Learning to Love Money

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- Money appeared in a multitude of cultures independently.
- Money facilitated trade across the Eurasia and beyond.

We want to study:

- Can learning from experience justify the appearance of interest in money?
- Can learning be used to select an equilibrium of the Kiyotaki-Wright model?

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- Evans and Honkapohja (2001): Learning Dynamics, Stochastic Approximation
- Williams (2002): Large Deviations

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Our paper: how people learn to use fiat money.

- Time is discrete and infinite.
- One coin in the economy.
- Finite number of agents; they specialize in what they produce.
- Goods and coin are indivisible, no storage costs.
- Some agents like some other agents' products.
- Agents meet randomly and anonymously each period.

- U is utility from consumption of good.
- c is disutility from production.
- δ is a time discount factor.
- p is the probability of mutual coincidence of wants; q is the probability that only one agent wants the other agent's good.

$$p + 2q \leq 1.$$

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- If both agents do not want their partner's good, they will depart.
- At the end of the day, everyone who has no good produces and pays cost c.

Universal Rejection Equilibrium No one takes money — always exists.
Universal Acceptance Equilibrium V_0 is value of not having money, V_1 is value of having a coin.

$$\frac{V_0^*}{\delta} = p\left(V_0^* + U - c\right) + \frac{1}{n-1} q\left(V_1^* - c\right) + \left(1 - p - \frac{q}{n-1}\right)\left(V_0^*\right)$$

$$\frac{V_1^*}{\delta} = p\left(V_1^* + U - c\right) + q\left(V_0^* + U\right) + (1 - p - q)\left(V_1^*\right)$$

This is an equilibrium when $V_1^* - V_0^* = V_\Delta^* > c$.

$$V_\Delta = V_1^* - V_0^* = \frac{qU + \frac{q}{n-1}c}{\frac{1}{\delta} - 1 + \frac{qn}{n-1}}$$

$$\Rightarrow \delta > \frac{c}{qU + (1-q)c}$$

Adaptive Learning

- Agents have a belief about value of money $\tilde{V}$.
- ...start with belief of 0.
- ...updating: if agent
  - took money at period T_1;
  - successfully traded it at moment T_2;
  - at beginning of period $T_2 + 1$ agent's $\tilde{V}$ becomes $\gamma(\delta^{T_2 - T_1} U) + (1 - \gamma)\tilde{V}$.
- Agents make a decision implementation error with probability ε.

- γ: Gain
- ε: Error probability

Beliefs follow the following dynamics:

$$V_{t+1} = V_t + \gamma(\delta^{\tau} U - V_t)$$

Where τ is the random wait time until successful trade. It is essentially an exponential wait time, depending on everyone's acceptance rules and the error probabilities.
Note that the time scale t refers to *learning* time.

We can write this as

$$\begin{aligned} V_{t+1} &= V_t + \gamma(\delta^\tau U - V_t) \\ \frac{V_{t+1} - V_t}{\gamma} &= \delta^\tau U - V_t \\ \dot{V} &= \delta^\tau U - V \end{aligned}$$

This is the *continuous approximation*. The equilibria of the learning dynamic satisfy $\mathbb{E}\dot{V} = 0$, or

$$\bar{V} = \mathbb{E}\delta^\tau U$$

There are two such equilibria:

- Under universal acceptance, the wait time τ can be approximated by an exponential distribution with wait time $q\varepsilon$, leading to equilibria value of money

$$\frac{\delta q\varepsilon}{1-\delta(1-q\varepsilon)}$$

- Under universal rejection, the wait time is is given by $q(1-\varepsilon)$, with value

$$\frac{\delta q(1-\varepsilon)}{1-\delta(1-q(1-\varepsilon))}$$

These correspond exactly to the rational equilibria.

Another way to view the dynamics:

$$\dot{V} = \delta^\tau U - V$$
$$\dot{V} = \delta^\tau U - V + \bar{V} - \bar{V}$$
$$\dot{V} = (\bar{V} - V) + (\delta^\tau U - \bar{V})$$
$$\dot{V} = (\bar{V} - V) + \xi$$

This consists of a drift towards the equilibrium and a bounded, mean zero error.

The dynamics of learning around the equilibria are described by the *mean dynamics* of the learning algorithm:

$$\dot{\tilde{V}} = \mathbb{E}((\bar{V} - V) - \xi)$$
$$\dot{\tilde{V}} = (\bar{V} - V)$$
$$\dot{\tilde{V}} = 0 \text{ at } V = \bar{V}$$

That is, both equilibria are stochastically stable.

- So, learning by itself does not select among rational equilibria; both equilibria can be learned.
- So long as errors are possible ($\varepsilon > 0$) and agents learn from the past ($\gamma > 0$), the learning dynamic will spend some time in both equilibria as $t \to \infty$.

But, it may be possible to characterize *how much time* in each and *how difficult* it is to leave each equilibrium.

We did 1000 simulations for 100 thousand periods, to get a sense of the behavior of the learning dynamic.
We use these values for the simulations:

- $\delta = 0.95$.
- $U = 1$.
- $c = 0.1$.
- $\gamma = 0.2$.
- $\varepsilon = 1/200$.

Particular Structure of Economy

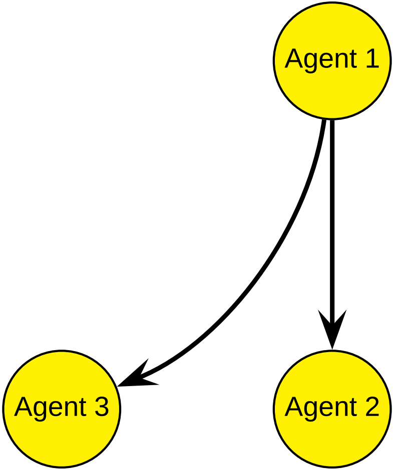

Figure: Agent 1 wants goods of 2 and 3.

Total of 8 agents.

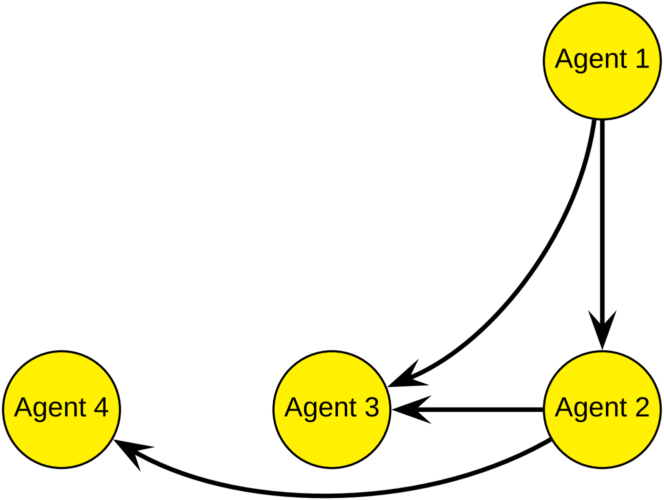

Figure: Agent 2 wants goods of 3 and 4.

Total of 8 agents.

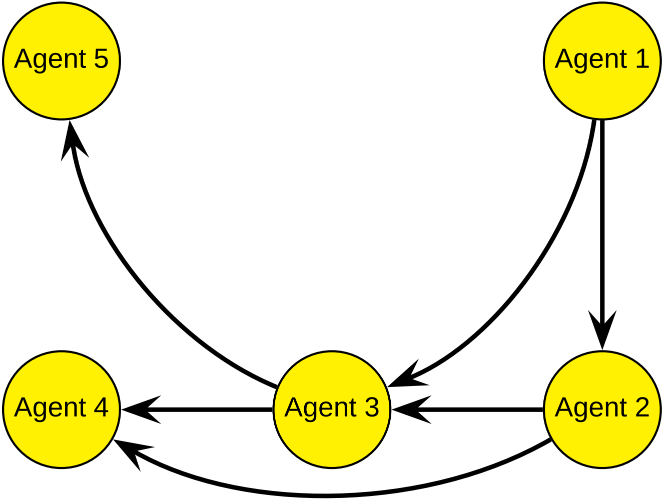

Figure: Agent 3 wants goods of 4 and 5.

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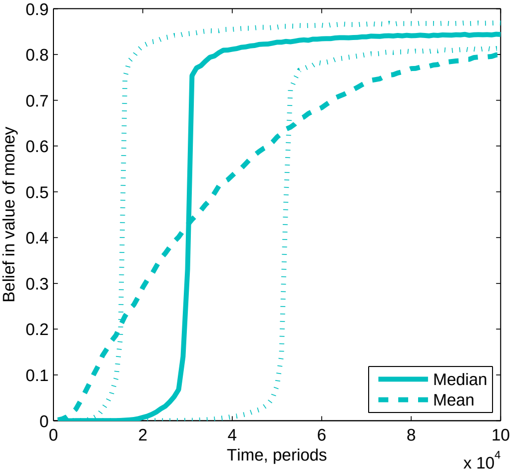


Figure: Mean Behavior of Learning.

- There is 185 observations of $\tilde{V} > c$ going to $\tilde{V} < c$.
- There is 7787 observations of $\tilde{V} < c$ going to $\tilde{V} > c$.
- These are not necessarily good estimates of equilibrium probabilities; these are extremely unlikely tail events

Assume agent 8 never ever ever accepts money.

- It will slow down learning of agent 7: he has only one channel for outflow of money.
- It will slow down learning of agent 5: he has to deal with agent 7.
- Will it prevent learning?
- Will it be strategically optimal?

Naturally, if both agents 7 and 8 never accepted money, money would never circulate.

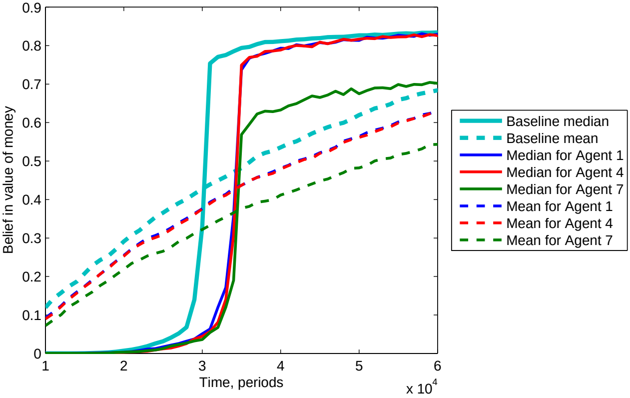
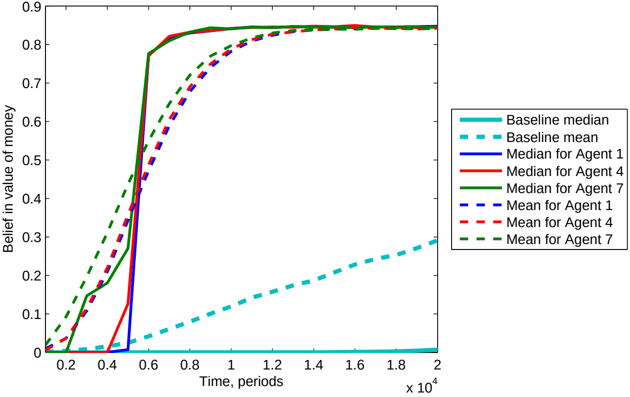


Figure: Mean Behavior of Learning.

Assume agent 8 always accepts money.

- It will speed up learning of agent 7: he has one channel that will always take money.
- It will speed up learning of agent 5: he can deal with agent 7.
- Will it expedite total learning?
- Will it be strategically optimal?



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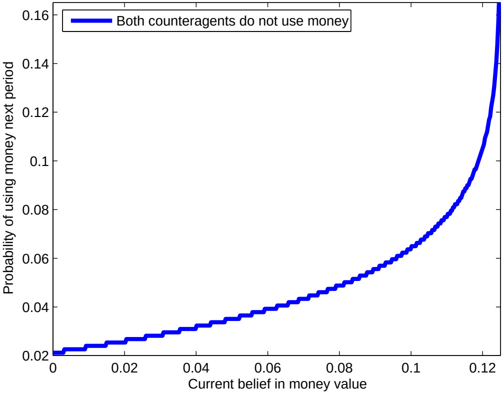


Figure: Your Counteragents Do Not Take Money.

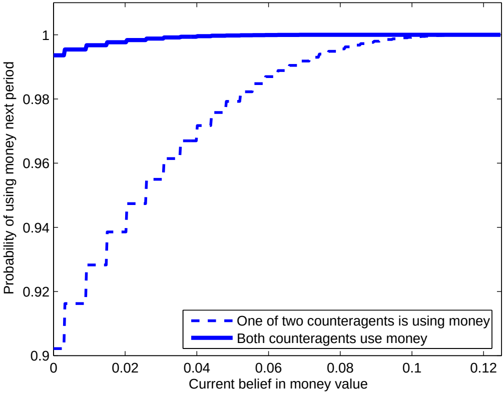
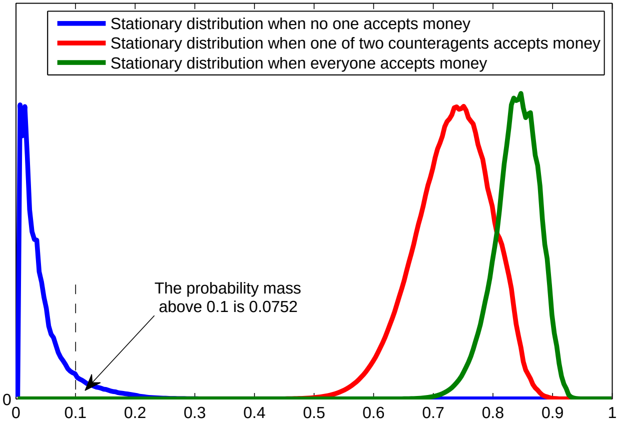


Figure: Your Counteragents Take Money.


Stationary distribution when no one accepts money
Stationary distribution when one of two counteragents accepts money
Stationary distribution when everyone accepts money
The probability mass above 0.1 is 0.0752
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

- It is easy to learn to accept money even when no one else does (probability of 7.52% in our parametrization).
- It is not easy to become disappointed in money:

Hitting value	0.8	0.7	0.6	0.5
Time to achieve	535	40K	51M	>2B

How to escape equilibrium

Consider the mean dynamics, holding every other agent's beliefs fixed:

$$\dot{\tilde{V}} = (\bar{V} - V)$$

In order for V to leave this equilibrium, forcing errors $s(t)$ must overcome the drift towards $\bar{V}$.

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We solve for the most likely way for this to happen.

The large deviations of these dynamics describe how beliefs escape these equilibria:

- it is unlikely, but must eventually happen due to the noise ξ;
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So we solve for the *rate function* that governs the likelihood of that happening.

The large deviation properties are entirely driven by the behavior of this random variable:

$$Z = \delta^{\tau} U, \ \tau \backsim \text{Exponential}(\lambda)$$

where λ is the probability of trading at equilibrium.

The distribution g(z) of this random variable is given by

$$g(z) = -\frac{\lambda \left(\frac{z}{U}\right)^{-1-\frac{\lambda}{\log \delta}}}{U \log \delta}.$$

for $z \in [0, U]$; essentially a truncated power-law.
The distribution of the mean zero error ξ is simply a shift of this distribution.

From this we calculate the culmulant generating function:

$$H(t) = \log \mathbb{E}_z e^{tz},$$

and the rate function I(x) is the Legendre transform of H:

$$I(x) = \sup_t xt - H(t).$$

This function I governs the large deviation properties of Z.

Punchline: I is asymmetric; it is easier to to increase than decrease, easier to learn money has value than to learn it does not.

The rate function I at the Universal Acceptance and Universal Rejection equilibria:

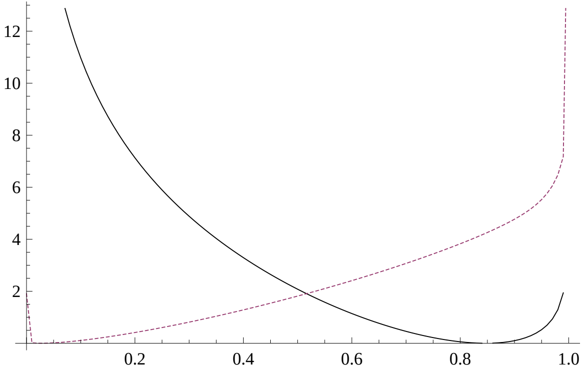
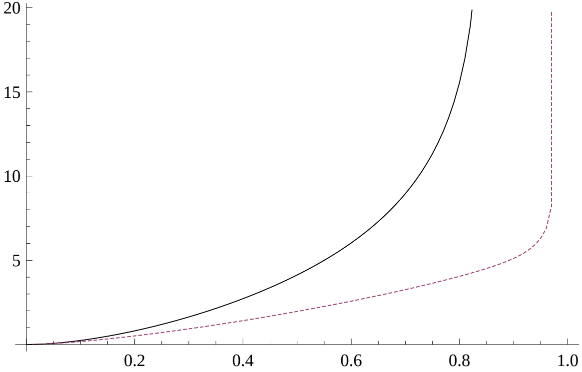


Figure: Solid - High Value Rate Function. Dashed - Low Value Rate Function

They are zero at their equilibrium.

Superimposed, the difference is clear:



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It is much lower cost to escape the low equilibrium.

To characterize the long run distribution over equilibria, consider first a single agent leaving the universal rejection equilibrium.

To do so, we must find a sequence of trading shocks $s(t)$ (which have distribution ξ) that will drive his estimate from $\bar{V}$ to c at time T:

$$\begin{aligned} V(0) &= \bar{V} \\ V(T) &= c \\ \dot{V} &= (\bar{V} - V(t)) + s(t) \end{aligned}$$

This condition has ODE form:

$$c = \bar{V} + e^{T} \int_0^T e^{-t}(\bar{V} + s(t))dt$$

We now solve the following minimization problem:

$$\min_{T,s(\cdot)} \int_0^T I(s(t))dt$$

subject to

$$c = \bar{V} + e^T \int_0^T e^{-t}(\bar{V} + s(t))dt$$

The rate function I can be interpreted as the "cost" of a shock of size $s(t)$, and we seek shocks of minimal cost that will force this agent across the boundary c.

This is the mostly likely way to escape the equilibrium, the *dominant escape path*.

The optimal shocks are constant in size. There are two effects:

- after you move closer to the boundary, you have less remaining distance to travel;
- but the drift back towards equilibrium is stronger.

The optimal shocks to travel to the boundary in time T is given by

$$s(t) = \frac{e^T(\bar{V} - c)}{e^T - 1}$$

which is constant in t.

For our parametrization, the minimized cost of escape as a function of T

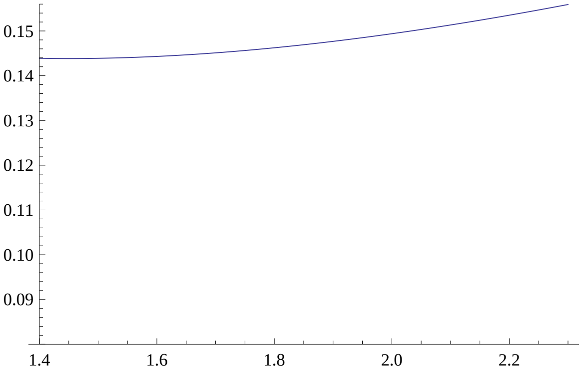

Figure: Low Value Equilibrium Action Functional

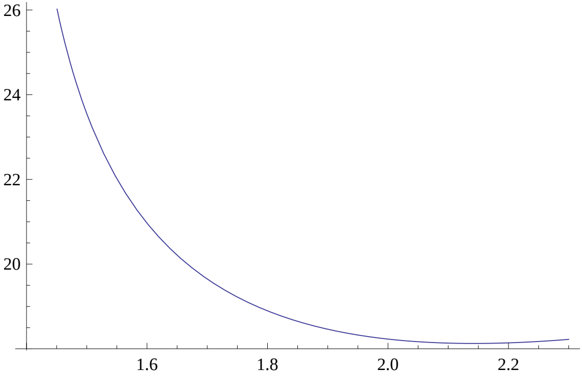

Figure: High Value Equilibrium Action Functional

There are two ways to escape: larger shocks alow one to escape in a shorter time, but are more costly. For the low equilibrium, since shocks are not so costly, the optimal T is smaller.

c = 0.1	Escape Time	Cost	Probability
Low Value Equilibrium	1.45098	0.143836	0.49438
High Equilibrium	2.14218	18.1264	$4.35614E-40$

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Note: this is not an artifact of c being closer to the Low equilibrium than to the High; with c equidistant we have

c=0.408	Escape Time	Cost	Probability
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It is driven by asymmetry in the escape probabilities.

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It is driven by asymmetry in the escape probabilities. Probabilities are equal at $c \approx 0.461$.

To calculate the probability of the system as whole escaping equilibrium, there are only two ways for an agent to leave one equilibrium:

- In the manner described above, on his own - this is costly
- If enough of his trading partners escape as above, the ODE governing his learning flips, so that the other equilibrium is attractive, and everyone may follow his new mean dynamics - this has cost zero.

We solve for the cost minimizing combination of agents going to the boundary on their own and “waiting” while other follow zero cost mean dynamics.

For our parametrization, the “low cost” way to escape the low equilibrium is for half to escape, half to wait. To escape the high equilibrium all agents must escape on their own. This is very costly.

	Transition Cost
Low Value Equilibrium	0.594863
High Equilibrium	145.011

Theorem (Williams 2002)

As the gain $\gamma \to 0$, the invariant distribution of beliefs are concentrated on the higher-cost equilibrium

Conclusion

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- We propose a way to choose a rational equilibrium in the Kiyotaki and Wright model.
- We show that the “good” rational equilibrium is quite prominent.
- In fact, it is the long run dominant equilibrium as $\gamma \to 0$.