Competitive Equilibrium in Asset Markets with Adverse Selection

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1 Introduction

This paper explores how adverse selection can create illiquidity in asset markets and how the extent of the illiquidity depends on the shocks hitting the economy. We study an economy in which there are two types of agents, investors and consumers, and two types of assets. At the start of each period, each agent owns some of each type of assets. The assets produce a dividend, units of a final good, which can be used both for consumption and investment within the period it is produced but cannot be stored. Assets are heterogeneous in terms of the amount of dividend that they produce, and only investors have access to the investment technology. After the dividend is produced, there is a competitive market in which investors can sell assets to consumers in return for the final good. Finally, investors divide their holdings of the final good between consumption and investment, while consumers must consume their holdings of the final good because they do not have direct access to the investment technology. Investments produce additional assets at the start of the following period.

We model adverse selection through an assumption that consumers are unable to distinguish between different types of assets in the competitive market. Our equilibrium concept is based on our previous work in Guerrieri, Shimer and Wright (forthcoming). In equilibrium, consumers anticipate that if they offer to purchase an asset at a price p, they will obtain each type of asset with some probability. That probability must be consistent with investors' rational decision to sell their assets at that price; for example, if investors would never want to sell a type j asset at price p, consumers cannot expect to buy one at that price. To keep the owners of bad assets out of the good asset market, there must be a shortage of consumers in that market. This means that the owner of a good asset may not be able to sell it within the period.

We find that the economy has a balanced growth path in which investment is unconstrained although there is no market for good assets. Instead, investors are able to fund all their investment from the dividend their assets produce and their sale of bad assets. If investors' wealth is initially too low relative to the wealth of consumers, the economy's growth and investment rates are below steady state; however, the adverse selection problem is ameliorated. During the transitional dynamics, the market for good assets is open and investors are able to gradually liquidate their good asset holdings in order to help finance investment.

Intuitively, when investors are constrained, failing to sell an asset is more costly. This reduces the incentive for investors to misrepresent their bad assets as good ones and so lessens the adverse selection problem. This effect is missing from models that treat the illiquidity of markets as a fixed parameter, e.g. Kiyotaki and Moore (2008).

We proceed in three steps. First, we analyze a model with only one type of asset which trades with probability one, $\theta = 1$. Our analysis of this case illustrates how the distribution of wealth between investors and consumers affects the economy's growth rate. Second, we consider a model with two types of assets. Type 1 assets produce fewer dividends per period but sell with probability 1. Type 2 assets sell with some time-varying probability $\theta_{2,t}$. We focus both on the case in which $\theta_{2,t}$ is constant and exogenous and the case in which it is time-varying. Finally, we consider a model with adverse selection, where the trading probabilities for type j asset, $\theta_{j,t}$, are determined by the Guerrieri, Shimer and Wright (forthcoming) notion of equilibrium. Right now the arguments in this last section are only sketched. In all cases, we focus on the full set of dynamic equilibria, not just the balanced growth path. To be concrete, we refer throughout the model to assets as "trees" and the perishable consumption-investment good as "fruit."

2 One Tree Model

We start with the case where there is only one type of tree. There are two types of agents, investors and consumers. All agents have period utility function $\log c$ and discount factor β. At the start of period t, a typical investor has k_t^i trees and a typical consumer has k_t^c units of trees. The following events then occur in sequence:

1. each tree produces δ fruit;
2. investors may sell trees for fruit; in equilibrium, trees sell for price p_t;
3. consumers consume their remaining fruit and investors divide their fruit between consumption and investment;

4. each unit of investment produces π trees in the following period.

We look for an equilibrium in which consumers choose how many trees to buy and how much to consume in order to maximize utility, taking as given the sequence of prices p_t; investors choose how many trees to sell and how much to consume and invest in order to maximize utility, taking as given the sequence of prices p_t, and the market for trees clears.

2.1 Consumers

We start by computing how consumption and tree purchases by consumers depends on the price of trees. Given our assumptions on preferences, this is a static problem. That is, consumption in period t, c_t^c, and tree holdings in period $t+1$, k_{t+1}^c depend only on the price of trees in period t, p_t, and tree holdings in period t, k_t^c.

To prove this, start by writing the sequence problem. A typical consumer who starts period t with k trees solves

$$V_t^c(k) = \max_{\{c_\tau^c, k_{\tau+1}^c\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \log c_\tau^c$$
$$\text{subject to } c_\tau^c + p_\tau(k_{\tau+1}^c - k_\tau^c) \leq \delta k_\tau^c$$
$$k_{\tau+1}^c \geq k_\tau^c$$
$$k_t^c = k.$$

The first constraint is a budget constraint, which says that the fruit produced in period τ can be used either for consumption or to purchase new trees. Tree purchases are constrained to be nonnegative.

A standard revealed-preference argument implies

$$V_t^c(k) = v_t^c + \frac{\log k}{1-\beta}$$

for all k and t. Basically a consumer with λk trees can consume λ times as much as a consumer with k trees, and similarly a consumer with k trees can invest and consume λ^{-1} times as much as a consumer with λk trees, proving that

$$V_t^c(\lambda k) = \frac{\log \lambda}{1-\beta} + V_t^c(k).$$

Substituting $\lambda = 1/k$ gives the desired result.

Now write the consumer's problem recursively:

$$V_t^c(k) = \max_{c,k'}[\log c + \beta V_{t+1}^c(k')]$$

subject to

$$\begin{aligned} c + p_t(k' - k) &\leq \delta k \\ k' &\geq k \end{aligned}$$

Assuming the last constraint is slack, the first order condition for k' is

$$\frac{p_t}{c_t^c} = \frac{\beta}{(1-\beta)k_{t+1}^c}.$$

Eliminate k_{t+1}^c using the budget constraint to get

$$c_t^c = (1-\beta)(\delta + p_t)k_t^c.$$

Then the budget constraint implies

$$k_{t+1}^c = \beta\left(\frac{\delta + p_t}{p_t}\right)k_t^c.$$

This is the solution if $\beta\left(\frac{\delta+p_t}{p_t}\right) \geq 1$, or equivalently $p_t/\delta \leq \beta/(1-\beta)$. Otherwise consumers do not purchase any trees, and so

$$\begin{aligned} c_t^c &= \delta k_t^c \\ k_{t+1}^c &= k_t^c. \end{aligned}$$

2.2 Investors

We next turn to investors. Again, this is a static problem, so consumption in period t, c_t^i, the fraction of trees sold to consumers in period t, s_t, the amount invested in period t i_t, and tree holdings in period $t+1$, k_{t+1}^i, depend only on the price of trees in period t, p_t, and the initial holdings of trees, k_t^i.

A typical investor who starts period t with k trees solves the sequence problem

$$
\begin{aligned}
V_t^i(k) = & \max_{\{c_\tau^i, i_\tau, s_\tau, k_{\tau+1}^i\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \log c_\tau^i \\
\text{subject to } & c_\tau^i + i_\tau \leq \delta k_\tau^i + p_\tau s_\tau k_\tau^i \\
& k_{\tau+1}^i = \pi i_\tau + (1 - s_\tau) k_\tau^i \\
& s_\tau \in [0, 1] \\
& i_\tau \geq 0 \\
& k_t^i = k.
\end{aligned}
$$

The first constraint states that the fruit produced in period τ plus the proceeds from the sale of a fraction s_τ of the trees are used for consumption and investment. The second constraint states that tree holdings next period are the sum of π times investment (since each fruit invested yields π trees) and the unsold trees from this period. In addition, investment is nonnegative and the fraction of trees sold must lie between 0 and 1.

The same revealed-preference argument implies

$$
V_t^i(k) = v_t^i + \frac{\log k}{1 - \beta}
$$

for all k and t. An investor with λk trees can invest, sell, and consume λ times as much as an investor with k trees, and similarly an investor with k trees can invest, sell, and consume λ^{-1} times as much as an investor with λk trees, proving that

$$
V_t^i(\lambda k) = \frac{\log \lambda}{1 - \beta} + V_t^i(k).
$$

Substituting $\lambda = 1/k$ gives the result.

Now write the investor's problem recursively, temporarily ignoring the nonnegativity constraint on investment:

$$
V_t^i(k) = \max_{s,k'} \left[\log \left(\left(\delta + s \left(p_t - \frac{1}{\pi} \right) \right) k - \frac{k' - k}{\pi} \right) + \beta V_{t+1}^i(k') \right].
$$

It follows that

$$
s_t \begin{cases} = 1 \\ \in [0, 1] & \text{if } p_t \gtreqless \dfrac{1}{\pi}. \\ = 0 \end{cases}
$$

In addition, the first order condition for k' implies

$$\frac{\frac{1}{\pi}}{\left(\delta + s_t\left(p_t - \frac{1}{\pi}\right)\right)k_t^i - \frac{k_{t+1}^i - k_t^i}{\pi}} = \frac{\beta}{k_{t+1}^i(1-\beta)}.$$

Solve for k_{t+1}^i:

$$k_{t+1}^i = \beta\left(1 + \pi\left(\delta + s_t\left(p_t - \frac{1}{\pi}\right)\right)\right)k_t^i.$$

Now check if this is consistent with $i_t \geq 0$, or equivalently $k_{t+1}^i \geq (1-s_t)k_t^i$. If $s_t = 1$, this holds trivially. If $s_t < 1$, $s_t\big(p_t - \frac{1}{\pi}\big) = 0$ from the first order condition for s_t. Then this holds if

$$\beta(1+\delta\pi) \geq 1 - s_t,$$

which is always true given our assumption that $\beta \geq 1/(1+\delta\pi)$. In summary,

$$\begin{aligned} k_{t+1}^i &= \beta\pi\big(\delta + \max\{p_t, 1/\pi\}\big)k_t^i \\ c_t^i &= (1-\beta)\big(\delta + \max\{p_t, 1/\pi\}\big)k_t^i, \end{aligned}$$

with all trees sold if $p_t > 1/\pi$, none sold if $p_t < 1/\pi$, and any fraction sold if $p_t = 1/\pi$.

2.3 Market Clearing

Next we turn to the market-clearing conditions. Because consumers' and investors' behavior is static, we can find a static market clearing condition which ensures that the tree market clears in each period.

There are three possible cases. First, suppose the investors sell all their trees in period t, $s_t = 1$. This requires $p_t \geq 1/\pi$. Consumers purchase $k_{t+1}^c - k_t^c$ trees. Using our formulae from the consumer's and investor's problems, equilibrium in the tree market requires

$$k_t^i = \left(\beta\left(\frac{\delta + p_t}{p_t}\right) - 1\right)k_t^c,$$

or equivalently

$$p_t = \frac{\beta\delta}{\kappa_t + 1 - \beta},$$

where

$$\kappa_t \equiv \frac{k_t^i}{k_t^c}.$$

Note that this defines $p_t \leq \beta\delta/(1-\beta)$, so consumers are indeed willing to purchase trees at this price. Therefore we have found a static equilibrium of the tree market if and only if this equation defines $p_t \geq 1/\pi$, or equivalently

$$\kappa_t \leq \bar{\kappa} \equiv \beta(1+\delta\pi) - 1.$$

Otherwise there is no equilibrium of this form.

Second, suppose investors sell some of their trees, $s_t \in (0,1)$. This requires $p_t = 1/\pi$. Equilibrium in the tree market that

$$s_t k_t^i = \left(\beta\left(\frac{\delta + p_t}{p_t}\right) - 1\right) k_t^c,$$

with $p_t = 1/\pi$, or equivalently

$$s_t = \frac{\bar{\kappa}}{\kappa_t}.$$

By assumption the numerator is positive. Moreover, this equation defines $s_t < 1$ if and only if $\kappa_t > \bar{\kappa}$, so there is no equilibrium with $s_t = 1$.

Finally, suppose $s_t = 0$. This requires $p_t \leq 1/\pi$ and $k_{t+1}^c = k_t^c$. From the consumer's problem, the latter condition holds if and only if

$$\beta\left(\frac{\delta + p_t}{p_t}\right) \leq 1.$$

Combining inequalities, this case reduces to

$$\beta(1+\delta\pi) \leq 1,$$

which we have ruled out by assumption. Thus there is always trade in the tree market in equilibrium. This completes the characterization of the static equilibrium in the tree market.

2.4 Dynamics

Finally we turn to the dynamics. There are two cases to study. First, suppose $\kappa_t \leq \beta(1+\delta\pi) - 1$, so $s_t = 1$ and $p_t = \frac{\beta\delta}{\kappa_t + 1 - \beta}$. Then the laws of motion for k^c and k^i,

$$k_{t+1}^c = \beta\left(\frac{\delta + p_t}{p_t}\right) k_t^c$$
$$k_{t+1}^i = \beta\pi\big(\delta + p_t\big) k_t^i,$$

imply

$$\kappa_{t+1} = p_t \pi \kappa_t = \frac{\beta\delta\pi\kappa_t}{\kappa_t + 1 - \beta}$$

It is straightforward to prove that for $\kappa_t < \bar{\kappa} = \beta(1+\delta\pi)-1$, this defines $\kappa_{t+1} \in (\kappa_t, \bar{\kappa})$, and so κ converges monotonically towards $\bar{\kappa}$. For $\kappa_t = \bar{\kappa}$, $\kappa_{t+1} = \bar{\kappa}$ as well.

On the other hand, if in period t, $\kappa_t > \bar{\kappa}$, $s_t < 1$ and $p_t = 1/\pi$. Although the laws of motion for k^c and k^i are unchanged,

$$k^c_{t+1} = \beta\left(\frac{\delta + p_t}{p_t}\right)k^c_t$$
$$k^i_{t+1} = \beta\pi\big(\delta + p_t\big)k^i_t,$$

this now implies $\kappa_{t+1} = \kappa_t$. That is, $\kappa_{t+1} = \kappa_t$ if and only if $\kappa_t \geq \bar{\kappa}$.

Intuitively, if the investors start with too little wealth, they have access to a scarce technology and so are able to earn a supernormal return. This leads to convergence of the wealth ratio to a level at which investors and consumers get the same return. Along the transition path, the growth rate is below its steady state value. With some algebra, we find that when $\kappa_t < \bar{\kappa}$,

$$\frac{(k^c_{t+1} - k^c_t) + (k^i_{t+1} - k^i_t)}{k^c_t + k^i_t} = \frac{\beta\delta\pi\kappa_t}{\kappa_t + 1 - \beta} = \kappa_{t+1} < \bar{\kappa}.$$

On the other hand, if investors start with too much wealth, they earn the same return as consumers and so the model does not have any tendency to converge to a lower wealth ratio. In this case, the inability of consumers to invest does not affect the equilibrium growth rate for a given aggregate tree endowment. Instead,

$$\frac{(k^c_{t+1} - k^c_t) + (k^i_{t+1} - k^i_t)}{k^c_t + k^i_t} = \bar{\kappa}$$

for any $\kappa_t \geq \bar{\kappa}$.

3 Two Tree Model

We now assume that there are 2 types of trees, $j = 1, 2$. A type j tree produces δ_j units of homogeneous fruit per period with $\delta_1 < \delta_2$. In addition, there is an ad hoc restriction on the liquidity of type 2 trees: investors can sell at most a fraction $\theta_{2,t}$ of their type 2 trees in period t. We define for notational convenience $\theta_{1,t} = 1$. Finally, a unit of fruit produces new trees in fixed proportions; π_j is the amount of type j trees produced per unit of fruit.

The notion of equilibrium is a natural extension of our earlier analysis: consumers choose how many trees of each type to buy and how much to consume in order to maximize utility, taking as given the sequence of prices $p_{j,t}$ and liquidity $\theta_{j,t}$ for type j trees; investors choose how many trees of each type to sell subject o the liquidity constraint and how much to consume and invest in order to maximize utility, taking as given the sequence of prices $p_{j,t}$ and liquidity $\theta_{j,t}$, and the market for trees clears.

3.1 Consumers

We again start by considering consumers. A typical consumer who starts period t with k_j type j trees solves

$$\begin{aligned}
V_t^c(k_1, k_2) = & \max_{\{c_\tau^c, k_{1,\tau+1}^c, k_{2,\tau+1}^c\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \log c_\tau^c \\
\text{subject to } & c_\tau^c + \sum_j p_{j,\tau}(k_{j,\tau+1}^c - k_{j,\tau}^c) \leq \sum_j \delta_j k_{j,\tau}^c \\
& k_{j,\tau+1}^c \geq k_{j,\tau}^c \\
& k_{j,t}^c = k_j.
\end{aligned}$$

The first constraint is a budget constraint, which says that the fruit produced in period τ can be used either for consumption or to purchase new trees.

An extension to our revealed preference argument implies that

$$V_t^c(k_1, k_2) = v_t^c + \frac{\log\left(\sum_j \delta_j k_j\right)}{1-\beta}$$

First, a consumer with $\{\lambda k_1, \lambda k_2\}$ trees can consume and invest λ times as much as a consumer with $\{k_1, k_2\}$ trees and earn additional utility $\log \lambda$ each period and vice versa. This implies

$$V_t^c(k_1, k_2) = V_t^c(\lambda k_1, \lambda k_2) - \frac{\log \lambda}{1-\beta}.$$

Second, two consumers with the same initial value of $\sum_j \delta_j k_{j,t}$ can afford the same consumption and same tree purchases in every period, which implies V_t^c depends only on $\sum_j \delta_j k_{j,t}$. Now let $\lambda = 1/\sum_j \delta_j k_j$; the result follows immediately.

Next write the consumer's problem recursively:

$$V_t^c(k_1, k_2) = \max_{c, k_1', k_2'} [\log c + \beta V_{t+1}^c(k_1', k_2')]$$

subject to

$$c + \sum_j p_{j,t}(k'_j - k_j) \leq \sum_j \delta_j k_j$$
$$k'_j \geq k_j.$$

The first order condition for a typical asset j is

$$\frac{p_{j,t}}{c^c_t} \geq \frac{\beta \delta_j}{(1-\beta)\sum_j \delta_j k^c_{j,t+1}},$$

with equality if $k^c_{j,t+1} > k^c_{j,t}$. It follows that the price-dividend ratio is the same for any asset j that is purchased at at time t, $p_{j,t} = \bar{p}_t \delta_j$, where

$$\bar{p}_t \equiv \frac{\beta c^c_t}{(1-\beta)\sum_j \delta_j k^c_{j,t+1}}.$$

For any asset that is not purchased at time t, $p_{j,t} \geq \bar{p}_t \delta_j$.

Rewrite the definition of $\bar{p}_t$ as

$$\frac{\beta}{1-\beta} c^c_t = \bar{p}_t \sum_j \delta_j k^c_{j,t+1}.$$

Add this to the binding budget constraint:

$$\frac{c^c_t}{1-\beta} + \sum_j (p_{j,t} - \bar{p}_t \delta_j) k^c_{j,t+1} = \sum_j (\delta_j + p_{j,t}) k^c_{j,t}.$$

Since $k^c_{j,t+1} = k^c_{j,t}$ whenever $p_{j,t} \neq \bar{p}_t \delta_j$, we can replace $k^c_{j,t+1}$ with $k^c_{j,t}$ on the left hand side. This implies

$$c^c_t = (1-\beta)(1+\bar{p}_t) \sum_j \delta_j k^c_{j,t}.$$

Consumption is a fraction $1-\beta$ of the value of a consumer's trees, where trees that are not traded are valued at the shadow price $\bar{p}_t \delta_j$ rather than at $p_{j,t}$. Conversely, the total fruit from next period's trees depends on this period's fruit and the price-divided ratio $\bar{p}_t$.

$$\sum_j \delta_j k^c_{j,t+1} = \beta \frac{1+\bar{p}_t}{\bar{p}_t} \sum_j \delta_j k^c_{j,t}.$$

This does not pin down the type of trees that consumers purchase except that $k^c_{j,t+1} = k^c_{j,t}$ if $p_{j,t} > \bar{p}_t \delta_j$.

3.2 Investors

Next turn to investors. A typical investor who starts period t with $\{k_1, k_2\}$ trees solves the sequence problem

$$
\begin{aligned}
V_t^i(k_1,k_2) = &\max_{\{c_\tau^i, i_\tau, s_{1,\tau}, s_{2,\tau}, k^i_{1,\tau+1}, k^i_{2,\tau+1}\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \log c_\tau^i \\
\text{subject to } c_\tau^i + i_\tau \leq &\sum_j \left(\delta_j + p_{j,\tau} s_{j,\tau}\right) k^i_{j,\tau} \\
&k^i_{j,\tau+1} = \pi_j i_\tau + (1 - s_{j,\tau}) k^i_{j,\tau} \\
&i_\tau \geq 0 \\
&s_{j,\tau} \in [0, \theta_{j,\tau}] \\
&k^i_{j,t} = k_j.
\end{aligned}
$$

The investor's consumption plus investment must come from the fruit her trees produces and the fruit she obtains by selling trees. Each unit of investment generates π_j type j trees, while investment is constrained to be nonnegative and the fraction of tree sold are constrained to lie between 0 and $\theta_{j,\tau}$.

While the basic homogeneity property of the value function carries over to this environment, $V_t^i(\lambda k) = V_t^i(k) + \log \lambda / (1-\beta)$, investors care about the distribution of their asset holdings. For example, the inability to sell one type of tree may be inframarginal to an investor who holds few of the illiquid trees, but may be costly to an investor who is endowed primarily with that type of tree.

We express the investor's problem recursively as

$$
V_t^i(k_1,k_2) = \max_{c, \{s_j\}, \{k'_j\}} \log c + \beta V^i_{t+1}(k'_1, k'_2)
$$

where

$$
k'_j = \pi_j \left(\sum_\ell \left(\delta_\ell + p_{\ell,t} s_\ell\right) k_\ell - c \right) + (1 - s_j) k_j,
$$

$s_j \in [0, \theta_{j,t}]$, and $c \leq \sum_j \left(\delta_j + p_{j,t} s_j\right) k_j$. We are interested primarily in situations where investment is constrained by investors' limited tree holdings and so investors sell the maximum possible amount of their trees, $s_{j,t} = \theta_{j,t}$, and the nonnegativity constraint on investment does not bind, $c < \sum_j \left(\delta_j + p_{j,t} s_j\right) k_j$. For now we solve the problem assuming these conditions hold. Later we will come back and discuss when this assumption is valid.

The first order condition for consumption is

$$\frac{1}{c_t^i} = \beta \sum_j \pi_j V_{j,t+1}^i,$$

where $V_{j,t+1}^i$ in the partial derivative of V_{t+1}^i with respect to j evaluated at $(k_{1,t+1}^i, k_{2,t+1}^i)$. In addition, the envelope conditions are

$$\begin{aligned} V_{j,t}^i &= (\delta_j + p_{j,t}\theta_{j,t})\beta \sum_\ell \pi_\ell V_{\ell,t+1}^i + (1-\theta_{j,t})\beta V_{j,t+1}^i \\ &= \frac{\delta_j + p_{j,t}\theta_{j,t}}{c_t^i} + (1-\theta_{j,t})\beta V_{j,t+1}^i, \end{aligned}$$

where the second equation uses the first order condition for consumption to simplify the first.

Since $\theta_{1,t} = 1$ by assumption, the envelope condition for $k_{1,t}^i$ reduces to

$$V_{1,t}^i = \frac{\delta_1 + p_{1,t}}{c_t^i} \text{ and } V_{1,t+1}^i = \frac{\delta_1 + p_{1,t+1}}{c_{t+1}^i}.$$

In addition, homogeneity of the value function implies

$$k_{1,t}^i V_{1,t}^i + k_{2,t}^i V_{2,t}^i = \frac{1}{1-\beta}.$$

Use this to eliminate $V_{2,t}^i$ from the envelope condition for $k_{2,t}^i$:

$$V_{2,t+1}^i = \frac{\frac{1}{k_{2,t}^i(1-\beta)} - \frac{k_{1,t}^i}{k_{2,t}^i} V_{1,t}^i - \frac{\delta_2 + p_{2,t}\theta_{2,t}}{c_t^i}}{\beta(1-\theta_{2,t})}$$

Then the first order condition for consumption implies

$$\frac{1}{c_t^i} = \beta\pi_1 \frac{\delta_1 + p_{1,t+1}}{c_{t+1}^i} + \pi_2 \frac{\frac{1}{k_{2,t}^i(1-\beta)} - \frac{k_{1,t}^i}{k_{2,t}^i}\frac{\delta_1 + p_{1,t}}{c_t^i} - \frac{\delta_2 + p_{2,t}\theta_{2,t}}{c_t^i}}{1-\theta_{2,t}},$$

or equivalently

$$c_t^i = \frac{(\delta_1 + p_{1,t})k_{1,t}^i + \left(\delta_2 + p_{2,t}\theta_{2,t} + \frac{1-\theta_{2,t}}{\pi_2}\right)k_{2,t}^i}{\frac{1}{1-\beta} + \frac{\beta\pi_1(\delta_1 + p_{1,t+1})(1-\theta_{2,t})k_{2,t}^i}{\pi_2 c_{t+1}^i}}.$$

This is the Euler equation for consumption. In the special case where $\theta_{2,t} = 1$, it reduces to

$$c_t^i = (1-\beta)\sum_j (\delta_j + p_{j,t})k_{j,t}^i,$$

but if investors are unable to sell all their trees in the period, consumption depends on both current and future tree prices.

3.3 Market Clearing

If investors sell both types of trees, consumers must buy both types. This implies that $p_{j,t} = \bar{p}_t \delta_j$ and the tree market clears,

$$\left(\beta \frac{1+\bar{p}_t}{\bar{p}_t} - 1\right) \sum_j \delta_j k_{j,t}^c = \sum_j \delta_j \theta_{j,t} k_{j,t}^i.$$

The left hand side is the increase in consumers' fruit production from period t to $t+1$. The right hand side is the fruit produced by the trees that investors sell. Solve to find the price-dividend ratio:

$$\bar{p}_t = \frac{\beta}{\kappa_t + 1 - \beta},$$

where

$$\kappa_t = \frac{\sum_j \delta_j \theta_{j,t} k_{j,t}^i}{\sum_j \delta_j k_{j,t}^c}$$

is the ratio of trees sold by investors to trees held by consumers, both measured in units of fruit.

3.4 Equilibrium

In equilibrium, investors' consumption Euler equation must hold. Using the fact that the price-dividend ratio is the same for both assets, this gives

$$c_t^i = \frac{\delta_1(1+\bar{p}_t)k_{1,t}^i + \left(\delta_2(1+\bar{p}_t)\theta_{2,t} + \frac{1-\theta_{2,t}}{\pi_2}\right)k_{2,t}^i}{\frac{1}{1-\beta} + \frac{\beta\pi_1\delta_1(1+\bar{p}_{t+1})(1-\theta_{2,t})k_{2,t}^i}{\pi_2 c_{t+1}^i}}.$$

In addition, the laws of motion for investors' tree holdings are

$$k_{j,t+1}^i = \pi_j \left(\sum_\ell \delta_\ell \big(1 + \bar{p}_t \theta_{\ell,t}\big)k_{\ell,t}^i - c_t^i\right) + (1-\theta_{j,t})k_{j,t}^i.$$

Similarly, consumers' tree holdings evolve as

$$k^c_{j,t+1} = k^c_{j,t} + \theta_{j,t} k^i_{j,t}.$$

Together with equilibrium price-dividend ratio, $\bar{p}_t = \beta/(\kappa_t + 1 - \beta)$, this determines the full dynamic equilibrium.

There are four state variables in this dynamic system, each type of tree holding by each type of agent. But as we previously discussed, only the consumers' total tree holdings, $k^c_t \equiv \delta_1 k^c_{1,t} + \delta_2 k^c_{2,t}$, is relevant for their behavior. In addition, the equilibrium is homogeneous of degree 1 in the four state variables. That is, doubling tree holdings at t simply doubles consumption at t and tree holdings at $t+1$, without affecting prices. This allows us to reduce the dimensionality of the state variables to two, say

$$\kappa_{j,t} \equiv \frac{\delta_j k^i_{j,t}}{k^c_t},$$

the fruit produced by investors' type j trees relative to the total fruit produced by consumers' trees. We can express the equilibrium as a dynamic system in $(\kappa_{1,t}, \kappa_{2,t})$. In particular, let $\tilde{c}_t \equiv c^i_t / k^c_t$. Since $k^c_{t+1}/k^c_t = 1 + \kappa_{1,t} + \theta_{2,t}\kappa_{2,t}$ and

$$\bar{p}_t = \frac{\beta}{1 + \kappa_{1,t} + \theta_{2,t}\kappa_{2,t} - \beta}, \tag{1}$$

the Euler equation reduces to

$$\tilde{c}_t = \frac{(1 + \bar{p}_t)\kappa_{1,t} + \left(1 + \bar{p}_t \theta_{2,t} + \frac{1-\theta_{2,t}}{\pi_2 \delta_2}\right)\kappa_{2,t}}{\frac{1}{1-\beta} + \frac{\pi_1 \delta_1 \bar{p}_t (1+\bar{p}_{t+1})(1-\theta_{2,t})\kappa_{2,t}}{\pi_2 \delta_2 (1+\bar{p}_t)\tilde{c}_{t+1}}}. \tag{2}$$

This relates current and future consumption to current and future $\kappa_{j,t}$. In addition, we can express these two state variables recursively:

$$\kappa_{j,t+1} = \frac{\bar{p}_t\left(\delta_j \pi_j \left((1 + \bar{p}_t)\kappa_{1,t} + (1 + \bar{p}_t \theta_{2,t})\kappa_{2,t} - \tilde{c}_t\right) + (1 - \theta_{j,t})\kappa_{j,t}\right)}{\beta(1 + \bar{p}_t)}. \tag{3}$$

For a given path of $\theta_{2,t}$, the Euler equation and state equations fully describe the dynamic behavior of the system.

To better understand the behavior of the economy, first suppose that $\theta_{2,t}$ is constant. Then the economy exhibits a balanced growth path in which consumption and each type of tree holding grows at a common rate $g = \beta\left(1 + \sum_j \delta_j \pi_j\right) - 1$ while prices are constant. Along the balanced growth path, investors' consumption relative to the fruit produced by consumers is

$$\tilde{c}^* = \frac{(1-\beta)g\big(g+1-\frac{\beta\delta_1\pi_1}{g+1-\beta}(1-\theta_2)\big)}{g+1-\beta-\beta\big(\frac{\delta_1\pi_1}{g+1}+\delta_2\pi_2\big)(1-\theta_2)}.$$

In addition, the investors' production of each type of fruit relative to the consumers' total production satisfies

$$\kappa_1^* = \frac{g}{1+\frac{\delta_2\pi_2\theta_2(1+g)}{\delta_1\pi_1(g+\theta_2)}}$$
$$\kappa_2^* = \frac{g}{\theta_2+\frac{\delta_1\pi_1(g+\theta_2)}{\delta_2\pi_2(1+g)}}$$

In particular, $\kappa \equiv \kappa_1 + \theta_2\kappa_2 = g$. This extends the results in the one tree economy in a simple way. In particular, investors are unconstrained by their inability to sell type-2 trees and the economy is unconstrained by investors' wealth.

If investors are endowed with relatively more trees (κ_t is larger) and with more type 1 trees ($\kappa_{1,t}/\kappa_{2,t}$ is larger), the equilibrium is unconstrained. We are interested in cases in which investors are poor or are endowed primarily with type 2 trees. In general, the equilibrium depends on the behavior of $\theta_{2,t}$. We make a particular assumption on how this behaves and justify it later through an adverse selection model.

3.5 Illiquidity

As our baseline, we assume that $\theta_{2,t}$ is set at a level such that investors are indifferent about selling their type 1 trees for sure at $p_{1,t}$ and selling them with probability $\theta_{2,t}$ at $p_{2,t}$. Later we show that this is essentially an incentive constraint in an adverse selection model, but for now we simply treat this as an ad hoc constraint on the liquidity of type 2 trees.

In particular, we impose

$$\frac{\delta_1\bar{p}_t}{c_t^i} = \theta_{2,t}\frac{\delta_2\bar{p}_t}{c_t^i} + (1-\theta_{2,t})\beta V_{1,t+1}^i.$$

If the tree fails to sell, the investor is left with a type 1 tree the following period. Eliminating $V_{1,t+1}^i$ using the envelope condition from the investor's problem and solving for $\theta_{2,t}$, this reduces to

$$\theta_{2,t} = \frac{\delta_1\left(\bar{p}_t c_{t+1}^i - \beta(1+\bar{p}_{t+1})c_t^i\right)}{\delta_2\bar{p}_t c_{t+1}^i - \beta\delta_1(1+\bar{p}_{t+1})c_t^i}.$$

We can rewrite this condition in terms of our stationary variables,

$$\theta_{2,t} = \frac{\delta_1\left(\frac{1+\bar{p}_t}{\tilde{c}_t} - \frac{1+\bar{p}_{t+1}}{\tilde{c}_{t+1}}\right)}{\delta_2\frac{1+\bar{p}_t}{\tilde{c}_t} - \delta_1\frac{1+\bar{p}_{t+1}}{\tilde{c}_{t+1}}}. \tag{4}$$

Along a balanced growth path, $\tilde{c}_t = \tilde{c}_{t+1}$ and $\bar{p}_t = \bar{p}_{t+1}$. This implies that $\theta_2 = 0$; investors are never able to resell their trees.

In general this will not be true away from the balanced growth path, however. Assuming

$$\frac{1+\bar{p}_t}{\tilde{c}_t} > \frac{1+\bar{p}_{t+1}}{\tilde{c}_{t+1}},$$

this defines $\theta_{2,t} \in (0,1)$. The numerator in the expression for $\theta_{2,t}$ is the difference between the value (dividend plus price) of a type 1 tree in the current period and next period, weighted by the marginal utility of consumption. The denominator is the the difference between the value of a type 2 tree in the current period and a type 1 tree next period. If a type 1 tree is worth more tomorrow than it is today, the opportunity cost of failing to sell at type 1 tree is zero and so it is impossible to keep those trees out of the type 2 tree market.

To characterize the equilibrium, solve equation (4) for c_{t+1} and use that to eliminate forward-looking variables from the Euler equation (2). This a static relationship between relative consumption and illiquidity.

$$\tilde{c}_t = (1-\beta)\left((1+\bar{p}_t)\kappa_{1,t} + \left(1 + \bar{p}_t\theta_{2,t}\left(1+\frac{\pi_1}{\pi_2}\right) + \frac{1-\theta_{2,t}-\pi_1\delta_1\bar{p}_t}{\pi_2\delta_2}\right)\kappa_{2,t}\right). \tag{5}$$

Using this, we can eliminate current and future consumption from the consumption Euler equation and obtain an Euler equation relating current and future illiquidity. We can similarly express the evolution of the relative capital stocks $\kappa_{j,t}$, given in equation (3), in terms of illiquidity. The resulting expressions are messy, but the point is that we have a dynamic system with two state variables and one control, $\theta_{2,t}$. We are interested in the behavior of this system at values of $(\kappa_{1,t}, \kappa_{2,t})$ such that $\theta_{2,t} > 0$.

We also consider a version of the model in which $\theta_{2,t} = 0$ for all t. We refer to this as a model with exogenous illiquidity. In this case, the equilibrium is characterized by the state equations for $\kappa_{j,t}$ and the Euler equation for $\tilde{c}_t$, equation (2), along with the restriction that $\theta_{2,t} = 0$.

3.6 Numerical Results

We characterize the equilibrium numerically. We think of a time period as representing one week and so set the discount factor to $\beta = 0.999$. We normalize the productivity of a type 1 tree to $\delta_1 = 1$ and assume type 2 trees are twice as productive, $\delta_2 = 2$. We also assume that half the fruit comes from each type of tree, which requires $\pi_1 = 2\pi_2$. Finally, we set $\pi_1 = 0.0008$ (and so $\pi_2 = 0.0004$) in order to pin down the growth rate of the economy, about 3.1 percent on an annual basis. In the baseline model, $\theta_{2,t}$ ensures that investors are indifferent between selling a type 1 tree for $p_{1,t}$ for sure and selling it for $p_{2,t}$ with probability $\theta_{2,t}$. Although these numbers are not completely random, the economy is sufficiently abstract that one might not want to take the numerical results too seriously.

In the steady state, $\kappa_2^*/\kappa_1^* = 1672$; this imbalance in investors' holdings of the two types of trees reflects their inability to resell type 2 trees. It is instructive to log-linearize the economy in a neighborhood of the steady state. More precisely, since the steady state value of θ_2 is zero, we use a first order Taylor approximation to express the deviation of $\log \kappa_{1,t+1}$, $\log \kappa_{2,t+1}$, and $\theta_{2,t+1}$ from their steady state values as a function of the deviation of $\log \kappa_{1,t}$, $\log \kappa_{2,t}$, and $\theta_{2,t}$ from steady state. The resulting system is saddle path stable: it has one eigenvalue that exceeds 1 (1.0026) and two that are smaller than 1 (0.9998 and 0.0005). The very small magnitude of the last eigenvalue implies that, regardless of the initial condition of the state variables, they quickly converge to the eigenvector associated with the larger stable eigenvalue. In this case, this implies $\kappa_{1,t} \approx \kappa_1^*(\kappa_{2,t}/\kappa_2^*)^{1.9987}$, effectively reducing the state space to a single dimension.

The economics for this result is informative. Suppose we start from an initial condition where $\kappa_{1,t}$ is small relative to $\kappa_{2,t}$. Since type 1 trees are more easily sold than type 2 trees, it follows that investment is tightly constrained. But this makes it less attractive to attempt to misrepresent a type 1 tree as a type 2 tree, which raises the equilibrium value of $\theta_{2,t}$. This then facilitates investment, which drives the economy towards the stable path.

Our nonlinear numerical solution verifies the relevance of this result.[1] We are unable to solve the model backwards from the neighborhood of the steady state because if we do not start exactly on the stable (κ_1, κ_2) locus, the values of κ_1 quickly explode. Instead we solve in forward by guessing an initial condition for $\theta_{2,t}$. We verify that θ_2 and κ_1 change quickly between the first and second periods as the economy moves onto the stable locus. Thereafter, the $\kappa_{j,t}$ gradually rise towards their steady state values, while $\theta_{j,t}$ slowly falls towards zero. Thus when $\kappa_{2,t}$ initially lies below its steady state value, the transitional dynamics take the economy through an extended period in which there is some market for type 2 trees.

[1] The local dynamics turn out to be quite accurate for all the outcomes except the investment rate, which

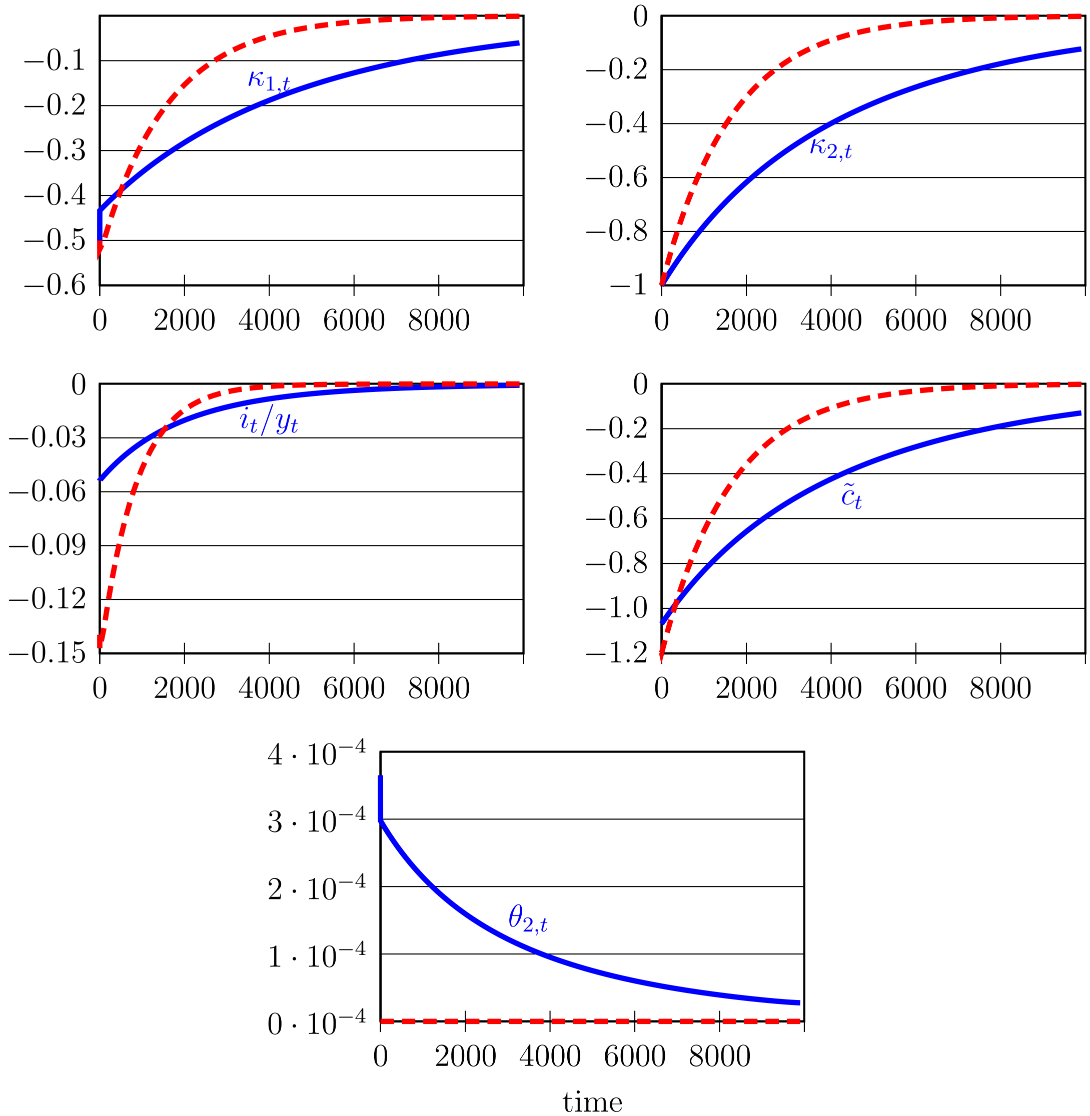

Figure 1: Transitional dynamics. Parameters given in text. In every panel, the solid blue line shows the model with $\theta_{2,t}$ endogenous and the dashed red lines shows the model with $\theta_{2,t} = 0$. The top row shows the percent deviation of the state variables $\kappa_{1,t}$ and $\kappa_{2,t}$ from steady state. The middle left figure shows the investment rate (investment divided by output) relative to steady state. This is equivalent to the growth rate relative to steady state. The middle right figure shows consumption of investors divided by income of consumers, relative to steady state. The bottom figure shows the liquidity of type 2 trees.

Figure 1 shows the nonlinear transitional dynamics. We start from an initial condition where $\kappa_{2,t}$ is 100 log points below steady state. It is possible to see the jump in $\kappa_{1,t}$ at the initial date, followed by the smooth convergence towards steady state. This is mirrored by the jump down in $\theta_{2,t}$. On the other hand, investment, consumption, and $\kappa_{2,t}$ are all much smoother. Here the investment rate is defined as the number of new trees created divided by total output,

$$\begin{aligned}\frac{i_t}{y_t} &= \frac{(\delta_1 + p_{1,t})k^i_{1,t} + (\delta_2 + \theta_{2,t}p_{2,t})k^i_{2,t} - c_t}{\delta_1(k^i_{1,t} + k^c_{1,t}) + \delta_2(k^i_{2,t} + k^c_{2,t})} \\ &= \frac{(1 + \bar{p}_t)\kappa_{1,t} + (1 + \theta_{2,t}\bar{p}_t)\kappa_{2,t} - \tilde{c}_t}{\kappa_{1,t} + \kappa_{2,t} + 1},\end{aligned}$$

where the second equation follows from the definition of our stationary objects. Note that the growth rate of the economy is proportional to the investment rate, $g_t = (\pi_1\delta_1 + \pi_2\delta_2)i_t/y_t$. Although the resale rate of type 2 trees is always very low, this still provides significant liquidity to investors because of the imbalance between holdings of the two types of trees.

If initially $\kappa_{1,t}$ is high and $\kappa_{2,t}$ is low, there may be a period where there is no market for type 2 trees but investment and consumption is low. Investors gradually liquidate their holdings of type 1 trees. Eventually the constraint on selling type 2 trees starts to bind and the economy converges to a steady state from below, as in Figure 1. This might represent the transitional dynamics following a shock that lowers the value of some of the investors' trees.

Figure 1 also compares these transitional dynamics to those in a model where $\theta_{2,t} = 0$ by assumption. The model is still saddle path stable and the eigenvalues are qualitatively similar, 2.3637, 0.9994, and 0.2651. In particular, $\kappa_{1,t}$ quickly moves onto the eigenvector associated with the dominant eigenvalue. The most obvious difference in the model's behavior lies in the magnitude of the response of investment. When investors cannot sell their type 2 trees, the investment rate starts 15 log point below steady state, instead of just 5 log points. Consumption also falls slightly more in the model with exogenous illiquidity. As a result of the low investment rate, $\kappa_{1,t}$ is initially pushed lower in the model without any resale market for type 2 trees, while $\kappa_{2,t}$ increases more rapidly. On the other hand, the inability of investors to sell their type 2 trees raises the speed at which the economy converges back to steady state.

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is severely understated by the log-linear approximation.

4 Competitive Equilibrium with Adverse Selection

This section formalizes our notion of equilibrium with adverse selection. We assume that investors know the type of each of their trees but consumers do not. After the trees bear fruit, a market opens at every price $p \geq 0$. Investors and consumers form expectations about both the type of tree that is traded and the buyer-seller ratio at the price. Let $\gamma_{j,t}(p)$ denote the fraction of type j trees trading at price p at time t and $\theta_t(p)$ denote the ratio of the fruit offered by consumers to the value of the trees sold by investors at price p and time t. That is, suppose at some price p and time t, consumers offer f units of fruit while investors offer k trees; then $\theta_t(p) = f/pk$. Investors assume that they are able to sell a tree at price p with probability $\min\{\theta_t(p), 1\}$, while consumers assume that they will be able to buy a type j tree with probability $\gamma_{j,t}(p)\min\{1/\theta_t(p), 1\}$. In equilibrium, expectations must be rational in the sense of Guerrieri, Shimer and Wright (forthcoming). At this point we describe only loosely what that means.

An investor can sell a type j tree at any price p. We impose that the functions $\theta_t(p)$ and $\gamma_{j,t}(p)$ are such that if $\gamma_{j,t}(p) > 0$, then p is among the optimal prices for an investor to sell a type j tree in period t, recognizing that the sale probability will only be $\min\{\theta_t(p), 1\}$. Moreover, we impose that any price p satisfies $p \geq \bar{p}_t \sum_j \gamma_{j,t}(p)\delta_j$, with equality if consumers actually purchase trees at the price p. This states that consumers buy trees at the price that gives them the highest expected value for their trees.

In equilibrium, type 1 trees are traded for $p_{1,t}$ and type 2 trees are traded for $p_{2,t}$. Moreover, consumers anticipate buying type 2 trees at any price $p \geq p_{2,t}$ and type 1 trees at $p \in [p_{1,t}, p_{2,t})$. Their beliefs are arbitrary at any lower price $p < p_{1,t}$. Finally, the trading probabilities $\theta_t(p)$ ensure that investors can always sell a tree at $p_{1,t}$, $\theta_t(p_{1,t}) = 1$, and are indifferent about selling a type 1 tree at any $p \in [p_{1,t}, p_{2,t}]$,

$$\frac{p_{1,t}}{c_t^i} = \theta_t(p)\frac{p}{c_t^i} + (1 - \theta_t(p))\beta V_{1,t+1}^i.$$

Similarly, they are indifferent about selling a type 2 tree at any $p \geq p_{2,t}$,

$$\theta_t(p_{2,t})\frac{p_{2,t}}{c_t^i} + (1 - \theta_t(p_{2,t}))\beta V_{2,t+1}^i = \theta_t(p)\frac{p}{c_t^i} + (1 - \theta_t(p))\beta V_{2,t+1}^i.$$

Since $V_{1,t+1}^i < V_{2,t+1}^i$, this condition implies that at any $p > p_{2,t}$,

$$\frac{p_{1,t}}{c_t^i} > \theta_t(p)\frac{p}{c_t^i} + (1 - \theta_t(p))\beta V_{1,t+1}^i,$$

so investors are not tempted to sell their type 1 trees at these high prices. Finally, $\theta_t(p) = \infty$

for any $p < p_{1,t}$, so it is impossible to buy a tree for less than $p_{1,t}$. One can confirm that an investor would rather sell a tree for $p_{1,t}$ with probability 1 than at a lower price.

Note that the conditions in the previous paragraph are all first order conditions. One must also check that an investor does not wish to misrepresent the types of all his trees, nor does he want to change his investment and consumption behavior and then misrepresent the types of his trees.

It is reasonably straightforward to prove that an equilibrium of this form exists. It is more work to prove that any equilibrium must take this form.

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