Private Money and Bank Runs*

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Abstract

This paper studies bank runs in a model with coexistence of fiat money and private money. When fiat money is the only medium of exchange, a bank run equilibrium coexists with an equilibrium that achieves the optimal risk-sharing. In contrast, when private money is also a medium of exchange, there exists a unique equilibrium where no one demands early withdrawals of fiat money and agents in need of liquidity only use private money to finance consumption. The unique equilibrium achieves the first-best outcome and eliminates bank runs without having to resort to any government intervention.

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1 Introduction

This paper examines bank runs in a model with coexistence of fiat money and private money. One of the most influential theories about bank runs, following Diamond and Dybvig (1983), is that banks are inherently unstable institutions. In particular, the sequential service rule of the demand deposits is considered a key element that causes potential banking panics. The goal of this paper is to study whether the allowing of demand deposits to circulate can become a natural mechanism to prevent bank runs. We argue that it is the lack of liquidity instruments (such as *private money*, i.e., circulating demand deposits), not the sequential service rule *per se*, that makes banks vulnerable to runs. We show the following results: when fiat money is the only medium of exchange, a bank run equilibrium coexists with a banking equilibrium that achieves the optimal risk-sharing. In contrast, when banknotes are also allowed to circulate, there exists a unique banking equilibrium which achieves the first-best outcome. Therefore, once private money is permitted, the sequential service rule of demand deposits no longer generates potential banking panics.

The model features overlapping generations of agents. There is heterogeneity within each generation in that agents face private, idiosyncratic liquidity shocks. Moreover, there is uncertainty regarding the aggregate liquidity needs. Banks offer demand deposit contracts and invest deposits of fiat money in nominal bonds. After depositing in the bank, agents observe liquidity shocks and choose when to withdraw fiat money from the bank. Withdrawal demand is served on a sequential basis. Early withdrawal of fiat money is costly in that the bank must liquidate investments before maturity and end up with a zero net return. We consider scenarios when private money is and is not allowed to be used as a medium of exchange. We examine the implications of these alternative mechanisms on the banking equilibrium.

When fiat money is the only medium of exchange, the optimal risk-sharing requires a gross rate of return $r > 1$ on early withdrawals. As a result, agents who need liquidity can have more fiat money to spend on consumption goods by depositing in the bank *ex ante*.

Nevertheless, the mechanism is vulnerable to bank runs in that the bank does not have enough assets to honour $r > 1$ if all agents decide to withdraw early. The pessimistic belief of bank runs is self-fulfilling.

When private money is also allowed to circulate, the banking equilibrium is unique and it achieves the first-best outcome. A critical feature of the equilibrium demand deposit contract is that it offers $r < 1$ on early withdrawals, i.e., redemption of private money for fiat money. It is this particular offer that eliminates the bank run equilibrium. With $r < 1$, the bank is actually charging transaction fees on early redemption. This has two direct implications: first, agents do not panic under any circumstances because the rate $r < 1$ guarantees a positive amount of residual bank assets after any volume of early redemption; second, depositors who need liquidity prefer to use private money rather than fiat money to buy goods because now early redemption is costly. Sellers are willing to accept private money because it is backed up by the bank's assets.

If the bank offers any $r > 1$, however, a bank run equilibrium always exists even though private money is allowed. With $r > 1$, the bank's assets will be depleted if all depositors try to redeem early. As a result, no one is willing to accept private money in trades because they expect zero assets backing up private money. A depositor can only use fiat money to buy goods. Thus it is the dominant strategy for a depositor to demand early redemption if all other depositors do so. The belief of bank runs is indeed self-fulfilling. *Ex ante*, since agents are aware of the potential bank runs associated with any offer of $r > 1$, they will choose to accept a contract with $r < 1$. Thus it is optimal for a bank to offer $r < 1$ by competitive banking.

As a result, in the unique banking equilibrium with private money, no one demands early withdrawals of fiat money and agents in need of liquidity only use private money to finance consumption. In effect, the bank manages to promote the use of private money as a medium of exchange by imposing costs on early redemption. The equilibrium is immune to bank runs and achieves the first-best outcome without having to resort to any government intervention. This result is robust to aggregate uncertainty.

Our model is based on the seminal work of Diamond and Dybvig (1983) [DD]. In contrast to DD, our paper presents a dynamic general equilibrium model of banking with serious roles for money. The key insight is that the form of money matters for bank runs. When fiat money is the only medium of exchange, the demand deposit contract suffers from inherent instability. On one hand, the mechanism is designed to provide liquidity for individual agents. On the other hand, the mechanism itself has inherent liquidity problems in that it does not have enough assets to serve if all depositors demand early redemption. Essentially, the demand deposit contract relies on fiat money to perform two potentially conflicting roles: the medium of exchange (i.e., a liquid asset) and the instrument for illiquid investments. It provides liquidity at the expense of terminating profitable illiquid investments.

However, when private money is allowed, the bank can modify the demand deposit contract to avoid any inherent instability. The modified contract promotes the use of private money as a medium of exchange by imposing costs on early withdrawals of fiat money. The key is to assign the two roles to different monies: private money as the medium of exchange and fiat money as the instrument for investment. Accordingly, the contract manages to provide liquidity without liquidating any existing profitable investments.

There have been previous papers that examine bank runs in a monetary context. For example, Champ, Smith and Williamson (1996) also show that bank runs occur if banks are restricted from issuing notes, but do not occur if banks are not restricted. Our paper differs from theirs in two ways. First, Champ et al. study fundamental-driven bank runs, i.e., banking panics triggered by shocks on information regarding fundamentals. In contrast, we focus on expectations-driven bank runs of the DD type. To date, it is by no means clear exactly what kind of shocks causes banking panics. Therefore, it is worth investigating both types. Second, we show that allowing private money changes the demand deposit contract significantly: the bank imposes an explicit cost on early redemption to discourage the use of fiat money. Accordingly, in equilibrium, depositors in need of liquidity choose to use only private money as a medium of exchange.

Our paper is also closely related to the recent research that studies banking and the co-circulation of fiat money and private money. For example, Sun (2007a,b) focuses on the roles of alternative media of exchange in a banking environment with aggregate uncertainty. Both papers show that private money improves the efficiency of banking and helps achieve higher welfare than fiat money does. Our paper follows the same agenda, but targets the issue of bank runs.

The remainder of the paper is organized as follows. Section 2 describes the environment of the model. Section 3 studies banking when fiat money is the only medium of exchange. Section 4 examines banking when private money is allowed. Section 5 compares the results of Sections 3–4 and Section 6 concludes the paper.

2 The environment

Time is discrete and has infinite horizons. Each period $t = 0, 1, \cdots, \infty$ consists of two sub-periods, *morning* and *afternoon*. The economy is populated by overlapping generations of agents. At the beginning of period t, a continuum of agents with mass one is born. Each generation of agents lives for two periods and is indexed by time of birth, t. We call those who are in the first period of life *young* agents and those in the second period of life *old* agents.

Agents are endowed with one unit of divisible goods when young and can costlessly harvest endowment goods. However, goods are perishable between sub-periods. Therefore, agents need to decide the amounts of goods to be harvested in the morning and in the afternoon, respectively. An agent only derives utility from consumption in old age. Moreover, in the second period of life, each agent receives a privately observed preference shock. With probability s, the agent is of type H and only consumes goods in the morning. With probability $1 - s$, the agent is of type L and only consumes goods in the afternoon. Let c_t^H

represent a type H agent's consumption in period t and c_t^L a type L agent's consumption in period t. An agent born in period t has the following lifetime utility function:

$$E_t\left[\theta_{t+1}U\left(c_{t+1}^H\right)+\left(1-\theta_{t+1}\right)U\left(c_{t+1}^L\right)\right],$$

where $U(\cdot)$ is twice continuously differentiable, strictly increasing and strictly concave. Also,

$$\theta_{t+1}=\begin{cases}1, & w.p. \quad s_{t+1}\\ 0, & w.p. \quad 1-s_{t+1}\end{cases}.$$

The variable s_{t+1} is also stochastic. It has support on $[\underline{s},\overline{s}]$ and is i.i.d. across time. Here $0<\underline{s}<\overline{s}<1$. Note that s_{t+1} is common to all agents of generation t. Thus, s_t also denotes the aggregate measure of type H old agents in period t.

2.1 The Planner's Problem

Suppose θ is costlessly observable to a benevolent planner. The planner assigns equal weights to all agents of the same generation and discounts across periods with factor β. Let E_t represent the aggregate amount of goods consumed in the morning of period t. Then, the planner chooses E_t to maximize social welfare:

$$W=\sum_{t=0}^{\infty}\beta^t\left[s_tU\left(\frac{E_t}{s_t}\right)+(1-s_t)U\left(\frac{1-E_t}{1-s_t}\right)\right]. \tag{1}$$

The measure of type H old agents is s_t. They share the amount E_t of goods in the morning. Similarly for type L agents, they have measure $1-s_t$ and share the amount $1-E_t$ in the afternoon. It is straightforward to show that the optimal solution to the planner's problem is $E_t^*=s_t$. Therefore, the planner allocates the following consumption levels: $C_t^{H*}=C_t^{L*}=1$. This is the first-best outcome, which achieves the full-information optimal risk-sharing.

3 Fiat Money and Bank Runs

3.1 The demand deposit contract

At the beginning of time, there lives a continuum of old agents with mass one, each of whom is endowed with M units of intrinsically worthless and durable objects, called fiat money. Throughout time, there also exists a large number of financial intermediaries, called banks. Banks have access to the tradings of nominal bonds, which are supplied by the government. The duration of bonds is one sub-period. In particular, bonds are issued at the end of period t and they mature at the *onset* of the afternoon of $t+1$. Nominal bonds are sold at par for fiat money. For each unit of nominal bonds, the government commits to a gross return of $\overline{R} > 1$ units of fiat money. Moreover, bonds can be liquidated early, i.e., in the morning of $t+1$. Nevertheless, early liquidation is costly in that it gives zero net interests. The interest payments on bonds are financed by a lump-sum tax T_t on generation-t agents' incomes in the first period of life. Let M_t denote the aggregate after-tax fiat money holdings of young agents at the end of period t.

At the end of period t, banks offer demand deposit contracts to the young agents of generation t. By accepting the contract, agents agree to deposit fiat money in a bank and the bank invests fiat money in nominal bonds. Assume the bank is mutually owned by depositors. A depositor has the liberty of withdrawing fiat money from the bank either in the morning or in the afternoon of $t+1$. Withdrawal demand is served according to the *sequential service rule*, i.e., on a first-come-first-served basis.

Upon withdrawal in the morning of $t+1$, a depositor is entitled to r units of fiat money for each unit of fiat money deposited in t. Upon withdrawal in the afternoon of $t+1$, depositors are entitled to the residual assets (returns to matured bonds) of the bank according to their relative shares of remaining deposits. If a bank goes bankrupt, i.e., does not enough assets to satisfy all withdrawal demand, the bank dies and will be replaced by a new-born bank. By competition, banks offer the same optimal contract in equilibrium. Without loss of generality, from now on we consider the banking sector as one representative bank.

The demand deposit contract is similar to the one studied by DD, except that here we have nominal deposits rather than real deposits. For now, we assume that s_t is not random and is publicly known. We will focus on the banking equilibrium where young agents choose to deposit the after-tax fiat money holdings in the bank. Let f_t^i denote the amount of withdrawals served before agent i as a fraction of the total demand deposits. Moreover, f_t denotes the total amount of withdrawals in the morning of t as a fraction of the total demand deposits. For any $0 \leq r \neq 1$, the rates of return to each unit of deposit withdrawn are defined as:

$$R_{1,t} = \begin{cases} r, & \text{if} \quad f_t^i < \frac{1}{r} \\ 0, & \text{if} \quad f_t^i \geq \frac{1}{r} \end{cases} \tag{2}$$

for morning withdrawals and

$$R_{2,t} = \max\left\{0, \frac{1 - r f_t}{1 - f_t}\overline{R}\right\} \tag{3}$$

for afternoon withdrawals. For $r = 1$, define $R_{1,t} = 1$ and $R_{2,t} = \overline{R}$. *Ex ante*, at the end of period $t-1$, a young agent has three options: (i) deposit fiat money in the bank; (ii) invest fiat money in nominal bonds (through the bank);[1] (iii) simply carry fiat money into period t. If $r = 1$, for an individual agent, the demand deposit contract is offering the same payoff schedule as the investment in nominal bonds. Without loss of generality, in this paper we focus on contracts with $r \neq 1$.

3.2 Timing

The following describes the detailed timing of events:

1. At the start of period t, young agents are born; they choose the amounts of goods to

[1] Here bonds are illiquid in the sense that individuals are not allowed to trade bonds for goods.

supply in the morning and in the afternoon, respectively.

2. By the end of t, young agents pay taxes T_t and decide whether to deposit fiat money income in the bank.

3. At the start of $t+1$, old agents of generation t observe their type $i = H, L$; old agents choose the fraction d_t^i of deposits to be withdrawn in the morning, depending on type. In the meantime, the morning market opens. Old agents trade fiat money for young agents' goods.

4. In the afternoon of $t+1$, nominal bonds mature. The bank liquidates assets. Old agents withdraw the remaining deposits, if any.

5. The afternoon market opens. Old agents trade fiat money for young agents' goods.

3.3 Equilibrium

In this section, the only medium of exchange is fiat money. Young agents supply goods to competitive markets to trade for fiat money. Type H old agents trade fiat money for young agents' goods in the morning. In the afternoon, type L old agents trade with young agents. Note that type L old agents do not trade for goods in the morning because goods are perishable.

Let $P_{1,t}$ denote the price of fiat money for goods in the morning market and $P_{2,t}$ the price in the afternoon market. Denote $e_t \in [0,1]$ as a young agent's supply of goods in the morning. We will focus on symmetric banking equilibria where agents of the same age and type apply the same strategy. Let m_t denote the amount of fiat money deposited in the bank by a young agent of generation-t. Let $d_t^i \in [0,1]$ denote the fraction of a type $i = H, L$ agent's deposits withdrawn in the morning of period t. Denote $D_t \in [0,1]$ as the expectation of the aggregate fraction of deposits to be withdrawn in the morning of t. In equilibrium, expectations are consistent with outcomes, i.e., $D_t = s_t d_t^H + (1-s_t)\, d_t^L$.

Proposition 1 *Provided that fiat money is the unique medium of exchange, there exists a banking equilibrium where there is no bank run and*

(i) young agents deposit all after-tax holdings of fiat money in the bank;

(ii) $d_t^H = 1$ *for all type* H *old agents;*

(iii) $d_t^L = 0$ *for all type* L *old agents.*

Proof. Suppose (i)-(iii) are all true. It is obvious that $d_t^H = 1$ is the dominant strategy for type H agents as long as they choose to deposit fiat money in the bank. By (i), each young agent of generation t deposits M_t units of fiat money in the bank. By (ii) and (iii), $f_t = s_t$. Suppose the bank chooses r such that $rs_t < 1$. By (2) and (3),

$$\begin{aligned} R_{1,t} &= r \\ R_{2,t} &= \frac{1 - rs_t}{1 - s_t}\overline{R}. \end{aligned}$$

Given the demand deposit contract $(R_{1,t}, R_{2,t})$ and prices $(P_{1,t}, P_{2,t})$, an old agent's expected utility before observing his type is given by

$$V(r) \equiv s_t U\left(\frac{rM_{t-1}}{P_{1,t}}\right) + (1 - s_t)U\left(\frac{1 - rs_t}{1 - s_t}\overline{R}\frac{M_{t-1}}{P_{2,t}}\right).$$

In equilibrium,

$$\begin{aligned} P_{1,t} &= \frac{rs_t M_{t-1}}{E_t} && (4) \\ P_{2,t} &= \frac{(1 - rs_t)\overline{R}M_{t-1}}{1 - E_t}. && (5) \end{aligned}$$

Recall that E_t is the aggregate supply of goods in the morning. Hence the above becomes

$$V(r) = s_t U\left(\frac{E_t}{s_t}\right) + (1 - s_t)U\left(\frac{1 - E_t}{1 - s_t}\right),$$

which is exactly the same as the objective of the planner in (1). Hence we know that $V(r)$

is maximized by a unique solution $E_t^* = s_t$.

Equilibrium requires no arbitrage opportunities across morning and afternoon markets, that is, $P_{1,t} = P_{2,t}$. Thus

$$\frac{rs_t M_{t-1}}{E_t} = \frac{(1-rs_t)\overline{R}M_{t-1}}{1-E_t}.$$

The above yields

$$E_t = \frac{rs_t}{rs_t + (1-rs_t)\overline{R}}. \tag{6}$$

Plugging $E_t^* = s_t$ into the above yields

$$s_t = \frac{s_t r}{s_t r + (1-s_t r)\overline{R}}.$$

Therefore, to maximize a young agent's expected utility of accepting the contract, the bank optimally chooses

$$r^* = \frac{\overline{R}}{1+\left(\overline{R}-1\right)s_t}. \tag{7}$$

Notice that $1 < r^* < \overline{R}$, and indeed $r^* s_t < 1$.

As mentioned before, it is optimal for a type H agent to choose $d_t^H = 1$ because he does not care about consumption in the afternoon. By (7), $R_{1,t} = R_{2,t} = r^*$. Therefore, given r^* and expectation $D_t = s_t$, a type L agent is indifferent between withdrawing in the morning and in the afternoon.[2] Thus it is also optimal for the type L agent to choose $d_t^L = 0$.

Ex ante, at the end of period $t-1$, a young agent has three options: (i) deposit fiat money in the bank; (ii) invest fiat money in nominal bonds (through the bank); (iii) simply carry fiat money into period t. It follows immediately that option (ii) strictly dominates option (iii). For each unit of fiat money, if the agent invests in bonds, he will have one unit of fiat money at disposal if he is a type H, and $\overline{R} > 1$ units of fiat money if type L.

The agent chooses between option (i) and option (ii), given $\left(R_{1,t}\left(r^*\right), R_{2,t}\left(r^*\right)\right)$ and the

[2] Unlike Diamond and Dybvig (1983), here agents do not discount future consumption. Otherwise, the optimal contract would have $R_{1,t} < R_{2,t}$ as in DD.

expectation that all other young agents deposit all their fiat money in the bank. Let δ denote the fraction of the agent's fiat money to be deposited in the bank. The agent's utility-maximizing problem is given by

$$\max_{\delta}\left\{s_t U\left[\frac{r^*\delta M_{t-1}+(1-\delta)\,M_{t-1}}{P_{1,t}}\right]+(1-s_t)U\left[\frac{r^*\delta M_{t-1}+\overline{R}\,(1-\delta)\,M_{t-1}}{P_{2,t}}\right]\right\}.$$

For each unit of fiat money, the demand deposit offers a payoff profile of $\{r^*, r^*\}$ for morning and afternoon, whereas the direct investment yields a profile of $\{1, \overline{R}\}$. Hence the above gives the expected utility of the young agent. By (4), (5), (7) and $E_t = s_t$, the above becomes

$$\max_{\delta}\left\{s_t U\left[\frac{(r^*-1)\,\delta+1}{r^*}\right]+(1-s_t)U\left[\frac{\left(r^*-\overline{R}\right)\delta+\overline{R}}{(1-r^*s_t)\overline{R}}\,(1-s_t)\right]\right\}.$$

The solution to the above problem is $\delta^* = 1$. Therefore, the agent optimally deposits all his fiat money in the bank.

Therefore, given contract $(R_{1,t}(r^*), R_{2,t}(r^*))$ there exists an equilibrium where (i)-(iii) are all satisfied. The belief of no type L agents withdrawing early is self-fulfilling. The equilibrium delivers the first-best outcome, $c_t^{H*} = c_t^{L*} = 1$. ■

Proposition 2 *Given r^*, suppose all other depositors withdraw fiat money in the morning, then it is optimal for a type $i = H, L$ agent to withdraw in the morning.*

Proof. It is obvious that a type H agent always withdraws all holdings of fiat money in the morning. Consider a type L agent. Given the belief that $D_t = 1$,

(i) if the agent waits till the afternoon, he receives $R_{2,t} = 0$ because $r^* f_t = r^* > 1$. Since all other agents withdraw in the morning, all the investments in bonds are liquidated early to meet the demand. There is no asset left in the bank by the end of the morning;

(ii) if the agent chooses to withdraw in the morning, with probability $\frac{1}{r^*}$ he can successfully receive $r^* M_{t-1}$ units of fiat money; otherwise, he receives zero money back.

Hence for the type L agent, the strategy $d_t^L = 1$ is stochastically dominant. That is, it is optimal to "run" along with other agents. ■

Propositions $1 - 2$ show that a good equilibrium that achieves the first-best outcome coexists with a bad (Pareto-inferior) equilibrium with bank runs. Beliefs about these equilibria are self-fulfilling. Which equilibrium arises will depend on the confidence level of the economy.

4 Private Money and Bank Runs

In this section, we consider the same environment as before, only that now demand deposits are also allowed to circulate as a medium of exchange. When an agent deposits a unit of fiat money in period $t - 1$, the bank issues him a banknote to be redeemed in period t. Any bearer of the banknote can redeem it at the bank for fiat money at the promised rates $(R_{1,t}, R_{2,t})$ given by (2) and (3). To finance consumption, old agents can redeem private money for fiat money and use fiat money to buy goods. Alternatively, they can use private money to buy goods directly. Therefore, in the morning of t, there will be two markets, where goods are respectively traded for fiat money and private money. Note that a private money market does not arise in the afternoon because the bank redeems its outstanding private money before any afternoon market opens.[3] This will become clear in the following section, where detailed timing is described. As before, we focus on symmetric equilibria where (i) all agents choose to accept the demand deposit contract; (ii) agents of the same type apply the same strategies; (iii) markets are all competitive. Note that now the aggregate state s_t is stochastic.

[3]Unredeemed (i.e., expired) private money does not circulate in future periods. That is, in any period t, private money issued prior to period $t-1$ will not be accepted as a medium of exchange. This is because fiat money dominates expired private money for two reasons. On one hand, fiat money can be used to purchase bonds, which offers a non-negative net return. On the other hand, banks accept fiat money as deposits.

4.1 Timing

Let Q_t represent the price of private money for goods in the morning of t. The timing of events is summarized as follows for a representative generation t of agents:

1. At the start of period t, young agents are born. They choose the amounts of goods to supply to the morning markets (fiat money and private money, respectively) and to the afternoon market (fiat money). Young agents also choose the fraction of private money holdings (if any) to be redeemed in the morning.

2. The bank starts morning redemption and markets open in the meantime.

3. In the afternoon of t, young agents redeem the remaining private money (if any) at the bank; young agents sell goods in the afternoon market for fiat money.

4. By the end of t, young agents pay taxes T_t and decide whether to deposit fiat money income (from redemption of private money and from proceeds of selling goods) in the bank.

5. At the start of $t+1$, old agents of generation t observe their type $i = H, L$; old agents choose the fraction d_t^i of private money holdings to be redeemed in the morning, depending on type. In the meantime, morning markets open. Old agents trade fiat money and unredeemed private money respectively for young agents' goods.

6. In the afternoon of $t+1$, nominal bonds mature. The bank liquidates assets. Young and old agents who are holding private money demand redemption at the bank.

7. The afternoon market opens where old agents trade fiat money for young agents' goods.

4.2 Equilibrium

We will focus on the banking equilibrium where young agents deposit all the after-tax fiat money holdings in the bank. Later we will prove that there is no profitable individual deviation from such a strategy.

4.2.1 The type H old agent's problem

Let m_t represent a young agent's after-tax fiat money holdings at the end of t. By depositing all fiat money in the bank, m_t is also an old agent's total private money holdings at the beginning of morning $t+1$. As mentioned before, there are two ways for a type H old agent to benefit from private money. First, the agent can use private money to purchase goods directly. Second, the agent can redeem private money for fiat money and use the latter to purchase goods.

At the beginning of period t, given prices $(P_{1,t}, P_{2,t}, Q_t)$, contracts $(R_{1,t}, R_{2,t})$ and expectation D_t, a type H old agent chooses the fraction of private money holdings to be redeemed in the morning, d_t^H, to maximize his expected utility:

$$W^H(m_{t-1}) = \max_{d_t^H} \left\{ \Delta_t U\left[\frac{rd_t^H m_{t-1}}{P_{1,t}} + \frac{(1-d_t^H)m_{t-1}}{Q_t}\right] + (1-\Delta_t)\, U\left(\frac{m_{t-1}}{Q_t}\right)\right\}. \tag{8}$$

If the agent successfully gets served for redemption, he spends fiat money $rd_t^H m_{t-1}$ and the remaining private money holdings $\left(1-d_t^H\right)m_{t-1}$ on goods. If the agent is not served for redemption, he uses all his private money m_{t-1} to buy goods. The first-order condition to the above maximization problem is given by:

$$\Delta_t U'\left[\frac{rd_t^H m_{t-1}}{P_{1,t}} + \frac{(1-d_t^H)m_{t-1}}{Q_t}\right] m_{t-1}\left(\frac{r}{P_{1,t}} - \frac{1}{Q_t}\right)\begin{cases} >0, & \text{if} \quad d_t^H = 1 \\ =0, & \text{if} \quad d_t^H \in [0,1] \\ <0, & \text{if} \quad d_t^H = 0 \end{cases}. \tag{9}$$

4.2.2 The type L old agent's problem

To maximize expected utility, a type L agent of generation $t-1$ chooses the fraction of private money to be redeemed in the morning, d_t^L. With probability Δ_t, he will be served for redemption. Then he redeems the remaining $(1-d_t^L)m_{t-1}$ units of private money in the afternoon, which yields him $R_{2,t}(1-d_t^L)m_{t-1}$ units of fiat money. The agent spends all his fiat money (received in the morning and the afternoon) in the afternoon goods market. With probability $1-\Delta_t$, the agent will not be served for morning redemption. In this case, the agent redeems all his private money in the afternoon. Note that the type L old agents do not participate in morning markets because goods are non-storable.

Taking prices $(P_{1,t}, P_{2,t}, Q_t)$, contracts $(R_{1,t}, R_{2,t})$ and expectation D_t as given, a type L old agent's problem is

$$W^L(m_{t-1}) = \max_{d_t^L} \left\{ \Delta_t U \left[\frac{r d_t^L m_{t-1} + R_{2,t}(1-d_t^L)m_{t-1}}{P_{2,t}} \right] + (1-\Delta_t) U \left(\frac{R_{2,t} m_{t-1}}{P_{2,t}} \right) \right\}. \tag{10}$$

The first-order condition to the above maximization problem is given by

$$\Delta_t U' \left[\frac{r d_t^L m_{t-1} + R_{2,t}(1-d_t^L)m_{t-1}}{P_{2,t}} \right] \frac{m_{t-1}}{P_{2,t}} (r - R_{2,t}) \begin{cases} > 0, & \text{if } d_t^L = 1 \\ = 0, & \text{if } d_t^L \in [0,1] \\ < 0, & \text{if } d_t^L = 0 \end{cases} . \tag{11}$$

4.2.3 The young agent's problem

Let $e_t^f \in [0, e_t]$ denote a young agent's supply of goods to the morning fiat money market. Denote $\pi_t \in [0,1]$ as the fraction of private money holdings that the agent redeems in the morning. Taking prices $(P_{1,t}, P_{2,t}, Q_t)$, contracts $(R_{1,t}, R_{2,t})$ and expectation D_t as given, a young agent of generation t seeks to maximize the expected lifetime utility:

$$\max_{\left(e_t^f, e_t, \pi\right)} E_t \left[s_{t+1} W^H(m_t) + (1-s_{t+1}) W^L(m_t) \right]. \tag{12}$$

The expectation is taken over s and

$$m_t = P_{1,t} e_t^f + Q_t \left(e_t - e_t^f \right) \left[\pi_t \Delta_t r + (1-\pi_t \Delta_t) R_{2,t} \right] + P_{2,t}(1-e_t) - T_t,$$

where $R_{2,t} = \max\left\{0, \frac{1-rD_t}{1-D_t}\overline{R}\right\}$ and Δ_t is the probability of the agent being served for redemption. The young agent sells e_t^f units of goods in the morning fiat money market, $e_t - e_t^f$ in the morning private money market and the rest $1 - e_t$ in the afternoon fiat money market. Young agents can only redeem private money either in the morning or in the afternoon, which earns an expected rate of return $\pi_t\Delta_t r + (1 - \pi_t\Delta_t) R_{2,t}$. If the total demand of redemption is no more than the maximal liquidation value of the bank, i.e., $rD_t \leq 1$, then the agent will be served with probability one and receive redemption rate r. Otherwise, the agent is served with probability $\frac{1}{rD_t}$. Thus

$$\Delta_t = \min\left\{1, \frac{1}{rD_t}\right\}. \tag{13}$$

By (8) and (10), it is straightforward that both W^H and W^L are strictly increasing in m_t. Therefore, the problem in (12) is simplified to maximize the expected money income:[4]

$$\max_{\left(e_t^f, e_t, \pi\right)} P_{1,t}e_t^f + Q_t\left(e_t - e_t^f\right)\left[\pi_t\Delta_t r + (1 - \pi_t\Delta_t) R_{2,t}\right] + P_{2,t}(1 - e_t).$$

The optimal solutions to the above problem are given by:

$$\pi_t^* \begin{cases} = 1, & \text{if} \quad r > R_{2,t} \\ \in [0, 1], & \text{if} \quad r = R_{2,t} \\ = 0, & \text{if} \quad r < R_{2,t} \end{cases} \tag{14}$$

$$e_t^{f*} \begin{cases} = e_t, & \text{if} \quad P_{1,t} > Q_t\widetilde{R}_t \\ \in [0, e_t], & \text{if} \quad P_{1,t} = Q_t\widetilde{R}_t \\ = 0, & \text{if} \quad P_{1,t} < Q_t\widetilde{R}_t \end{cases} \tag{15}$$

[4] When production is costly, agents face a trade-off between marginal cost and marginal benefit of holding money. Agents choose the output level to maximize the expected value of money holdings. Since production is costless in this model, the young agent's utility-maximizing problem is equivalent to maximizing money holdings. Nevertheless, it can be shown that the main results of this paper also hold when costly production is introduced.

$$e_t^* \begin{cases} = 1, & \text{if} \quad Q_t \widetilde{R}_t > P_{2,t} \\ \in [0,1], & \text{if} \quad Q_t \widetilde{R}_t = P_{2,t} \\ = 0, & \text{if} \quad Q_t \widetilde{R}_t < P_{2,t} \end{cases}, \tag{16}$$

where

$$\widetilde{R}_t \equiv \pi_t^* \Delta_t r + (1 - \pi_t^* \Delta_t) R_{2,t}. \tag{17}$$

4.2.4 Equilibrium Definition

Definition 1 *A symmetric banking equilibrium with coexistence of fiat money and private money consists of the demand deposit contract* $(R_{1,t}(r), R_{2,t}(r))$*, prices* $(P_{1,t}, P_{2,t}, Q_t)$*, individual choices* $\left(e_t^f, e_t, \pi_t, m_t, d_t^H, d_t^L\right)$ *and aggregate variables* $\left(E_t^f, E_t, D_t, M_t, s_t\right)$ *such that*

(i) given $(R_{1,t}, R_{2,t}, P_{1,t}, P_{2,t}, Q_t, D_t)$*, all individuals chooses quantities and strategies to maximize expected utility;*

(ii) the bank chooses r *to maximize the expected utility of a depositor;*

(iii) consistency: $e_t^f = E_t^f, e_t = E_t, D_t = s_t d_t^H + \left(1 - d_t^H\right) s_t \pi_t + (1 - s_t) d_t^L, m_t = M_t$*;*

(iv) $T_t = M_{t-1} \max\left[0, \left(\overline{R} - 1\right)(1 - rD_t)\right]$*;*

(v) all markets clear.

The above definition is mostly self-explanatory. Condition (iii) characterizes the consistency between individual choices and their aggregate counterparts. Note particularly that the aggregate fraction of withdrawals in the morning, D_t, is equal to the sum of the fractions of withdrawals by type H old agents $s_t d_t^H$, by young agents who have traded goods for private money $\left(1 - d_t^H\right) s_t \pi_t$, and by type L old agents $(1 - s_t) d_t^L$. Condition (iv) specifies that the government aims to maintain a constant end-of-period money supply. The government collects no tax if the demand of morning redemption forces the bank to liquidate all bonds early, i.e., $rD_t \geq 1$. Otherwise, the amount of the lump-sum tax is equal to the positive net returns on bonds, $\left(\overline{R} - 1\right)(1 - rD_t) M_{t-1}$.

4.2.5 Characterization of Equilibrium

Proposition 3 *Provided that* $0 \leq r < 1$*, there exists a unique banking equilibrium where no one redeems private money for fiat money in the morning, i.e.,* $d_t^i = \pi_t = 0$ *for all young agents and type* $i = H, L$ *old agents. The unique equilibrium achieves the first-best outcome, i.e.,* $c_t^i = 1$ *for all type* $i = H, L$ *agents.*

Proof. Given $r \in [0, 1)$, we have $rD_t < 1$ for any given D_t. It follows that $\Delta_t = 1$ and $R_{1,t} = r < \overline{R} \leq R_{2,t}$. The first-order condition (11) implies that $d_t^L = 0$, because the left-hand side of (11) is strictly negative. By (14), $\pi_t^* = 0$ and $\widetilde{R}_t = R_{2,t}$. It also follows that $rQ_t \leq \widetilde{R}_t Q_t$, where the equality holds if and only if $Q_t = 0$.

(i) Suppose $Q_t > 0$,

– if $rQ_t < \widetilde{R}_t Q_t < P_{1,t}$, then $rQ_t < P_{1,t}$ implies $d_t^H = 0$ by (9). Moreover, $\widetilde{R}_t Q_t < P_{1,t}$ implies $e_t^f = e_t$ by (15). This means a type H old agent does not redeem private money for fiat money yet young agents do not sell goods for private money in the morning. It follows that type H agents do not consume in the morning, which cannot be optimal. Thus $rQ_t < \widetilde{R}_t Q_t < P_{1,t}$ cannot be an equilibrium outcome;

– if $rQ_t < \widetilde{R}_t Q_t = P_{1,t}$, then (9) implies $d_t^H = 0$ and (15) implies $e_t^f \in [0, e_t]$. Therefore, the only symmetric equilibrium is $d_t^L = d_t^H = \pi_t = 0$, and $e_t^f = 0$;

– if $P_{1,t} < rQ_t < \widetilde{R}_t Q_t$, then (9) implies $d_t^H = 1$ and (15) implies $e_t^f = 0$. This means young agents do not sell goods for fiat money yet all type H old agents redeem private money for fiat money. It follows that type H agents do not consume in the morning. Thus $P_{1,t} < rQ_t < \widetilde{R}_t Q_t$ cannot be an equilibrium outcome;

– if $P_{1,t} = rQ_t < \widetilde{R}_t Q_t$, then (9) implies $d_t^H \in [0, 1]$ and (15) implies $e_t^f = 0$. Again, the only symmetric equilibrium is $d_t^L = d_t^H = \pi_t = e_t^f = 0$;

– if $rQ_t < P_{1,t} < \widetilde{R}_t Q_t$, then (9) implies $d_t^H = 0$ and (15) implies $e_t^f = 0$. Again, the symmetric equilibrium is $d_t^L = d_t^H = \pi_t = e_t^f = 0$;

(ii) Suppose $Q_t = 0$. Then it must be true that $d_t^H = 0$ for all type H old agents by (9) and $e_t^f = e_t = 0$ by (15) and (16). This means all type H old agents hope to trade private money yet young agents supply no goods to the private money market. Obviously, this cannot be an equilibrium outcome. The price Q_t must be adjusting upwards till demand equals supply.

To summarize, when $r \in [0, 1)$, any symmetric equilibrium must have $d_t^L = d_t^H = \pi_t = e_t^f = 0$. Note that in the morning no one demands redemption and only private money is traded for goods. Accordingly, the equilibrium prices are $P_{2,t} = \frac{(1-s_t)M_{t-1}\overline{R}}{1-E_t}$, $Q_t = \frac{s_t M_{t-1}}{E_t}$ and $P_{1,t} \geq rQ_t$. No arbitrage condition requires that $P_{1,t} = \widetilde{R}_t Q_t = P_{2,t}$. Thus $e_t = s_t$ and $c_t^i = 1$ for all type $i = H, L$ agents, which coincide with the planner's optimal choices. Equilibrium consistency is satisfied: $D_t = s_t d_t^H + (1 - s_t)\, d_t^L + \left(1 - d_t^H\right) s_t \pi_t = 0$. Therefore, given $r \in [0, 1)$ there exists a unique symmetric equilibrium with $d_t^{i*} = 0$ for all type $i = H, L$ old agents, and $\left(\pi_t = e_t^{f*} = 0, e_t^* = s_t\right)$ for all young agents.

The last step is to prove that there is no profitable deviation for a young agent from depositing all after-tax fiat money holdings in the bank. Given the contract $(R_1(r), R_2(r))_{r<1}$ and the expectation that all other young agents deposit all their fiat money in the bank, a young agent of generation-t decides how to invest in bonds and in banking.[5] Let δ denote the fraction of the agent's fiat money to be deposited in the bank. The agent's utility-maximizing problem is given by

$$\max_{\delta} E \left\{ s_{t+1} U \left[\frac{\delta M_t}{Q_{t+1}} + \frac{(1-\delta)\, M_t}{P_{1,t+1}} \right] + (1 - s_{t+1}) U \left[\frac{\overline{R} M_t}{P_{2,t+1}} \right] \right\}.$$

The expectation is taken over s. If the agent turns out to be of type H, he trades for goods using the private money holdings of δM_{t-1} units and the fiat money holdings of δM_t units (obtained from early liquidation of bonds). If the agent is of type L, he redeems private money for fiat money, which provides a gross rate of return $\overline{R}$. Also, government bonds mature and yield the same return $\overline{R}$. Altogether, the agent will have $\overline{R} M_t$ units of fiat money to spend on goods. As mentioned before, no arbitrage implies that $P_{1,t+1} = \widetilde{R}_{t+1} Q_{t+1} = \overline{R} Q_{t+1} > Q_{t+1}$. It follows immediately that it is optimal to choose $\delta^* = 1$. In

[5]Same as in the previous section, investment in bonds strictly dominates no investment whatsoever.

other words, it is optimal for a young agent to deposit all fiat money holdings in the bank.

■

Proposition 4 *Provided that $r > 1$, there always exists a self-fulfilling bank run equilibrium.*

Proof. Given the expectation that $D_t = 1$, we have $R_{2,t} = 0$. Thus $d_t^L = 1$ by (11), $\pi_t = 1$ by (14). It follows that indeed $D_t = s_t d_t^H + (1 - s_t) d_t^L + \left(1 - d_t^H\right) s_t \pi_t = 1$ for any $\left(s_t, d_t^H\right)$. Therefore, $\Delta_t = \frac{1}{r}$ by (13) and $\widetilde{R}_t = 1$ by (17). No arbitrage requires that $P_{1,t} = P_{2,t} = \widetilde{R}_t Q_t = Q_t$. By (9), we have $d_t^H = 1$ as $rQ_t > P_{1,t}$. Therefore, there exists an equilibrium where $d_t^H = d_t^L = \pi_t = 1$ and $e_t^f = e_t = 1$. Indeed, the expectation of bank runs is self-fulfilling. Due to the pessimistic belief of $D_t = 1$, type H and L agents choose to redeem all private money in the morning. Accordingly, young agents do not value private money because they expect there is zero asset to back it up (i.e., available for redemption).

■

According to Proposition 3, when $r^* < 1$ there exists a unique equilibrium and it achieves the first-best outcome. In this case, bank runs never occur. In contrast, Proposition 4 shows that there always exists a self-fulfilling bank run equilibrium provided that $r^* > 1$. With pessimistic beliefs ($D_t = 1$), young agents will not value private money because they expect that the bank will have no remaining assets for redemption. In the meantime, type L old agents redeem private money for fiat money even if they do not have pressing needs for consumption. By doing so, they try to get hold of fiat money before the bank runs completely out of assets.

If $r^* < 1$, however, type L agents do not panic over any given belief of D_t in any aggregate state s_t. With $r^* < 1$, the bank is essentially imposing transaction fees on withdrawal demands. This guarantees that the bank's assets will never be depleted, i.e., $r^* D_t < 1$ for any given D_t. Therefore, agents do not panic over any volume of redemption demand. Furthermore, since $r^* < 1$, redemption in the morning offers a strictly lower return than redemption in the afternoon, i.e., $R_{1,t} = r^* < \overline{R} \leq R_{2,t}$. Therefore, a type L old agent, or any young agent who has sold goods for private money, has no incentive to redeem private money in the morning. Moreover, none of the type H agents has the incentive to redeem private money in the morning because redemption is costly by $r^* < 1$. They are better off buying goods with private money, which is fully supported by the demand deposit contract ($R_{2,t} \geq \overline{R}$). As a result, only private money is traded in the morning.

Since Proposition 3 applies to any aggregate state s_t, a contract with $r^* < 1$ guarantees the first-best outcome for any realization of s_t. Therefore, it dominates any contract with $r^* > 1$, which is plagued by potential bank runs. Through competitive banking, a bank optimally offers the contract that maximizes agents' expected utilities. Hence follows the corollary:

Corollary 1 *With coexistence of fiat money and private money, the banking equilibrium is unique and the optimal demand deposit contract offers* $r^* \in [0, 1)$. *The unique equilibrium achieves the first-best outcome.*

5 Private Money vs. Fiat Money

Recall that $r < 1$ cannot be an equilibrium bank offer if agents are not allowed to trade demand deposits for goods. The equilibrium bank offer is given by $r^* = \frac{\overline{R}}{1-s_t+s_t\overline{R}} > 1$, which provides the optimal risk-sharing. Intuitively, this offer helps buffer the liquidity shock by injecting more liquidity (money) when needed. Nevertheless, it is also $r^* > 1$ that creates the vulnerability to bank runs. The demand deposit contract becomes inherently unstable. On one hand, the mechanism is designed to provide liquidity for individual agents. On the other hand, the mechanism itself has inherent liquidity problems in that it does not have enough assets to serve if all depositors demand early redemption.

The inherent instability disappears if private money is allowed. The bank offers $r^* < 1$, essentially charging transaction fees to discourage early redemption. This effectively prevents the depletion of assets due to panicking withdrawals. Furthermore, offering $r^* < 1$ does not compromise risk-sharing as it does when fiat money is the only medium of exchange. Now the bank can conveniently provide liquidity through circulation of private money. As a result, agents no longer rely on fiat money to purchase consumption goods in that private money is just as good a medium of exchange.

Another striking feature of the equilibrium with private money is that it is robust to aggregate uncertainty. The demand deposit contract delivers the first-best outcome for any realization of the stochastic aggregate state. Recall that when private money is restricted, preventing bank runs requires government intervention. With aggregate uncertainty, it becomes more problematic in that intervention itself may be costly. In contrast, once private money is allowed in trades, bank runs are no longer an issue. Achieving the first-best outcome requires no intervention whatsoever.

6 Conclusion

We have built a simple banking model with micro-foundations of money. There are overlapping generations of agents with idiosyncratic liquidity risks. Banks offer demand deposit contracts and invest deposits of fiat money in nominal bonds. When fiat money is the only medium of exchange permitted, a bank run equilibrium exists along with a "good" equilibrium where the first-best outcome is achieved. The optimal risk-sharing requires a gross rate of return $r > 1$ on early withdrawals. As a result, agents who face liquidity shocks can have more money to spend on consumption goods by depositing in the bank *ex ante*. Nevertheless, the mechanism is vulnerable to bank runs in that the bank does not have enough assets to honour $r > 1$ should all agents decide to withdraw early.

In contrast, when private money (i.e., banknotes) is allowed to circulate, banks no longer rely on $r > 1$ to provide liquidity in need. Agents in need of liquidity can simply use private money to buy goods instead of redeeming private money for fiat money at the bank. Moreover, an offer of $r < 1$ can help prevent bank runs. The bank is essentially charging transaction fees on early withdrawals of fiat money. This guarantees a positive amount of residual bank assets for any volume of early demand of withdrawals. In effect, agents do not form any panic beliefs. *Ex ante*, to choose from the demand deposit contracts, agents are aware of the potential bank runs associated with any contract offering $r > 1$. Thus agents would accept any contract with $r < 1$ rather than those with $r > 1$. Through competitive banking, it is optimal for a bank to offer $r < 1$.

Consequently, in the unique banking equilibrium with private money, no one demands early withdrawals of fiat money and agents in need of liquidity use private money to finance consumption. This result is robust to aggregate uncertainty. The economy manages to eliminate bank runs and achieves the first-best outcome without having to resort to any government intervention.

Finally, our model is a simple mechanism of money and banking. For future research, it will be interesting to embed the mechanism in environments with more sophisticated banking activities or alternative monetary environments with trading frictions (e.g., Shi [1997] and Lagos and Wright [2005]), so as to study other relevant issues on money and banking.

References

[1] Champ, B., B. Smith and S. Williamson, 1996, "Currency Elasticity and Banking Panics: Theory and Evidence," *Canadian Journal of Economics* **29(4)**, 828-864.

[2] Diamond, D. and P. Dybvig, 1983, "Bank Runs, Deposit Insurance and Liquidity," *Journal of Political Economy* **91**, 401-419.

[3] Kocherlakota, N., 2003, "Societal Benefit of Illiquid Bonds," *Journal of Economic Theory* **108**, 179-193.

[4] Lagos, R. and R. Wright, 2005, "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy* **113(3)**, 463-484.

[5] Shi, S, 1997, "A Divisible Search Model of Fiat Money," *Econometrica* **65**, 75-102.

[6] Sun, H., 2007, "Aggregate Uncertainty, Money and Banking," *Journal of Monetary Economics* **54**, 1929-1948.

[7] Sun, H., 2007, "Banking, Inside Money and Outside Money," Queen's University, QED Working Paper No. 1146.