Equilibrium selection in a fundamental model of money*

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June 2011

Abstract

We introduce a small modification to a fundamental model of money in order to select among equilibria. It turns out that intrinsic and extrinsic properties of money have implications for equilibrium selection that are absent from other settings where coordination matters. In particular, the time discount factor matters not only for determining whether money is better than autarky, but also to pin down the conditions under which money is the unique equilibrium. As the time discount factor approaches one, the economy tends to the efficient outcome.

JEL Codes: E40, D83

Keywords: Money, search, equilibrium selection.

"For the importance of money essentially flows from its being a link between the present and the future."

Keynes, The General Theory of Employment, Interest and Money (1936)

1 Introduction

Fundamental models of money (e.g., search, overlapping generations, turnpike) always exhibit equilibria where money has no value. Such equilibria might suggest that valued money is a tenuous phenomenon, for depending on a particular coordination of beliefs. In practice, though, the existence of valued money is quite robust. Since ancient times, money is present in most economies, and even under conditions where the suppliers of money behave in quite erratic ways, people seem

*We thank Federico Bonetto, Gabriele Camera, Raoul Minetti, Stephen Morris, Randy Wright, Tao Zhu and seminar participants at ESSIM 2011, Michigan State, Rochester, SAET Meeting 2011 and São Paulo School of Economics-FGV for their comments and suggestions.

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to always coordinate on accepting money in exchange for goods. This suggests that the models are missing something, but what is it?

In this article, we argue that what is missing in those models is a better understanding of the distinctiveness of the coordination problem that involves the use of money. Two features of money are particularly important in this respect. First, money is a durable asset. Thus, even if money is not used in the near future because agents coordinate on not accepting it, agents might eventually coordinate on its use. Second, money is a medium of exchange. Hence the benefit of accepting money today depends not on whether agents coordinate on the use of money today, but on whether agents coordinate on the use of money in the future. Moreover, the decision to accept money is not made once but many times, thus the sooner the agent accepts and uses money, the sooner he will be able to accept and use it again.

It turns out that the combination of these elements creates additional incentives to coordinate on the use of money that are not present in standard coordination problems. To make this point, we apply techniques from the literature on equilibrium selection in coordination games to a monetary model. The literature on global games (Carlsson and Van Damme (1993), Morris and Shin (2000, 2003), Frankel, Morris and Pauzner (2003)) shows that the multiplicity of equilibria disappears once the information structure of the game is slightly perturbed. A related argument applies to dynamic games with complete information where a state variable is subject to shocks (Frankel and Pauzner (2000), Burdzy, Frankel, and Pauzner (2001)). The key elements in those papers are: the existence of strategic complementarities; the existence of dominant regions where one action is strictly dominant; and a friction (asymmetric information in global games or staggered moves in dynamic models).

We consider a monetary model that incorporates these features. Precisely, we cast our analysis in a standard search model of money along the lines of Kiyotaki and Wright (1993) (KW). Strategic complementarities are present in KW (and in any monetary model, since the value of money intrinsically relies on coordination). We introduce dominant regions by assuming that the economy experiences different states according to a random walk and might reach remote areas where either accepting money or not accepting money is a dominant strategy. No other friction is added.

The model considers an environment where agents have complete information. Hence our setup is quite distant from the global games literature and closer to Burdzy, Frankel, and Pauzner (2001) (BFP). The difference is that while BFP analyses an economy where agents meet randomly and repeatedly play a 2x2 coordination game, we consider a dynamic monetary model.

Our main results are as follows. First, as in the literature on equilibrium selection, there is a unique equilibrium. Money is always accepted if there are enough gains from trade, and is never accepted if gains from trade are small. However, there is a subtle difference in the assumptions leading to equilibrium uniqueness: a crucial assumption in BFP is that each agent has only a small chance of changing his action in between matches, so that an agent may be locked into an action when he enters a match. This prevents agents from shifting from one action to another purely for coordination reasons and leads to a unique equilibrium. In contrast, in our environment, even though we get a unique equilibrium, agents are always free to choose their actions. The reason is that payoffs are determined by other players' future actions. Since the value of money today comes from its future use as a medium of exchange, an agent deciding about accepting money has to take into account whether other agents will accept money tomorrow, or at some point in the future, but not whether money will change hands today.

Our second, most interesting, result is that the region where money is the unique equilibrium expands with agents' discount factor. In general, the time discount factor influences an agent's decision about accepting money through two distinct channels. First, there is a "fundamental" channel. Any fundamental model of money exhibits a delay between production and consumption and an agent must be relatively patient if he is to incur the production cost. This channel is well-understood and determines whether a monetary equilibrium is better than autarky, but not if it will be selected. We unveil an additional channel, the "coordination" channel, which pins down the equilibrium selection. This channel operates through the extrinsic property of money as a medium of exchange and through its intrinsic property of durability. Durability implies that the agent can both delay the use of money earned today and defer the decision to accept money. The role of money as a medium of exchange implies that if an agent earns money today, he will be able to spend it sooner, and consequently further opportunities to accept and spend money will also come sooner. The larger the discount factor, the more important the role of money as a medium of exchange and the easier it is for agents to coordinate on its use.

As the time discount factor approaches one, the economy tends to the efficient outcome: if there are gains from trade, money is the unique equilibrium. That is in sharp contrast to BFP, where a larger discount factor does not help selecting the efficient outcome. In their model, equilibrium selection depends on history and risk-dominance considerations, and the time discount factor determines the relative importance of each of them. If the time discount factor is large enough, the risk-dominant equilibrium is selected regardless of whether it is efficient.

There is a strand of literature that studies how the addition of an intrinsic value to money may help to reduce the set of equilibria. In overlapping generations models, the focus is on the elimination of monetary equilibria that exhibit inflationary paths (Brock and Scheinkman (1980), Scheinkman (1980)). In search models of money, the objective is to characterize the set of fiat money equilibria that are limits of commodity-money equilibria when the intrinsic value of money converges to zero (Zhou (2003), Wallace and Zhu (2004), Zhu (2003, 2005)). A result that comes out of this literature is that, as long as goods are perfectly divisible and the marginal utility is large at zero consumption, autarky is not the limit of any commodity money equilibria. This result critically depends on the assumption that there is a sufficiently high probability that the economy reaches a state where fiat money acquires an intrinsic value. In contrast, our results hold even if the probability that money ever acquires an intrinsic value is arbitrarily small and even if the economy is eventually in states where money is not accepted. Finally, perhaps most importantly, this literature does not deal with the relation between standing properties of money (medium of exchange, durability) and the coordination involved in its use.

The paper is organized as follows. In section 2 we present the model and deliver our main result. Some examples are presented in section 3 and in section 4 we conclude.

2 Model

Our environment is a version of Kiyotaki and Wright (1993).[1] Time is discrete and indexed by t. There are k indivisible and perishable goods, and the economy is populated by a unit continuum of agents uniformly distributed across k types. A type i agent derives utility u per unit of consumption of good i and is able to produce good $i+1$ (modulo k) at a unit cost of c, with $u > c$. Agents maximize expected discounted utility with a discount factor $\beta \in (0,1)$. There is also a storable and indivisible object, which we denote as money. An agent can hold at most one unit of money at a time, and money is initially distributed to a measure m of agents.

Trade is decentralized and agents face frictions in the exchange process. We formalize this idea by assuming that there are k distinct sectors, each one specialized in the exchange of one good. In every period, agents choose which sector they want to join but inside each sector they are anonymously and pairwise matched under a uniform random matching technology. Each agent faces one meeting per period, and meetings are independent across agents and independent over

[1] Precisely, the environment is basically the same as the one in Araujo and Camargo (2006).

time. For instance, if an agent wants money he goes to the sector which trades the good he produces and searches for an agent with money. If he has money he goes to the sector that trades the good he likes and searches for an agent with the good. Due to the unit upper bound on money holdings, a transaction may happen only when an agent with money (buyer) meets an agent without money (seller).

We depart from the standard search model of money by assuming that, in any given period, the economy is in some state $z \in \mathbb{R}$. States evolve according to a random process $z_t = z_{t-1} + \Delta z_t$, where Δz_t follows a continuous probability distribution that is independent of t, with expected value $E(\Delta z)$ and variance $V(\Delta z)$. The fundamentals of the economy describing preferences (u and β) and technology (k and c) are invariant across states. However, there exists a state $\widehat{z} > 0$ such that accepting money is a strictly dominant strategy if and only if $z \geq \widehat{z}$, and a strictly dominated strategy if and only if $z \leq -\widehat{z}$. Throughout, we think of $\hat{z}$ as being finite but very large, and we are interested in describing how agents behave in the region $z \in (-\widehat{z}, \widehat{z})$.

2.1 Discussion

There are several ways in which one can motivate the existence of the dominant regions. One possibility runs as follows. Assume that exchange is only viable if there exists a special agent in the economy (say, the government) that provides a safe environment for trade. In states $z \leq -\widehat{z}$, there is no trade because the government does not exist. In states $z \geq \widehat{z}$, the government exists and it has a technology that enforces the use of money in all transactions. Finally, in states $z \in (-\widehat{z}, \widehat{z})$, the government exists but it has no technology that enforces the use of money.

A slightly different environment that would generate essentially the same results would be like this: even though money is completely fiat if $z \in (-\widehat{z}, \widehat{z})$, it may acquire a positive intrinsic value if $z \geq \widehat{z}$ and a negative intrinsic value if $z \leq -\widehat{z}$.[2]

Irrespective of the interpretation, the key implication of the existence of remote regions is that it imposes a condition on beliefs held by agents. In one direction, it rules out the belief that money is always going to be employed in all states of the world. In another direction, it rules out the

[2] That could be modelled as follows: in any state $z \geq \widehat{z}$, if an agent holds one unit of money at the beginning of the period, he can choose between keeping this unit throughout the period and obtaining a positive flow payoff γ; and bringing this unit into a trading post, in which there is no flow payoff but the unit can be used as a medium of exchange. In turn, in any state $z \leq -\widehat{z}$, if an agent holds one unit of money at the beginning of the period, he can choose between keeping this unit throughout the period, obtaining a flow payoff zero; and bringing this unit into a trading post, in which case he can use the unit as a medium of exchange but he obtains a negative flow payoff $-\xi$. We need $\frac{\beta}{1-\beta}\gamma > c$, to ensure that for large enough z, an agent always produces in exchange for money; and $\xi > u$ to ensure that, for small enough z, an agents never uses money as a medium of exchange.

belief that money is never going to be employed in any state of the world. In our view, equilibria that depend on such extreme beliefs are tenuous for relying on agents being sure about how they will coordinate on the use of money in all possible states at every point in time. As discussed by Morris and Shin (2000), in models with multiple equilibria, it is not at all clear why agents would be certain that everyone would always coordinate in a particular set of beliefs. Interestingly, the existence of remote but attainable regions has a very large impact on the equilibrium set.

As it will become clear in what follows, we can make the probability of ever reaching either of the remote regions arbitrarily small with virtually no effect in any of our results. The probability that $-\widehat{z}$ and $\widehat{z}$ will ever be reached depends on the stochastic process of Δz. If the expected value of Δz is zero (and its variance is positive), both $-\widehat{z}$ and $\widehat{z}$ will eventually be reached with probability one. If $E(\Delta z)$ is positive, $\widehat{z}$ will eventually be reached with probability 1 regardless of how far $z = 0$ is from $\widehat{z}$, but the probability that $-\widehat{z}$ will ever be reached depends on the distance between the initial state $z = 0$ and $-\widehat{z}$. Likewise, if $E(\Delta z)$ is negative, the probability that $\widehat{z}$ will ever be reached depends on how far $z = 0$ is from $\widehat{z}$. If this distance is large enough, the probability that $\widehat{z}$ is ever reached can be arbitrarily small.

2.2 Benchmark

We initially consider the problem of an agent when the initial state $z \in (-\widehat{z}, \widehat{z})$, $E(\Delta z) = 0$ and $V(\Delta z) = 0$. In this case, the economy never reaches a state where money has intrinsic value. First, there always exists an equilibrium where an agent does not accept money simply because he believes no other agent will ever accept money. In this case, the economy is in permanent autarky. Now, assume that an agent believes that all other agents always accept money. Let V_0 be his value function if he does not have money, and let V_1 be the corresponding value function if he has money. We have

$$V_1 = m\beta V_1 + (1-m)(u + \beta V_0),$$

and

$$V_0 = m[\sigma(-c + \beta V_1) + (1-\sigma)\beta V_0] + (1-m)\beta V_0,$$

where $\sigma \in [0, 1]$ is the probability that the agent accepts money. For example, if an agent has money he goes to the sector that trades the good he likes. In this sector, there is a probability m that he meets another agent with money and no trade happens. There is also a probability $(1-m)$ that he meets an agent without money. In this case they trade, the agent obtains utility u, and moves to the next period without money. A similar reasoning holds for an agent without money.

Assume that $\sigma = 1$. This implies that

$$V_1 - V_0 = (1-m)\,u + mc.$$

It is indeed optimal to always accept money as long as $-c + \beta V_1 \geq \beta V_0$, i.e.,

$$\beta\left[(1-m)\,u + mc\right] \geq c. \tag{1}$$

Since agents with money always find it optimal to spend it, as long as (1) holds, the economy exhibits multiple equilibria.[3]

2.3 General case

We now consider the case where $V(\Delta z) > 0$ and $z \in \mathbb{R}$. The economy starts at $z = 0$. Suppose the economy in period s is in state z^* and denote by $\varphi(t)$ the probability that any state $z \geq z^*$ will be reached at time $t + s$, and not before.[4] We are ready to present our first result.

Proposition 1 *There is a unique equilibrium. For all states $z \in (-\widehat{z}, \widehat{z})$, money is always accepted if*

$$\left(\sum_{t=1}^{\infty} \beta^t \varphi(t)\right)\left[(1-m)\,u + mc\right] > c, \tag{2}$$

and is never accepted if the inequality is reversed.

Proof. See Appendix. ■

In what follows we provide an informal proof of our result. By assumption, agents accept money in states $z \geq \widehat{z}$ and do not accept money in states $z \leq -\widehat{z}$. We are interested in the behavior of an agent in states $z \in (-\widehat{z}, \widehat{z})$.

It turns out that the equilibrium conditions depend solely on the choices of an agent in a hypothetical state $z^* \in (-\widehat{z}, \widehat{z})$ that divides the state space in two regions: everyone accepts money for all $z > z^*$ and nobody accepts money for all $z < z^*$. If an agent in this state z^* strictly prefers to accept money, then in the unique equilibrium of the model, money is always accepted (unless $z \leq -\widehat{z}$). Conversely, if an agent in this hypothetical state prefers not to accept money, then money is not accepted in the unique equilibrium of the model (unless $z \geq \widehat{z}$).

[3] Kiyotaki and Wright (1993) prove that there exists an equilibrium where agents accept money with probability between zero and one. A similar equilibrium also exists here.

[4] Note that $\varphi(t)$ is not a function of z^*.

The proof uses an induction argument, where at each step strictly dominated strategies are eliminated. The idea runs as follows. Suppose an agent in the hypothetical state z^* strictly prefers to accept money. Then, by continuity, there exists some ϵ such that the agent strictly prefers to accept money in state $z^* - \epsilon$. Moreover, incentives for an agent to accept money are increasing in the likelihood that others will accept it later. Thus incentives for accepting money in the hypothetical state z^* could only become stronger if agents were to accept money for some $z < z^*$. In consequence, once not-accepting money has been ruled out for all $z \geq z^*$, accepting money is a dominant strategy for an agent in any state $z > z^* - \epsilon$ that strictly prefers to accept money in the hypothetical state z^*. Note that this argument applies for any state $z^* \in (-\widehat{z}, \widehat{z})$ satisfying the property that money is accepted in any $z > z^*$ and not accepted in any $z < z^*$.

Now, by assumption, all agents accept money for any $z \geq \widehat{z}$. The above argument implies that if an agent at $\widehat{z}$ receives a positive expected payoff from accepting money, than not accepting money for all $z \geq \widehat{z} - \epsilon$ is a strictly dominated strategy, and can be eliminated. The argument can then be repeated assuming that all agents accept money for all $z \geq \widehat{z} - \epsilon$ and so on. Successive iterations of this argument lead to the conclusion that accepting money is the only strategy that survives iterative elimination of dominated strategies for all $z \in (-\widehat{z}, \widehat{z})$. An agent in state $-\widehat{z}$ would also like to accept money, but external conditions (say, a war) prevent money from circulating.

Conversely, suppose that an agent in the hypothetical state z^* prefers not to accept money. An analogous argument implies that he will also refuse to accept money at $z^* + \epsilon$ if nobody accepts money at $z \leq z^*$, regardless of their actions at $z > z^*$. Thus an iterative process of elimination of strictly dominated strategies starting from $-\widehat{z}$ rules out accepting money in all states $z \in (-\widehat{z}, \widehat{z})$.[5]

Now, what is the optimal choice of an agent in the hypothetical state $z^* \in (-\widehat{z}, \widehat{z})$ at time 0? There are two differences in payoffs of accepting money ($V^a_{z^*}$) and not accepting money ($V^n_{z^*}$): (i) an agent that accepts money at z^* pays the cost c; and (ii) as soon as the economy crosses the state z^*, an agent will spend or accept money depending on whether he had accepted money at time 0 (money is a durable good, agents can wait to spend it). Hence the difference between payoffs $V^a_{z^*}$

[5] In standard global games as described in Morris and Shin (2003), the equilibrium depends on the optimal choice of a player with a uniform belief over the proportion of his opponents choosing each action. That could be interpreted as the maximum degree of uncertainty about others' behavior. Here, equilibrium depends on the optimal choice of a player at a state that divides the state space in two regions: at the right agents accept money and at the left agents refuse to do so. In a dynamic game, where agents accept money if the state is above a certain threshold, that can also be seen as the maximum amount of uncertainty about others' actions.

and $V_{z^*}^n$ is given by

$$V_{z^*}^a - V_{z^*}^n = -c + \sum_{t=1}^{\infty} \beta^t \varphi(t) \left[\int_z (V_{1,z} - V_{0,z})\, dF(z|t) \right] \tag{3}$$

where the second term is the difference between the payoffs of having and not having money in state z $(V_{1,z} - V_{0,z})$ averaged across states z and then multiplied by the probability the economy will reach a state larger than z^* at time t (and not before) times the discount factor.

The value of having money in some state $z > z^*$ is given by

$$V_{1,z} = m\beta E_z V_1 + (1-m)\left(u + \beta E_z V_0\right),$$

where the first term is the value of having money in the next period if money cannot be spent today and the second term is the benefit of using money. The term $(1-m)\beta E_z V_0$ corresponds to the benefit of being able to accept and use money again, related to the fact that money is a medium of exchange and future opportunities of accepting and using money arise when money is spent.

The value function of an agent without money in some state $z < z^*$ is

$$V_{0,z} = m\left(-c + \beta E_z V_1\right) + (1-m)\beta E_z V_0,$$

where the first term is the cost paid by selling a good plus the expected benefit of using money in the future and the second term is the opportunity value of trading in the future. Hence, for any $z > z^*$

$$V_{1,z} - V_{0,z} = (1-m)u + mc, \tag{4}$$

All terms depending on β cancel out. That is only true because money can be accepted and spent many times. If money could only be accepted and spent once, the term $(1-m)\beta E_z V_0$ would be absent from the expression for $V_{1,z}$ and thus $V_{1,z} - V_{0,z}$ would depend negatively on β and results would be very different. Combining equations (3) and (4) yields the condition for indifference between accepting money or not that determines the equilibrium condition in (2).

The result in Proposition 1 holds for any value of $\widehat{z}$, no matter how large it is. Indeed, the role of the regions $z \geq \widehat{z}$ and $z \leq \widehat{z}$ is simply to rule beliefs that money will never be accepted in any state, and money will always be accepted in any state. This allows us to rule out either autarky or money in every state $z \in (-\widehat{z}, \widehat{z})$ by iterative deletion of strictly dominated strategies. There is no mention of the (positive or negative) intrinsic value of money in condition (2).

Remark 1 *Proposition 1 holds even if the probability that $z \geq \widehat{z}$ is ever reached is arbitrarily small.*

Starting from a given state z, if $E(\Delta z)$ is negative, the probability of reaching the region where accepting money is a dominant strategy can be arbitrarily small by choosing a large enough $\widehat{z}$. Likewise, if $E(\Delta z)$, the probability of reaching the region where accepting money is a dominated strategy can be made arbitrarily small. Such long term probabilities are not important in the computation for the condition in (2). All that matters is the set of probabilities $\varphi(t)$ of reaching a nearby state in the following periods, while the discount rate is still not too low. Hence, two very similar stochastic processes, one with $E(\Delta z) = 0$ and another with a slightly negative $E(\Delta z)$ will yield very similar conditions for equilibria, although the difference between the probabilities of ever reaching $\widehat{z}$ can be arbitrarily close to 1.[6]

The assumption that z follows a random walk implies that in the long run the economy will usually be at states outside the $(-\widehat{z}, \widehat{z})$ interval. However, a small modification of the random process could rule out this outcome without significantly affecting our results. For instance, consider a process such that $E(\Delta z) = -\eta$ for any $z > 0$ and $E(\Delta z) = \eta$ for any $z < 0$. For η sufficiently small, the set of probabilities of reaching a nearby state in the following periods would not be substantially affected, and thus the condition for a unique monetary equilibrium would be very similar to (2). We can then make sure that the economy will rarely be outside of the $(-\widehat{z}, \widehat{z})$ interval by choosing a large enough $\widehat{z}$.

2.4 Convenient parametrization

In order to make easy the comparison between the condition for existence of the monetary equilibrium in the benchmark model (1) and the conditions for uniqueness of the monetary equilibrium in this model, it is worth rewriting the condition in (2) as

$$\lambda\beta\left[(1-m)\,u + mc\right] > c$$

so that the only difference between the condition for existence of a monetary equilibrium in (1) and the condition in (2) is the factor λ, given by

$$\lambda = \sum_{t=1}^{\infty} \beta^{t-1}\varphi(t). \tag{5}$$

[6] Under the usual assumption of common knowledge of rationality, the distance between the current state z and $\widehat{z}$ can be disregarded from the analysis. That distance could have some effect on the conditions if boundedly rational agents were not able to think too far ahead, for example.

The factor λ is a number between 0 and 1. If $\lambda = 0$, condition (6) is never satisfied and autarky is always the unique equilibrium. The larger the value of λ, the larger the region where money is the unique equilibrium. If $\lambda = 1$, money is the unique equilibrium whenever it is an equilibrium in the benchmark case. Thus λ provides us with a convenient way to describe the changes on the set of equilibria. The key question is whether λ is closer to 0 or to 1.

2.5 The role of the time discount factor

The main distinguishable results of this model concern the effects of the time discount factor, which can be interpreted as representing the frequency of meetings in the economy. In the benchmark model, money can only be an equilibrium if $\beta\left[(1-m)\,u+mc\right] > c$: since the benefits of selling a good are only enjoyed in the future, an increase in the time discount factor effectively implies larger gains from trade and thus raises the incentives for accepting money. This effect is present here as well as is any other monetary model.

In the model with $V(\Delta z) > 0$, money is an equilibrium if and only if $\lambda\beta\left[(1-m)\,u+mc\right] > c$. The key result here regards the effects of β on λ, which can be seen as a corollary of Proposition 1 using the normalization that leads to Equation (5) and are absent from other models.

Proposition 2 *The time discount factor β affects λ in the following way:*

1. *λ is increasing in β.*
2. *If $E(\Delta z) = 0$, as $\beta \to 0$, $\lambda \to \frac{1}{2}$.*
3. *If $E(\Delta z) = 0$, as $\beta \to 1$, $\lambda \to 1$.*

Proof. *(1) Equation (5) shows that λ is increasing in β. (2) As $\beta \to 0$, $\lambda \to \varphi(1)$. If $E(\Delta z) = 0$, $\varphi(1) = 1/2$. (3) If $E(\Delta z) = 0$, $\sum_{t=1}^{\infty}\phi(t) = 1$ (since there is no drift, the threshold z^* will eventually be crossed). As $\beta \to 1$, the value of λ in equation (5), converge to 1.* ■

The proposition shows that β affects the set of equilibria not only by effectively increasing gains from trade but also through the *coordination channel*: larger values of β imply larger values of λ. In order to understand this channel, we have to understand the effect of β on the behavior of an agent in the hypothetical state z^* such that money is accepted if and only if $z > z^*$. For this agent, the cost of accepting money is c and the (relative) benefit is given by the average difference between V_{1z} and V_{0z} when the economy reaches a state $z > z^*$ for the first time, discounted by $\sum_{t=1}^{\infty} \beta^t \varphi(t)$.[7] Equation (4) shows that the $V_{1z} - V_{0z}$ is positive and independent of β.

The key intuition lies then on the reason for why $V_{1z} - V_{0z}$ is independent of β. As shown in Section 2.3, if money could be accepted and spent only once, $V_{1,z} - V_{0,z}$ would depend negatively on β. An agent that has not accepted money can sell his good at cost c once $z > z^*$ and enjoy the benefits later (discounted by a function of β), but since agents always have further opportunities of using money, an agent that has accepted money will also be able to sell his good again. It turns out that both effects cancel out and $V_{1,z} - V_{0,z}$ is independent of β.

Then, it all boils down to how patient an agent is to reap the rewards. For more patient agents, the cost of waiting until money can be spent is smaller. Knowing everyone will think like that, an agent will be more willing to accept money, hence patience helps agents to coordinate in the money equilibrium. In the limit $\beta \to 1$, as meeting between agents becomes more frequent, even small gains from trade imply that there will be money in equilibrium, money is the unique equilibrium in the whole region where money is an equilibrium in the benchmark model.

It is possible to construct an example where the probability of ever getting to the region where holding money is a dominant strategy is arbitrarily small and still, as $\beta \to 1$, the above result holds. Consider $E(\Delta z) = \eta$ for some $\eta < 0$. As $\eta \to 0_-$, $\lambda \to 1$ (result of the proposition plus continuity). But for any $\eta < 0$ there exists a large enough $\widehat{z}$ so that the probability of ever reaching the region where money has positive intrinsic value is arbitrarily small. Note that in this case money is the unique equilibrium as long as $u > c$; the probability of ever reaching $\widehat{z}$ is arbitrarily small; and the probability of ever reaching $-\widehat{z}$ is one.

It is clear from (5) that λ is increasing in β regardless of the process for Δz that determines $\varphi(.)$. Hence a larger β also helps agents to coordinate on the monetary equilibrium in the case $E(\Delta z) < 0$, even though it raises the importance of future payoffs, when the economy is more likely to be in a region where accepting money is a dominated strategy. The likelihood of reaching the regions where accepting money is a dominant or a dominated strategy has no effect on the results. The equilibrium condition depends on the behavior of an agent close to the hypothetical state that determines whether money is accepted or not. The role of the dominant regions is solely to exclude the beliefs that money will never ever be accepted and that money will alwaya be accepted.

[7] Implicitly in this argument is the fact that agents have an option to sell his good or use money in all future periods.

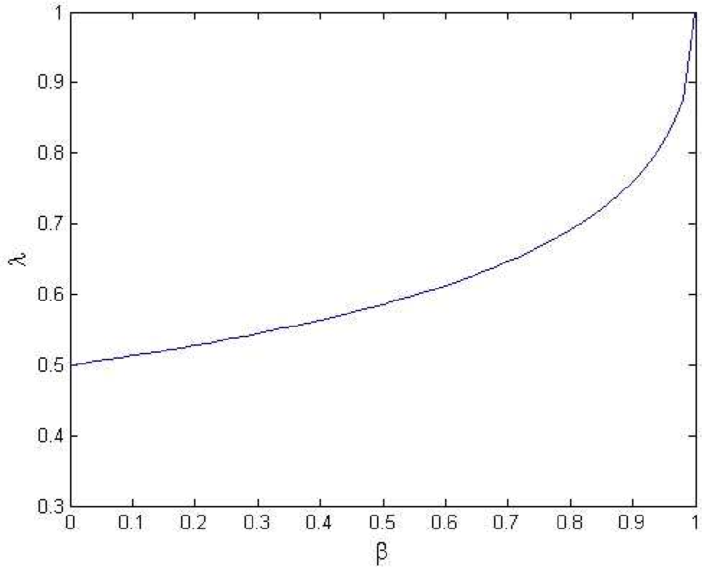

Figure 1: Normal case

2.6 Example: normal case

Figure 1 shows λ for a normal process assuming $E(\Delta z) = 0$. The probabilities $\varphi(t)$ for the normal case are obtained from Monte Carlo simulations (they do not depend on the variance $V(z)$).

The conditions for existence of a monetary equilibrium in the benchmark model (1) and the conditions for existence and uniqueness of a monetary equilibrium in our model (2) depend on β, m, u and c. Normalizing $c = 1$ and assuming $m = 1/2$, which maximizes the amount of exchanges, the possible equilibria are drawn in figure 2. The solid curve depicts the condition for existence of a monetary equilibrium in the benchmark model (a version of Kiyotaki and Wright (1993)): autarky is the unique equilibrium in the region below the solid curve, and there are multiple equilibria above the solid curve. The dotted curve shows the equilibrium condition in our model. Autarky is the unique equilibrium below the dotted line and money is the unique equilibrium above the dotted line. The distance between both lines decreases with β and vanishes if agents meet often enough (β is close to 1).

As $\beta \to 1$, money is an equilibrium in the benchmark model if $u > c$. In the model with $E(\Delta z) = 0$ and $V(\Delta z) > 0$, money is the only equilibrium if $u > c$. However, things are very different for lower values of β. Assuming $m = 1/2$, if $\beta = 0.5$, money is an equilibrium in the

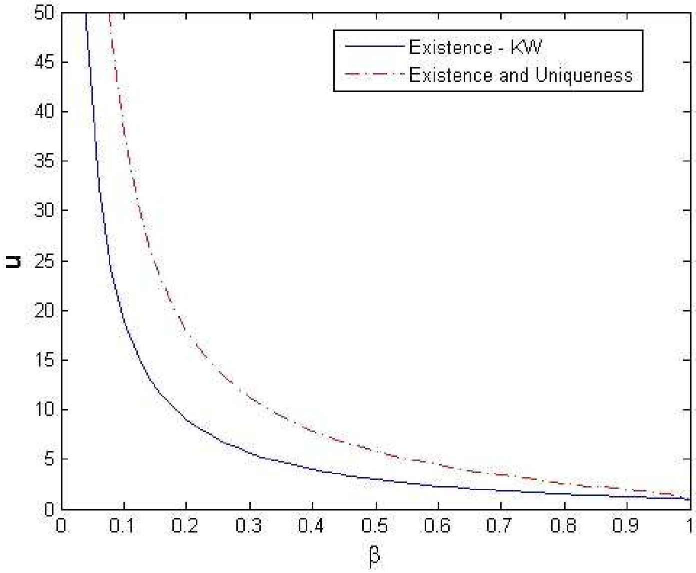

Figure 2: Equilibrium conditions in (1) and (2)

benchmark model if $u > 3c$, but in the case $V(\Delta z) > 0$, money is an equilibrium if and only if $u > 5.67c$.

2.7 Discrete state space

The analysis up to now has considered a continuous state space, but the results are easily extended to a discrete state space. Consider $V(\Delta z) > 0$ as before but now $z \in \mathbb{Z}$. Suppose the economy in period s is in state z^*. Denote by $\phi(t)$ the probability of reaching any state strictly larger than z^* at time $s+t$ and not before; and by $\phi_+(t)$ the probability of reaching any state larger or equal than z^* at time $s+t$ and not before. An argument similar to Proposition 1 yields the following result.

Proposition 3 *There exists a unique equilibrium. For all states $z \in (-\widehat{z}, \widehat{z})$,*

1. *If*
$$\left(\sum_{t=1}^{\infty} \beta^t \phi(t)\right) [(1-m)u + mc] > c, \tag{6}$$
then money is always accepted.

2. If

$$\left(\sum_{t=1}^{\infty}\beta^{t}\phi_{+}(t)\right)[(1-m)u+mc]<c, \tag{7}$$

then money is never accepted.

Proof. See Appendix. ■

Equations (6) or (7) are versions of (2) with $\phi(t)$ or $\phi_{+}(t)$ instead of $\varphi(t)$. In the continuous case we consider the probabilities of reaching z^* when the economy starts in a state that is arbitrarily close to z^*. Here, we have to start from the closest state where money is not accepted or the closest state where money is accepted, depending on which strategies we want to eliminate. As there is some distance between them, the probabilities $\phi(t)$ and $\phi_{+}(t)$ will not be the same. Hence there will be a region with multiple equilibria. But as the support of Δz increases, the discrete distribution gets closer to a continuous distribution, and $\phi(t)$ and $\phi_{+}(t)$ get closer and closer to each other.

As in the continuous case, we can rewrite condition (6) as

$$\lambda_M\beta[(1-m)u+mc]>c,$$

and condition (7) as

$$\lambda_A\beta[(1-m)u+mc]<c,$$

where

$$\lambda_M = \sum_{t=1}^{\infty}\beta^{t-1}\phi(t) \tag{8}$$

$$\lambda_A = \sum_{t=1}^{\infty}\beta^{t-1}\phi_{+}(t) \tag{9}$$

The benchmark case corresponds to $\lambda_M=0$, which means autarky is always an equilibrium, and $\lambda_A=1$, which means money is an equilibrium as long as it is an equilibrium in the benchmark model. As λ_M increases and λ_A decreases, the multiple equilibrium shrinks, so one question is whether they are close to each other (implying a small multiple-equilibrium region).

The effect of β on equilibrium selection is exactly the same as in the continuous case and are summarized in the next proposition.

Proposition 4 *The time discount factor β affects λ in the following way:*

1. *λ_A and λ_M are increasing in β.*

2. *If $E(\Delta z) = 0$, as $\beta \to 0$, λ_A and λ_M converge to $\frac{1}{2}$.*

3. *If $E(\Delta z) = 0$, as $\beta \to 1$, λ_A and λ_M converge to 1.*

Proof. *(1) Equations (8) and (9) show that λ_M and λ_A are increasing in β. (2) As $\beta \to 0$, $\lambda_A \to \varphi(1)$ and $\lambda_M \to \varphi(1)$. If $E(\Delta z) = 0$, $\varphi(1) = 1/2$. (3) If $E(\Delta z) = 0$, $\sum_{t=1}^{\infty} \phi_+(t) = 1$ and $\sum_{t=1}^{\infty} \varphi(t) = 1$. As $\beta \to 1$, $\lambda_A \to 1$ and $\lambda_M \to 1$.* ■

2.7.1 Example: Binary case

Consider a simple stochastic process where

$$\Pr(\Delta z = 1) = p \text{ and } \Pr(\Delta z = -1) = 1 - p.$$

The process is illustrated in figure 3. Departing from state $z^* - 1$ in period s, the probability of reaching state z^* in period $s+1$ is p. Otherwise, the economy moves to state $z^* - 2$. Then, state z^* can only be reached in period $s+3$. The stochastic process until state z^* is reached is illustrated in Figure 3. The probabilities that state z^* will be reached for the first time at time $s+t$ are given by (for all $i \geq 0$)

$$\begin{aligned} \phi(2i+1) &= \frac{(2i)!}{i!\,(i+1)!} p^{i+1}(1-p)^i, \\ \phi(2i) &= 0. \end{aligned}$$

Remember that $\phi(t)$ is the probability of reaching for the first time state z^* at time $s+t$, when the initial state is $z^* - 1$. The formula for $\phi(2i+1)$ resembles a binomial distribution, but the usual combination is replaced with the Catalan numbers.[8] The value of λ_M is given by

$$\lambda_M = \sum_{i=0}^{\infty} \beta^{(2i)} \left(\frac{(2n)!}{n!\,(n+1)!} p^{i+1}(1-p)^i \right). \tag{10}$$

Departing from state z^* in period s, the probability of reaching a state larger than z^* in period $s+1$ is p. Otherwise, the economy moves to state $z^* - 1$, which happens with probability $1-p$. At $z^* - 1$, we are at the previous case. Thus (for all $i \geq 0$)

$$\begin{aligned} \phi_+(1) &= p, \\ \phi_+(2i+2) &= (1-p)\frac{(2i)!}{i!\,(i+1)!} p^{i+1}(1-p)^i, \\ \phi_+(2i+3) &= 0. \end{aligned}$$

[8] See, e.g., http://mathworld.wolfram.com/CatalanNumber.html.

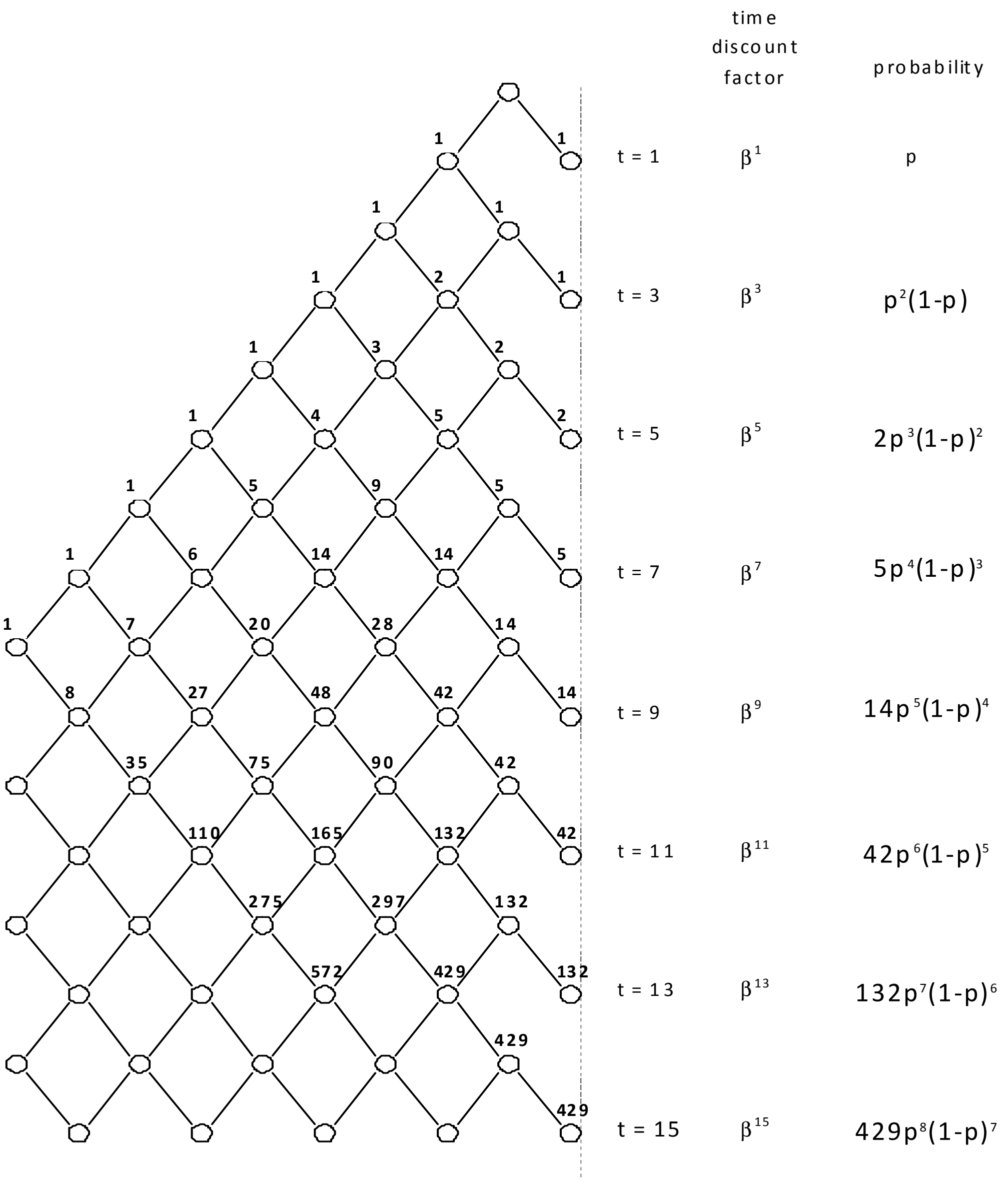

Figure 3: Binary process

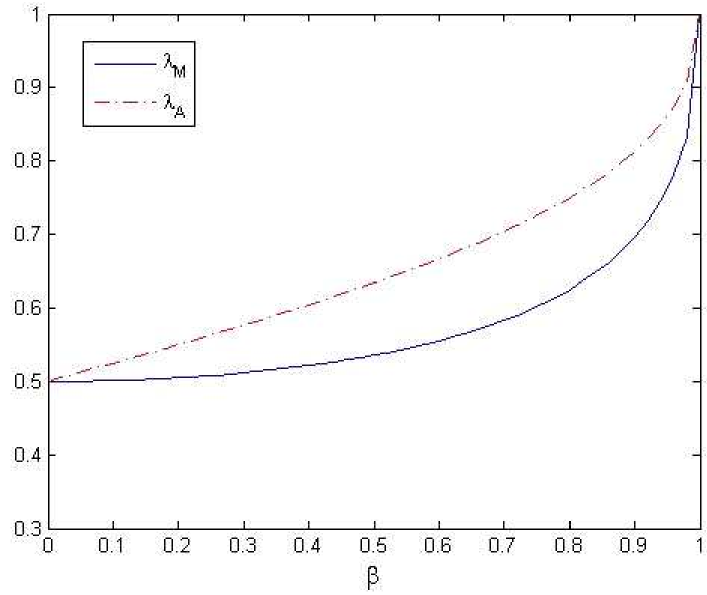

Figure 4: Binary case: λ_M and λ_A

Hence

$$\begin{aligned} \lambda_A &= p + (1-p)\sum_{i=0}^{\infty} \beta^{(2i+1)} \left(\frac{(2n)!}{n!\,(n+1)!} p^{i+1}(1-p)^i \right) \\ \lambda_A &= p + (1-p)\beta\lambda_M \end{aligned} \tag{11}$$

The case $p = 0.5$ If $p = 0.5$, λ_M becomes:

$$\lambda_M = \sum_{i=0}^{\infty} \beta^{(2i)} \left(\frac{(2i)!}{i!\,(i+1)!} \left(\frac{1}{2}\right)^{2i+1} \right) \tag{12}$$

which is a function of β only, and λ_A is then

$$\lambda_A = \frac{1 + \beta\lambda^M}{2}$$

Figure 4 shows λ_A and λ_M as a function of β. It turns out that the factor λ for a normal distributions lies between the lines for λ_A and λ_M as one would expect. Both λ_M and λ_A converge to 1 as β approaches 1 and converge to 0.5 as β approaches 1. The multiple-equilibrium region is larger for intermediate values of β.

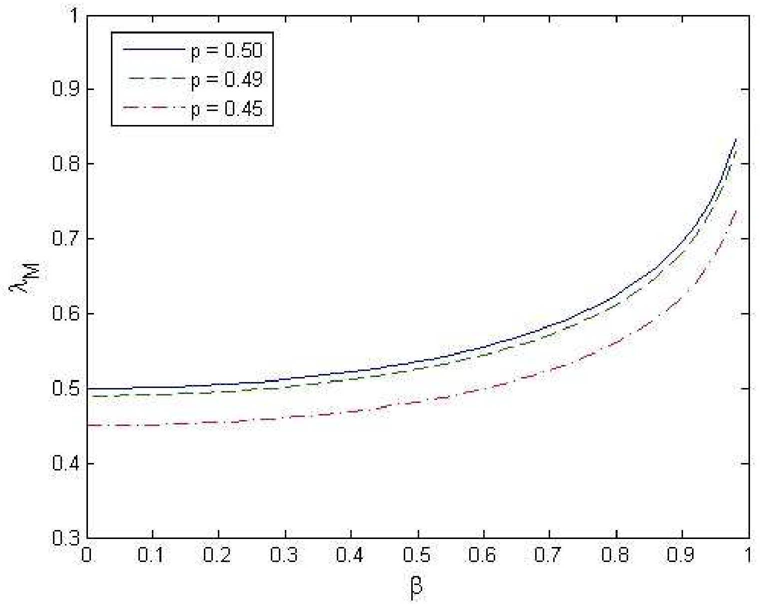

Figure 5: Case $p < 0.5$

The case $p < 0.5$ Assume now that $p = 0.5 - \varepsilon$. For a sufficiently small ε, a value of λ_M very similar to the one implied by (12) would be obtained, but the probability that $\widehat{z}$ would ever be reached could be made arbitrarily close to 0 for some $\widehat{z}$. For lower values of p, the factor λ_M is given by equation (10). Figure 5 shows the relation between β and λ_M for different values of p. As before, as β approaches 0, λ approaches p. However, as β approaches 1, λ_M does not approach 1 since there is a positive probability, bounded away from zero, that z^* will never be reached by a process departing from $z^* - 1$. But the results are similar, the factor λ_M is increasing in β, since late arrivals at z^* are worth more for larger values of β, and is not far from 1 for high values of β.

3 Concluding remarks

We have conveyed our message in a search model of money along the lines of Kiyotaki and Wright (1993), but we believe that our results might be more general than that and arise in other settings that meet two requirements. First, there must exist some states of the world where accepting money is a dominant strategy and some other states where accepting money is a dominated strategy. These states might be as unlikely as we want, and their unique role is to rule out extreme beliefs about the value of money. Second, money has to be a link between the present and the future, that is, the value of money must come from its future use as a medium of exchange. This last requirement is satisfied by other fundamental models of money such as turnpike models, and by variants of search models (such as Trejos and Wright (1995) and Lagos and Wright (2005)). It is also satisfied by overlapping generations models. However, while in turnpike and search models money earned today can be spent at any time in the future, in overlapping generations models an agent has fewer opportunities to spend his money. In particular, in a two-period overlapping generations model, a young agent is willing to produce in exchange for money only if he believes that he will be able to spend his money with a high probability when old. This difference should not matter for our uniqueness result. However, it should matter for our result on the equivalence between the condition for uniqueness of the monetary equilibrium and the condition for existence of the monetary equilibrium. Intuitively, money becomes more risky and thus autarky becomes more likely if there are fewer opportunities for money to be spent.

Finally, since the focus of our analysis was on the selection between autarky and money, we have considered an economy with only one variety of money (say, seashells). In reality, many varieties may be available at any point in time (e.g., seashells, stones, salt, gold). We believe that the selection mechanism proposed in this paper can also be extended to such environments. However, such extension is beyond the objectives of the present paper and is left for future work.

References

[1] Araujo, L. and Camargo, B., 2006, Information, Learning and the Stability of Fiat Money, *Journal of Monetary Economics* 53, 1571-1591.

[2] Brock, W. and Scheinkman, J., 1980, Some Remarks on Monetary Policy in an Overlapping Generations Model, in Models of Monetary Economies, edited by John Kareken and Neil Wallace, *Federal Reserve Bank of Minneapolis*, Minneapolis.

[3] Burdzy, K., Frankel, D., and Pauzner A., 2001, Fast Equilibrium Selection by Rational Players Living in a Changing World, *Econometrica* 68, 163-190.

[4] Carlsson, H. and van Damme, E., 1993, Global games and equilibrium selection, *Econometrica* 61, 989-1018.

[5] Frankel, D., Morris, S., and Pauzner, A., 2003, Equilibrium Selection in Global Games with Strategic Complementarities, *Journal of Economic Theory* 108, 1-44.

[6] Frankel, D. and Pauzner, A., 2000, Resolving indeterminacy in dynamic settings: the role of shocks, *Quarterly Journal of Economics* 115, 283-304.

[7] Kiyotaki, N. and R. Wright, 1993, A Search-Theoretic Approach to Monetary Economics, *American Economic Review* 83, 63-77.

[8] Lagos, R. and Wright, R., 2005, A Unified Framework for Monetary Theory and Policy Analysis, *Journal of Political Economy* 113, 463-484.

[9] Morris, S. and Shin, H., 2000, Rethinking Multiple Equilibria in Macroeconomic Modeling, *NBER Macroeconomics Annual* 15, 139-161.

[10] Morris, S. and Shin, H., 2003, Global Games: Theory and Applications, in Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society), edited by M. Dewatripont, L. Hansen and S. Turnovsky; Cambridge University Press.

[11] Obstfeld, M. and K. Rogoff, 1983, Speculative Hyperinflations in Maximizing Models: Can We Rule Them Out?, *Journal of Political Economy* 91, 675-87.

[12] Scheinkman, J., 1980, Discussion, in Models of Monetary Economies, edited by John Kareken and Neil Wallace, *Federal Reserve Bank of Minneapolis*, Minneapolis.

[13] Trejos, A. Wright, R., 1995, Search, Bargaining, Money, and Prices, *Journal of Political Economy* 103, 118-141.

[14] Wallace, N. and T. Zhu, 2004, A Commodity-Money Refinement in Matching Models, *Journal of Economic Theory* 117, 246-258

[15] Zhou, R., 2003, Does Commodity Money Eliminate the Indeterminacy of Equilibria?, *Journal of Economic Theory* 110, 176–190.

[16] Zhu, T., 2003, Existence of a Monetary Steady State in a Matching Model: Indivisible Money, *Journal of Economic Theory* 112, 307-324.

[17] Zhu, T., 2005, Existence of a Monetary Steady State in a Matching Model: Divisible Money, *Journal of Economic Theory* 123, 135-160.

A Proofs

A.1 Proof of Proposition 1

Proof. First, we prove that money is accepted if (2) holds. Fix $\widehat{z} > 0$. For any $z > \widehat{z}$, an agent will find it optimal to always produce in exchange for money. The proof is done by induction, where at each step strictly dominated strategies are eliminated. First, fix $z^* \in (-\widehat{z}, \widehat{z})$ and assume that all agents accept money if and only if $z \geq z^*$. We need to check an agent's incentives to accept money in some state $z = z^* - \epsilon$ for some $\epsilon > 0$. If an agent accepts money in exchange for his good in state $z^* - \epsilon$, he obtains

$$-c + \sum_{t=1}^{\infty} \beta^t \varphi^\epsilon(t) \left(\int_0^\infty f(z^* + s | t_\phi = t) V_{1,z^*+s} ds \right) \equiv V^a_{z^*-\epsilon},$$

where $\varphi^\epsilon(t)$ is the probability that a state $z \geq z^*$ will be reached at time t, and not before, when z departs from $z^* - \epsilon$; t_ϕ denotes the period a state larger than or equal to z^* is reached; and $f(z|t_\phi = t)$ denotes the probability density function that the state z is reached conditional on t_ϕ equal to t. Since no agent is accepting money when $z < z^*$, the money received by the agent will not be useful (or harmful) until a state $z \geq z^*$ is reached. When such a state is reached, the agent's value function is $V_{1,z}$. The term in brackets is the average of such value functions, weighted by their densities. The expected payoff of an agent that accepts money equals the discounted value of such averages, weighted by their own probabilities, minus c. In turn, if an agent does not accept money in state $z^* - \epsilon$, he obtains

$$\sum_{t=1}^{\infty} \beta^t \varphi^\epsilon(t) \left(\int_0^\infty f(z^* + s | t_\phi = t) V_{0,z^*+s} ds \right) \equiv V^n_{z^*-\epsilon}.$$

This implies that the agent accepts money in state $z^* - \epsilon$ as long as

$$V^a_{z^*-\epsilon} - V^n_{z^*-\epsilon} = -c + \sum_{t=1}^{\infty} \beta^t \varphi^\epsilon(t) \left(\int_0^\infty f(z^* + s | t_\phi = t) \left[V_{1,z^*+s} - V_{0,z^*+s} \right] ds \right) > 0. \tag{13}$$

Now, since all other agents are accepting money in any state $z \geq z^*$, the value function of an agent with money in some state $z \geq z^*$ is

$$V_{1,z} = m\beta E_z V_1 + (1-m)\left(u + \beta E_z V_0\right),$$

while the value function of an agent without money in some state $z \geq z^*$ is

$$V_{0,z} = m\left(-c + \beta E_z V_1\right) + (1-m)\beta E_z V_0,$$

where $\beta E_z V_1$ and $\beta E_z V_0$ are the expected value of holding, respectively, one and zero unit of money at the end of the period, when the current state is z. Subtracting $V_{0,z}$ from $V_{1,z}$ yields equation (4):

$$V_{1,z} - V_{0,z} = (1-m)u + mc.$$

Substituting (4) into (13) yields

$$V^a_{z^*-\epsilon} - V^n_{z^*-\epsilon} = -c + \sum_{t=1}^{\infty} \beta^t \varphi^\epsilon(t) \left[(1-m)u + mc\right],$$

which used the fact that $\int_0^\infty f(z^* + s|t_\phi = t)ds = 1$ and $(1-m)u + mc$ is a constant. Therefore, $V^a_{z^*-\epsilon} - V^n_{z^*-\epsilon} > 0$ if

$$\sum_{t=1}^{\infty} \beta^t \varphi^\epsilon(t) \left[(1-m)u + mc\right] > c.$$

This argument holds for any $\epsilon > 0$. As $\epsilon \to 0$, $\varphi^\epsilon(t) \to \varphi(t)$, and we obtain the expression in (2).

The argument has assumed that agents will not accept money in states smaller than z^*, but if that were not the case, incentives for holding money would only increase, owing to the strategic complementarities in using money. Hence, if condition (2) holds, accepting money in state $z^* - \epsilon$ is a strictly dominant strategy given that all agents are accepting money in states larger than or equal to z^*. Thus, as (i) it is a strictly dominant strategy to accept money if $z \geq \widehat{z}$, and (ii) for all $z^* > -\widehat{z}$, if all agents accept money whenever $z \geq z^*$, accepting money at $z = z^* - \epsilon$ is a strictly dominant strategy, accepting money is the only strategy that survives iterative elimination of strictly dominated strategies for all $z \in (-\widehat{z}, \widehat{z})$.

It remains to show that money is not accepted if the inequality in (2) is reversed. The argument is analogous to the one above. For any $z < -\widehat{z}$, an agent will find it optimal to never bring money into the trading post. Again, we proceed by induction. Fix $z^* \in (-\widehat{z}, \widehat{z})$ and suppose that all agents accept money if and only if $z \geq z^*$. We need to compare payoffs from accepting and not accepting money at state $z^* + \epsilon < \widehat{z}$, for some $\epsilon > 0$. An agent that accepts money in exchange for his good in state $z^* + \epsilon$ obtains

$$-c + \sum_{t=1}^{\infty} \beta^t \varphi_\epsilon(t) \left(\int_0^\infty f(z^* + s|t_\phi = t) V_{1,z^*+s} ds \right) \equiv V^a_{z^*+\epsilon},$$

where $\varphi_\epsilon(t)$ is the probability that a state $z \geq z^*$ will be reached at time t, and not before, when z departs from $z^* + \epsilon$; t_ϕ denotes the period a state larger than or equal to z^* is reached and $f(z|t_\phi = t)$ denotes the probability density function that the state z is reached conditional on t_ϕ equal to t. If an agent does not accept money in state $z^* + \epsilon$, he obtains

$$\sum_{t=1}^{\infty} \beta^t \varphi_\epsilon(t) \left(\int_0^\infty f(z^* + s|t_\phi = t) V_{0,z^*+s} ds \right) \equiv V^n_{z^*+\epsilon}.$$

This implies that the agent does not accept money in state z^* as long as

$$V^n_{z^*+\epsilon} - V^a_{z^*+\epsilon} = \sum_{t=1}^{\infty} \beta^t \varphi_\epsilon(t) \left(\int_0^\infty f(z^* + s | t_\phi = t) \left[V_{0,z^*+\epsilon} - V_{1,z^*+\epsilon} \right] ds \right) + c > 0.$$

Following the same steps as above, we obtain that not accepting money is optimal if

$$\sum_{t=1}^{\infty} \beta^t \varphi_\epsilon(t) \left[(1-m)u + mc \right] < c.$$

Taking the limit $\epsilon \to 0$, $\varphi_\epsilon(t)$ converges to $\varphi(t)$. Following the same reasoning as above, we get the claim. ■

A.2 Proof of Proposition 3

Proof. The proof is very similar to the proof of Proposition 1. First, we prove the first statement. Now, starting from a threshold $z^* \in (-\widehat{z}, \widehat{z})$, it is shown that an agent finds it optimal to accept money in state $z^* - 1 > -\widehat{z}$. Suppose that all agents accept money if and only if $z \geq z^*$. We need to compare the payoff of such an agent with the one received by someone who accepts money if $z \geq z^* - 1$. An agent that accepts money in exchange for his good in state $z^* - 1$ obtains

$$-c + \sum_{t=1}^{\infty} \beta^t \phi(t) \left(\sum_{i=0}^{\infty} \pi(z^* + i | t_\phi = t) V_{1,z^*+i} \right) \equiv V^a_{z^*-1},$$

and

$$\sum_{t=1}^{\infty} \beta^t \phi(t) \left(\sum_{i=0}^{\infty} \pi(z^* + i | t_\phi = t) V_{0,z^*+i} \right) \equiv V^n_{z^*-1}.$$

and following the reasoning in the proof of Proposition, 1, we get the first statement.

The proof of the second statement is also analogous: for any $z < -\widehat{z}$, it is a dominant strategy for all agents not to accept money. Now, suppose that all agents accept money if and only if $z \geq z^*$, where $z^* < \widehat{z}$. We need to compare the payoff of accepting and not accepting money at state z^*. An agent that accepts money in exchange for his good in state z^* obtains

$$-c + \sum_{t=1}^{\infty} \beta^t \phi_+(t) \left(\sum_{i=0}^{\infty} \pi(z^* + i | t_\phi = t) V_{1,z^*+i} \right) \equiv V^a_{z^*},$$

If an agent does not accept money in state z^*, he obtains

$$\sum_{t=1}^{\infty} \beta^t \phi_+(t) \left(\sum_{i=0}^{\infty} \pi(z^* + i | t_\phi = t) V_{0,z^*+i} \right) \equiv V^n_{z^*}.$$

and following the reasoning in the proof of Proposition, 1, we get the proof of the second statement. ■