# Knowledge Growth and the Allocation of Time

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- New model of endogenous growth

- Productivity related knowledge held by large number of agents, each with his own productivity level

- Agents meet stochastically, learn from others, improve productivity

- Mechanics adapted from Kortum (1997), Eaton and Kortum (1999), elaborated in Alvarez, Buera, Lucas (2007), Lucas (2009)

- New feature here is agents' allocation of time between production and learning

# Plan of talk

1 – Dynamics of the productivity distribution

2 – Decentralized economy with time-allocation decisions

3 – An optimally planned economy

4 – Alternative learning technologies

# 1 – A technology of learning

- Kortum (1997), Eaton and Kortum (1999)
- Consider economy with continuum of infinitely-lived agents
- All have one unit of labor per unit of time (year). Only factor of production.
- Produce same, single, non-storable good: GDP
- Productivity of each is a random variable, $Z$
- Entire distribution of productivities is state variable of economy

- Convenient to work with "cost" rather than productivity

- Let $Z = z^{-\theta}, \;\; \theta \in (0,1)$, where $z \sim$ density $f$, cdf $F$

- Think: smaller $z$ is better

- Per capita GDP is

$$y\,(t) = \int_0^\infty z^{-\theta} f(z,t)dz$$

- How do individual productivities evolve?

- Every agent has some cost $z$ at date $t$: a draw from $F(\cdot, t)$

- Over $(t, t+h)$ everyone gets one new draw from same cdf $F$ with probability $\alpha h$

- If lower cost, he switches; if draw $z' > z$ or if no draw, stays with $z$

- Assumptions imply law of motion of state variable $F$:
$$\frac{\partial F(z,t)}{\partial t} = \alpha F(z,t)\left[1 - F(z,t)\right]$$

- Treat as ODE for each fixed $z$. $F(z,0)$ is given. Solve for
$$F(z,t) = \frac{F(z,0)}{F(z,0) + \left[1 - F(z,0)\right] e^{-\alpha t}}$$

- Define a balanced growth path—BGP—a cdf $\Phi$ such that

$$F(z,t) = \Phi(ze^{\alpha t}) \quad \text{for all } t.$$

- Can show there is a one-parameter family of BGP densities:

$$\phi(x) = \frac{\lambda}{(1+\lambda x)^2}$$

- If $\theta \in (0,1)$, model gives us a theory of gdp growth:

$$y(t) = \lambda^{\theta} e^{\alpha\theta t} \int_0^{\infty} z^{-\theta} \frac{1}{(1+z)^2} dz$$

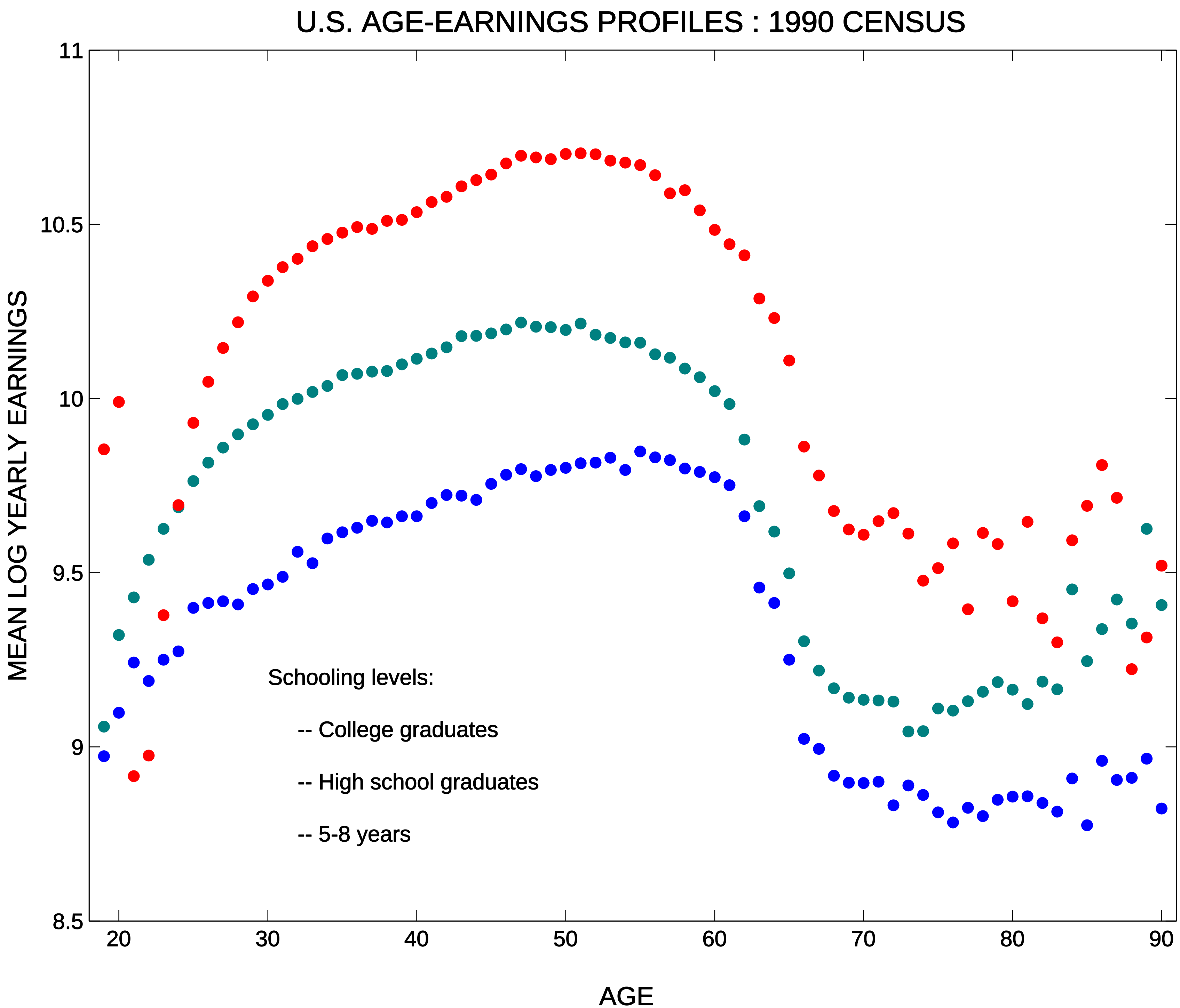
- Also a theory of income distribution: Lorenz curves to come

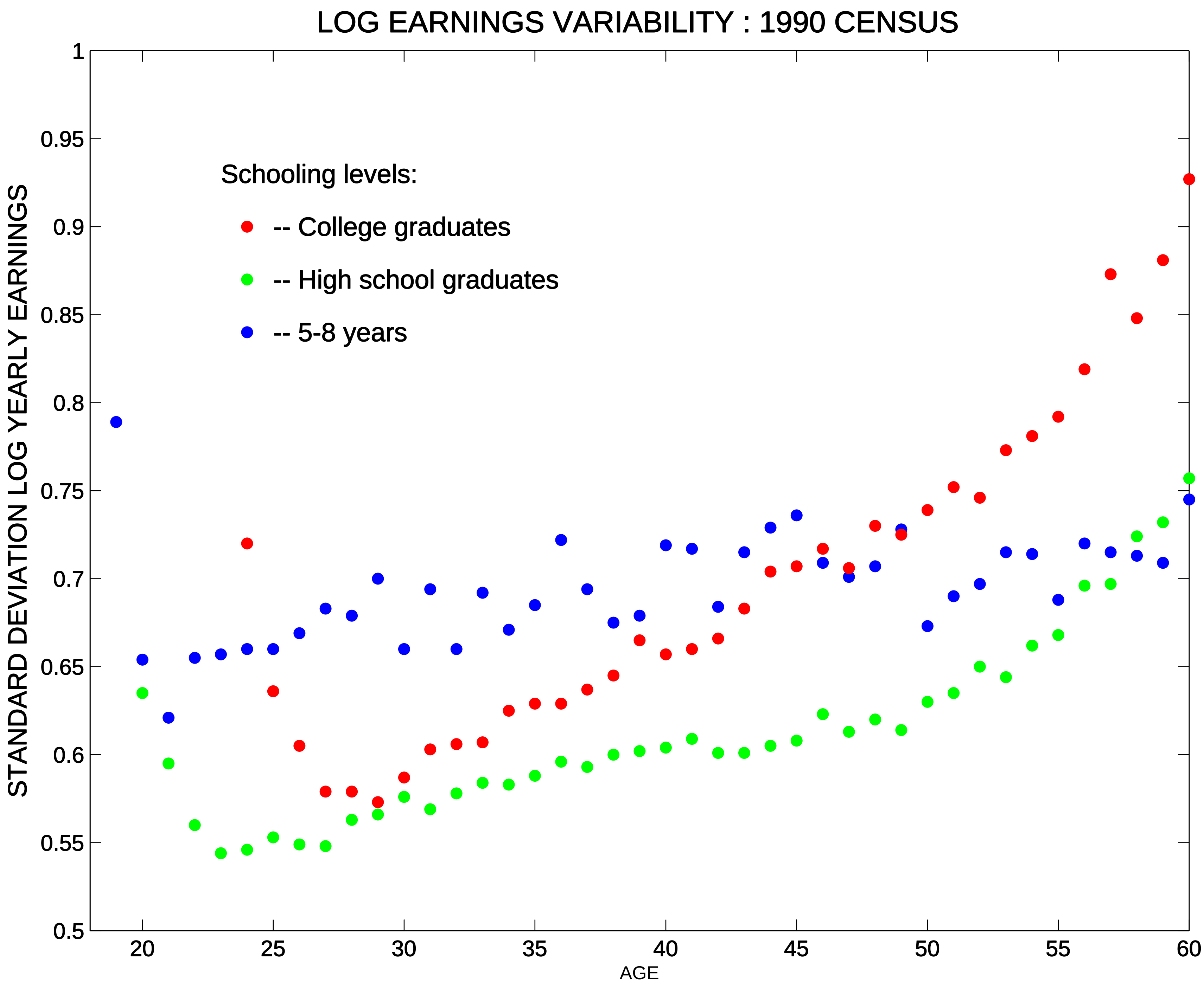
- Objects in model have natural counterparts in data

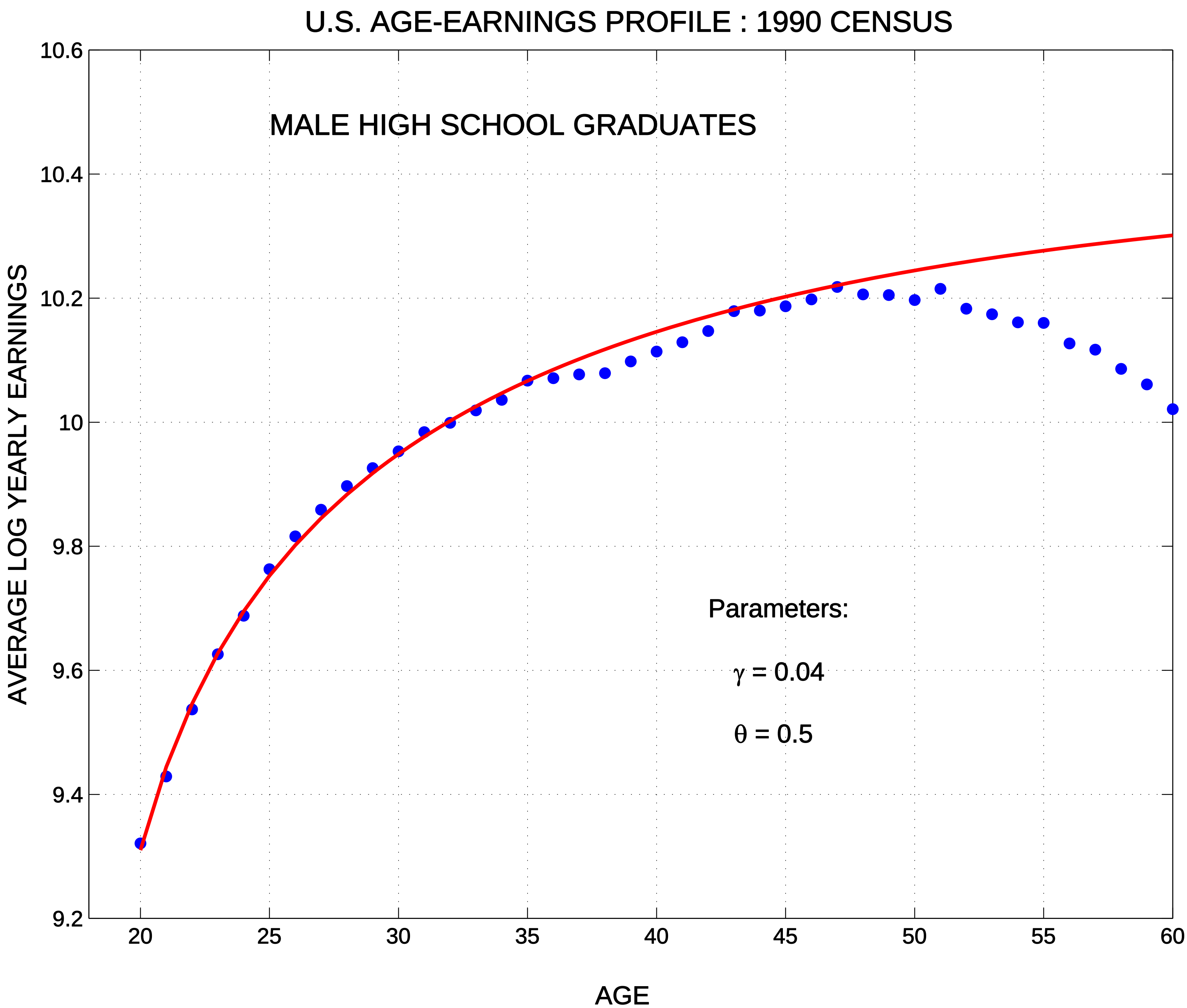
- Can use aggregate growth rate to estimate $\alpha\theta$ and observations on variance of individual earnings to estimate $\theta$

- See Lucas, *Economica*, 2009

- This paper a purely theoretical inquiry: attempt to see how matching/learning technology just described interacts with an individual control problem


U.S. AGE-EARNINGS PROFILES : 1990 CENSUS
MEAN LOG YEARLY EARNINGS
AGE
11
10.5
10
9.5
9
8.5
20
30
40
50
60
70
80
90
Schooling levels:
-- College graduates
-- High school graduates
-- 5-8 years


LOG EARNINGS VARIABILITY : 1990 CENSUS
Schooling levels:
-- College graduates
-- High school graduates
-- 5-8 years
STANDARD DEVIATION LOG YEARLY EARNINGS
AGE
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
20
25
30
35
40
45
50
55
60


U.S. AGE-EARNINGS PROFILE : 1990 CENSUS
MALE HIGH SCHOOL GRADUATES
Parameters:
γ = 0.04
θ = 0.5
AVERAGE LOG YEARLY EARNINGS
AGE
10.6
10.4
10.2
10
9.8
9.6
9.4
9.2
20
25
30
35
40
45
50
55
60

## 2 – Decentralized Time Allocation Decisions

- Now we drop the assumption that new ideas are just a by-product

- Replace it with the assumptions that

  – ...everyone divides his time into $1-s$ spent producing goods, so

$$\text{earnings} = (1-s)z^{-\theta}$$

  – ...and $s$ spent seeking new ideas, with Poisson arrival rate a function of $s$ :

$$\alpha(s) = ks^{\eta}, \quad \eta \in (0,1)$$

- This change requires us to specify
  - the way agents determine their time allocations $s$, given the distributions $F(z,t)$
  - a new law of motion for $F(z,t)$, given agents' policy functions are $s(z,t)$
- We assume that agents maximize expected PV of earnings, discounted at given $\rho > 0$
- No risk aversion, no theory of real interest rates

- Individual preferences are

$$V(z,t) = \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\tau - t)} \left[1 - s(\tilde{z}(\tau), \tau)\right] \tilde{z}(\tau)^{-\theta} d\tau \middle| z(t) = z \right\}$$

- Associated Bellman equation is

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} =$$

$$\max_{s \in [0,1]} \left\{ (1-s) z^{-\theta} + \alpha(s) \int_0^z [V(y,t) - V(z,t)] f(y,t) dy \right\}$$

- Description of problem solved by atomistic agent taking environment $f$ as given

- Next, formulate law of motion for density $f$

$$\frac{\partial f(z,t)}{\partial t} = \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} + \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}}$$

- Consider first the outflow. The $f(z,t)$ agents at $z$ have meetings at the rate $\alpha(s(z,t))f(z,t)$

- A fraction $F(z,t)$ of these draws satisfy $y < z$ and lead agents to leave $z$. Hence

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} = -\alpha(s(z,t))F(z,t)f(z,t)$$

- Next, consider inflow. Agents with cost $y \geq z$ have meetings at the rate $\alpha(s(y,t))f(y,t)$

- Each of these meetings yields a draw $z$ with probability $f(z,t)$. Hence

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}} = f(z,t) \int_z^\infty \alpha(s(y,t))f(y,t)dy$$

- Combining ins and outs we have law of motion

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t))F(z,t)f(z,t) + f(z,t) \int_z^\infty \alpha(s(y,t))f(y,t)dy$$

- A Boltzmann equation! Sum up on next slide

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} =$$

$$\max_{s\in[0,1]} \left\{(1-s)z^{-\theta} + \alpha(s)\int_0^z [V(y,t)-V(z,t)]f(y,t)dy\right\} \quad \text{(BE)}$$

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t))F(z,t)f(z,t) + f(z,t)\int_z^\infty \alpha(s(y,t))f(y,t)dy \quad \text{(LM)}$$

**Definition**: An *equilibrium,* given the initial distribution $f(z,0)$, is a triple $(f,s,V)$ of functions on $\mathrm{R}_+^2$ that such that (LM), (BE) are satisfied

- A "mean field game" (Lasry and Lions (2007)) See also Perla and Tonetti (2011).

**Definition**: A *balanced growth path (BGP)* is a number $\gamma$ and a triple of functions $(\phi, \sigma, v)$ on $\mathbf{R}_+$ such that

$$f(z,t) = e^{\gamma t}\phi(ze^{\gamma t}),$$

$$V(z,t) = e^{\theta\gamma t}v(ze^{\gamma t}),$$

$$s(z,t) = \sigma(ze^{\gamma t})$$

and $(f, s, V)$ is an equilibrium.

- On a BGP output growth is $\theta\gamma$
- Lorenz curves are constant

- For a BGP with $\gamma > 0$ to exist, need an assumption to ensure that the stock of good ideas waiting to be discovered is inexhaustible

- (With lower bound $z_0 > 0$ to cost, probability mass would pile up at $z_0$, return to search would go to 0, growth would cease)

- Throughout paper we assume that $f(0,0) = \phi(0) > 0$

- Model does not involve distinctions between learning/discovery or imitation/innovation

- A serious limitation?

- Our view expressed well by following passage:

No sharp line divides learning things that are already known to others from learning things that are new to the world. What constitutes novelty depends on what knowledge is already in the mind of the problem solver and what help is received from the environment in adding to this knowledge. We should expect therefore that processes very similar to those employed in learning systems can be used to construct systems that discover new knowledge.

Herbert A. Simon, *The Sciences of the Artificial*, Third Edition. p. 105.

- Restate (BE), (LM) for BGP only. Use $x = ze^{\gamma t}$ : relative cost

$$(\rho - \theta\gamma)\, v(x) - v'(x)\gamma x =$$

$$\max_{\sigma \in [0,1]} \left\{ (1-\sigma)x^{-\theta} + \alpha(\sigma) \int_0^x [v(y) - v(x)]\phi\,(y)\, dy \right\}$$

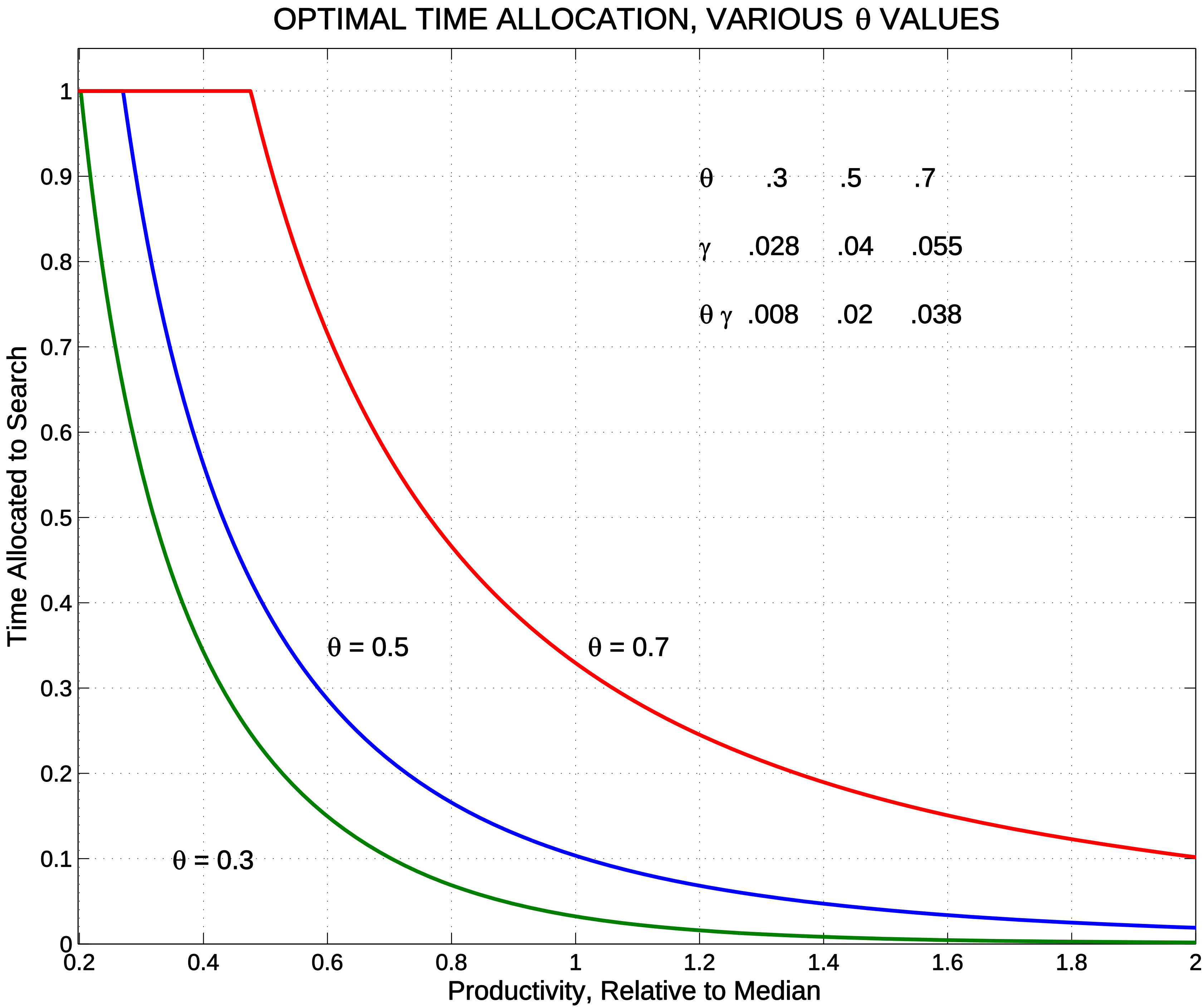
$$\phi(x)\gamma + \phi'(x)\gamma x = \phi(x) \int_x^\infty \alpha(\sigma(y))\phi(y)dy - \alpha(\sigma(x))\phi(x) \int_0^x \phi(y)dy.$$

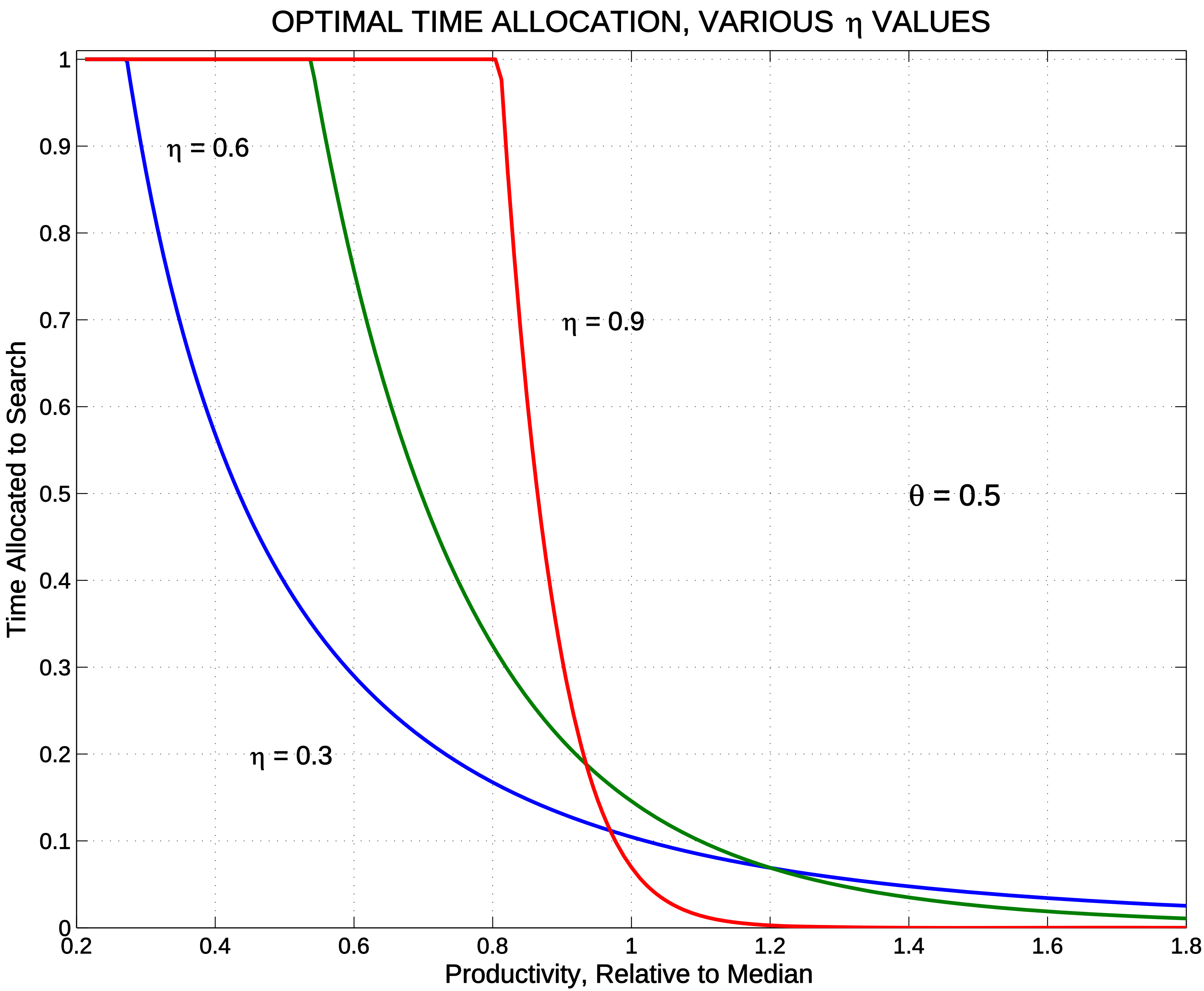
$$\phi(0)\gamma = \phi(0) \int_0^\infty \alpha(\sigma(y))\phi(y)dy.$$

$$\gamma = \int_0^\infty \alpha(\sigma(x))\phi(x)dx$$

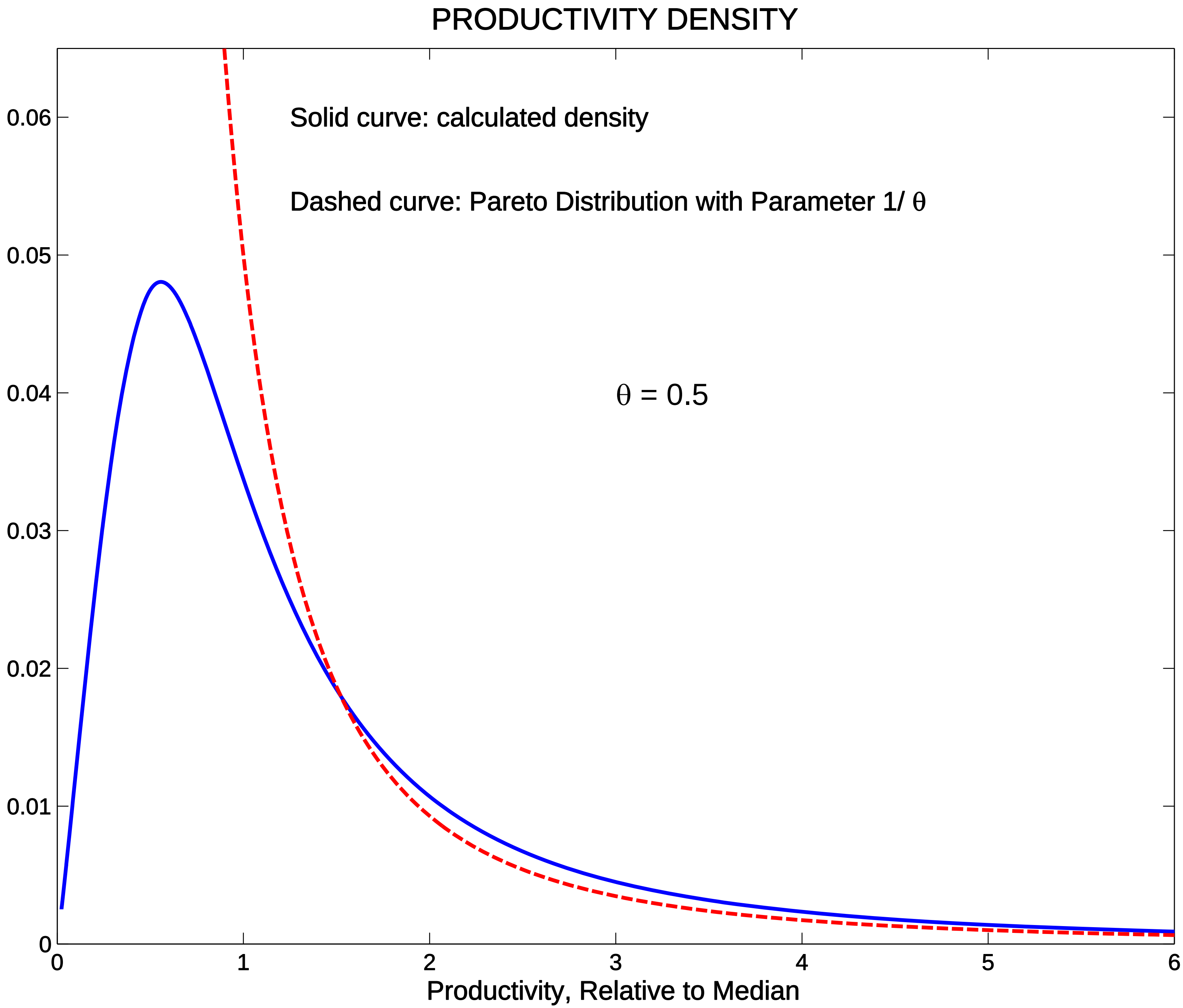
- Talk through algorithm: $(\phi_n, \gamma_n) \rightarrow (v_n, \sigma_n) \rightarrow \left(\phi_{n+1}, \gamma_{n+1}\right)$

- Next slides present results from simulations of BGP
  - Initial cost distribution is exponential: $\phi_0(x) = \lambda e^{-\lambda x}$
  - Function $\alpha(\sigma)$ given by $\alpha(\sigma) = k\sigma^\eta$
- Baseline parameter values:
  - $\theta = 0.5$ (variations 0.3 and 0.7)
  - $\eta = 0.3$ (variations 0.6 and 0.9)
  - $\gamma = .04$ ($k$ adjusted to get growth rate $\theta\gamma = .02$)
  - $\rho = .06$

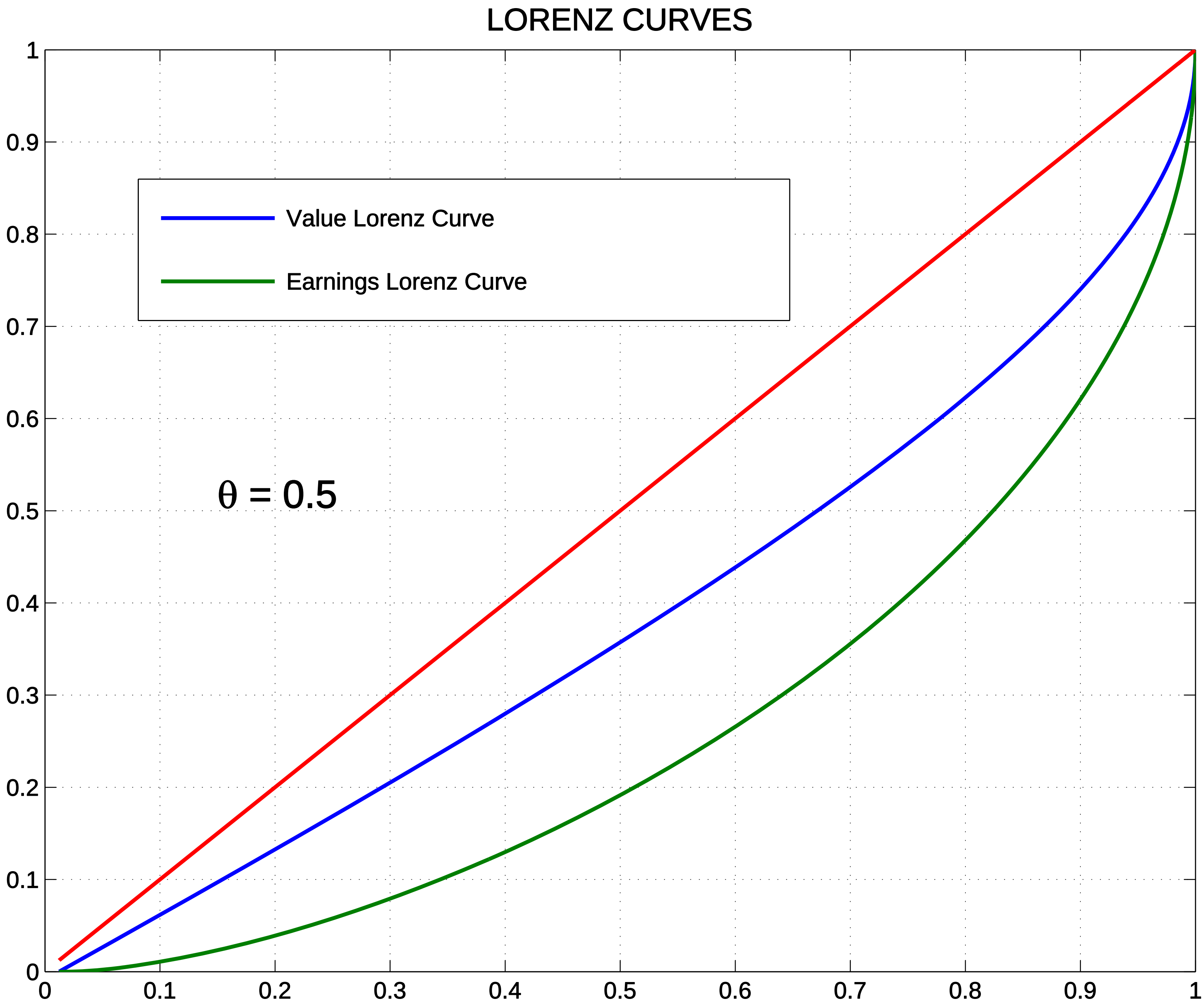

OPTIMAL TIME ALLOCATION, VARIOUS θ VALUES
θ .3 .5 .7
γ .028 .04 .055
θ γ .008 .02 .038
θ = 0.3
θ = 0.5
θ = 0.7
Time Allocated to Search
Productivity, Relative to Median
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



OPTIMAL TIME ALLOCATION, VARIOUS η VALUES
Time Allocated to Search
Productivity, Relative to Median
η = 0.6
η = 0.9
η = 0.3
θ = 0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8



PRODUCTIVITY DENSITY
Solid curve: calculated density
Dashed curve: Pareto Distribution with Parameter 1/ θ
θ = 0.5
0.06
0.05
0.04
0.03
0.02
0.01
0
0
1
2
3
4
5
6
Productivity, Relative to Median


LORENZ CURVES
Value Lorenz Curve
Earnings Lorenz Curve
θ = 0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

## 3 – An optimally planned economy

- Knowledge acquisition—search—in this economy has an external effect

- Decentralized allocation will not be efficient

- A central feature of any model of technological change

- Think this is why successful societies subsidize schooling, create patent- and copyright-protected monopoly rents, subsidize basic research

- Can we work out the economically efficient time allocation? Implement it with taxes/subsidies?

- Formulate planning problem

$$W[f(z,t)] = \max_{s(\cdot,\cdot)} \int_t^{\infty} e^{-\rho(\tau-t)} \int_0^{\infty} [1 - s(z,\tau)]\, z^{-\theta} f(z,\tau) dz d\tau$$

subject to the law of motion for $f$ :

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t)) f(z,t) \int_0^z f(y,t)dy + f(z,t) \int_z^{\infty} \alpha(s(y,t)) f(y,t) dy.$$

and with $f(z,t)$ given.

- Here $W$ maps a set $\mathbf{S}$ of density functions into $\mathbb{R}$

- Can define directional derivative of $W$,

$$\tilde{w}(z, f) = \frac{\delta W(f)}{\delta f(z)}$$

  on $\mathbb{R} \times \mathbf{S}$. See Appendix.

- Now define

$$w(z, t) = \tilde{w}(z, f(z, t))$$

  on $\mathbb{R}^2$: derivative valued along an optimal trajectory

- Idea familar: If policy is optimal, it can't be improved by telling any single individual to deviate at any time

- Now write a Bellman equation for the marginal value $w(z,t)$ to the planner of a single type $z$ agent at date $t$

$$\rho w(z,t) - \frac{\partial w(z,t)}{\partial t} =$$

$$\max_{s \in (0,1)} \; (1-s)\, z^{-\theta} + \alpha(s) \int_0^z \left[w(y,t) - w(z,t)\right] f(y,t)dy$$

$$- \int_z^\infty \alpha(s\,(y,t)) \left[w(y,t) - w(z,t)\right] f(y,t)dy$$

- Have reduced planner's problem from infinite-dimensional to two dimensional problem

- Last term in maximand is new: expected value from improvements in the cost of other types $y > z$ to $z$ in case they should meet $z$.

- Planner values this external benefit; agent $z$ himself does not

- First order condition is

$$z^{-\theta} = \alpha'(s(z,t)) \int_0^z [w(y,t) - w(z,t)] f(y,t) dy$$

- Why not third term? Because changing $s(z,t)$ has no *direct* effect on the distribution at $y > z$ which only depends on the search intensities $s(y,t)$ of those with costs $y > z$

- As in decentralized problem, can restate the equations in terms of relative productivities

$$f(z,t) = e^{\gamma t}\phi(ze^{\gamma t}), \quad w(z,t) = e^{\theta\gamma t}\omega(ze^{\gamma t})$$

- Letting $x = ze^{\gamma t}$ we obtain a BGP Bellman equation

$$(\rho - \theta\gamma)\,\omega(x) - \omega'(x)\gamma x =$$

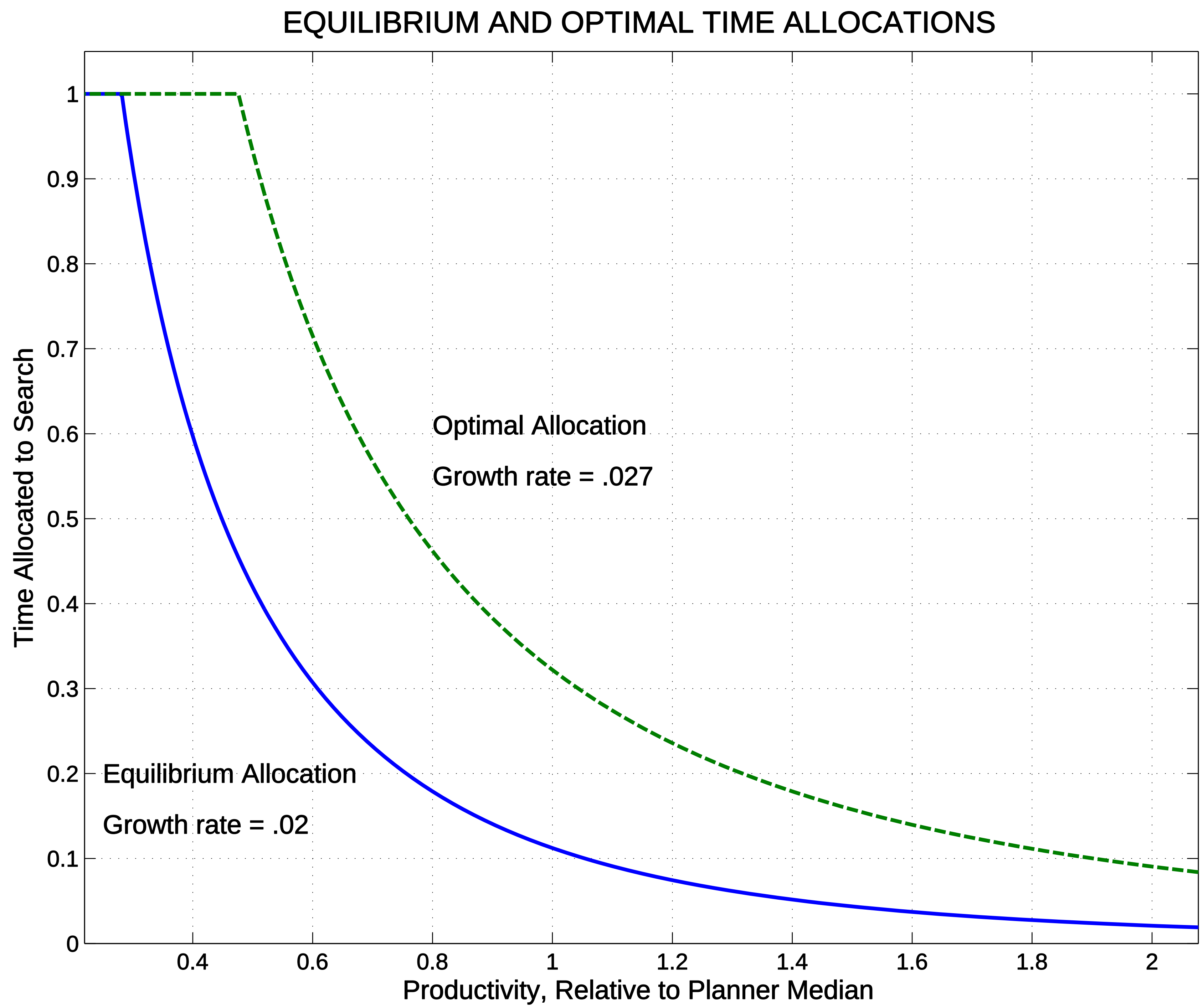
$$\max_{\sigma\in[0,1]} \left\{(1-\sigma)\,x^{-\theta} + \alpha\,(\sigma)\int_0^x [\omega(y) - \omega(x)]\,\phi(y)dy\right\}$$

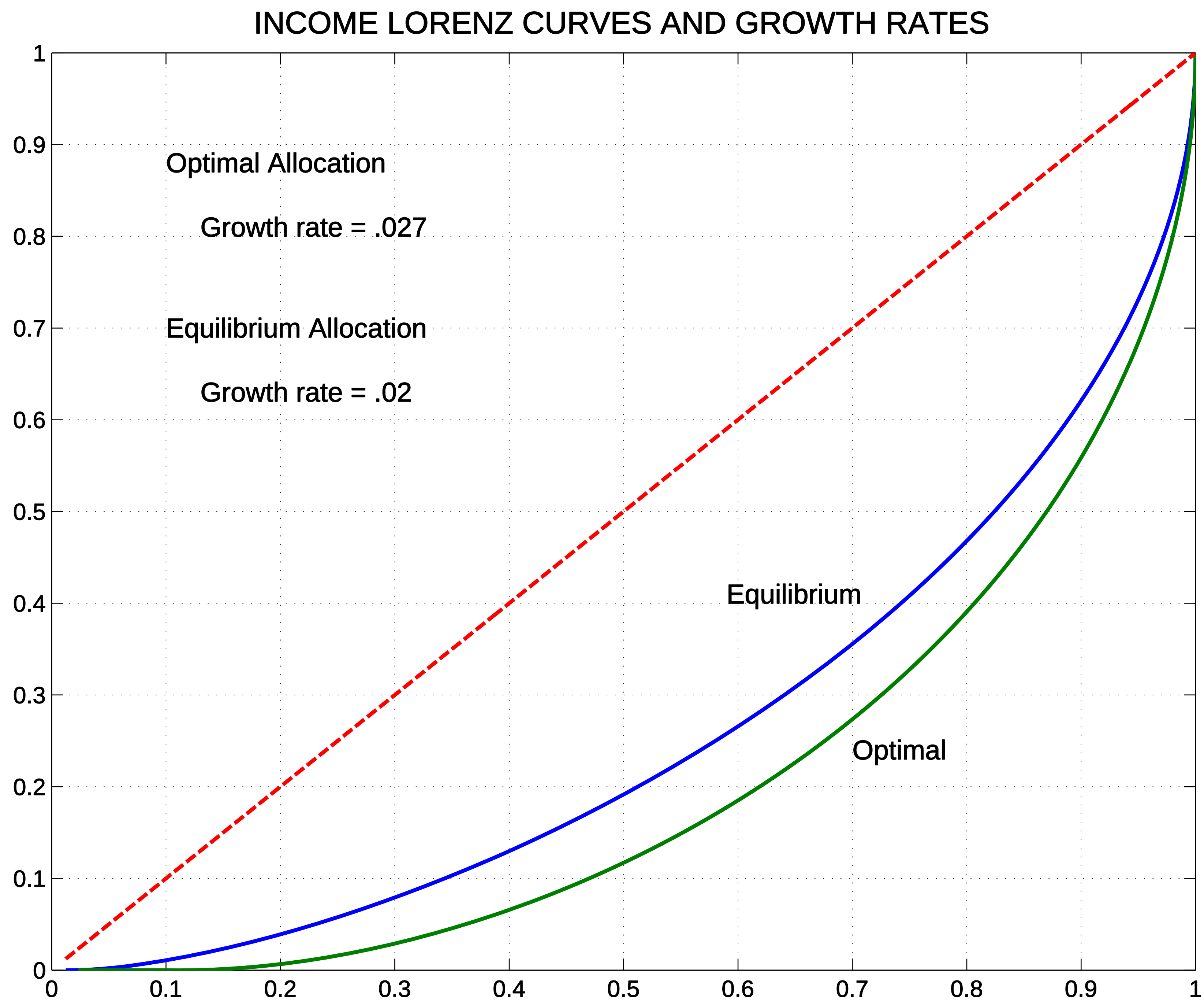
$$-\int_x^\infty \alpha\,(\sigma(y))\,[\omega(y) - \omega(x)]\,\phi(y)dy$$

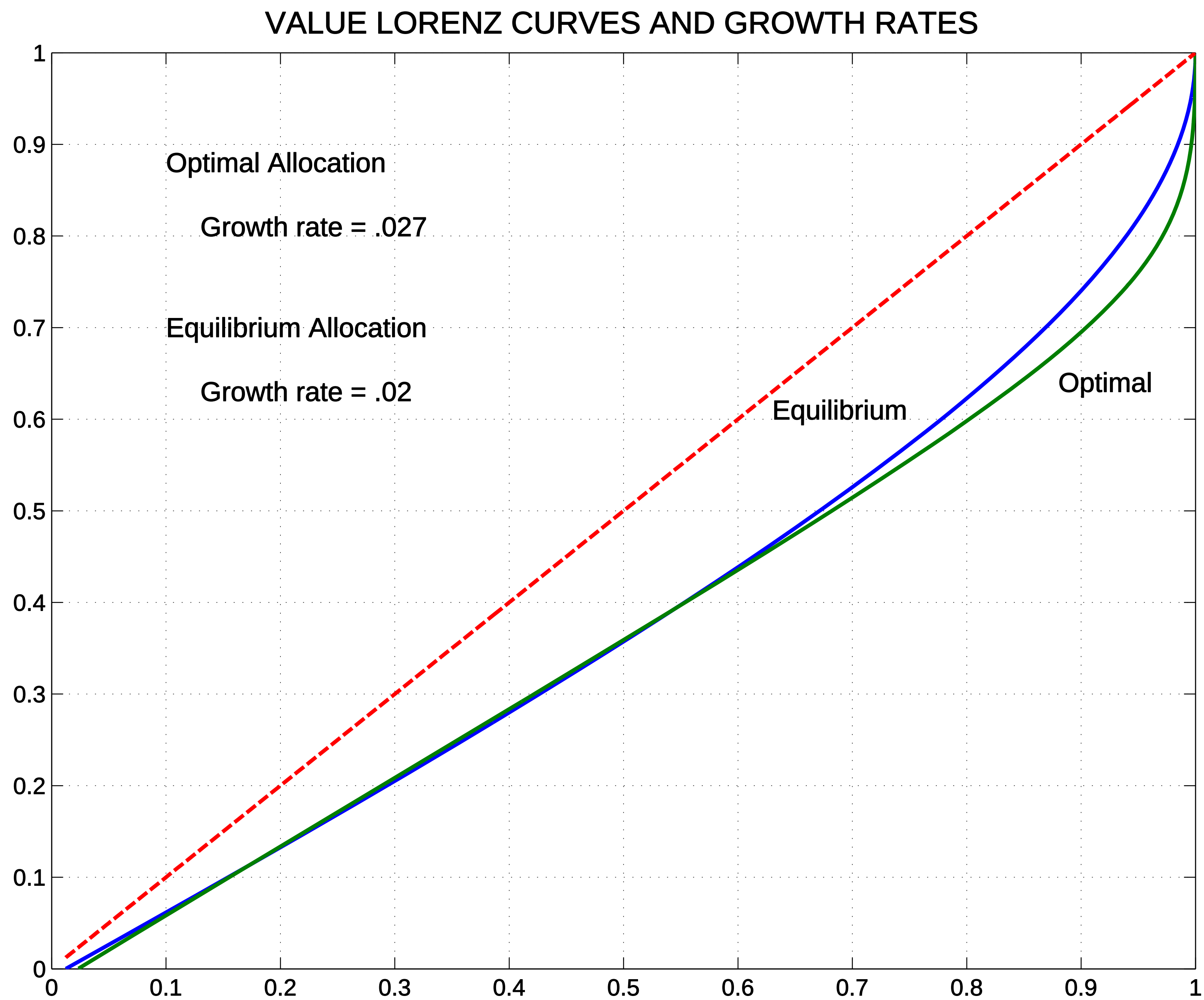
- Law of motion for density $\phi$ and equation for $\gamma$ same as in decentralized equilibrium

- Algorithmic strategy same as in decentralized case, except that (BE) is different

- Next slides compare features of BGP of optimal allocation to the BGP of the decentralized equilibrium studied earlier


EQUILIBRIUM AND OPTIMAL TIME ALLOCATIONS
Time Allocated to Search
Productivity, Relative to Planner Median
Optimal Allocation
Growth rate = .027
Equilibrium Allocation
Growth rate = .02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


INCOME LORENZ CURVES AND GROWTH RATES
Optimal Allocation
Growth rate = .027
Equilibrium Allocation
Growth rate = .02
Equilibrium
Optimal
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


VALUE LORENZ CURVES AND GROWTH RATES
Optimal Allocation
Growth rate = .027
Equilibrium Allocation
Growth rate = .02
Equilibrium
Optimal
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

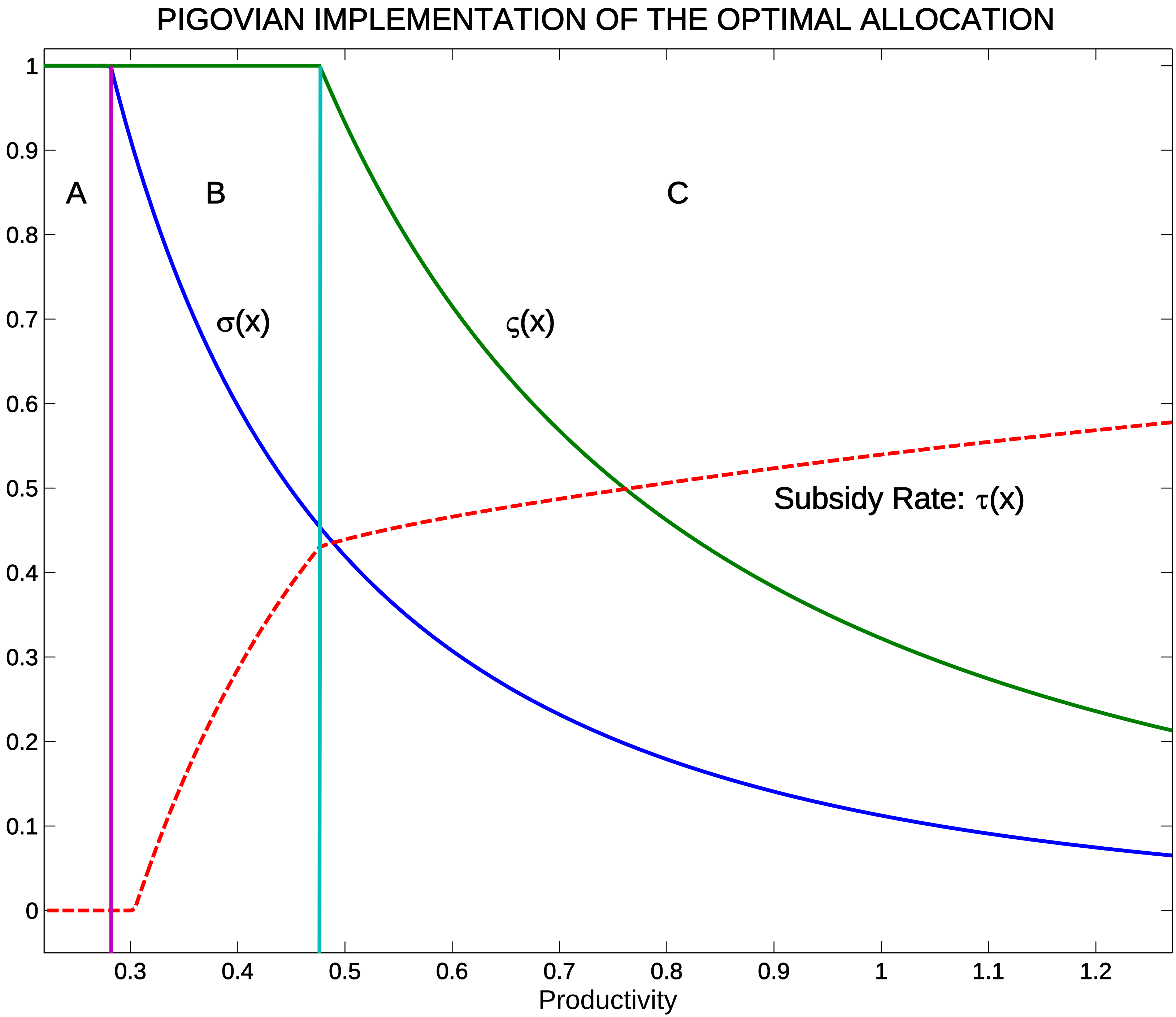
- Seek Pigovian tax structure that implements optimal allocation by setting taxes that equate, on the margin, private and social returns to work and search

- Simple one is direct subsidy to search, yielding individual (BE):

$$(\rho - \theta\gamma)\, v_n(x) - v_n'(x)\gamma x =$$

$$\max_{\sigma \in [0,1]} \left\{ (1-\sigma)x^{-\theta} + \tau(x)x^{-\theta}\sigma + \alpha(\sigma) \int_0^x [v_n(y) - v_n(x)]\phi\,(y)\, dy \right\}$$

- Includes neutral flat tax to balance budget: Multiplies both sides by a constant


PIGOVIAN IMPLEMENTATION OF THE OPTIMAL ALLOCATION
A
B
C
σ(x)
ς(x)
Subsidy Rate: τ(x)
Productivity
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2

## 4 – Other learning technologies

- Learning technology described above involves
  - probabilistic model of agents' meetings
  - description of effects of meetings on agents' knowledge
- Easy to think of modifications, other Boltzmann equations
- Algorithm very flexible. Beginning to explore possible variations

- Consider imposing an order to learning, limits on intellectual range
  - If $y$ meets $z < y$ at $t$, he can adopt $z$ with given probability $K(|y - z|, t)$
  - w/prob. $1 - K(|y - z|, t)$ he cannot do this; retains cost $y$
- Equivalently, think of meeting probabilities as depending on $|y - z|$ : stratification or segregation
- Looking ahead to BGP, want $K$ to depend on **quantile** differences: $\left|ye^{\gamma t} - ze^{\gamma t}\right|$

- We tried $K(y,z,t) = \exp(-\kappa \left| ye^{\gamma t} - ze^{\gamma t} \right|)$.

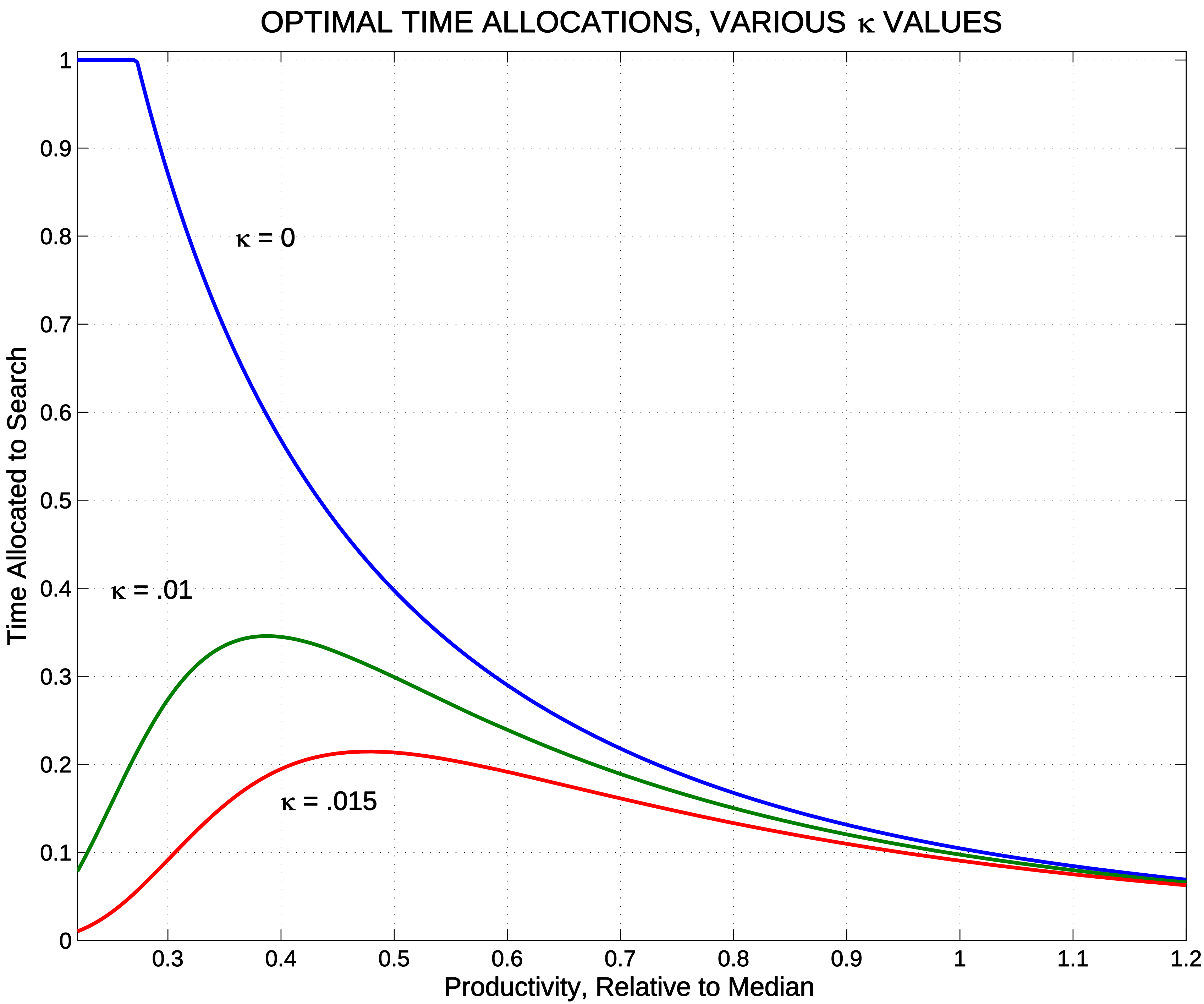
- In this case, BGP equations become

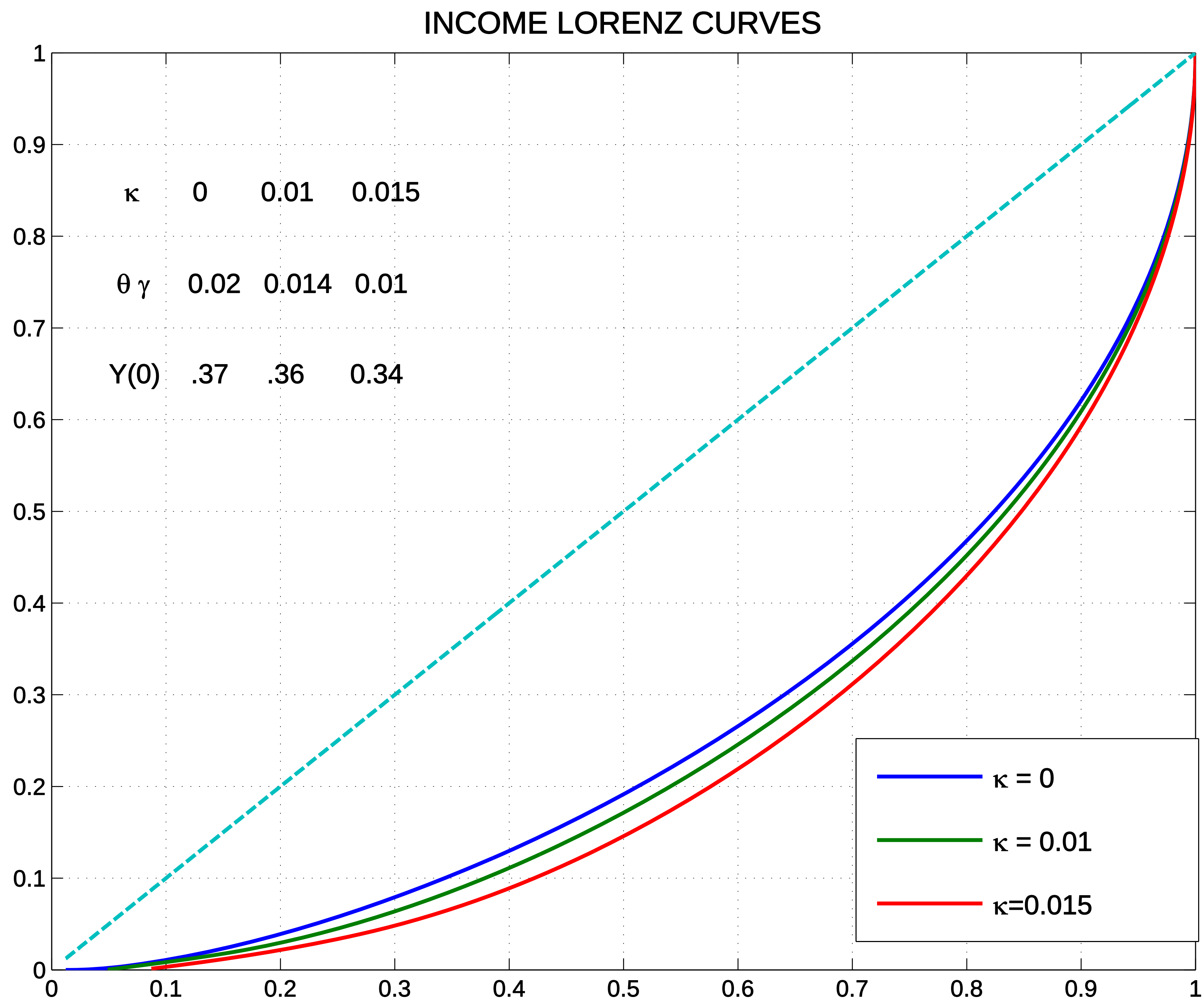
$$(\rho - \theta\gamma)\, v(x) - v'(x)\gamma x =$$

$$\max_{\sigma\in[0,1]} \left\{ (1-\sigma)x^{-\theta} + \alpha(\sigma) \int_0^x [v(y) - v(x)]\phi\,(y)\, e^{-\kappa(x-y)} dy \right\}$$

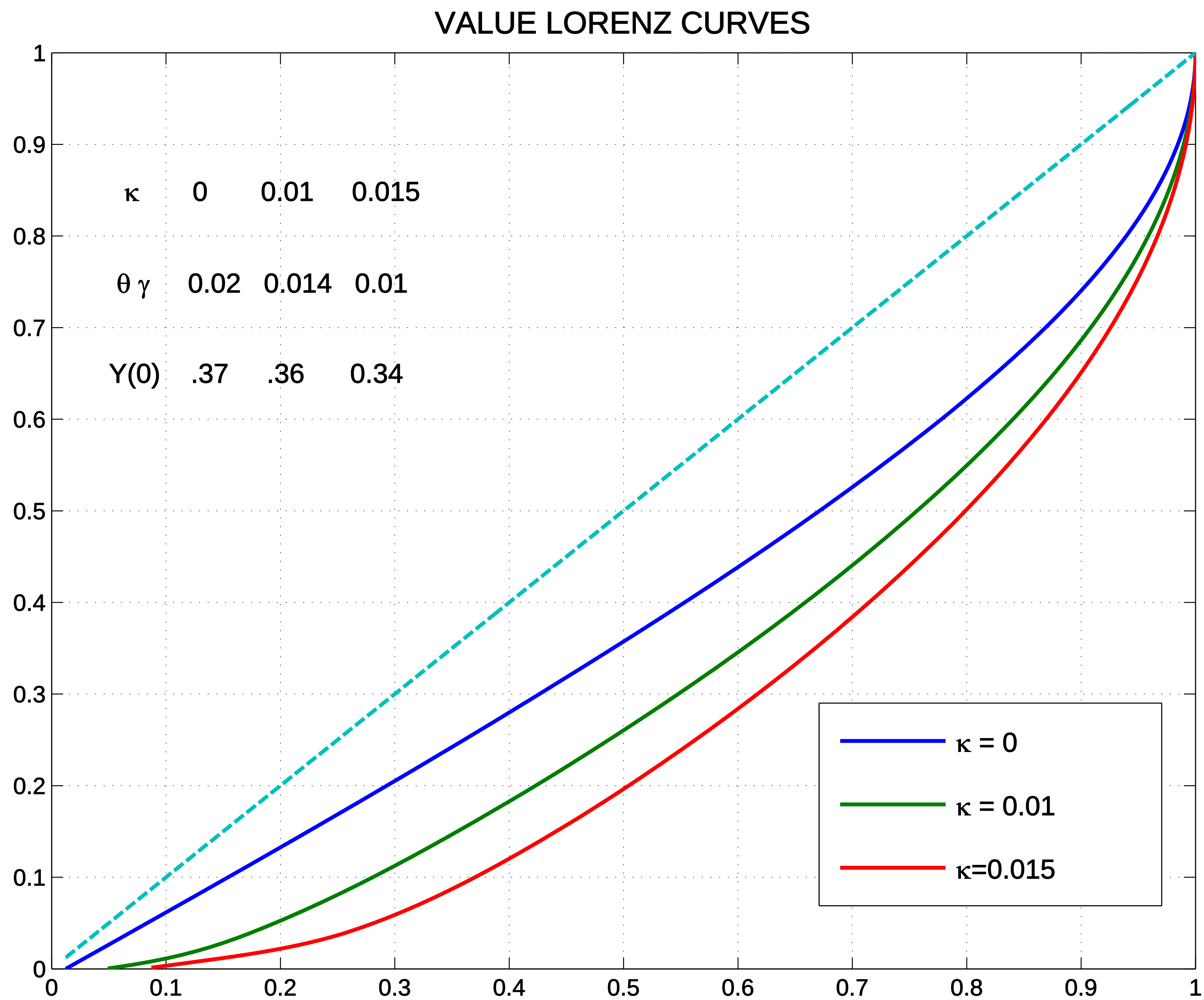
$$\begin{aligned} \phi(x)\gamma + \phi'(x)\gamma x \;&=\; \phi(x) \int_x^\infty \alpha(\sigma(y))\phi(y)e^{-\kappa(x-y)} dy \\ &\quad -\alpha(\sigma(x))\phi(x) \int_0^x \phi(y)e^{-\kappa(x-y)} dy. \end{aligned}$$

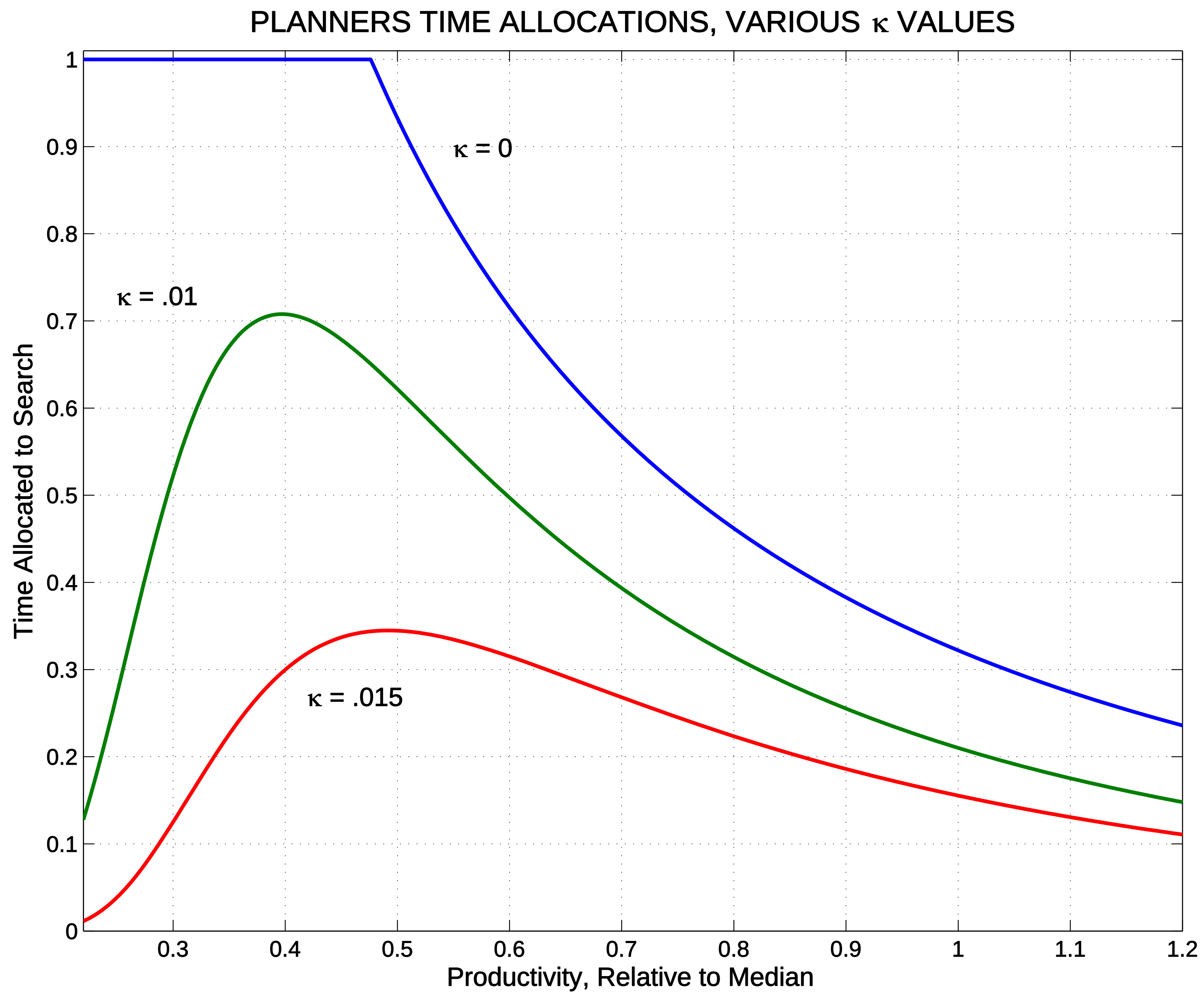
$$\gamma = \int_0^\infty \alpha(\sigma(x))\phi(x)e^{-\kappa x} dx$$


OPTIMAL TIME ALLOCATIONS, VARIOUS κ VALUES
Time Allocated to Search
Productivity, Relative to Median
κ = 0
κ = .01
κ = .015
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2



INCOME LORENZ CURVES
κ 0 0.01 0.015
θ γ 0.02 0.014 0.01
Y(0) .37 .36 0.34
κ = 0
κ = 0.01
κ=0.015
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


VALUE LORENZ CURVES
κ 0 0.01 0.015
θ γ 0.02 0.014 0.01
Y(0) .37 .36 0.34
κ = 0
κ = 0.01
κ=0.015
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


PLANNERS TIME ALLOCATIONS, VARIOUS κ VALUES
Time Allocated to Search
Productivity, Relative to Median
κ = 0
κ = .01
κ = .015
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2