac{v_a^b}{v_m^b} > \frac{v_a^s}{v_m^s},$$

which clearly implies that the surplus of the buyer is larger when he uses money. Hence, it is only if the buyer does not have enough money in his portfolio to purchase the desired quantity from the seller that he will consider using the asset as a means of payment. In particular, if the buyer has in his portfolio a quantity of money $m \geq m^*(\epsilon_m)$, where $m^*(\epsilon_m)$ the quantity of money for which the buyer is not constrained, then the buyer obtains $y^*(\epsilon_m)$ by transferring $m^*(\epsilon_m)$ to the seller. If, however, the buyer has marginally less than $m^*(\epsilon_m)$, he will use all of his money but he will not be using his asset regardless of how much asset he has. The reason for this is that since his relative valuation of the asset exceeds that of the seller, i.e., $v_a^s/v_m^s < v_a^b/v_m^b$, and for m close enough to $m^*(\epsilon_m)$ the buyer's instantaneous marginal utility gain from using the asset to purchase some extra good $u'(y)$ is less than the discounted marginal utility loss from transferring the asset to pay for the additional good purchased, which is $(v_a^s/v_a^b)c'(y)$, because for y marginally less than $y^*(\epsilon_m)$ we have that

$$\frac{u'(y)}{c'(y)} = \frac{v_m^b}{v_m^s} + \eta < \frac{v_a^b}{v_a^s},\ \eta > 0 \text{ small.}$$

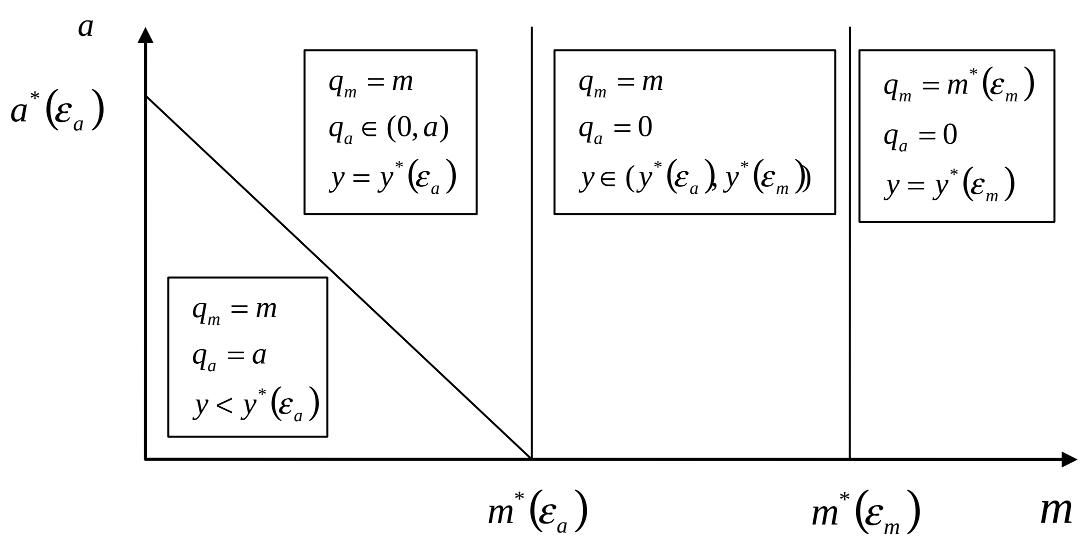

Figure 2

Then as the money holdings of the buyer decrease further, the quantity he can buy from the seller decreases. When m reaches $m^*(\epsilon_a)$, the quantity that can be purchased by the buyer becomes $y^*(\epsilon_a)$ such that $u'(y^*(\epsilon_a))/c'(y^*(\epsilon_a)) = v_a^s/v_a^b$, i.e., the marginal benefit of consumption becomes equal to the marginal cost of financing additional consumption with the asset. Hence, when m falls below $m^*(\epsilon_a)$ a buyer of type j will use all of his money and some of his asset to purchase consumption from the seller up to the point where the cost of using the asset equals marginal utility of consumption. The quantity of asset transferred is given by (28), i.e., the quantity of asset transferred exactly complements the money holdings that the buyer transferred to the seller in order to buy $y^*(\epsilon_a)$. Naturally, the buyer might not have enough asset holdings to purchase all of these extra units of consumption, in which case he will give all his portfolio to the buyer, i.e., $q_a = a$ and $q_m = m$, and in exchange he obtains $y < y^*(\epsilon_a)$ such that $c(y) = \beta(v_a^s a + v_m^s m)$. The fact that money can increase the quantity traded in the DM by increasing the size of the surplus, which here is captured entirely by buyers, is reminiscent of Engineer and Shi (1998) and Berentsen and Rocheteau (2003).

Symmetrically, when the buyer values the asset relative to money less than the seller, the buyer has a strict preference in using the asset as a means of payment, and he will therefore first use the asset to buy from the seller, and he will use money as a means of payments only if his portfolio does not contain enough asset to buy the desired quantity of good from the seller. The results in this case are perfectly symmetric to the previous case, and we have the following lemma illustrated with Figure 3.

Lemma 5 *If $\epsilon_a < \epsilon_m$ and the buyer's portfolio is (a, m), there exist $a^*(\epsilon_a)$ and $a^*(\epsilon_m)$ such that the terms of trade (y, q_a, q_m) are:*

$$(y, q_a, q_m) = \begin{cases} (y^*(\epsilon_a), a^*(\epsilon_a), 0) & , \text{ if } a \geq a^*(\epsilon_a); \\ (y(a, 0), a, 0) & , \text{ if } a \in [a^*(\epsilon_m), a^*(\epsilon_a)); \\ (y^*(\epsilon_m), a, q_m(a; \epsilon_m)) & , \text{ if } a < a^*(\epsilon_m) \text{ and } \omega(a, m) \geq \omega^*(\epsilon_m); \\ (y(a, m), a, m) & , \text{ if } a < a^*(\epsilon_m) \text{ and } \omega(a, m) < \omega^*(\epsilon_m), \end{cases}$$

where $a^*(\epsilon) = c(y^*(\epsilon))/(\beta v_a^s)$, and

$$q_m(a; \epsilon_m) = \frac{c(y^*(\epsilon_m))}{\beta v_m^s} - \frac{v_a^s}{v_m^s} a. \tag{29}$$

From lemmas 4 and 5, note that when the buyer and the seller have different relative valuations of asset to money the quantity of goods the buyer purchases from the seller can exceed the efficient quantity y^*. This inefficiency is coming from the fact that the buyer and the seller can value differently the means of payment. For instance, when the buyer values the asset relative to money more than the seller, the buyer purchase first by using money. And if the buyer values money less than the seller, i.e., if $v_m^b < v_m^s$, then $\epsilon_m < 1$, and therefore the quantity of good traded if the buyer is not constrained by his portfolio is $y^*(\epsilon_m) > y^*$. This type of inefficiency does not exist in general in search-theoretic models of money because even though the buyer and seller value the good exchanged differently, they value the means of payment the same way. A notable exception is Berentsen and Rocheteau (2003).

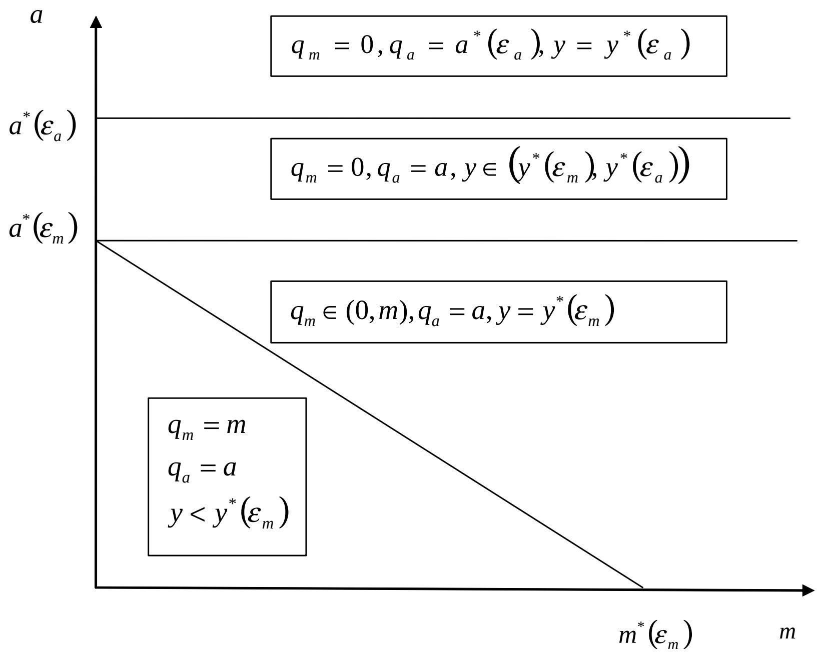

Figure 3

4.2 The Optimal Portfolio Choices

Now that we have established the terms of trade for the different possible types of meetings in the DM, we turn our attention to the agents' choice of portfolio (a, m) in the first CM of their lives. The first order conditions for a type-j agent's problem in either state are

$$a: \quad \psi \geq W_a^j(a,m), \quad = \text{ if } a > 0;$$
$$m: \quad \phi \geq W_m^j(a,m), \quad = \text{ if } m > 0.$$

There are potentially three cases to consider depending on whether buyers value the asset relatively to money more than sellers. However, we will be considering only one of the three possible cases, that is the case where buyers value the asset relative to money more than sellers. There are at least two reasons to focus on this case. First, for the two other cases to exist requires that the price of the asset is sufficiently negatively correlated with its dividend, which seems unnatural. In fact, if the price of the assets state-independent, then buyers value the asset more than sellers. Second and foremost, when buyers value the asset relative to money less than or the same as buyers, then buyers will use the asset first as a means of payment which implies that money will be essential only if the stock of asset is not large enough for buyers to purchase the desired quantity of goods in the DM. Since this paper aims at showing that fiat money can be valued even when the stock of asset is large, we focus on the case where this can happen, and this requires buyers to value the asset relative to money more than sellers.

4.2.1 Sellers

When the agent is a seller, his portfolio has no incidence on the terms of trade in the DM, and

$$W^s(a,m) = -c(y) + \beta \mathbb{E}_{\widehat{z}} \left[v^{s,\widehat{z}}(a+q_a, m+q_m) \right],$$

where $\widehat{z}$ denotes the state next period. The expected marginal values of each portfolio component are therefore given by

$$\begin{aligned} a: \quad & W_a^s(a,m) = \beta v_a^s; \\ m: \quad & W_m^s(a,m) = \beta v_m^s. \end{aligned}$$

The marginal value of money for sellers is therefore simply the expected discounted value of money next period. This is because when an agent is a seller in the DM he does not need money nor the asset to carry out transactions there, and therefore neither money nor the asset provide him with any liquidity services. Hence, a seller's valuations of both elements of his portfolio are based solely on their payouts.

4.2.2 Buyers

When an agent is a buyer, the quantities of money and asset he has brought with him into the DM matter because he will need either money, or both money and the asset, to purchase from the seller he has been matched with. The marginal value of the component i of the portfolio, $i \in \{a,m\}$, is given by

$$W_i^b(a,m) = \beta \mathbb{E}_{\widehat{z}} v_i^{b,\widehat{z}} + \frac{dy}{di} u'(y) + \frac{dq_a}{di} \beta \mathbb{E}_{\widehat{z}} \frac{\partial v^{b,\widehat{z}}}{\partial q_a} + \frac{dq_m}{di} \beta \mathbb{E}_{\widehat{z}} \frac{\partial v^{b,\widehat{z}}}{\partial q_m}.$$

The first component of the marginal value of component i of the portfolio is the discounted expected marginal price of the component next period. But the extra units of i being brought into the DM can increase the quantity of the good the buyer can purchase from the seller, in which case the buyer's marginal value is enhanced by $(dy/di)\, u'(y)$, minus the utility cost of these extra units of the special good [*explain*]

$$(dq_a/di) \times \beta \mathbb{E}_{\widehat{z}} \left[\partial v^{b,\widehat{z}} / \partial q_a \right] + (dq_m/di) \times \beta \mathbb{E}_{\widehat{z}} \left[\partial v^{b,\widehat{z}} / \partial q_m \right].$$

If the buyer already has enough money in his portfolio to purchase the desired quantity of good from the seller, the additional quantity of money or asset brought in the DM will not be used there and will instead be brought forward to the next CM. If instead the buyer was constrained by the value of his money holding, then bringing more money or asset can help in buying more from the seller. But as established in lemma 4, it is not always true that bringing more of the asset will translate into a greater quantity of good transacted. In fact, we have shown that when the buyer's relative valuation of asset to money is greater than that of the seller, then the buyer strictly prefers using money as a means of payment. In particular, if his money holding is not too small he might prefer not to use the asset to purchase from the seller even though he is constrained by his money holdings. Hence, we should expect that $dq_m/dm \geq 0$, and likewise $dq_a/da \geq 0$.

We have moreover shown that in some cases the buyer will use all of his money, and use some of his asset, in order to purchase a specific quantity of the good. In this case, when the buyer brings more money into the DM he will use this extra quantity of money to purchase the same quantity of good from the seller, and money will therefore be used as a substitute for the asset. Hence, we should expect that $dq_a/dm \leq 0$ and $dq_m/da = 0$.

Lemma 6 *When $\epsilon_a > \epsilon_m$ and the buyer's portfolio is (a, m):*

$$W_a^b(a,m) = \beta v_a^b \left[1 + \ell(y;\epsilon_a) \cdot 1\left\{m < m^*(\epsilon_a) \text{ and } \omega(a,m) < \omega^*(\epsilon_a)\right\}\right], \textit{ and}$$

$$W_m^b(a,m) = \beta v_m^b \left[1 + \ell(y;\epsilon_m) \cdot 1\left\{m < m^*(\epsilon_m)\right\}\right].$$

Proof. In the Appendix. ■

The detailed expression for $W_m^b(a,m)$ is

$$\begin{aligned} W_m^b(a,m) \quad = \quad & \beta v_m^b [1 + \ell(y;\epsilon_m) \cdot 1\left\{m \in [m^*(\epsilon_a), m^*(\epsilon_m))\right\} \\ & + \underbrace{\left(\frac{v_m^s}{v_m^b}\frac{v_a^b}{v_a^s} - 1\right)}_{\ell(y^*(\epsilon_a);\epsilon_m)} \cdot 1\left\{m < m^*(\epsilon_a) \textit{ and } \omega(a,m) \geq \omega^*(\epsilon_a)\right\} \\ & + \ell(y;\epsilon_m) \cdot 1\left\{m < m^*(\epsilon_a) \textit{ and } \omega(a,m) < \omega^*(\epsilon_a)\right\}]. \end{aligned}$$

Note that for the sake of notation we have not indexed the quantity of good exchanged, but naturally the quantity changes with the portfolio composition and value that the buyer brings in the DM. When the value of the buyer's money holdings is more than $m^*(\epsilon_m)$, then the buyer can purchase the desired quantity $y^*(\epsilon_m)$ from the seller, and therefore neither money nor the asset bear any liquidity premium. If the buyer holds less money, but more than the quantity $m^*(\epsilon_a)$, then he is not able to purchase $y^*(\epsilon_m)$ but he is still not willing to use the asset as a means of payment yet. This is because the cost to him is too large since the seller values the asset less than him. In this situation the buyer increases the amount he can purchase from the seller only if he brings extra money in. That is why only money bears a liquidity premium $\ell(y;\epsilon_m)$ in this case. The same reasoning applies if the portfolio of the buyer is such that $m < m^*(\epsilon_a)$ *and* $\omega(a,m) \geq \omega^*(\epsilon_a)$ since in this case he purchases $y^*(\epsilon_a)$ from the seller, and if he brings extra cash he will use it to reduce his use of the asset but not to increase the quantity of special good consumed. And if he brings some extra asset, it will not change the terms of trade. However, if the portfolio the buyer is holding on to has a total value, in the eye of the seller, equal to $\omega^*(\epsilon_a)$, then he can purchase only $y^*(\epsilon_a)$, which is such that $u'(y^*(\epsilon_a)) = c'(y^*(\epsilon_a)) \times (v_a^b/v_a^s)$. Hence, if $\omega(a,m) < \omega^*(\epsilon_a)$, we have that even if the buyer uses all his portfolio he can at most purchase y such that $u'(y) > c'(y) \times \epsilon_a^{j,k}$. Therefore, the buyer is happy to use the asset to purchase from the seller and both the asset and money bear a liquidity premium. It is worth noting that, even though both money and the asset bear a liquidity premium when $\omega(a,m) < \omega^*(\epsilon_a)$, the premium is not the same for both. In fact, $\ell(y;\epsilon_a) < \ell(y;\epsilon_m)$, and therefore money's liquidity premium is larger than that of the asset, which is quite intuitive: the buyer prefers to use money as a means of payment because of the difference in their relative marginal values for the asset, and therefore money dominates the asset in the provision of liquidity services.

4.2.3 The FOCs

The first order conditions governing the optimal portfolio decisions for seller and buyers in either state of the world can therefore be written respectively as

$$\begin{aligned} a: \quad & \psi \geq \beta v_a^s, \quad = \text{ if } a^s > 0; \\ m: \quad & \phi \geq \beta v_m^s, \quad = \text{ if } m^s > 0, \end{aligned} \tag{30}$$

and

$$\begin{aligned} a: \quad & \psi \geq \beta v_a^b \left[1 + L(y;\epsilon_a)\right], \quad = \text{ if } a^b > 0; \\ m: \quad & \phi \geq \beta v_m^b \left[1 + L(y;\epsilon_m)\right], \quad = \text{ if } m^b > 0. \end{aligned} \tag{31}$$

The presence of an aggregate shock to the economy means that there might be a role for a monetary policy that is not stationary in the sense that it induces a constant inflation rate. In this paper we restrict ourselves to stationary policies in that the growth of the money stock is constant across periods and states, inducing a constant inflation rate.

5 Stationary Monetary Policy

In this section we consider the standard monetary policy of a constant money growth rate. Let us denote by τ the growth rate of the money stock. Since we focus on equilibria where real balances are constan, i.e., $\phi M = \widehat{\phi}\widehat{M}$, this implies that τ is also the inflation rate , i.e., $1+\tau = \phi/\widehat{\phi}$. In this case, we have that, given a current value for money and the money stock of ϕ and M , the value of money and money stock next period are given by $\widehat{\phi}^1 = \widehat{\phi}^2 = \widehat{\phi} = \phi/(1+\tau)$ and $\widehat{M} = (1+\tau) M$, so that

$$v_m^b = v_m^s = \widehat{\phi}.$$

If follows straightforwardly from the FOCs (30) and (31) for agent's first CM that the fact that the inflation rate is constant and state-independent also implies that the price of the asset is constant and state-independent. This in turn implies that $v_a^b = \psi + \overline{d}$, and that $v_a^s = \psi + \overline{d}^s$. The first order conditions for the portfolio decisions for sellers and buyers are therefore state independent and can be rewritten for sellers as

$$\begin{aligned} a: \quad & \psi \geq \beta\left(\psi + \overline{d}^s\right), \quad = \text{ if } a^s > 0; \\ m: \quad & 1+\tau \geq \beta, \qquad = \text{ if } m^s > 0, \end{aligned} \tag{32}$$

and for buyers as

$$\begin{aligned} a: \quad & \psi \geq \beta\left(\psi + \overline{d}\right)[1 + L(y;\epsilon_a)], \quad = \text{ if } a^b > 0; \\ m: \quad & 1+\tau \geq \beta\,[1 + L(y;\epsilon_m)], \qquad = \text{ if } m^b > 0. \end{aligned} \tag{33}$$

However, $v_a^b = \psi + \overline{d} > v_a^s = \psi + \overline{d}^s$, implying that $a^b = A$ and $a^s = 0$. Moreover, in any monetary equilibrium such that $\tau > \tau^F = \beta - 1$, if such a monetary equilibrium exists, then we must have that $m^b = M$ and $m^s = 0$. For $\tau = \tau^F$ we consider the limit as $\tau \to \tau^F$, and therefore at the Friedman rule we also obtain that $m^b = M$ and $m^s = 0$. We then have the following proposition.

Proposition 2 *(i) For any stock of asset A there exists an inflation rate $\overline{\tau}(A) > \tau^F$ such that for all $\tau \in \left(\tau^F, \overline{\tau}(A)\right)$ a (unique) monetary equilibrium exists. $\overline{\tau}(A)$ is continuous and such that*

$$\overline{\tau}(A) = \begin{cases} \overline{\tau}^* > \tau^F, & \text{for all } A \geq \widetilde{A}, \text{ and} \\ \overline{\tau}(A) > \overline{\tau}^*, \text{ with } \overline{\tau}'(A) \leq 0, & \text{for all } A < \widetilde{A}; \end{cases}$$

(ii) For all $\tau \in [\tau^F, \overline{\tau}^*)$ money serves as the unique *means of payment and $\psi(\tau) = \widetilde{\psi}^*$. And for all $\tau \in [\overline{\tau}^*, \overline{\tau}(A)]$ both money and the asset serve as means of payments, and for all $\tau \in (\overline{\tau}^*, \overline{\tau}(A)]$, $\psi(\tau) > \widetilde{\psi}^*$, with $\partial\psi(\tau)/\partial\tau > 0$.*

Proof. In the Appendix. ■

The first part of the proposition says that the maximum inflation rate for which a monetary equilibrium exists, $\overline{\tau}(A)$, is strictly decreasing in the stock of asset A, for A less than the cutoff value $\widetilde{A}$ that is sufficient for buyers to purchase the quantity desired $\widetilde{y}^*$ in a non-monetary equilibrium. And for A equal or greater than the cutoff value $\widetilde{A}$ the maximum possible inflation rate is $\overline{\tau}^* > \tau^F$. (see figure 4).

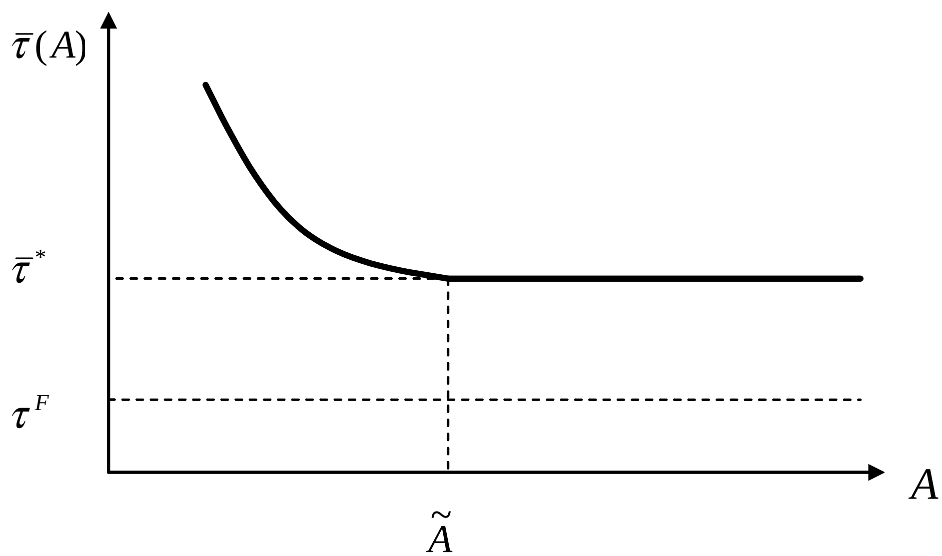

Figure 4

This is in contrast with the results in Lagos and Rocheteau (2007) and Geromichalos et al. (2007). They obtain respectively that if the efficient stock of capital for production in the CM or the stock of Lucas trees are not large enough for these assets to fully play the role of means of payment, then the upper bound on the maximum inflation rate to exist is strictly greater than that of the Friedman rule, and this upper bound increases as the efficient stock of capital or stock of Lucas trees decreases. However, if the efficient stock of capital for production in the CM or the stock of Lucas trees are large enough for these assets to fully play the role of means of payment, then a monetary equilibrium exists only at the Friedman rule, and hence, money is not essential.

Hence, contrary to these papers we obtain that for *any* level of stock of asset a monetary equilibrium exists *away* from the Friedman rule. The reason why a monetary equilibrium exists away from the Friedman rule even when the stock of asset is large enough for buyers to purchase the desired quantity $\widetilde{y}^*$ in the DM in a non-monetary equilibrium is as follows. The quantity $\widetilde{y}^*$ that buyers desire to purchase in the DM in a non-monetary equilibrium is less than the efficient quantity y^*. The inefficiency of the non-monetary equilibrium lies in the different valuations that buyers and sellers have of the asset because of the different correlation properties of their disutility of effort and the rate of return of the asset. When the monetary authority follows a policy delivering a state-independent inflation rate, buyers and sellers value money the same way, and hence buyers value the asset relative to money more than sellers. At the Friedman rule it is costless for buyers to bring money into the DM and they bring enough real balances to purchase the efficient quantity y^*. When the inflation rate increases and lies in the interval $\left(\tau^F, \overline{\tau}^*\right)$, although it becomes costly to carry money for buyers, the cost of inflation is not too large in that the level of real balances that buyers are bringing into the DM enables them to purchase a quantity $y(\tau)$ greater than $\widetilde{y}^*$. Hence, for levels of inflation less than the cutoff value $\overline{\tau}^*$ buyers are better off using money than not. Note that, as highlighted above, this is true for any size of the stock of asset. This is because in the DM buyers will use only money as a means of payment.

The cutoff inflation rate $\overline{\tau}^*$ is such that buyers bring into the DM real balances to purchase the quantity $\widetilde{y}^*$, at which point buyers become indifferent between money or the asset in the DM. If the inflation rate increases further, then a monetary equilibrium no longer exists when the stock of asset is large, i.e., greater than $\widetilde{A}$, because the cost of inflation becomes too large and buyers are better off using the asset as the unique means of payment. When the stock of asset is less than $\widetilde{A}$, then a monetary equilibrium exists for inflation rates greater than $\overline{\tau}^*$ because the stock of asset is not large enough for buyers to be able to purchase $\widetilde{y}^*$ in the DM. Hence, buyers are willing to accept a larger inflation rate.

Regarding the asset pricing implication of our model, we obtain that if the inflation rate is no more than $\overline{\tau}^*$ the asset does not carry any liquidity premium, and therefore its price is equal to its fundamental value, that of the expected discounted value of dividends. It is worth noting that this result holds no matter how large the stock of the asset is. This is because when τ is no greater than $\overline{\tau}^*$ only money is used as a means of payment and therefore only money can carry a liquidity premium. Inflation matters for the price of the asset only if its serves as a means of payment through its impact on the liquidity premium. This happens only when the inflation rate is greater than $\overline{\tau}^*$, which itself requires the stock of asset to be smaller than $\widetilde{A}$.

There is another important result that our model delivers: a form of Rate of Return Dominance, RRD hereafter. In fact, the rate of return on money, excluding the liquidity premium, is

$$R_m(\tau) = \frac{\widehat{\phi} - \phi}{\phi} = \frac{-\tau}{1+\tau}, \tag{34}$$

whereas the expected rate of return on the asset is

$$R_a(\tau) = \frac{\left(\psi(A,\tau) + \overline{d}\right) - \psi(A,\tau)}{\psi(A,\tau)} = \frac{\overline{d}}{\psi(A,\tau)}. \tag{35}$$

When the inflation rate is less than $\overline{\tau}^*$ the price of the asset is $\widetilde{\psi}^* = \beta\overline{d}/(1-\beta)$, in which case (35) simplifies to

$$R_a(\tau) = \frac{1-\beta}{\beta}. \tag{36}$$

It is clear that for $\tau \in \left(\tau^F, \overline{\tau}^*\right)$ we have that $R_m < R_a$, i.e., we obtain RRD in expected terms. When $\tau > \overline{\tau}^*$, using the FOCs (30) and (31) for the agents' portfolio decisions, we obtain that the price of the asset is given by

$$\psi(\tau) = \frac{1+\tau}{-\tau} \times \overline{d}^s, \tag{37}$$

and therefore for $\tau \in [\overline{\tau}^*, \overline{\tau}(A)]$ we have that

$$R_a(\tau) = \frac{-\tau}{1+\tau} \times \frac{\overline{d}}{\overline{d}^s} > R_m(\tau). \tag{38}$$

We summarize this in the following proposition.

Proposition 3 *For any inflation rate $\tau > \tau^F$, we have that $R_a(\tau) > R_m(\tau)$. Moreover, for all $\tau \in (\tau^F, \tau^*)$, $R_a'(\tau) = 0$, and for $\tau > \tau^*$, $R_a'(\tau) < 0$.*

This is a form of RRD because buyers hold money which is dominated by the asset. We would like to highlight that, although the asset dominates money only in expected terms and not for every state, buyers are risk neutral agents with regards to the risk related to the asset, and in that sense the result is really a form of RRD. In fact, the asset's dividends and what they obtain from the sale of assets enable buyers to work less in the last CM of their lives. And since buyers do not face any variability in the disutility of effort, they care only about the expected value of the dividend they will receive. We obtain a RRD in this model because the asset is an inefficient means of payment due to the difference in valuations of buyers and sellers. Buyers are therefore willing to hold on to a dominated asset because it is a better means of payment. In our model, money is a better means of payment because buyers and sellers value it the same way, and therefore buyers can purchase a greater quantity in the DM that they would if they were to use the asset.

For the reader who is not convinced that we have obtained a RRD, consider our model but replace sellers by agents with a marginal disutility of effort of 1 in each state in the last CM of their lives. The portfolio decision of buyers in this modified version of the model is almost identical to that of agents in Geromichalos et al. (2007), and money is essential only if the stock of asset is not large enough for buyers to purchase the efficient quantity y^* in the DM. In any case the rate of return on the asset is equal to the rate of return of money. This shows that the stochastic nature of the dividend is of no importance by itself.

6 Conclusion

We have developed a model where money and an asset endogenously coexist despite money being return dominated by the asset. Agents carry money for it endogenously offers better liquidity properties than the asset for decentralized trades. This is because buyers value the asset relative to money more than sellers, and they therefore strictly prefer to use money rather than the asset as a means of payment.

It was shown that for low levels of inflation a dichotomy is obtained: money is used as the only means of payments in the DM, and the asset is used solely to transfer across periods. Hence, the asset does not carry any liquidity premium, and its price and rate of return are independent of the inflation rate. When the inflation rate increases enough, buyers' real balances are so low that buyers start using the asset as a means of payment in addition to money despite sellers not valuing it as much as buyers. In this case the asset carries a liquidity premium as well, albeit smaller than that of money, and the asset price and real rate of return are affected by the inflation rate: the price of the asset increases with inflation, and therefore its real rate of retrun is negatively correlated with the inflation rate.

Appendix A

Proof of Lemma 1

The first order conditions for problem (8), which are necessary and sufficient since the Hessian for the Lagrangian is negative semi-definite, are (9) and (10). Hence, if $\mu_a = 0$, that is the buyer is not constrained, combining the two FOCs and defining $\epsilon_a = v_a^b/v_a^s$, we obtain that $y = y^*(\epsilon_a)$ such that $u'(y^*(\epsilon_a))/c'(y^*(\epsilon_a)) = \epsilon_a$. This holds if and only if $a \geq a^*(\epsilon_a) \equiv c(y^*(\epsilon_a))/\beta v_a^s$.

If the buyer is constrained, that is, $\mu_a > 0$, then we obtain from the constraint that the seller's surplus should be non-negative that that $q_a = a$ and $y = c^{-1}(\beta v_a^s a) < y^*(\epsilon_a)$. This holds if and only if $a < a^*(\epsilon_a)$.■

Proof of Lemma 2

From (2) we have that

$$W_a^b(a) = u'(y)\frac{dy}{da} - \beta v_a^b \frac{dq_a}{da} + \beta v_a^b.$$

From lemma 1 we know that if $a \geq a^*(\epsilon_a)$ then $dy/da = dq_a/da = 0$, implying that $W_a^b(a) = \beta v_a^b$. When $a < a^*(\epsilon_a)$, then we have that $dq_a/da = 1$ and from the seller's zero surplus condition $dy/da = \beta v_a^s/c'(y)$. Hence, in this case we have

$$W_a^b(a) = u'(y)\frac{\beta v_a^s}{c'(y)} - \beta v_a^b + \beta v_a^b = \beta v_a^s \frac{u'(y)}{c'(y)},$$

which can be rewritten as

$$W_a^b(a) = \beta v_a^b \left(\frac{1}{\epsilon_a} \frac{u'(y)}{c'(y)} \right) = \beta v_a^b [1 + \ell(y; \epsilon_a)].$$

Defining $L(y; \epsilon_a) \equiv \max\{0; \ell(y; \epsilon_a)\}$ and combining the two expressions for $W_a^b(a)$ for a greater and smaller than $a^*(\epsilon_a)$, $W_a^b(a)$ can be rewritten as (15).■

Proof of Proposition 1

We will prove existence of a steady-equilibrium by construction. Uniqueness follows.

(i) When the buyer is not constrained the price of the asset does not carry any liquidity premium, and since the buyer holds all of the asset its price solves $\psi = \beta\left(\psi + \overline{d}\right)$, which yields

$$\psi = \frac{\beta}{1-\beta}\overline{d}.$$

This implies that

$$v_a^b = \frac{\overline{d}}{1-\beta}, \text{ and } v_a^s = \frac{\beta\overline{d} + (1-\beta)\overline{d}^s}{1-\beta},$$

so that

$$\epsilon_a = \widetilde{\epsilon}_a^* \equiv \frac{\overline{d}}{\beta\overline{d} + (1-\beta)\overline{d}^s}.$$

Hence, $y = \widetilde{y}^* = y^*(\widetilde{\epsilon}_a^*)$ is such that

$$\frac{u'(y)}{c'(y)} = \widetilde{\epsilon}_a^*,$$

and is unique for $\lim_{y\to 0} u'(y)/c'(y) = \infty$, $u'(y)/c'(y)$ is strictly decreasing and $\lim_{y\to\infty} u'(y)/c'(y) = 0$. And since sellers obtain no surplus, i.e., $\beta v_a^s q_a = c(\widetilde{y}^*)$, we have that

$$q_a = \widetilde{A} \equiv \frac{1-\beta}{\beta} \frac{c(\widetilde{y}^*)}{\beta\overline{d} + (1-\beta)\overline{d}^s}.$$

This holds if and only if the stock of asset in the economy is large enough, that is, if and only if $A \geq \widetilde{A}$.

(ii) When buyers are constrained, i.e., when $A < \widetilde{A}$, then buyers give away their entire asset holding to the seller they are matched with, and therefore $y < \widetilde{y}^*$ solves $c(y) = \beta v_a^s A$. Replacing v_a^s by its expression yields

$$c(y) = \beta\left(\psi + \overline{d}^s\right) A. \tag{A1}$$

Since $y < \widetilde{y}^*$, we have that $\ell(y; \epsilon_a) > 0$, and therefore the price of the asset is given by

$$\psi = \beta v_a^b \times \frac{1}{\epsilon_a} \frac{u'(y)}{c'(y)}.$$

Since $\epsilon_a = v_a^b / v_a^s$, and that $v_a^s = \psi + \overline{d}^s$, the above expression simplifies to

$$\frac{\psi}{\psi + \overline{d}^s} = \beta \frac{u'(y)}{c'(y)}. \tag{A2}$$

(A1) and (A2) form a system of two equations in the two unknowns ψ and y. (A1) gives $y^1(\psi)$ such that $y^1(0) = c^{-1}(\beta\overline{d}^s A) > 0$ for $A > 0$, and $y^1(\psi)$ is strictly increasing in ψ with $\lim_{\psi\to\infty} y^1(\psi) = \infty$. (A2) gives $y^2(\psi)$ such that $\lim_{\psi\to 0} y^2(\psi) = \infty$, $y^2(\psi)$ is strictly decreasing and converges to some strictly positive value. This implies that there is a unique solution (y, ψ) to the two equations.

Moreover, (A2) is unaffected by changes in A whereas (A1) shifts up as A increases. This implies that y and ψ are strictly increasing and strictly decreasing respectively in A. It is easy to verify that when $A = \widetilde{A}$, we obtain $y = \widetilde{y}^*$ and $\psi = \widetilde{\psi}^*$.■

Proof of Lemma 3

The first order conditions for problem (24), which are necessary and sufficient for the Hessian for the Lagrangian is negative semi-definite, are (25), (26), and (27). First, when $\epsilon_a = v_a^b/v_a^s = v_m^b/v_m^s = \epsilon_m$, we have that $\mu_a > 0$ if and only if $\mu_m > 0$. We then have two cases:

• Case 1: $\mu_m = \mu_a = 0$. Then from (26) and (27) we have that $\lambda = \epsilon_m$. This, together with (25) implies that $y = y^*(\epsilon_m)$ with $y^*(\epsilon_m)$ such that $\ell(y^*(\epsilon_m);\epsilon_m) = 0$. Hence, we have that the transfer (q_a, q_m) is such that $v_a^s q_a + v_m^s q_m = \omega^*(\epsilon_m) = c(y^*(\epsilon_m))/\beta$. And since $\mu_m = \mu_a = 0$, this holds if and only if $\omega(a,m) = v_a^s a + v_m^s m \geq \omega^*(\epsilon_m)$.

• Case 2: $\mu_m > 0$ and $\mu_a > 0$. In this case we have that $q_a = a$, $q_m = m$ and since $S^s(y) = 0$, we have that

$$y(a,m) = c^{-1}[\beta(v_m^s m + v_a^s a)].$$

T holds if and only if $\omega(a,m) = v_a^s a + v_m^s m < \omega^*(\epsilon_m)$.■

Proof of Lemma 4

The first thing to note is that if $\epsilon_a = v_a^b/v_a^s > v_m^b/v_m^s = \epsilon_m$, we have that $q_a > 0$ implies that $\mu_m > 0$, i.e., if money is transferred from the buyer to the seller it is only because the buyer has run out of money. We therefore have four cases to consider.

• Case 1: $q_m > 0, \mu_m = 0$, and therefore $q_a = \mu_a = 0$. From (25) and (27), we have that $y = y^*(\epsilon_m)$ with $y^*(\epsilon_m)$ such that $\ell(y^*(\epsilon_m);\epsilon_m) = 0$. Hence, we have that $q_m = m^*(\epsilon_m) = c(y^*(\epsilon_m))/\beta v_m^s$ and since $\mu_m = 0$, this holds if and only if $m \geq m^*(\epsilon_m)$.

• Case 2: $q_m > 0, \mu_m > 0$, and $q_a = \mu_a = 0$. In this case, since $q_a = 0$, we have from (26) and (27) that $\lambda > \epsilon_m$. Moreover, $\mu_m > 0$ implies $q_m = m$, and therefore $y = c^{-1}(\beta v_m^s m)$. And $\lambda \in (\epsilon_m, \epsilon_a)$ implies that this holds if and only if $m \in (m^*(\epsilon_a), m^*(\epsilon_m))$ where $m^*(\epsilon_a) = c(y^*(\epsilon_a))/\beta v_m^s$ and $y^*(\epsilon_a)$ solves $\ell(y^*(\epsilon_a);\epsilon_a) = 0$.

• Case 3: $q_m > 0, \mu_m > 0$, $q_a > 0$ and $\mu_a = 0$. Then one has $y = y^*(\epsilon_a)$, $q_m = m$, and from (26), q_a is such that

$$q_a(m;\epsilon_a) = \frac{c(y^*(\epsilon_a))}{\beta v_a^s} - \frac{v_m^s}{v_a^s} m.$$

This holds if and only if $m < m^*(\epsilon_a)$ and $\beta(v_m^s m + v_a^s a) \geq c(y^*(\epsilon_a))$.

• Case 4: $q_m > 0, \mu_m > 0$, $q_a > 0$ and $\mu_a > 0$. In this case we have that $q_a = a$, $q_m = m$ and since $S^s(y) = 0$, we have that

$$y(a,m) = c^{-1}[\beta(v_m^s m + v_a^s a)].$$

This holds if and only if $m < m^*(\epsilon_a)$ and $\beta(v_m^s m + v_a^s a) < c(y^*(\epsilon_a))$.■

Proof of Lemma 5

It is symmetric to that of lemma 4 and is therefore omitted.

Proof of Lemma 6

From lemma 4 we have that

$$\frac{dy}{dm} = \begin{cases} 0 & , \quad \text{if } m \geq m^*(\epsilon_m); \\ \frac{\beta v_m^s}{c'(y)} & , \quad \text{if } m \in [m^*(\epsilon_a), m^*(\epsilon_m)); \\ 0 & , \quad \text{if } m < m^*(\epsilon_a) \;\; and\; \omega(a,m) \geq \omega^*(\epsilon_a); \\ \frac{\beta v_m^s}{c'(y)} & , \quad \text{if } m < m^*(\epsilon_a) \;\; and\; \omega(a,m) < \omega^*(\epsilon_a), \end{cases},$$

$$\frac{dq_m}{dm} = \begin{cases} 1 & , \quad \text{if } m < m^*(\epsilon_m); \\ 0 & , \quad \text{otherwise}, \end{cases}, \text{ and } \frac{dq_a}{dm} = \begin{cases} -\frac{v_m^s}{v_a^s} & , \quad \text{if } m < m^*(\epsilon_a) \;\; and\; \omega(a,m) \geq \omega^*(\epsilon_a); \\ 0 & , \quad \text{otherwise}. \end{cases}$$

Similarly,

$$\frac{dy}{da} = \begin{cases} \frac{\beta v_a^s}{c'(y)} & , \quad \text{if } m < m^*(\epsilon_a) \;\; and\; \omega(a,m) < \omega^*(\epsilon_a), \\ 0 & , \quad \text{otherwise}. \end{cases},$$

$$\frac{dq_a}{da} = \begin{cases} 1 & , \quad \text{if } m < m^*(\epsilon_a) \;\; and\; \omega(a,m) < \omega^*(\epsilon_a), \\ 0 & , \quad \text{otherwise}, \end{cases}, \text{ and } \frac{dq_m}{da} = 0.$$

And since

$$\mathbb{E}_{\hat{z}} \frac{\partial v^{b,\hat{z}}}{\partial q_a} = -v_a^b, \text{ and } \mathbb{E}_{\hat{z}} \frac{\partial v^{b,\hat{z}}}{\partial q_m} = -v_m^b,$$

we have that

$$W_a^b(a,m) = \beta v_a^b + \beta v_a^b \left[\frac{v_a^s}{v_a^b} \frac{u'(y)}{c'(y)} - 1 \right] \cdot 1\{m < m^*(\epsilon_a) \;\; and\; \omega(a,m) < \omega^*(\epsilon_a)\},$$

and

$$\begin{aligned} W_m^b(a,m) \quad = \quad & \beta v_m^b + \beta \left[v_m^s \frac{u'(y)}{c'(y)} - v_m^b \right] \cdot 1\{m < m^*(\epsilon_a) \;\; and\; \omega(a,m) < \omega^*(\epsilon_a)\} + \\ & \beta \left[-v_m^b + v_a^b \times \frac{v_m^s}{v_a^s} \right] \cdot 1\{m < m^*(\epsilon_a) \;\; and\; \omega(a,m) \geq \omega^*(\epsilon_a)\} + \\ & \beta \left[v_m^s \frac{u'(y)}{c'(y)} - v_m^b \right] \cdot 1\{m \in [m^*(\epsilon_a), m^*(\epsilon_m))\}. \end{aligned}$$

Since for $m < m^*(\epsilon_a)$ *and* $\omega(a,m) \geq \omega^*(\epsilon_a)$ we have that $y = y^*(\epsilon_a)$ solving $\ell(y;\epsilon_a) = 0$, the last expression can be simplified to

$$W_m^b(a,m) = \beta v_m^b + \beta v_m^b \left[\frac{v_m^s}{v_m^b} \frac{u'(y)}{c'(y)} - 1 \right] \cdot 1\{m < m^*(\epsilon_m)\}. \blacksquare$$

Proof of Proposition 2

The FOCs for buyers are given by (33), and since buyers hold all the money and asset, we have that in a monetary equilibrium both equations in (33) hold with equality.

(i) In a monetary equilibrium with stock of asset A and with rate of inflation τ, the output in the DM is $y(\tau)$ such that $1+\tau = \beta[1+L(y(\tau);1)]$, which for $\tau > \tau^F = \beta - 1$ can be rewritten as

$$1+\tau = \beta \frac{u'(y(\tau))}{c'(y(\tau))}. \tag{39}$$

However, in a non-monetary equilibrium the level of output is given by $\widetilde{y}(A)$ solving (A1) and (A2) for $A < \widetilde{A}$ and is $\widetilde{y}^*$ otherwise. Hence, since for a monetary equilibrium to exist agents must be better off holding money and the asset than using only the asset, in a monetary equilibrium we must have that $y(\tau) \geq \min\{\widetilde{y}(A); \widetilde{y}^*\}$. This implies that for $A \geq \widetilde{A}$ we must have $\tau \leq \overline{\tau}^*$, where $\overline{\tau}^*$ solves

$$1+\overline{\tau}^* = \beta \frac{u'(\widetilde{y}^*)}{c'(\widetilde{y}^*)}.$$

And for $A < \widetilde{A}$ the supremum inflation rate is $\overline{\tau}(A) = \min\{\widetilde{\tau}(A), 0\}$,[15] with $\widetilde{\tau}(A)$ such that

$$1+\widetilde{\tau}(A) = \beta \frac{u'(\widetilde{y}(A))}{c'(\widetilde{y}(A))} = \beta \widetilde{\epsilon}_a^*.$$

We leave for later the proof that τ must be strictly less than 0. Since $u'(y)/c'(y)$ is strictly decreasing in y we have that $\overline{\tau}'(A) \leq 0$.

(ii) For $\tau \in \left[\tau^F, \overline{\tau}^*\right]$ we have from (A3) that $u'(y(\tau)/c'(y(\tau)) \in [1, \widetilde{\epsilon}_a^*]$. This implies that for $\tau \in \left[\tau^F, \overline{\tau}^*\right]$ that $L_a(y(\tau); \widetilde{\epsilon}_a^*) = 0$, and, using (33), we therefore obtain that $\psi(\tau) = \beta v_a^b = \beta\left(\psi(\tau) + \overline{d}\right)$. That is,

$$\psi(\tau) = \widetilde{\psi}^*.$$

And we can check that when $\psi(\tau) = \widetilde{\psi}^*$, then $\epsilon_a = \widetilde{\epsilon}_a^*$. Moreover, then

$$\beta \widetilde{\epsilon}_a^* = \frac{\beta \overline{d}}{\beta \overline{d} + (1-\beta)\overline{d}^s} < 1,$$

which implies that $\overline{\tau}^* < 0$.

For $\tau \in (\overline{\tau}^*, \overline{\tau}(A)]$, we have from (A3) that $u'(y(\tau)/c'(y(\tau)) \in (\widetilde{\epsilon}_a^*, u'(\widetilde{y}(A))/c'(\widetilde{y}(A)))$. This implies that for $\tau \in (\overline{\tau}^*, \overline{\tau}(A)]$ that

$$L_a(y(\tau); \epsilon_a) + 1 = \frac{1}{\epsilon_a}\left[L_m(y(\tau); 1) + 1\right] > 1.$$

[15] The supremum is a maximum when $\overline{\tau}(A) = \widetilde{\tau}(A) < 0$.

We therefore obtain that the price of the asset is given by

$$\psi(\tau) = \beta v_a^b \times \frac{1}{\epsilon_a} \frac{u'(y(\tau))}{c'(y(\tau))},$$

which, using the facts that $\beta u'(y(\tau)/c'(y(\tau)) = 1 + \tau$ and $\epsilon_a = v_a^b / v_a^s$, yields

$$\psi(\tau) = v_a^s (1+\tau) = \left(\psi(\tau) + \overline{d}^s\right)(1+\tau), \text{ that is}$$

$$\psi(\tau) = \frac{(1+\tau)\,\overline{d}^s}{-\tau}.$$

It is clear that we must have $\tau < 0$ for otherwise the price of the asset is unbounded, which cannot be true in equilibrium. And therefore for $\tau \in (\overline{\tau}^*, \overline{\tau}(A)]$ we have that $\psi'(\tau) > 0$.■

Appendix B

If both buyers and sellers have a constant disutility of effort equal to one in the last subperiod of their lives, then, when the monetary policy is stationary, we have that $v_a^s = v_a^b = \psi + \overline{d}$ and $v_m^s = v_m^b = \widehat{\phi}$, and therefore $\epsilon_a = \epsilon_m = 1$. The first order conditions for the portfolio decisions for sellers and buyers are

$$\begin{aligned} a: \quad & \psi \geq \beta\left(\psi + \overline{d}\right), \quad && = \text{ if } a^s > 0; \\ m: \quad & 1 + \tau \geq \beta, && = \text{ if } m^s > 0, \end{aligned}$$

and

$$\begin{aligned} a: \quad & \psi \geq \beta\left(\psi + \overline{d}\right)[1 + L(y;1)], \quad && = \text{ if } a^b > 0; \\ m: \quad & 1 + \tau \geq \beta\,[1 + L(y;1)], && = \text{ if } m^b > 0. \end{aligned} \tag{A4}$$

For $\tau > \tau^F$, if money is to be held it will be held by buyers, in which case it follows that from the buyers' FOC wrt money that $y < y^*$. However, since buyers and sellers value the asset the same way, in an equilibrium without money buyers can purchase the quantity y^* if the stock of asset in the economy is

$$A \geq \widehat{A} \equiv \frac{1-\beta}{\overline{d}} \frac{c(y^*)}{\beta}.$$

It follows that if the stock of asset is large enough, i.e., $A \geq \widehat{A}$ then $\overline{\tau}(A) = \tau^F$, in which case money is not essential. If $A < \widehat{A}$, then buyers can purchase less than $y(A) < y^*$ in the DM, and therefore then, following the line of proof for proposition2, one can show that $\overline{\tau}(A) > \tau^F$, and that for all $\tau \in [\tau^F, \overline{\tau}(A))$ the quantity produced in the DM is strictly greater than y^*, and therefore money is essential.

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