Liquidity, Productivity and Efficiency*

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Abstract

This paper studies the optimality of the private liquidity provision in a model of overlapping generations of entrepreneurs. The model features endogenous liquidity choice due to heterogeneous investment opportunities. One type of investment is more liquid but less productive and the other is less liquid but more productive. I prove existence and uniqueness of equilibria for all parameter values and study their properties. I show that share of investment in each type, equilibrium interest rate and leverage are non-monotone in the pledgeabilities of investment types and respond asymmetrically to changes in the pledgeability of the two types. I characterize regions of parameter space which give rise to constrained inefficient competitive equilibria and show that the inefficiency is precisely due to the endogeneity of entrepreneurs' choice between liquidity and return. This endogeneity leads to a pecuniary externality in portfolio decisions of entrepreneurs which works through the interest rate. When pledgeabilities are low (high), this externality leads to inefficiently liquid (inefficiently illiquid) equilibria with strictly negative (positive) interest rate. I show that in the class of inefficient equilibria, a negative (positive) interest rate indicates that the interest rate is in fact too high (low). Hence the level of the interest rate can be a misleading indicator of the type of the inefficiency in the economy: a Pareto improvement in an inefficiently liquid equilibrium involves lowering the negative interest rate even further. Finally, I show that government bonds can be helpful for a subset of inefficiently illiquid equilibria as suggested by previous works, but may not help much in case of inefficiently liquid equilibria.

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1 Introduction

There is a substantive literature on the efficiency of liquidity provision by the private sector and possible role of the government in improving the market outcomes. This strand of works is generally concerned with how market provides firms and consumers with liquidity, whether there are any market failures in doing so, and whether and how government can make Pareto improvements in the allocations induced by market.[1] Diamond and Dybvig (1983) and Holstrom and Tirole (1998) are two seminal examples each of which has initiated a series of related papers. The latter has laid out a microfoundation for the inefficiencies in private liquidity provision by the production sector due to moral hazard and limited commitment, while the former shows how intermediation can achieve an (constrained) efficient allocation, when idiosyncratic liquidity needs of consumers are private information.

This paper studies the efficiency of private liquidity provision in a model of overlapping generations of entrepreneurs, when there are heterogeneous investment opportunities. The source of heterogeneity is a trade off between liquidity and return across different types of investments where more productive investments have less pledgeability in their return. In this environment, the level of liquidity and productivity of aggregate investment is endogenously chosen by the production sector. I show that this endogenous choice of liquidity and productivity of aggregate investments lead to pecuniary externalities which, in turn, can lead to an (constrained) inefficient allocation. I also investigate whether and how government can improve social welfare by regulating the private sector or by other means such as government bonds when the competitive equilibrium fails to achieve efficiency.

The model in this paper can also be seen as a model of endogenous choice of liquidity. In this sense, it is a complement to the previous works by Kiyotaki and Moore and Farhi and Tirole where liquidity of the investment is not a decision variable. In fact, the model structure is similar to Farhi and Tirole (2010a) and the only difference is that this paper features heterogenous investment opportunities that leads to endogenous choice of liquidity. I show that this endogeneity is a new source of inefficiency, and therefore is important to consider in macroeconomic policy making.

My contribution is threefold. First, I present a general equilibrium model that features a trade off between liquidity and productivity by introducing two types of investments; one type is more productive and less liquid and the other type is less productive but more liquid.

[1] The term "liquidity" here refers to the ability to transfer wealth across different time periods by pledging the returns to a real or financial investment. In other words, my focus is on an intertemporal notion of liquidity or, using the terminology of Brunnermeier and Pedersen (2009), funding liquidity.

I fully characterize the competitive equilibrium and the steady state and prove their existence and uniqueness for the space of parameter values. I show that the steady state equilibria have some interesting and counterintuitive properties. I show that the share of investment in each type, the steady state interest rate and the aggregate leverage are non-monotone in the liquidity of the investment types and respond asymmetrically to changes in the liquidity of the two types. These comparative statics have positive implications about how changes in institutional environment, i.e. contract enforcement, corporate governance, bankruptcy laws etc, affect aggregate macroeconomic variables. For instance an increase in the pledgeability of the less (more) pledgeable type of investment leads to a lower (higher) steady state interest rate. This is consistent with the decline of the real interest rates in US since 1990s as documented by Caballero et al. (2008), and a series of financial innovations and regulatory reforms leading to a dramatic expansion in the mortgage market and securitization and sale of bank loans in the secondary market which in essence made less pledgeable investments more pledgeable.

Second, the trade off between liquidity and return presented in this paper is, to the best of my knowledge, a novel source of inefficiency in private liquidity provision which have not been explored and studied explicitly in previous works. I show that the trade off between liquidity and return gives rise to an externality in the private portfolio choices of the entrepreneurs that works through the interest rate. When entrepreneurs have relatively low initial wealth, they might end up investing too much in the more liquid type of investment while the efficient allocation requires investing only in the more productive type. Investment in the more liquid type when entrepreneurs have low initial wealth does not bid up the equilibrium interest rate sufficiently to make this inefficient investment unprofitable. In contrast, when entrepreneurs start with relatively high initial wealth, they might invest too much in the more productive type while efficiency requires investing only in the more liquid type. In this case, the equilibrium interest rate is bid up too much which renders the more liquid type unprofitable. In general equilibrium, however, the initial wealth of each generation of investing entrepreneurs is endogenously determined. Low (high) initial wealth is an equilibrium outcome when pledgeabilities of both types are relatively low (high) since a low (high) fraction of the returns to investment in any period can be invested by future entrepreneurs. Hence, when pledgeabilities are low this externality can lead to *inefficiently liquid* equilibria and when pledgeabilities are high it can lead to *inefficiently illiquid* equilibria.

Similar to Lorenzoni (2008) this externality is pecuniary and acts through the borrowing constraints. One difference is that the externality in this paper works through an intertemporal price (i.e. the interest rate). Moreover the inefficient sale of productive assets in an environment with aggregate uncertainty, is the key in Lorenzoni (2008) that leads to an externality, while here, it is the demand for investible resources in an economy without any uncertainty that entails an inefficient outcome.

I further study some important properties of inefficient steady state equilibria. I show that in an inefficiently liquid steady state, interest rate is strictly negative and in an inefficiently illiquid steady state, the equilibrium interest rate is strictly positive.[2] This is because, the inefficiently liquid steady state equilibria arise when both types of investment have relatively low pledgeability and the inefficiently illiquid equilibria arise when both types have relatively high pledgeability.[3] This implies that the level of the interest rate can be a misleading indicator in determining the type of inefficiency in the economy: a Pareto improvement in an inefficiently liquid (inefficiently illiquid) equilibrium induces an even more negative (more positive) interest rate. In traditional overinvestment models, e.g. Diamond (1965), an interest rate below the growth rate of the output must be raised to make a Pareto improvement. In contrast, a negative interest rate in this paper indicates that the interest rate is in fact *too high*. Hence, a Pareto improvement leads to an even more negative interest rate.

My third contribution is concerning the role of macroeconomic policy. I study the welfare effects of an important class of public liquidity, i.e. government bonds. I show that government bonds have limited power in the model economy: it enhances the welfare for a subset of inefficiently illiquid equilibria as suggested by Holstrom and Tirole (1998) and Woodford (1990), while it can not help much for a subset of inefficiently liquid equilibria. The reason is that the source of inefficiency in this paper, which is the trade off between liquidity and return to investment, is absent in Holstrom and Tirole (1998) and Woodford (1990). I further show a class of inefficiently illiquid (inefficiently liquid) equilibria where at the steady state, outside liquidity, e.g. government bonds, only crowds out inside liquidity without affecting the steady state interest rate and with a negative (positive) effect on welfare.

The trade off between liquidity and return is the key to the results of this paper which can be motivated in a simple example. Suppose you have only two types of investments with constant return to scale in a one period economy subject to credit market frictions. Con-

[2]There is a zero measure set of parameters, however, that corresponds to inefficiently liquid equilibria with a zero interest rate in steady state. Moreover, there is no multiplicity of equilibria in the model studied in this paper. The two types of inefficiency happen in different parts of the parameter space.

[3]Note that in any of these cases, one type has more pledgeability, i.e., is more liquid, and the other is more productive regardless of the type of inefficiency.

sumption goods are invested at $t = 0$ to produce more consumption at $t = 1$. Let $(R_1, \theta_1 R_1)$ and $(R_2, \theta_2 R_2)$ denote the total return and the pledgable return per unit of investment for type one and two investments respectively. Pledgeability of the return is the fraction of the return to investment that can be credibly promised to be paid to the investors. I use the concept of pledgeability of return in the sense used in Farhi and Tirole (2009, 2010a,b). This concept is also close to the extent of frictions in bilateral commitments in Kiyotaki and Moore (2002, 2005, 2008). Let $R_1 > R_2 > 1$ so that the investment in type one has a higher return and both types are socially desirable and that, $\theta_1 R_1$ and $\theta_2 R_2$ are both strictly less than one. Suppose an entrepreneur in this economy has initial endowment of $e > 0$ which can be used to raise funds and for simplicity set the interest rate to zero. If an entrepreneur is given an opportunity to invest only in type $i \in \{1, 2\}$, she can raise at most $d_i = \frac{\theta_i R_i}{1-\theta_i R_i} e$[4] in a competitive credit market and so her total profits (net of debt payment) would be $\pi_i = (R_i - 1)d_i + R_i e$. If type one has also more pledgable return, i.e. $\theta_1 R_1 \geq \theta_2 R_2$, it is easy to see that no investment in type two is made. The reason is that in this case and if there is any positive investment in type two, the entrepreneur can raise her profits by diverting a unit of investment in type two and investing it in type one. In other words type one not only has a higher profits per unit of investment, i.e. $(R_1 - 1) > (R_2 - 1)$, but also allows an investment size that is no less attractive, i.e. $d_1 \geq d_2$. Hence, in any equilibrium of this economy if a positive investment in type two is observed, one can infer that $\theta_1 R_1 < \theta_2 R_2$. In this sense, in any equilibrium of this economy where there are investments with heterogenous return and liquidity, there is a trade off between liquidity and return: less liquid investments (type one in the example) must command a higher return.

The trade off between liquidity and return can be observed in both real and financial sectors. First, consider investments in the real sector. Bigger and mature firms tend to have lower cost of external financing, i.e., more pledgeability of return, due, for instance, to their reputation and higher value of their collateral while smaller firms within the same industry grow faster and face costlier external financing. High tech startups, e.g., IT ventures, can provide another example in this regard; they have higher returns relative to less knowledge intensive firms but are subject to many kinds of agency problems because of the very new and advanced nature of their technology which generally result in a very low borrowing capacity.[5]

[4]The entrepreneur maximizes $R_i x_i - d_i$, subject to the resource constraint, $x_i \leq d_i + e$, and the borrowing (pledgeability) constraint, $d_i \leq \theta_i R_i x_i$. It is easy to check that the solution to this problem under the mentioned assumptions, is exactly what is given in the text.

[5]Of course ventures have much riskier payoff as well, but note that riskiness per se does not lead to illiquidity of the investment.

The same trade off can be observed when a firm can invest in different types of projects: to build a new plant, i.e., accumulate more physical capital, or to invest more in the specific human capital in the same plant, i.e., accumulate more organizational capital. The latter option creates less collateralizable assets[6] and is therefore less liquid but may have a higher return to investment. Financial sector can also provide several examples of this trade off. For example more liquid bonds tend to have lower yields. Another example is a bank that can choose the intensity and effectiveness of monitoring of an investment loan. The more effective the monitoring, the higher the value that can be extracted from the investment and the lower the liquidity of the loan.[7]

1.1 Related Literature

(To Be Completed)

2 Model

2.1 Agents, Preferences and Technology

The model economy is an overlapping generations of entrepreneurs without any kind of uncertainty. Each individual lives for three periods and there is a unit measure of each of young, middle aged and old cohorts in each period. Each entrepreneur receives a fixed endowment $e > 0$ of non-storable and homogenous consumption good when young and no endowment thereafter and consumes only when she is old.

At any period the middle aged cohort has the opportunity to invest in two types of investments which pay off the next period. Investments differ in their return and liquidity. Investment of type $i \in \{1, 2\}$ has a constat return to scale of R_i and a fraction $\theta_i R_i$ of the return can be pledged to the investors. Limited pledgeability of return can arise in many contexts and for a number of different reasons including moral hazard and limited commitment. Following Kiyotaki and Moore (2002, 2005, 2008) and Farhi and Tirole (2009, 2010a,b) I summarize all these frictions in the variable $\theta_i, i \in \{1, 2\}$. I refer to $\theta_i R_i$ as "pledgeability" of the type i investment. However when (R_1, R_2) are fixed and within the

[6]This can be due to more specificity of organizational capital or because of the inalienability of the human capital as suggested by Hart and Moore (1994).

[7]This is the case because monitoring makes the loan more specific to the bank or it can reveal the type of investment to the bank and hence lead to adverse selection. For an explicit model of the latter effect see Parlour and Plantin (2008).

appropriate region of parameters, I refer to $\theta_i, i \in \{1,2\}$ as well as $\theta_i R_i, i \in \{1,2\}$ as pledgeability of return to investments. I make the following assumption about the return and pledgeability of investments:

Assumption 1. $R_1 > R_2 > 1$ and $\theta_1 R_1 < \theta_2 R_2 < 1$.

Assumption 1 is made for three purposes. First, it guarantees the existence of the competitive equilibrium and the stability of the steady state. Second, it captures the trade off between liquidity and return across the two types of investments; type one investment is more productive but less liquid. Finally, it makes sure that both types of investments are socially efficient in the sense that if you have an economy investing only in one type, say, $j \in \{1,2\}$, any competitive equilibrium of this economy would be efficient.[8] I will return to each of the above statements later in the paper.

2.2 Problem of the Middle Aged Entrepreneurs

In each period a credit market opens up where young and middle aged entrepreneurs can lend and borrow. Every young born at period $t > 0$ inelastically supplies all her endowments in the capital market. The middle aged at time t who have transferred funds from period $t-1$ by investing in the projects of middle aged at $t-1$, demands additional liquidity from the young given the limited pledgeability of their optimal investment portfolio. She simultaneously chooses her optimal investment portfolio given the ongoing interest rate $1+r_t$, the resources of the young cohort of period t and the liquid resources that has been transferred from period $t-1$ to period t.

Let x_{1t} and x_{2t} denote investments in types 1 and 2 and i_t denotes the new funds raised by the middle aged at t using the resources of young entrepreneurs in period t. Given the interest rate $1+r_t$, a middle aged entrepreneur at t solves the following problem:

$$\begin{aligned} c^o_{t+1} \equiv \max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1+r_t) i_t \\ s.t. \quad & x_{1t} + x_{2t} \leq (1+r_{t-1}) i_{t-1} + i_t \, , \\ & (1+r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \, . \end{aligned} \tag{I}$$

The first constraint in the maximization above is the resource constraint of the middle aged entrepreneur. $(1+r_{t-1})i_{t-1}$ is the liquidity that is transferred from period $t-1$ to t by the

[8] I will define the term "efficiency" more precisely in Section 3

middle aged entrepreneur by investing in the projects of middle aged of period $t-1$. The second term, i_t, is the total external funds that the middle aged at t raises by borrowing from the young entrepreneurs at t. The second constraint is the manifestation of the limited pledgeability of the investments; the middle aged entrepreneur can not borrow more than what she can promise or commit to pay in period $t+1$. For type $j \in \{1,2\}$ the maximum that can be credibly promised to the lenders is $\theta_j R_j x_{jt}$ and so the total amount of pledgeable return is given by the right hand side of the second constraint. Finally, c^o_{t+1} denotes the consumption of the old entrepreneur in period $t+1$.

In any equilibrium the resource constraint binds and so I can solve for the value of x_{2t} in the above and rewrite the problem as:

$$\begin{aligned} \max_{i_t, x_{1t} \geq 0} \quad & (R_1 - R_2)x_{1t} + \left(R_2 - (1+r_t)\right) i_t + R_2(1+r_{t-1})i_{t-1} \\ s.t. \quad & (1+r_t - \theta_2 R_2)i_t \leq (\theta_1 R_1 - \theta_2 R_2)x_{1t} + \theta_2 R_2 (1+r_{t-1})i_{t-1}\,, \\ & 0 \leq x_{1t} \leq (1+r_{t-1})i_{t-1} + i_t\,. \end{aligned} \tag{II}$$

I can immediately see from the above that $1 + r_t > \theta_2 R_2$. Otherwise it must be that $1 + r_t \leq \theta_2 R_2 < R_2$ in which case i_t can be raised without any bound and there would not be any maximum to the objective function. I must also have $1 + r_t \leq R_1$, otherwise the optimal solution to the problem requires that $i_t = 0$. To see why rewrite problem II with x_{2t} in the objective function. If $1 + r_t > R_1$ then by Assumption 1 both coefficients of x_{2t} and i_t are strictly negative and so the best an entrepreneur can do is to set both to zero. This can not be an equilibrium since market clearing in the capital market can not be satisfied. From now on I only consider equilibria where the strict inequality holds, i.e., $1 + r_t < R_1$. With the strict inequality I can state the following lemma:

Lemma 1. *In any competitive equilibrium where $1 + r_t < R_1$ holds for all t, the borrowing constraint of the middle aged entrepreneur binds in every period. Moreover the problem of the middle aged entrepreneur can be written in the following form:*

$$\begin{aligned} \max_{i_t} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) i_t + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) i_{t-1} \\ s.t. \quad & \left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t - \theta_1 R_1}\right) i_{t-1} \leq i_t \leq \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t - \theta_2 R_2}\right) i_{t-1}\,, \end{aligned} \tag{III}$$

where,

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) \equiv \left(\frac{(\theta_2 - \theta_1)R_1R_2}{\theta_2R_2 - \theta_1R_1}\right) - \left(\frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{\theta_2R_2 - \theta_1R_1}\right)(1 + r_t)\,,$$
$$\Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) \equiv \left(\frac{(\theta_2 - \theta_1)R_1R_2}{\theta_2R_2 - \theta_1R_1}\right)(1 + r_{t-1})\,.$$

The bold symbols $(\boldsymbol{\theta}, \boldsymbol{R})$ is the vector of pledgeabilities and returns of the two types of investments, i.e., $(\theta_1, \theta_2, R_1, R_2)$.

Proof of Lemma 1. Suppose that the borrowing constraint does not bind for some $t \geq 0$. Then since the coefficient of x_{1t} in the objective function of problem II which is $R_1 - R_2$ is strictly negative by Assumption 1, x_1t must be at the highest possible value which is $(1 + r_{t-1})i_{t-1} + i_t$. At this value the objective function in II can be written as $(R_1 - (1 + r_t))i_t + D_{t-1}$ where D_{t-1} is determined at $t - 1$. Moreover the borrowing constraint at this value of x_{1t} is:

$$(1 + r_t - \theta_1R_1)i_t < \theta_1R_1(1 + r_{t-1})i_{t-1}\,.$$

Since the constraint is not binding one can raise i_t by an small amount $\epsilon > 0$ so that the constraint is still satisfied and the value of objective function is increased by $(R_1 - (1 + r_t))\epsilon$. This contradiction shows that the borrowing constraint has to be always binding. The rest of the lemma is straight forward by using the borrowing constraint to eliminate x_{1t}. ■

In Lemma 1 the two terms Λ and Φ are in fact the marginal (net) return of period t and period $t-1$ liquidity for the middle aged entrepreneurs. In problem III, it is obvious that the middle aged entrepreneur demands strictly positive or zero liquidity in period t, i.e., $i_t > 0$ or $i_t = 0$, depending on the sign of the marginal return Λ and she would be indifferent when $\Lambda = 0$.

The two bounds in the constraint of problem III corresponds to the two limits; when i_t hits the lower bound, the entrepreneur invests only in type one or the more productive and when it hits the upper bound only in type two which is the more liquid investment. Define $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ as the (gross) interest rate in period t that makes Λ equal to zero:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2 - \theta_1)R_1R_2}{(1-\theta_1)R_1 - (1-\theta_2)R_2}\,. \tag{1}$$

Then the entrepreneurs' optimal liquidity demand is characterized as follows:

$$\begin{cases} i_t = \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) i_{t-1}\,, & \text{if } 1+r_t < 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})\,, \\ \\ i_t \in \left[\left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) i_{t-1}, \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) i_{t-1}\right]\,, & \text{if } 1+r_t = 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})\,, \\ \\ i_t = \left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) i_{t-1}\,, & \text{if } 1+r_t > 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})\,. \end{cases} \tag{2}$$

Figure 1 is an illustration of middle aged demand for fund given by 2 and the inelastic supply of fund by the young entrepreneurs at time t. Wealth of the middle aged at t is $w_{t-1} = (1+r_{t-1})i_{t-1}$ which is determined in period $t-1$. As the figure shows, middle aged entrepreneurs specializes in type one or two when the supply curve crosses the demand on its left or right arm respectively. Entrepreneurs mix if the intersection happens to be on the flat part of the demand curve which is characterized by $1+r_t = 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$. A higher period $t-1$ interest rate, i.e. higher w_{t-1}, leads to an upward shift in the two arms of the demand curve but has no effect on its flat segment.

2.3 Competitive Equilibrium

In each period there is fixed supply of liquidity e. Hence the market clearing condition dictates:

$$i_t = e, \quad \forall t \geq 0\,. \tag{3}$$

Combining market clearing 3 and the optimal investment decision of the middle aged entrepreneur 2, I derive the optimal path of the interest rates at any competitive equilibrium:

$$\begin{cases} 1+r_t = \theta_2 R_2 (2+r_{t-1}) & \text{if } 1+r_t < 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})\,, \\ \\ 1+r_t = \theta_1 R_1 (2+r_{t-1}) & \text{if } 1+r_t > 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})\,. \end{cases} \tag{4}$$

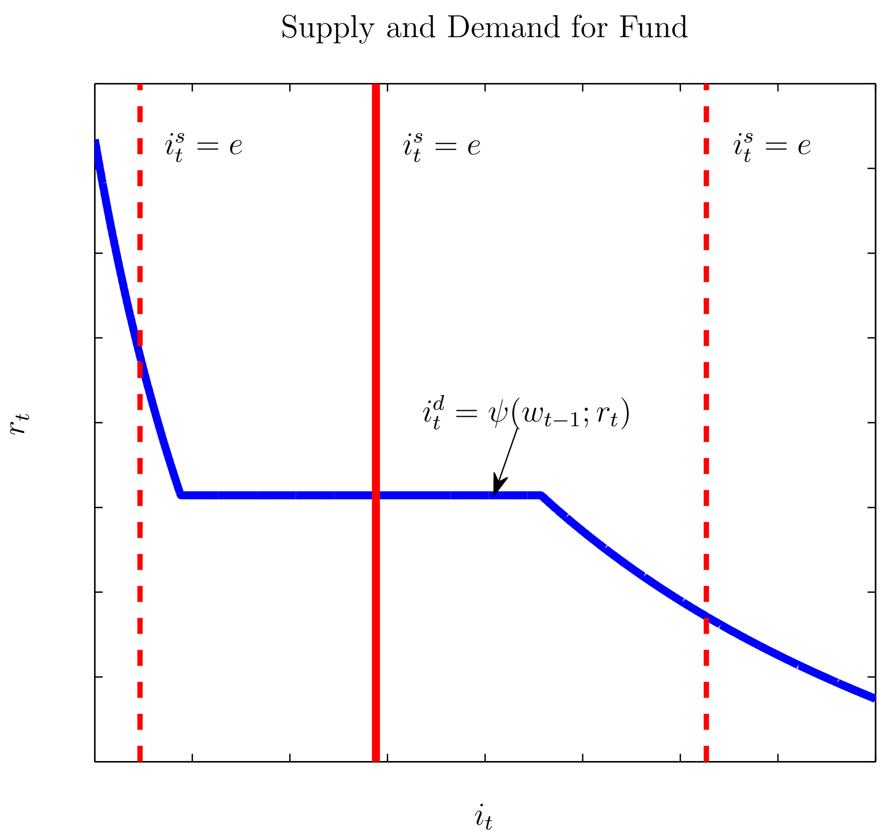

Figure 1: Supply (red) and demand (blue) for fund at any period t as a function of the interest rate. w_{t-1} denotes the wealth of the middle aged, i.e. $(1 + r_{t-1})i_{t-1}$. The two arms on the demand curve correspond to investing only in type 1 or 2. The flat segment in between corresponds to $1 + r_t = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where entrepreneurs mix.

Using the middle condition in 2 I can state the dynamic upper and lower bounds on the interest rate:

$$\theta_1 R_1(2 + r_{t-1}) \leq 1 + r_t \leq \theta_2 R_2(2 + r_{t-1}) \,. \tag{5}$$

Now, I can define a competitive equilibrium:

Definition 1. A competitive equilibrium is a sequence $\{i_t, x_{1t}, x_{2t}, r_t\}_{t=0}^{\infty}$ of investments and interest rates and an initial value of r_{-1} that satisfy conditions 1 to 5, x_{1t} and x_{2t} solve problem II and $1 + r_t < R_1$ and $1 + r_{-1} > 0$ for all $t > 0$.

By market clearing 3, the equilibrium values of liquidity demand is fixed at e in every period. Using this fact and the optimal investment decisions in 2 and the borrowing constraint, one can compute the composition of the aggregate investment portfolio of the middle aged entrepreneurs at any date as follows:

$$\begin{cases} x_{1t} = 0, x_{2t} = (2 + r_{t-1})e & \text{if } 1 + r_t < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ \\ x_{1t} = \left(\frac{\theta_2 R_2 (2 + r_{t-1}) - (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{\theta_2 R_2 - \theta_1 R_1}\right) e, \\ x_{2t} = \left(\frac{(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1 (2 + r_{t-1})}{\theta_2 R_2 - \theta_1 R_1}\right) e & \text{if } 1 + r_t = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ \\ x_{1t} = (2 + r_{t-1})e, x_{2t} = 0 & \text{if } 1 + r_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}). \end{cases} \tag{6}$$

The above results might seem rather counterintuitive at first. When interest rate is lower (higher) than $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, market is more (less) liquid and borrowing is less (more) costly. Hence one expects the entrepreneurs to specialize in the more (less) productive but less (more) liquid investment project i.e x_{1t} (x_{2t}). However, as seen above, exactly the opposite is the case. Entrepreneurs invest in the *more productive* but *less liquid* type of investment when the market is *less liquid* and they invest in the *more liquid* type when the market is *more liquid*. When $1 + r_t$ is lower not only the marginal return to an additional unit of liquidity Λ is higher but the upper and lower bounds on i_t in problem III are both higher.

The problem with this intuition is that it is based on a partial equilibrium analysis. The interest rate $1 + r_t$ is an endogenous variable in general equilibrium. In fact knowing only a single interest rate, say $1 + r_t$ in period t, I can compute the whole trajectory of the interest rates of the past and future periods given $(\boldsymbol{\theta}, \boldsymbol{R})$. This means that the two interest rates $1 + r_t$ in the two upper and lower conditions in 6 are corresponding to two competitive equilibria that are fundamentally different from each other and so such a comparison is not warranted.

Note that by assuming a constant endowment and that entrepreneurs consume only when old, I have eliminated any possible supply responses in general equilibrium, i.e., the sum $x_{1t} + x_{2t}$ is predetermined at time $t - 1$.

Before showing the existence of uniqueness of the competitive equilibrium it is useful to first look at the steady state equilibria. To this end, let me define the following three regions of the parameter space:

Definition 2. Define F as the set of $(\boldsymbol{\theta}, \boldsymbol{R})$ that satisfies Assumption 1 and also $\frac{\theta_1 R_1}{1 - \theta_1 R_1} < R_1$. Then the three regions of F are defined as follows:

Liquid Region is defined as $F_\ell = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) \leq (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\}$.

Mixed Region is defined as $F_m = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

Illiquid Region is defined as $F_i = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) \leq \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

Notice that all three regions, F_ℓ, F_m and F_i have nonempty interiors. I later show an example to illustrate that all the three are nonempty and hence using continuity I can conclude that the interiors are nonempty as well. In the definition above I require the elements of F to satisfy an additional condition namely, $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$. This condition is to ensure that the steady state interest rate in the *illiquid region* is strictly below R_1 so that the competitive equilibrium exists in the *illiquid region* according to Definition 1. [9] This particular partition of F is important because each region has a unique and different steady state equilibrium with different characteristics as suggested by the following lemma:

Lemma 2. Each of the three regions in Definition 2 has a unique and different stable steady state equilibrium. More specifically:

$$\begin{cases} 1 + r_\ell^{ss} = \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell\,. \\ 1 + r_m^{ss} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_m\,. \\ 1 + r_i^{ss} = \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_i\,. \end{cases}$$

Moreover at the steady state, the entrepreneurs specialize in the more liquid and more productive type of investments in regions F_ℓ and F_i respectively. Entrepreneurs invest in both types in F_m where the amounts of each type is given by 6.

Having the steady state equilibria characterized in Lemma 2, I close this subsection by the following proposition that establishes the existence and uniqueness of the competitive equilibrium:

Proposition 1. Given any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, and an initial condition $1+r_{-1} < R_1$, there exists a

[9] When $\frac{\theta_1 R_1}{1-\theta_1 R_1} \geq R_1$, an steady state equilibria exists in the illiquid region where the borrowing constraint does not bind. In this steady state equilibrium $1 + r_i^{ss} = R_1$.

unique competitive equilibrium that converges to the steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$, according to Lemma 2.

2.4 Properties of Equilibria

Figure 2 is an illustration of the three regions for $R_1 = 4$ and $R_2 = 3$ and different values of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. The light gray, medium gray and dark gray indicate F_ℓ, F_m and F_i respectively. The white area below the positively sloped straight line is the region where $\theta_2 R_2 < \theta_1 R_1$ which violates Assumption 1. In this region type one investment dominates type two investment both in terms of return and liquidity. Hence, no entrepreneur invests in type two projects in this region. This is the economy of Farhi and Tirole (2010a) when there is no bubbles or outside liquidity. This is a useful benchmark for my analysis of the efficiency in Section 3.

The figure shows that close to the origin, i.e., when θ_i , $i \in \{1, 2\}$ are both very low, one can have all three kinds of steady state equilibria but this is not the case when pledgeabilities are high. In the latter case no steady state equilibrium lies in the liquid region. The following lemma shows that in fact Figure 2 represents more general properties of the three regions for any given vector of returns:

Proposition 2. For a given vector of returns $\boldsymbol{R}$ satisfying Assumption 1, when $\boldsymbol{\theta}$ is small enough (close to the origin) one can have all three types of steady state equilibria. Given the vector of the returns $\boldsymbol{R}$, for any $\boldsymbol{\theta}$ in the liquid region $\boldsymbol{\theta} \leq (\frac{1}{1+R_1}, \frac{1}{1+R_1})$. For any value of θ_1 the values of θ_2 for which (θ_1, θ_2) belongs to the liquid region lies strictly above the respective values of θ_2 for which (θ_1, θ_2) belongs to the illiquid region. Moreover the boundary of the liquid region is a non-monotonic curve cutting $\theta_1 = 0$ line twice; once at the origin and again at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$. In contrast the inner boundary of the illiquid region is an increasing function of θ_1 which reaches the maximum possible of $\theta_2 = \frac{1}{R_2}$. Finally, the top right corner of F in the space of pledgeabilities that is $\boldsymbol{\theta} = (\frac{1}{1+R_1}, \frac{1}{R_2})$ belongs to the illiquid region.

A few observations are in order. Suppose for the moment that θ_2 is fixed at $\theta_2 = \bar{\theta}_2$, where $\bar{\theta}_2 \leq \frac{1}{1+R_1}$. As one moves along θ_1 dimension in Figure 2, one first passes the liquid region at lower values of θ_1 and then enters the mixed region for medium values of the θ_1 and finally leaves the mixed region to enter into the illiquid region. This part is very intuitive as for low values of θ_1 the pledgeability cost of investing in type one is very high. As the value

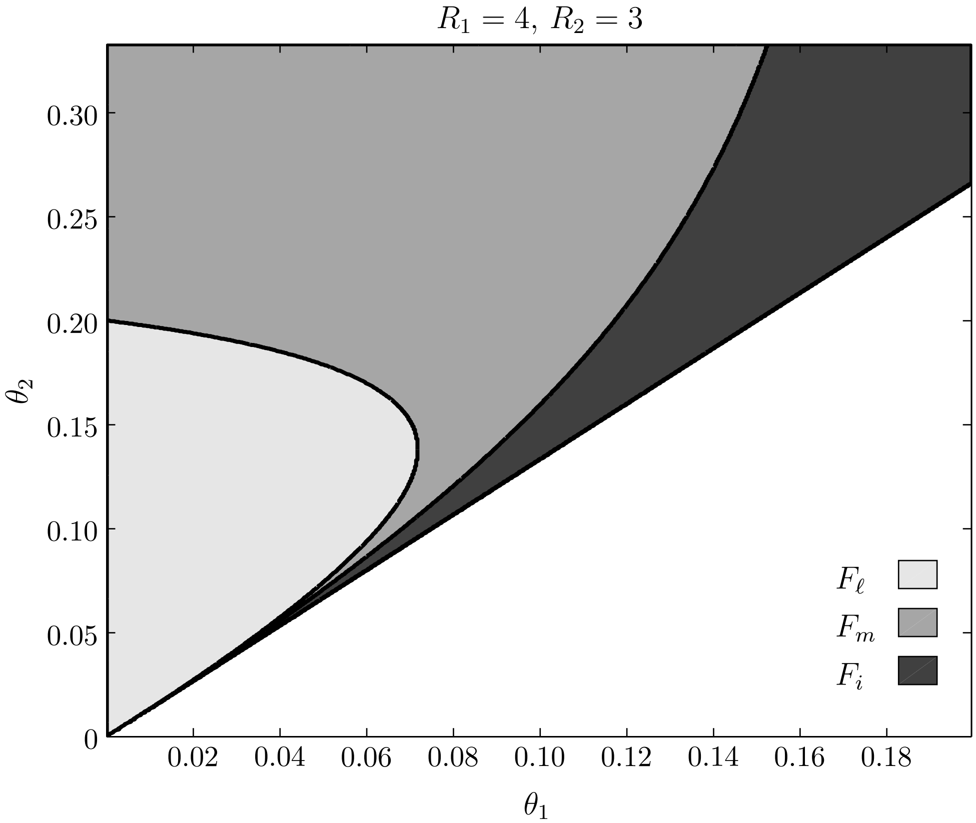

Figure 2: Image of F_ℓ, F_m and F_i for $R_1 = 4$ and $R_2 = 3$ over the space of (θ_1, θ_2).

of θ_1 increases and gets closer to its upper bound the pledgeability gains of investing in type two diminishes further and so it becomes very costly to invest in type two, in terms of the forgone returns to investment.

Let the thresholds of θ_1 after which one enters the mixed and illiquid regions be $\theta_1^m(\bar{\theta}_2)$ and $\theta_1^i(\bar{\theta}_2)$. By Lemma 2, $\theta_1^i(\bar{\theta}_2)$ is increasing in $\bar{\theta}_2$ which is intuitive since higher values of $\bar{\theta}_2$ implies higher cost of forgone pledgeability per unit of investment in type one. However according to Lemma 2, $\theta_1^m(\bar{\theta}_2)$ is not monotonic. At first when $\bar{\theta}_2$ is raised, things look normal as the threshold to enter the mixed region increases but beyond some value of $\bar{\theta}_2$, $\theta_1^m(\bar{\theta}_2)$ begins to decrease and even for $\bar{\theta}_2 \geq \frac{1}{1+R_1}$ the economy fails to have any liquid steady state equilibria. This is a rather counterintuitive observation.

Now let me do the same comparative statics fixing $\theta_1 = \bar{\theta}_1$. As shown in the bottom of Figure 3, there is no liquid steady states for high values of $\bar{\theta}_1$ since type one investment has enough pledgeability itself. For these values of $\bar{\theta}_1$, low θ_2 leads to an illiquid equilibrium but as one increases θ_2 and after some threshold, one enters the mixed region where the entrepreneurs invest in both types in the steady state. This is because higher θ_2 naturally makes type two investment more and more attractive relative to type one. The threshold to enter the mixed region is increasing in $\bar{\theta}_1$ by Lemma 2 due to the fact that it gets more difficult for type two to compete with type one when type one has higher pledgeable returns.

The pattern is, however, different for lower values of $\bar{\theta}_1$ where liquid steady states exist, as illustrated in the top of Figure 3. In this case when θ_2 is increased, one first passes the illiquid region and then enters the mixed and liquid regions subsequently similar to the previous case. If one increases θ_2 even further, one enters the mixed region for the second time and entrepreneurs start investing in type one again. One can notice a similar pattern for high $\theta_1 = \bar{\theta}_1$ at the bottom of Figure 3 where the ratio of liquid investment to the total in the mixed region is a humped shaped curve. This pattern is stated formally in the following lemma:

Lemma 3. *Given any $\boldsymbol{R}$ satisfying Assumption 1, the ratio of investment in the more liquid type to the total investment at the steady state, i.e. $\frac{x_2^{ss}}{x_1^{ss}+x_2^{ss}}$, is non monotone in θ_2 and has a unique interior maximum for relatively low values of θ_1. In contrast, this ratio is always weakly decreasing in θ_1 and strictly decreasing in θ_1 when $\boldsymbol{\theta} \in F_m$. Moreover, the steady state interest rate, i.e. $1 + r_z^{ss}$ for $z \in \{\ell, m, i\}$, is non monotone in θ_1 while it is weakly increasing in θ_2 and strictly increasing in θ_2 when $\boldsymbol{\theta} \in F_m$.*

To understand Lemma 3, suppose θ_j increases while θ_{-j} is held constant, where $\{j, -j\} = \{1, 2\}$. On one hand, this increase makes type j investment more attractive and so incentivizes the middle aged entrepreneurs to raise the share of type j in total investment. This is a partial equilibrium effect. On the other hand, this increase in pledgeability of type j affects the interest rate at the steady state which is a general equilibrium effect. If this general equilibrium effect leads to a lower (higher) interest rate, it encourages more investment in the more liquid (more productive) type. Hence the general equilibrium effect can potentially reinforce or undermine the partial equilibrium effect. Note that the mixed region, F_m, is the interesting region to look at the direction of the general equilibrium effect, since share of liquid investment is simply zero or one in the other two regions. Now observe that:

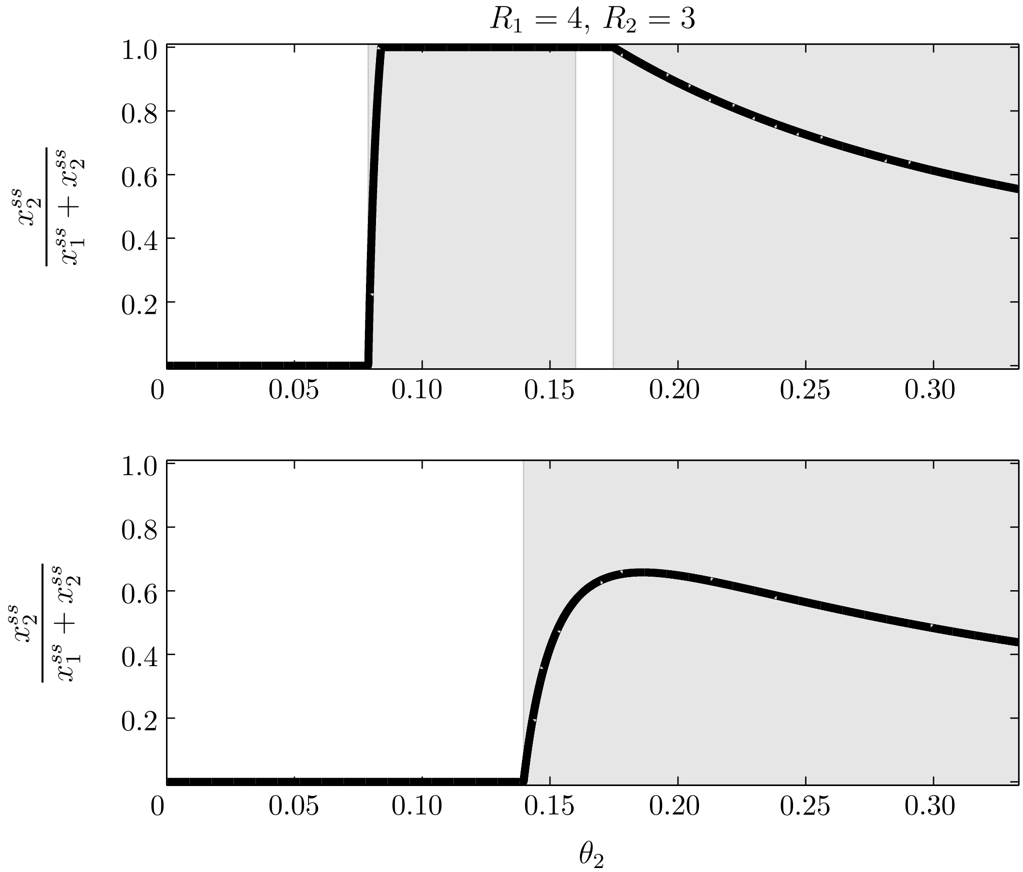

Figure 3: Two illustrations of the fraction of liquid investment to the total investment at the steady state for two values of θ_1, and different values of θ_2. The top and the bottom figures correspond respectively to $\theta_1 = 0.055$ and $\theta_1 = 0.09$. Shaded regions indicate inefficient steady state equilibria.

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} = \frac{(1-\theta_2)R_1R_2(R_2-R_1)}{\Big((1-\theta_1)R_1-(1-\theta_2)R_2\Big)^2} < 0\,, \tag{7}$$

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(1-\theta_1)R_1R_2(R_1-R_2)}{\Big((1-\theta_1)R_1-(1-\theta_2)R_2\Big)^2} > 0\,. \tag{8}$$

8 is intuitive but 7 is less straightforward. An increase in θ_1 makes the more productive type more attractive. This, in turn, encourages the entrepreneurs to a disproportionately invest in the more productive type. Since the more productive type is still the less liquid one, this results in a lower interest rate. Therefore, the general equilibrium effect reinforces

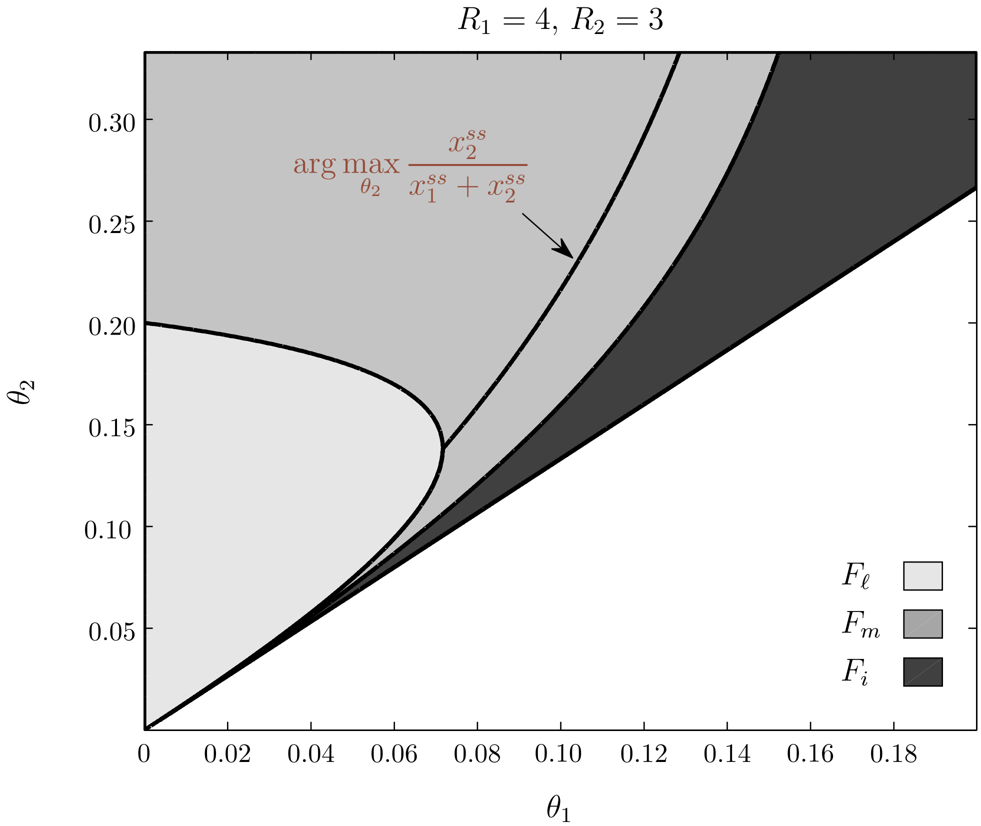

Figure 4: Locus of values of θ_2 that maximizes the fraction of total investment in liquid type, i.e. $\frac{x_2^{ss}}{x_1^{ss}+x_2^{ss}}$, for a given θ_1.

the partial equilibrium effect for θ_1 but weakens it for θ_2 and becomes the dominant effect for high values of θ_2.

As discussed above the behavior of interest rate is asymmetric with respect to θ_1 and θ_2. Figure 5 is a contour plot of the steady state interest rate in the three regions. As shown in the figure, interest rate is weakly increasing in θ_2 but non monotone in θ_1.

3 Welfare and Efficiency

In this section I study the welfare properties of the steady state competitive equilibria in the three regions of Definition 2. In this regard, it is important to be explicit about the welfare measures based on which I want to assess the efficiency of equilibria. First, I state a basic definition that will be helpful later in this section.

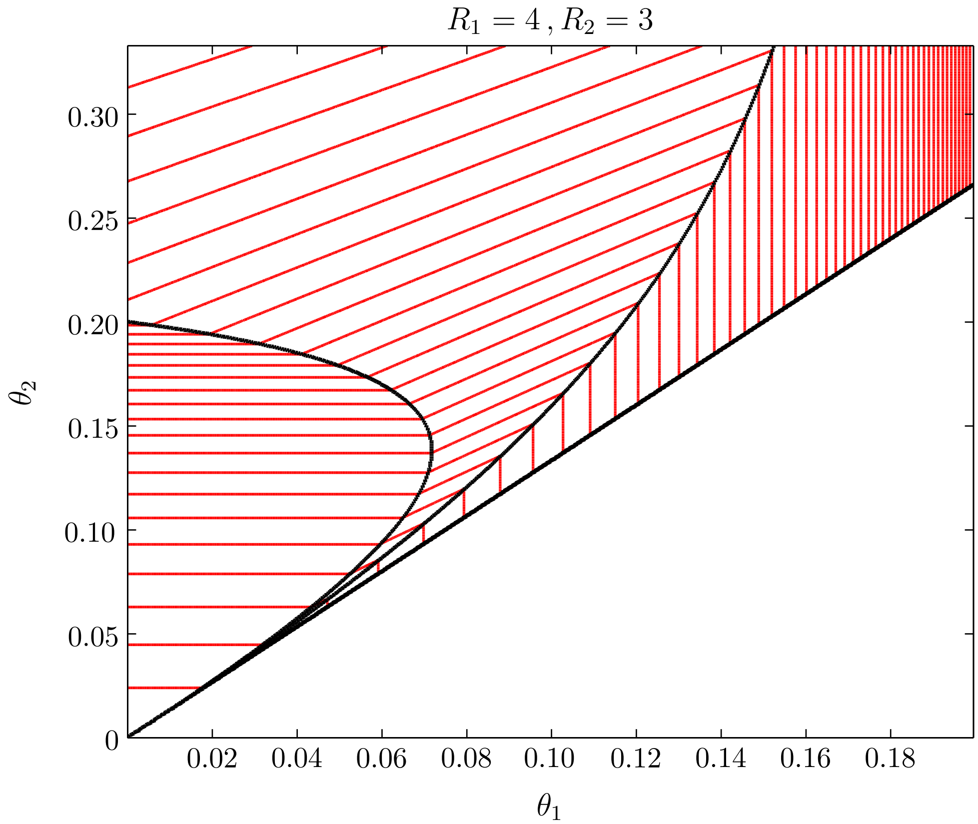

Figure 5: Contour plot of the steady state interest rates (red lines) for the three regions. Interest rate is highest at the top left corner.

Definition 3. An allocation in the overlapping generations economy is called **Constrained Pareto Efficient** if a social planner cannot reallocate the resources to make at least one entrepreneur strictly better off while keeping all others at least as well off *and* the reallocation respects the pledgeability constraint in problems I, II and III. More formally, an allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient if it is feasible, i.e. it satisfies the following series of constraints for all $t \geq 0$:

$$\begin{cases} c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e\,, \\ \\ x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e\,. \end{cases} \tag{9}$$

and there does not exist any feasible allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ such that $c_t \geq c_t^*$ for all $t \geq 0$ with at least one strict inequality, given initials $x_{j,-1} = x_{j,-1}^*$ for $j \in \{1, 2\}$.

Equipped with the above definition, I can evaluate the steady state equilibrium welfare of the model economy for different parameter values. A benchmark helps to understand the model in a more subtle manner. Therefore it is useful as a first step to look at the welfare consequences of the model in the absence of the liquidity-return trade off. In this case when type one investment has also higher pledgeability so that $\theta_1 R_1 > \theta_2 R_2$, Assumption 1 is violated and the model collapses to a model with only one type of investment as in the Farhi and Tirole (2010a). Below I restate their result about efficiency of the competitive equilibrium when type one dominates type two in liquidity and return to the investment:

Proposition 3. (Farhi and Tirole (2010a)) Suppose $R_1 > R_1 > 1$ and $1 > \theta_1 R_1 > \theta_2 R_2$. Then entrepreneurs invest only in type one. Moreover the allocation of resources in any competitive equilibrium is constrained Pareto efficient.

3.1 Efficiency of Competitive Equilibria

In this section I study constrained Pareto efficiency of competitive equilibria in the overlapping generations economy. Typical models of overlapping generations in which Pareto inefficiency arises feature overinvestment. Diamond (1965) presents an environment where such an overinvestment takes place and shows that it leads to inefficiently high capital stock. Abel et al. (1989) characterize, in a more general setup, testable conditions under which this overinvestment leads to Pareto inefficiency. A celebrated test of inefficiency due to overinvestment, at least in theory, is whether the prevailing interest rate is less than the growth rate of the output. Investing too much reduces the return to capital to a level below the growth rate of aggregate output. Similar to Samuelson (1958), Pareto inefficiency induced by overinvestment implies an interest rate that is too low.

In contrast with the benchmark economy where there is only one type of investment and in the absence of the liquidity and return trade off, I show that competitive equilibria can be constrained Pareto inefficient in regions of parameters characterized by Assumption 1. I further show in Section 3.2, that the interest rate in a constrained Pareto inefficient equilibrium *is too high despite being negative* and so a Pareto improvement is compatible with a lower (and even more negative) interest rate.[10] This implies that the cause of inefficiency in this model is different from that of the overinvestment models. I prove that all equilibria in $F_\ell \cup F_m$ where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$, are constrained Pareto inefficient.

[10] I show below that the Pareto improving reallocation is equivalent to a regulated steady state where the interest rate is lower than the unregulated steady state equilibrium.

For the moment consider only steady state equilibria. Let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$, where investment in the liquid type is strictly positive in steady state. Suppose the planner reduces the aggregate (debt) payments of all middle aged entrepreneurs in every generation to the young, $(1 + r_z^{ss})e, z \in \{m, \ell\}$, by an amount $\delta > 0$. The pledgeability constraint will be slack as a consequence of this reduction and the planner can raise the more productive investment, x_1, at the expense of a reduction in x_2. Let the increase in x_1 be $\epsilon > 0$, then the resource constraint implies that x_2 has to be reduced by $\epsilon + \delta$ to make this increase possible. Now given δ, the maximum possible ϵ is determined when the pledgeability constraint binds:

$$\delta = (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta \,,$$

$$\epsilon = \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\delta \,.$$

The two sides of the equation above is simply changes in the two sides of the pledgeability constraint. The change in the consumption level of the old is simply $R_1\epsilon - (\epsilon + \delta)R_2 + \delta$ and hence for $z \in \{m, \ell\}$ is equal to:

$$\Delta V_z^{ss} = \left(\frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1} R_1 - (1 + \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}) R_2 + 1 \right) \delta \,. \tag{10}$$

Note that the change in consumption of the initial middle aged is always strictly positive:

$$\Delta V_0 = \left(1 + \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1} \right) \delta > 0 \,. \tag{11}$$

The initial middle aged is strictly better off because the planner reduces her debt payments to the young by substituting the more productive type for the more liquid type while, in contrast with future middle aged, her receipts from the initial old do not change.[11] Therefore if $\Delta V_z^{ss} \geq 0$, the steady state allocation is constrained Pareto inefficient. Now one has:

$$\Delta V_z^{ss} \geq 0 \;\Leftrightarrow\; r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0 \,. \tag{12}$$

Hence, the region of $(\boldsymbol{\theta}, \boldsymbol{R})$ where this reallocation leads to a higher consumption for at least one generation (every generation when $\Delta V_z^{ss} > 0$) coincides with the steady state equilibria of $F_\ell \cup F_m$ where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, that is, the light gray region in Figure 6.

[11] Note that the initial old pays off her debt, which is given as an initial condition and is not changed by

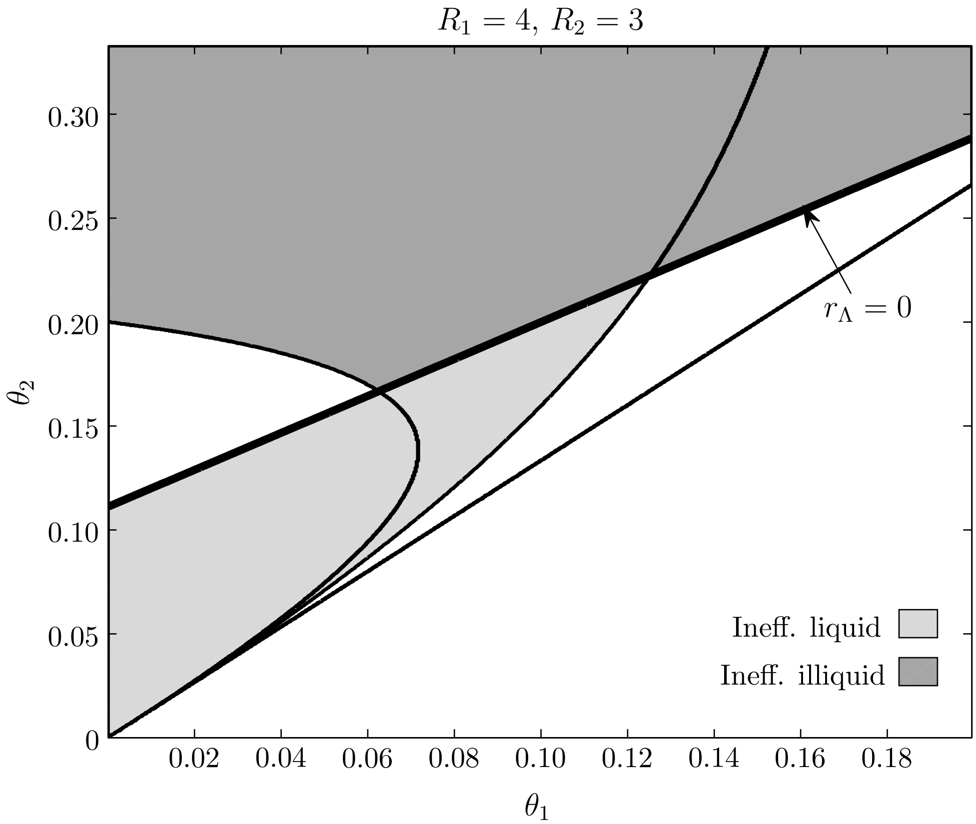

Figure 6: An illustration of inefficiently liquid and inefficiently illiquid equilibria and the line corresponding to $r_\Lambda = 0$, for $R_1 = 4$ and $R_2 = 3$.

Note that the above reallocation can also work outside the steady state. In any competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$, condition 12 determines whether a similar reallocation can lead to a Pareto improvement. This is of course not the case for $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_i$ because by Lemma 2, any competitive equilibrium converges to the illiquid steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_i$ where there is no investment in the liquid type, i.e., $x_2^{ss} = 0$. I can summarize the results in the following proposition:

Proposition 4. Consider any competitive equilibrium with pledgeabilities and returns given by $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, the competitive equilibrium is constrained Pareto inefficient. Moreover, the equilibrium interest rate at the steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$ is strictly negative when $(\boldsymbol{\theta}, \boldsymbol{R})$ lies in the interior of the inefficient region, i.e. $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0\}$, and zero on its boundary.

This is in sharp contrast with the efficiency results in Farhi and Tirole (2010a). In an

the planner, to the middle aged at $t = 0$, consumes and dies.

economy where the private entrepreneurs have access only to one type of investment, say the first type (θ_1, R_1), there will be no inefficiency by Proposition 3 as long as $R_1 > 1$. In this paper, I have added an endogenous choice between more liquidity and higher return. The results above shows that this endogeneity can lead to inefficient private allocations. I show below that typically there are subregions in both liquid and illiquid regions where the steady state equilibria are constrained Pareto efficient.

The social planner can also do the opposite by increasing the aggregate debt payments by $\delta > 0$ when there is strictly positive investment in the productive type along the equilibrium path. In that case condition 12 will be reversed to $\Delta V_z^{ss} > 0, z \in \{m, i\}$, if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. When $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i$ and $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, this reallocation increases the utility of every generation starting from $t = 1$ but is not a Pareto improvement because the initial middle aged would suffer.

Note, however, that the overlapping generations economy can be reinterpreted as follows. There is a continuum of each of three types of individuals living forever with the same technology and asynchronicity between receiving endowment, production and consumption opportunities. More explicitly type $i \in \{0, 1, 2\}$ receives endowment $e > 0$ in periods $(i + 1 \bmod 3) + 3k$, produces in $(i + 2 \bmod 3) + 3k$ and consumes in periods $(i \bmod 3) + 3k$ for all $k \in \{0, 1, 2, ...\}$. Let all individuals discount future consumption at a constant rate $\beta \in (0, 1)$ so that the utility function of type $j \in \{0, 1, 2\}$ is:

$$U_j = \sum_{k=0}^{\infty} \beta^k c_{j+3k}$$

This economy is similar, though not isomorphic, to the one in Kiyotaki and Moore (2002) and Woodford (1990).[12] With this interpretation, the proposed reallocation above would be a Pareto improvement, i.e. at least one type is made strictly better off while the other two are at least as well off, for a high enough discount factor β. This is stated in the following lemma.

Lemma 4. Let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i$ such that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. Given any competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$ reinterpreted as above, a planner can make a Pareto improvement by increasing the intertemporal debt payments by some amount $\delta > 0$, while respecting the pledgeability constraints, if $1 > \beta \geq \bar{\beta}(\boldsymbol{\theta}, \boldsymbol{R})$ for some threshold $\bar{\beta}(\boldsymbol{\theta}, \boldsymbol{R}) \in (0, 1)$. Moreover

[12] In Woodford (1990) it is shown that the infinite horizon model is equivalent to the overlapping generations presented in Diamond (1965) along the relevant dimensions.

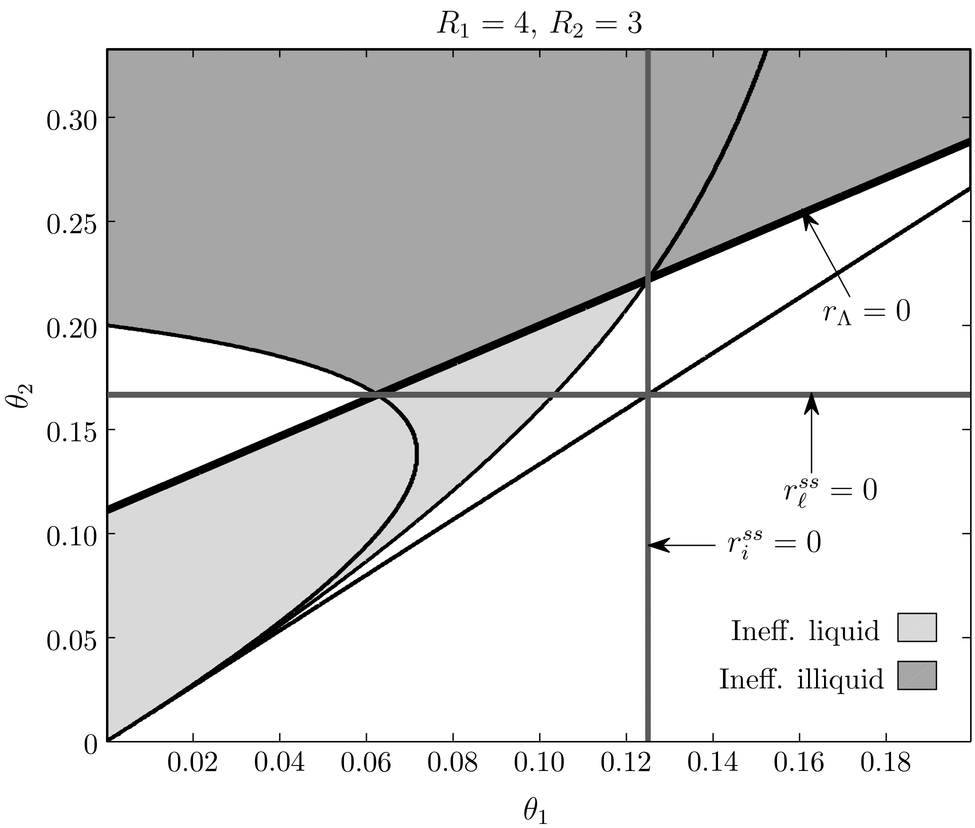

Figure 7: The insufficiency of the interest rate for determining the constrained steady state efficiency.

the steady state interest rate in $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0\}$ is strictly positive.

The set $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \cup F_i \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0\}$ is the region which lies above $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ line in Figure 6 with dark gray color. As the figure suggests, the two regions that give rise to constrained Pareto inefficiency in Proposition 4 and Lemma 4 can be called *inefficiently liquid* and *inefficiently illiquid* regions respectively. As the above Pareto improving reallocation illustrates, there is too much investment in the more liquid type (more productive type) in competitive equilibria of the light gray region (dark gray region).

Sign of the interest rate, as a standard indicator of inefficiency, is not sufficient to determine the inefficiency in the two regions. Figure 7 is an illustration of this insufficiency. The bold gray lines (one horizontal and the other vertical) correspond to the zero steady state interest rate, i.e. $r_\ell^{ss} = 0$ and $r_i^{ss} = 0$, in the figure. The $r_\ell^{ss} = 0$ line cuts the liquid region F_ℓ into two parts with negative and positive interest rates in steady state equilibrium. The inefficiently liquid equilibria in F_ℓ is a proper subset of F_ℓ with strictly negative interest rate.

Hence there is a nonempty subset of F_ℓ featuring negative interest rate in steady state that are not inefficiently liquid. The same arguments apply to the case of illiquid region; there is a nonempty subset of equilibria in F_i featuring strictly positive interest rate which are not inefficiently illiquid.[13]

I show in Section 3.2 that the above reallocations can be implemented by regulating the portfolio choice of the middle aged entrepreneurs. The interest rate under this Pareto improving regulation is shown to be lower (higher) than the competitive equilibrium interest rate in the inefficiently liquid (inefficiently illiquid) region. In this sense, inefficiently illiquid equilibria are similar to the more traditional type of inefficient equilibria with credit market frictions, as in Woodford (1990). Section 4 studies the effects of public liquidity in the form of government bonds in F and shows these effects to be different in the two regions.

Figure 6 suggests that for every $\boldsymbol{R} = (R_1, R_2)$, the set of inefficient equilibria is nonempty. The straight line $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ has interior intersections with the two vertical boundaries of the rectangular region namely $\theta_1 = 0, \theta_1 = \frac{1}{1+R_1}$ and Proposition 2 implies that the points close to the origin belong to the liquid region while the points close to the top right corner of the F belong to the illiquid region. The following lemma summarizes some of the properties of the two inefficient regions in F:

Lemma 5. For any $\boldsymbol{R}$ satisfying Assumption 1 the following are correct. The sets of inefficiently liquid and inefficiently illiquid competitive equilibria in F are nonempty with strictly positive measures. There are inefficiently liquid equilibria in any arbitrarily small neighborhood of the origin. The top right corner of F, i.e., $\boldsymbol{\theta} = (\frac{1}{1+R_1}, \frac{1}{R_2})$, corresponds to inefficiently illiquid equilibria. The set of inefficiently illiquid equilibria belonging to F_i (call it A_i) is a proper subset of F_i. The set of inefficiently liquid equilibria in F_ℓ (call it B_ℓ) is a proper subset of F_ℓ if and only if $\frac{R_1 - R_2}{R_2 - 1} \leq 1$. Finally, if $(\theta_1, \theta_2) \in A_i$ and $(\theta_1', \theta_2') \in B_\ell$ then one must have $\theta_1' < \theta_1$ and $\theta_2' < \theta_2$.

By Proposition 5, the inefficiently illiquid part of F_i lies to the top right of the inefficiently liquid part of F_ℓ. This implies that, only looking at F_ℓ and F_i, the economy becomes too liquid (too illiquid) when the pledgeabilities are relatively low (high).[14] Another implication of Proposition 5 is illustrated in Figure 8. When the difference between returns to the two

[13] One other point in Figure 7 is the fact that the zero interest rate lines, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and the boundaries of the liquid and illiquid regions intersect at two points which is not a coincidence.

[14] First, note that in one sense, pledgeabilities, i.e. $\boldsymbol{\theta}$, are low for all $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and that is why the borrowing constraint is binding. Hence "high" and "low" should be understood in relative terms when comparing different regions of F. Second, this statement is correct only considering F_ℓ and F_i and *not* F_m.

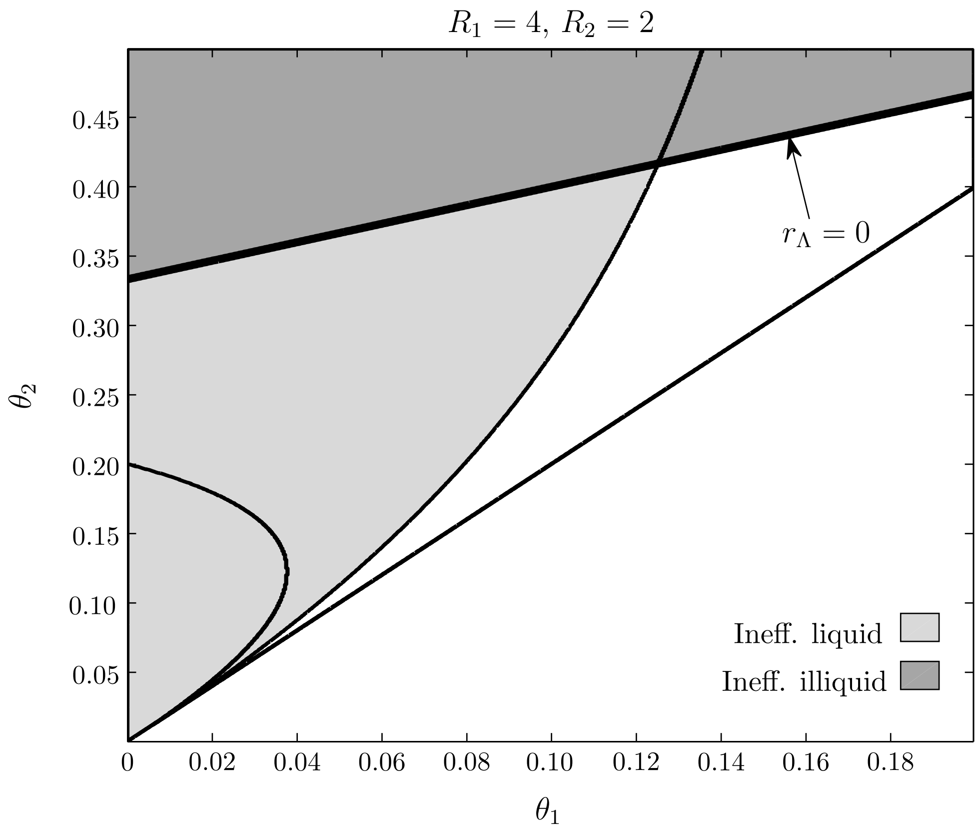

Figure 8: This figure shows the expansion of the inefficiently liquid region when $\boldsymbol{R} = (4, 2)$. In contrast with the case of $\boldsymbol{R} = (4, 3)$, all competitive equilibria in liquid region F_ℓ are constrained Pareto inefficient.

types of investment goes beyond some threshold, all equilibria in F_ℓ becomes inefficiently liquid. In other words, an increase in the difference between returns to the two types expands the inefficiently liquid region in F. I close this section by characterizing the set of constrained Pareto efficient allocation in the following proposition.

PROPOSITION 5. Let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, any allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$ is constrained Pareto efficient. Consequently, any competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$ is constrained Pareto efficient. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, any allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$ *and* has $x_{2t} = 0, t \geq T$ for some $T \geq 0$ is constrained Pareto efficient. Consequently, any competitive equilibrium in F_i is constrained Pareto efficient.

3.2 Regulated Economy

(Under Revision)

In this section, I study a regulation that can implement the Pareto improving reallocation proposed in Section 3.1. Consider the steady state equilibria of the three regions and suppose that a social planner (e.g. government) can regulate investment portfolios of the middle aged entrepreneurs. More precisely suppose that the social planner can regulate the fraction α_ℓ of total investible funds, $(1+r_{t-1})i_{t-1}+i_t$, at the hands of middle aged entrepreneurs, that is invested in the more liquid type. Therefore:

$$\alpha_\ell = \frac{x_{2t}}{i_t + (1+r_{t-1})i_{t-1}} .$$

In this case the entrepreneur only chooses the level of new funds raised i_t and so the maximization problem of the middle aged entrepreneurs takes the following form:

$$\begin{aligned} \max_{i_t \geq 0} \quad & ((1-\alpha_\ell)R_1 + \alpha_\ell R_2)\left(i_t + (1+r_{t-1})i_{t-1}\right) - (1+r_t)i_t \\ s.t. \quad & (1+r_t)i_t \leq (\theta_1(1-\alpha_\ell)R_1 + \theta_2\alpha_\ell R_2)\left(i_t + (1+r_{t-1})i_{t-1}\right) \end{aligned} \tag{IV}$$

The only constraint in the above problem is the pledgeability constraint when the choice of liquidity-return is regulated by the planner. Let $R_\alpha = (1-\alpha_\ell)R_1 + \alpha_\ell R_2$ and $\gamma_\alpha = \theta_1(1-\alpha_\ell)R_1 + \theta_2\alpha_\ell R_2$ be the marginal product and marginal pledgeable return of the regulated portfolio. Problem IV is the maximization problem of an entrepreneur that has access only to one type of investment project with return R_α and pledgeability of γ_α. The optimal solution to IV is:

$$\begin{cases} i_t = \left(\frac{(1+r_{t-1})\gamma_\alpha}{(1+r_t)-\gamma_\alpha}\right) i_{t-1} & \text{if } R_\alpha \geq 1+r_t \,, \\ i_t = 0 & \text{if } R_\alpha < 1+r_t \,. \end{cases}$$

Notice that $\gamma_\alpha < 1$ by Assumption 1. Also note that for any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, there exists an $\epsilon > 0$ such that $\left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right) < R_\alpha$ for all $\alpha_\ell \in [0, \epsilon)$. This is because the inequality holds for $\alpha_\ell = 0$ according to the definition of F and so by continuity it holds in a neighborhood of zero. When α_ℓ is in this neighborhood the steady state equilibrium of the regulated economy is:

$$1 + r_\alpha^{ss} = \left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right) . \tag{13}$$

For these values of α_ℓ, one can simply compute the steady state utility level using IV and the market clearing conditions. Using the objective function in problem III, Lemma 2 and the market clearing conditions I can obtain the steady state utility levels for the three regions. These values are stated in the following lemma:

Proposition 6. The steady state level of utility for any values of α_ℓ for which $\left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right) < R_\alpha$ is given by:

$$V_\alpha^{ss} = \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right) e = \left(\frac{(1-\alpha_\ell)(1-\theta_1)R_1 + \alpha_\ell(1-\theta_2)R_2}{(1-\alpha_\ell)(1-\theta_1 R_1) + \alpha_\ell(1-\theta_2 R_2)}\right) e\,.$$

The steady state utility levels for the three regions in Definition 2 are, $V_\ell^{ss} = \left(\frac{(1-\theta_2)R_2}{1-\theta_2 R_2}\right) e$, $V_i^{ss} = \left(\frac{(1-\theta_1)R_1}{1-\theta_1 R_1}\right) e$ and:

$$V_m^{ss} = \left(\frac{(\theta_2-\theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)}\right) e\,.$$

Moreover suppose $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and that $\widetilde{\alpha}_\ell = \left(\frac{x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{x_1^{ss}(\boldsymbol{\theta},\boldsymbol{R}) + x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}\right)$. Then in the regulated economy one has:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 1 + r_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)\theta_1 R_1 + \widetilde{\alpha}_\ell \theta_2 R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right)\,,$$

$$V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = V_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)(1-\theta_1)R_1 + \widetilde{\alpha}_\ell(1-\theta_2)R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) e\,.$$

The last part of Proposition 6 reads as follows. Consider an steady state of the mixed region where entrepreneurs invest in both types of investment projects. Suppose the social planner sets $\alpha_\ell = \widetilde{\alpha}_\ell$, that is, equal to the ratio of the liquid investment to total investment that prevails in the steady state competitive equilibrium given by 6. Then the resulting interest rates and utility levels of the steady state equilibrium and the regulated economy will be the same. In this sense, there is a close relationship between the ratio of liquid investment to the total investment, the interest rate and utility level at any steady state equilibrium.

Note that the numerator and denominator of V_α^{ss} are weighted averages of those of V_ℓ^{ss}

and V_i^{ss}. This implies that V_α^{ss} always lies between the two values of V_ℓ^{ss} and V_i^{ss}. The following lemma provides conditions under which a social planner can achieve a higher level of steady state consumption by regulating the mixture of liquid and productive types of investments in each of the three regions:

LEMMA 6. For any values of $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, the following statements are correct. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ and $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ have the same sign. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is negative if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$.

The expressions of the type $Z(\boldsymbol{\theta}, \boldsymbol{R})$ for variable Z should be considered solely as functions of parameters $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ in Lemma 6 which may or may not be an equilibrium utility level or interest rate for the particular vector $(\boldsymbol{\theta}, \boldsymbol{R})$. For instance a vector $(\boldsymbol{\theta}, \boldsymbol{R})$ may correspond to a liquid steady state where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is not the value of the interest rate but nonetheless sign of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is still the right criterion for welfare comparisons according to Lemma 6. Henceforth when I specify $(\boldsymbol{\theta}, \boldsymbol{R})$ in front of any variable, I imply taking that variable solely as a function which may not be an equilibrium value for $(\boldsymbol{\theta}, \boldsymbol{R})$.

Lemma 6 shows a close connection between sign of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and the possibility of raising the steady state utility via regulation. This is in fact what one could expect from the results obtained in Section 3.1. An immediate implication of Lemma 6 is that the planner can achieve a higher steady state utility for all $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ except a measure zero set defined by $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$. For instance suppose that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and so according to Lemma 6 one must have $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$. Let $\widetilde{\alpha}_\ell$ be the ratio of liquid investment to the total funds invested for the particular $(\boldsymbol{\theta}, \boldsymbol{R})$. Then the planner can set the fraction of liquid investment to total equal to $\widetilde{\alpha}_\ell - \epsilon$ for $\epsilon > 0$ and raise the steady state utility of the entrepreneurs.

Notice that when $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$, the interest rate in the regulated economy will be lower than the competitive equilibrium. The reason is that the planner reduces the pledgeability of return to the aggregate investment portfolio which reduces the liquidity demand of the middle aged entrepreneurs. Although the competitive equilibrium interest rate is *strictly negative* in this case, the planner can increase the steady state welfare by *reducing* the equilibrium interest rate to an even more negative level. This is in contrast with the type of inefficiency in the overlapping generations models (e.g. Samuelson (1958)), where a negative interest rate has to be *raised* by a planner via intergenerational transfers to achieve efficiency.

In the opposite case where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and hence $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$, the planner can raise the fraction of liquid investments $\widetilde{\alpha}_\ell$, to $\widetilde{\alpha}_\ell + \epsilon$ for $\epsilon > 0$ to raise the steady state welfare. In this case the interest rate in the steady state competitive equilibrium is too low despite being strictly positive. Therefore, the planner must raise the interest rate, by raising the liquidity demand, to an even more positive level in order to raise utility. In this sense, this case is similar to Woodford (1990) where the social planner (i.e. government) can raise the steady state welfare by issuing more government liabilities to the private sector.

Regulating the investment portfolio of middle aged entrepreneurs works outside the steady state as well. The following proposition shows that the reallocation studied in Section 3.1 can be implemented by this regulation. It also shows that the same effects on the interest rates as above can be observed in all inefficient competitive equilibria:

Proposition 7. Any Pareto improving reallocation of the type analyzed in Section 3.1, when δ is small enough in absolute value, can be implemented by regulating the investment portfolios of the entrepreneurs. In an *inefficiently liquid* equilibrium, planner chooses a lower liquid investment to total investment ratio, the steady state interest rate is *negative* and the regulated interest rate is *lower* than in the unregulated equilibrium. In an *inefficiently illiquid* equilibrium, planner chooses a higher liquid investment to total investment ratio, the steady state interest rate is *positive* and the regulated interest rate is *higher* than in the unregulated equilibrium. Moreover, given any inefficiently liquid equilibria, this regulation can implement a Pareto improvement reallocation that results in a constrained Pareto efficient allocation.

3.3 Discussion

The reallocations considered in this section clearly show that the aggregate liquidity that is transferred between entrepreneurs from different generations may not be socially optimal. Entrepreneurs either provide too much liquidity for the next generation by investing too much of their resources in liquid type, or provide too little liquidity by investing too much in the more productive type. The inefficiency of this intertemporal liquidity transfer is confirmed by the observation in Section 3.2 that a social planner can raise the utility by changing the investment composition. To see how portfolio choices can entail an externality consider the problem of the middle aged entrepreneurs again:

$$\begin{aligned}
\max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1+r_t) i_t \\
s.t. \quad & x_{1t} + x_{2t} \leq w_{t-1} + i_t \,, \\
& (1+r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \,.
\end{aligned}$$

$w_{t-1} = (1 + r_{t-1})i_{t-1}$ is initial wealth of the middle aged which is determined at $t-1$. Suppose for the moment $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ and let $r_{t-1} < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or equivalently $w_{t-1} < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))e$. In other words, entrepreneurs have a relatively low initial wealth in period t. Given these assumptions on model parameters and initial wealth of the middle aged, the static competitive equilibrium in the credit market at t is:

$$\begin{cases} i_t = e \\ 1 + r_t = \theta_2 R_2 (2 + r_{t-1}) \\ V = (1 - \theta_2) R_2 (w_{t-1} + e) \end{cases}$$

In this equilibrium all middle aged invest only in type two that is the more liquid type. V is the maximized value of the objective function of the middle aged in the competitive equilibrium at t. Now consider a government which prohibits any investment in type two, i.e. it forces the middle aged to set $x_{2t} = 0$. It is easy to see that given the same parameters and initial condition on wealth of the middle aged, the competitive equilibrium in the regulated economy is given by:

$$\begin{cases} \widetilde{i}_t = e \\ 1 + \widetilde{r}_t = \theta_1 R_1 (2 + r_{t-1}) \\ \widetilde{V} = (1 - \theta_1) R_1 (w_{t-1} + e) \end{cases}$$

Since $(1 - \theta_1)R_1 > (1 - \theta_2)R_2$, the regulation raises the consumption of the middle aged, i.e. $\widetilde{V} > V$. Absent regulation, middle aged entrepreneurs have more flexibility in their investment but this flexibility in portfolio decisions can lead to an excessively high level of interest rate, $1 + r_t$.

As shown in Section 3.1, when $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, the equilibrium is constrained Pareto inefficient if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$. Note that by Proposition 6, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ is equivalent to $\frac{(1-\theta_1)R_1}{1-\theta_1 R_1} > \frac{(1-\theta_2)R_2}{1-\theta_2 R_2}$, where the two sides of the latter inequality are steady state utility levels when entrepreneurs are allowed to invest only in type one and two respectively.[15] Under Assumption 1, one always have $(1-\theta_1)R_1 > (1-\theta_2)R_2$. One needs, however, a stronger assumption, i.e., $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$, to have inefficiency in the fully dynamic setting because one

[15] To be more precise, the steady state utility level of investing only in type $j \in \{1, 2\}$ is $\frac{(1-\theta_j)R_j}{1-\theta_j R_j} e$.

should consider the welfare of future middle aged at $t+1, t+2, \ldots$ as well.

In contrast with the above, inefficient investment in the more productive type may occur when the initial wealth of the middle aged is high. More specifically, suppose $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_i$ and middle aged entrepreneurs have relatively high initial wealth, i.e. w_{t-1} is high. In this case, a regulation that forces the middle aged to invest only in the liquid type can improve the utility. When $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, a low initial wealth does not allow the interest rate to go up sufficiently to make entrepreneurs' investment in the more liquid type unprofitable. Similarly, when $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_i$, a high initial wealth bids up the interest rate too much even when all middle aged invest only in the more productive type.

One should note that in general equilibrium the initial wealth w_{t-1} is determined endogenously. Low (high) w_{t-1} is an equilibrium outcome when pledgeabilities of the two types are both low (high). This is in fact what Proposition 2 shows and Figure 6 suggests. The inefficiently illiquid equilibria in F_i has relatively higher pledgeabilities, i.e. $\boldsymbol{\theta}$, than the inefficiently liquid equilibria in F_ℓ.[16] That is why the inefficiently liquid steady states have strictly negative interest rate while the inefficiently illiquid steady state equilibria have strictly positive interest rate.

One important aspect of the inefficiencies discussed so far is that it does not rely on the presence of multiple equilibria and the two types of inefficient equilibria occur in different regions of parameter space. While multiple equilibria can be a relevant result in some environment, it usually makes welfare analysis more difficult.

As mentioned in Section 3.2 and Section 3.1, a Pareto improvement (by regulating the mix of investment types) would raise the interest rate in the inefficiently illiquid and reduce it in the inefficiently liquid equilibria. This is optimal despite the fact that interest rate is strictly negative and strictly positive in inefficiently liquid and illiquid regions respectively. In this sense, inefficiently illiquid equilibria are similar to the more traditional type of inefficient equilibria with credit market frictions, as in Woodford (1990). Moreover and in contrast with overinvestment models, a low interest rate is a sign that the interest rate is *too low* while here a negative interest rate indicates that the interest rate is in fact *too high*. Hence, the level of interest rate can be a misleading factor in determining and understanding of the nature of inefficiencies present in the economy.

[16]The whole region of inefficiently liquid steady state equilibria in F_ℓ lies down and to the left of all inefficiently illiquid steady state equilibria in F_i

4 Public Liquidity

In this section I study the effects of public liquidity provision in the form of government bonds by performing an exercise similar to Diamond (1965). Theoretically, as it is argued in Holstrom and Tirole (1998), government bonds are liquid due to the exclusive ability of the government to tax the private sector. Krishnamurthy and Vissing-Jorgensen (2010) provide convincing evidence that the low yield on government treasuries is essentially due to their liquidity services to the private agents.

I analyzed the competitive equilibria of a model without any government in previous sections. I showed that for certain regions in the space of parameter values, the competitive outcome fails to achieve efficiency. Now I want to see how the introduction of publicly supplied liquidity by a government can affect the welfare in the steady state equilibria in different regions of the parameter space. Intuitively, one expects to see that, public liquidity has a positive effect on the welfare when the steady state equilibrium is inefficiently illiquid, as it is an extra means to create more pledgable income for the middle aged entrepreneurs. One may not, however, see any benefits to public liquidity provision in an inefficiently liquid equilibrium. This is simply because in an inefficiently liquid equilibrium, the private sector already provides more liquidity than the socially optimal level. This intuition turns out to be correct for the inefficient equilibria in F_ℓ and F_i but matters are different for the mixed region, F_m.

4.1 Competitive Equilibrium with Bonds

Consider the model in Section 2 only with one difference: the young and middle aged entrepreneurs at any time $t \geq 0$ can purchase a one period and risk free government bond sold at par, denoted by b_t^y and b_t^t. A unit of bond purchased at time t is a promise by the government to deliver one unit of consumption good plus the interest in period $t+1$. Since I have stated all the steps to solve the maximization problem of the middle aged in Section 2, I only show the final form of the maximization that can be compared to III:

$$\max_{i_t, b_t^m \geq 0} \quad \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t)(i_t - b_t^m) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1})(i_{t-1} + b_{t-1}^y) - \tau_{t+1}^o \tag{III$_b$}$$

$$s.t. \quad \left(\frac{\theta_1 R_1(1+r_{t-1})}{1+r_t-\theta_1 R_1}\right)(i_{t-1}+b_{t-1}^y) \leq (i_t - b_t^m) \leq \left(\frac{\theta_2 R_2(1+r_{t-1})}{1+r_t-\theta_2 R_2}\right)(i_{t-1}+b_{t-1}^y).$$

For convenience I restate the expressions for Λ and Φ below:

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) \equiv \left(\frac{(\theta_2 - \theta_1)R_1R_2}{\theta_2R_2 - \theta_1R_1}\right) - \left(\frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{\theta_2R_2 - \theta_1R_1}\right)(1 + r_t)\,,$$
$$\Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) \equiv \left(\frac{(\theta_2 - \theta_1)R_1R_2}{\theta_2R_2 - \theta_1R_1}\right)(1 + r_{t-1})\,.$$

Problem III_b is very similar to III. The main difference is that the middle aged entrepreneur have a new opportunity to invest in b_t^m units of government bond. The important assumption here is that the total return to investment in bonds is pledgable so that bond purchases essentially reduce the total debt payment by the same amount to $(1 + r_t)(i_t - b_t^m)$. The middle aged entrepreneur also receives the principle and interests on her bond purchases in the previous period namely $(1 + r_{t-1})b_{t-1}^y$. Finally τ_{t+1}^o denotes the lump sum tax that is levied on the old entrepreneurs before consumption takes place.[17] I suppose that government balances its budget every period so that for all $t \geq 0$:

$$(1 + r_t)b_t = b_{t+1} + \tau_{t+1}^o\,. \tag{14}$$

Market clearings dictate that for all $t \geq 0$:

$$i_t + b_t^y = e\,, \tag{15}$$
$$b_t^m + b_t^y = b_t\,. \tag{16}$$

Since I am mainly interested in the steady states equilibria, I fix the supply of government bonds at a constat level or $b_t = b \geq 0$ for all $t \geq 0$. Let $\sigma = \frac{b}{e}$, then I need the following assumption on σ:

Assumption 2. $\sigma < \min(1 - \theta_2R_2, 1 - \frac{\theta_1R_1}{1-\theta_1})$.

Now I need to redefine the three regions:

Definition 4. Define $F(\sigma)$ as the set of $(\boldsymbol{\theta}, \boldsymbol{R})$ that satisfies Assumption 1 and Assumption 2. Then the three regions of $F(\sigma)$ are defined as follows:

[17] I choose to levy the tax on the old for two reasons. First, if I impose the tax on young and middle aged entrepreneurs, the value of the tax collected would affect the equilibrium conditions in a more subtle way. I want to avoid this since fiscal policy is not the focus of my study here and so I want to have the taxation as neutral as possible. Nonetheless, I see that even in this case, taxation plays a role in the mixed region. Second, imposing the tax on the young and middle aged complicates the problem; for example if I impose it on the young I would obtain multiple steady state equilibria.

Liquid Region: $F_\ell(\sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) \leq (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\}$.

Mixed Region: $F_m(\sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.

Illiquid Region: $F_i(\sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\sigma) | (1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) \leq \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.

Assumption 2 has the following interpretation. σ has to be smaller than $1 - \theta_2 R_2$, otherwise no steady state with positive investment in type one or two investments exist.[18] I also need $\sigma < 1 - \frac{\theta_1 R_1}{1-\theta_1}$ to ensure that there is positive investment in steady state and the borrowing constraint is binding. Now I can characterize the steady states in the following lemma:

Lemma 7. In the model with government bond, each region in Definition 4 has a unique and stable steady state equilibrium as follows:

$$\begin{cases} 1 + r_\ell^{ss}(\sigma) = \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell(\sigma)\,. \\ \\ 1 + r_m^{ss}(\sigma) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_m(\sigma)\,. \\ \\ 1 + r_i^{ss}(\sigma) = \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_i(\sigma)\,. \end{cases}$$

Moreover at the steady state, the entrepreneurs specialize in the more liquid and more productive type of investments in regions $F_\ell(\sigma)$ and $F_i(\sigma)$ respectively but invest strictly positive amounts in both types in $F_m(\sigma)$.

Note that an increase in σ, i.e., supply of government bonds, has two effects. First, given a fixed $\boldsymbol{R}$ and as can be seen in Definition 4, it rotates the boundaries of $F_i(\sigma)$ and $F_\ell(\sigma)$ counterclockwise around the origin in the space of $\boldsymbol{\theta}$. Looking at Figure 6, it is easy to see that as a result of an increase in σ, both $F_\ell(\sigma)$ and the inefficiently liquid region get smaller. $F_i(\sigma)$ and the inefficiently illiquid region, however, shrink at their north east and upper boundaries (due to Assumption 2) but expand into the previously mixed and liquid regions respectively.[19] In this sense, government bonds eliminate part of the inefficiently

[18] Note that in this case and in contrast with the equilibrium with no bond, one can have equilibria with no production where...

[19] Notice that I am using the fact that the locus of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$, which is the boundary between inefficiently liquid and inefficiently illiquid regions, does not depend on the value of σ.

liquid region as well as part of the liquid region. Second, by Lemma 7, an increase in σ increases the steady state interest rate in $F_i(\sigma)$ and $F_\ell(\sigma)$ for any given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\sigma)$ but has no effects on the interest rate in $F_m(\sigma)$ which is fixed at $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$.

4.2 Welfare Effects of Government Bond

I showed in the previous section that an increase in supply of government bonds raises the interest rate in $F_i(\sigma)$ and $F_\ell(\sigma)$ but has no effects on the interest rate in the mixed region. I further showed in Section 3.2 and Section 3.1 that a Pareto improvement would raise the interest rate in the inefficiently illiquid region while it would reduce the interest rate in the inefficiently liquid region. These two observations suggest that government bond might be beneficial for the inefficiently illiquid region but should not help much in the case of the inefficiently liquid equilibria. It is not clear, a priori, how government bonds affect the welfare in the mixed region. Hence the following proposition:

Proposition 8. Let $V_z^{ss}(\sigma)$ denotes the steady state utility level for region $z \in \{\ell, m, i\}$ given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_z(\sigma)$, when the supply of government bonds is σ. Then the welfare effects of an increase in σ are given as follows:

$$\begin{aligned} \frac{dV_\ell^{ss}(\sigma)}{d\sigma}\Big|_{\sigma=0} &= \left(\frac{R_2 - 1}{1 - \theta_2 R_2}\right) r_\ell^{ss} e\,, \\ \frac{dV_i^{ss}(\sigma)}{d\sigma}\Big|_{\sigma=0} &= \left(\frac{R_1 - 1}{1 - \theta_1 R_1}\right) r_i^{ss} e\,, \\ \frac{dV_m^{ss}(\sigma)}{d\sigma}\Big|_{\sigma=0} &= -r_m^{ss} e\,. \end{aligned}$$

r_z^{ss} denotes the steady state interest rate for region $z \in \{\ell, m, i\}$ when there is no government bonds in the economy.

Suppose $\sigma = 0$, so that $F_\ell(\sigma) = F_\ell$, $F_i(\sigma) = F_i$ and $F_m(\sigma) = F_m$. I am interested in the effect of a small increase in the supply of government bonds from zero to $\sigma = \epsilon > 0$. By Proposition 8, the welfare effect is positive in either of F_ℓ or F_i, if and only if the equilibrium interest rate is *positive*. This immediately implies that government bonds reduce the steady state welfare in the inefficiently liquid part of F_ℓ and enhances the welfare in the inefficiently illiquid part of F_i. This is because by Proposition **??**, the interest rate is strictly negative in the inefficiently liquid part of F_ℓ and strictly positive in the inefficiently illiquid part of F_i.

Therefore, effects of publicly supplied liquidity in F_ℓ and F_i are in conformance with the intuition. Government bonds can potentially be beneficial when the private sector generates less than enough, i.e., socially optimal, pledgable returns. This happens in an inefficiently illiquid steady state, as observed in Section 3.1, where the interest rate is strictly positive. This positive welfare effect is in line with the effects of government bond in Woodford (1990) and Holstrom and Tirole (1998). On the other hand, government bonds can be harmful to the steady state welfare, when the private sector generates more than the socially optimal pledgable returns, since government bond would encourage the private sector to generate even more pledgable returns. This is the case in an inefficiently liquid steady state, as observed in Section 3.1.

The situation is rather different in the mixed region. Proposition 8 implies that the supply of government bonds enhances the steady state welfare if and only if the prevailing interest rate is *negative*. By Proposition **??**, this implies that the welfare effect is positive in the inefficiently liquid part and negative in the inefficiently illiquid part of F_m. This is in fact the opposite of what happens in the case of two other regions, F_ℓ and F_i. To see why, let me express the steady state utility, using problem III_b and conditions 14 and 15 for $z \in \{\ell, m, i\}$, as follows:

$$V_z^{ss}(\sigma) = \big((1-\sigma)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) - \sigma r_z^{ss}(\sigma)\big)e \,. \tag{17}$$

If I take the derivative of $V_z^{ss}(\sigma)$, collect the terms and evaluate them at $\sigma = 0$, I get the following decomposition:

$$\frac{dV_z^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \Delta_{1z}^{ss} + \Delta_{2z}^{ss} + \Delta_{3z}^{ss} \,, \tag{18}$$

$$\Delta_{1z}^{ss} = \big((1-\sigma)\frac{d\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma))}{dr_z^{ss}(\sigma)} + \frac{d\Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma))}{dr_z^{ss}(\sigma)}\big)\frac{dr_z^{ss}(\sigma)}{d\sigma}e|_{\sigma=0} \tag{19}$$

$$\Delta_{2z}^{ss} = -\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma))e|_{\sigma=0} \,, \tag{20}$$

$$\Delta_{3z}^{ss} = -\frac{d\sigma r_z^{ss}(\sigma)}{d\sigma}e|_{\sigma=0} \,. \tag{21}$$

(To Be Completed)

5 Conclusion

(To Be Completed)

A Appendix: Proofs

Proof of Lemma 2. First, I show that these are the only steady state equilibria for the three regions. Suppose that $1 + r_z^{ss}$ is a steady state interest rate for $z \in \{\ell, m, i\}$. Consider an steady state of the liquid region. If $\frac{\theta_2 R_2}{1-\theta_2 R_2} < (1 + r_\ell^{ss})$, by 5 both of the upper and lower bounds on the next period interest rate will be strictly smaller than $1 + r_\ell^{ss}$. It cannot be that $(1 + r_\ell^{ss}) < \frac{\theta_2 R_2}{1-\theta_2 R_2}$ either. In that case using 5, the upper bound for the next period interest rate $\theta_2 R_2(2 + r_\ell^{ss})$, will be strictly bigger than $1 + r_\ell^{ss}$ but strictly less than $\frac{\theta_2 R_2}{1-\theta_2 R_2}$ and so strictly less than $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (since $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$). Hence by 4 the next period interest rate will be the upper bound itself which is a contradiction given that it is strictly bigger than $1 + r_\ell^{ss}$. Hence one must have $(1 + r_\ell^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. In a similar fashion I can show that if there exists an steady state for F_i, it must be $(1 + r_i^{ss}) = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Finally, suppose that $(1 + r_m^{ss}) < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$ in the mixed region, then using 4 and the fact that the economy is at the steady state, one must have $(1 + r_m^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$ which gives $\frac{\theta_2 R_2}{1-\theta_2 R_2} < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$ that is a contradiction given that the economy is in F_m. Similarly one cannot have $(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < (1 + r_m^{ss})$ and so $(1 + r_m^{ss}) = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$.

It only remains to check that these steady states exist. As I showed above the trajectories of the interest rates are consistent with the equilibrium conditions 4 and 5 given an initial interest rate $1 + r_{-1}$ equal to the steady state. The values of i_t are exogenously given and equal to e and the values of x_{1t} and x_{2t} can be derived from 6. The only condition that remains is that $1 + r_t < R_1$ for all $t \geq 0$. To see this note that under Assumption 1:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2)\,.$$

Hence, the remaining condition is satisfied for the liquid and mixed regions. The condition is also satisfied in the illiquid region since I assumed that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$ in Definition 2. Local stability of the steady states in F_ℓ and F_i, follows from the fact that $\theta_1 R_1 < \theta_2 R_2 < 1$ by Assumption 1. If $1 + r_z^{ss} \neq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$, suppose without loss of generality that $1 + r_z^{ss} - \epsilon < 1 + r_t < 1 + r_z^{ss}$. For small enough $\epsilon > 0$, the whole interval $[1 + r_z^{ss} - \epsilon, 1 + r_z^{ss}]$ is either strictly below or above $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. In either case, 4 and 5 imply $1 + r_t < 1 + r_{t+1} = \theta_z R_z(2 + r_t) < 1 + r_z^{ss}$, and hence by Assumption 1, the interest rates starting from a point in the interval $[1 + r_z^{ss} - \epsilon, 1 + r_z^{ss}]$, converge to the steady state value. For the case where $1 + r_z^{ss} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$ or the mixed region, F_m, where the steady state interest rate is $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, suppose without loss of generality that $1 + r_{t-1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (the proof for the case $1 + r_{t-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar). If

$1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) \in [\theta_1 R_1(2+r_{t-1}), \theta_2 R_2(2+r_{t-1})]$, then 4 gives $1+r_t = 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$. Otherwise suppose that $\theta_2 R_2(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$. Then 5 implies $1+r_{t-1} < 1+r_t = \theta_2 R_2(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$. Hence, $1+r_{t+k}, k=1,2,3,...$ converges to $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$. The proof is very similar when $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) < \theta_1 R_1(2+r_{t-1})$ and so this completes the proof. ■

Proof of Lemma 3. Claims about the steady state interest rate are established by 7 and 8 in the text. Let $s_j(\boldsymbol{\theta},\boldsymbol{R}) = \frac{x_j^{ss}}{x_1^{ss}+x_2^{ss}}$ be share of type $j \in \{1,2\}$ in total investment at the steady state. By 6:

$$s_1(\boldsymbol{\theta},\boldsymbol{R}) = \frac{\theta_2 R_2 - (1-\theta_2 R_2)(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))},$$

$$s_2(\boldsymbol{\theta},\boldsymbol{R}) = \frac{(1-\theta_1 R_1)(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})) - \theta_1 R_1}{(\theta_2 R_2 - \theta_1 R_1)(2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}.$$

Now one can rewrite $s_1(\boldsymbol{\theta},\boldsymbol{R})$ as:

$$s_1(\boldsymbol{\theta},\boldsymbol{R}) = \frac{\frac{1}{2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})} - (1-\theta_2 R_2)}{\theta_2 R_2 - \theta_1 R_1}.$$

Numerator of the above is strictly increasing in θ_1 by 7 and the denominator is strictly decreasing in θ_1. This implies that $s_1(\boldsymbol{\theta},\boldsymbol{R})$ is strictly increasing in θ_1 when $(\boldsymbol{\theta},\boldsymbol{R}) \in F_m$ and hence monotone in θ_1 in all three regions. For $s_2(\boldsymbol{\theta},\boldsymbol{R})$ one has:

$$\frac{\partial s_2(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} = \frac{(\theta_2 R_2 - \theta_1 R_1)\frac{\partial r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} - \big((1-\theta_1 R_1)(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})) - \theta_1 R_1\big) R_2 (2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}{\Big((\theta_2 R_2 - \theta_1 R_1)(2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))\Big)^2}.$$

Arranging terms in the numerator, the above can be written as:

$$\frac{\partial s_2(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} = \frac{\big(a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)\big) R_2}{\Big((\theta_2 R_2 - \theta_1 R_1)(2+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))((1-\theta_1)R_1 - (1-\theta_2)R_2)\Big)^2}.$$

where $a(\theta_1)$, $b(\theta_1)$ and $c(\theta_1)$ are:

$$
\begin{aligned}
a(\theta_1) =& - R_1R_2^2(1+R_1)(1-(1+R_1)\theta_1)\,,\\
b(\theta_1) =& R_1R_2^2(1+R_1) + R_1R_2\big((R_1-R_2)(1+2R_1) - R_1(R_1R_2-1)\big)\theta_1\\
&- R_1^2R_2(1+R_1)(2+R_2)\theta_1^2\,,\\
c(\theta_1) =& R_1R_2(R_1-R_2) - R_1R_2(R_1(2+R_1)-R_2)\theta_1 + R_1^2(1+R_2)(R_1R_2-R_1+R_2)\theta_1^2\\
&+ R_1^3(1+R_1)\theta_1^3\,.
\end{aligned}
$$

To show that $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum, it is enough to show that given any θ_1, $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root as a quadratic polynomial of θ_2 inside F. By Proposition 2, $\theta_1 \leq \frac{1}{1+R_1}$ and therefore $a(\theta_1) \leq 0$. In the next step, I show that $c(\theta_1) > 0$ for all θ_1 by proving that $\widetilde{c}(\theta_1) = R_1^-1\big(c(\theta_1) - R_1^3(1+R_1)\theta_1^3\big) > 0$ inside F. If $\widetilde{c}(\theta_1)$ has no roots then $\widetilde{c}(\theta_1) > 0$ since $\widetilde{c}(0) > 0$. Therefore suppose θ_1^* is the smallest root of $\widetilde{c}(\theta_1) = 0$:

$$
\begin{aligned}
\theta_1^* =& \frac{R_2(R_1(2+R_1)-R_2) - \sqrt{\Delta}}{2R_2(R_1-R_2)}\,,\\
\Delta =& (R_2(R_1(2+R_1)-R_2))^2 - 4R_1R_2(R_1-R_2)(1+R_2)(R_1R_2-R_1+R_2)\,.
\end{aligned}
$$

Now one has:

$$
\begin{aligned}
&\theta_1^* = \frac{R_2(R_1(2+R_1)-R_2) - \sqrt{\Delta}}{2R_2(R_1-R_2)} > \frac{1}{1+R_1} \;\Leftrightarrow\\
&(1+R_1)^2\Big(R_2(R_1(2+R_1)-R_2))^2 - 4R_1R_2(R_1-R_2)(1+R_2)(R_1R_2-R_1+R_2)\Big) <\\
&\Big(R_2(1+R_1)(R_1(2+R_1)-R_2)) - 2R_2(R_1-R_2)\Big)^2 \;\Leftrightarrow\\
&(1+R_1)\Big(R_1(1+R_1)(1+R_2)(R_1(R_2-1)+R_2) - R_2(R_1(2+R_1)-R_2)\Big) > -R_2(R_1-R_2)\,.
\end{aligned}
$$

The last inequality holds since:

$$
\begin{aligned}
&R_1(1+R_1)(1+R_2)(R_1(R_2-1)+R_2) - R_2(R_1(2+R_1)-R_2) >\\
&R_1R_2(1+R_1)(1+R_2) - R_1R_2(2+R_1) > R_1^2R_2 > 0 > -R_2(R_1-R_2)\,.
\end{aligned}
$$

Hence $\theta_1^* > \frac{1}{1+R_1}$ and since $\theta_1 \leq \frac{1}{1+R_1}$ in F one must have $c(\theta_1) > 0$ in F. Now since $a(\theta_1) \leq 0$ and $c(\theta_1) > 0$ in F, at least one root of $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ for any given θ_1 has to be non positive. Therefore $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root in F for any

θ_1 and consequently $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum. Note that when $\boldsymbol{\theta}$ is on the boundary of F_m and F_i, $s_2(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and hence $\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} > 0$ given any θ_1. Now suppose $\widetilde{\theta}_1$ is the value for which the vertical line $\theta_1 = \widetilde{\theta}_1$ is tangent to the boundary of F_ℓ. Observe that when θ_2 increases along $\theta_1 = \widetilde{\theta}_1$ line, $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ reaches the maximum of one at the point of tangency. Therefore beyond the point of tangency $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ must be strictly decreasing in θ_2. This implies that for the particular value of $\theta_1 = \widetilde{\theta}_1$, there is a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$. Hence by continuity, there must be a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ over the range of θ_2 given any θ_1 in a neighborhood of $\widetilde{\theta}_1$ which completes the proof. ■

Proof of Proposition 1. First I show that for all $t \geq 0$, $1+r_t < R_1$. Suppose $1+r_{t-1} < R_1$ for some $t \geq 0$. Consider the window defined by 5, where $1 + r_t \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$. If $\theta_2 R_2(2+r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, 4 implies $1+r_t = \theta_2 R_2(2+r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. The last inequality holds by Assumption 1. If $\theta_1 R_1(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2+r_{t-1})$, by 4 I get $1+r_t = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. Finally, consider the case $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_1 R_1(2+r_{t-1})$. 4 implies that $1 + r_t = \theta_1 R_1(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_1 R_1}{1-\theta_1 R_1}) < R_1$. The first inequality holds because the value of $\theta_1 R_1(2 + r_{t-1})$ is always between $1 + r_{t-1}$ and $\frac{\theta_1 R_1}{1-\theta_1 R_1}$. The second inequality is obtained by the assumption that $1+r_{t-1} < R_1$ and definition of F. By induction, $1 + r_{-1} < R_1$ implies $1 + r_t < R_1$ for all $t \geq 0$. This proves the necessary condition for the interest rates in the competitive equilibrium.

In the second step, I prove the existence and uniqueness. I show that given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and the initial condition $1+r_{-1}$, a unique path of interest rates are defined by 4 and 5. Note that given the path of interest rates I can simply solve for (x_{1t}, x_{2t}, i_t) for all $t \geq 0$ using 3 and 6 in each period. Suppose I have determined the unique interest rate $1 + r_{t-1}$ for $t - 1$. Consider the window, defined by 5, where $1+r_t \in [\theta_1 R_1(2+r_{t-1}), \theta_2 R_2(2+r_{t-1})]$. If $\theta_2 R_2(2+r_{t-1}) \leq 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq \theta_1 R_1(2+r_{t-1})$, using 4 gives $1+r_t = \theta_2 R_2(2+r_{t-1})$ and $1+r_t = \theta_1 R_1(2+r_{t-1})$ respectively. Finally, suppose $\theta_1 R_1(2+r_{t-1}) < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2+r_{t-1})$. Then, if $1 + r_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1 + r_t = \theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Similarly if $1+r_t < 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1+r_t = \theta_2 R_2(2+r_{t-1}) > 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. The two contradictions show that one must have $1+r_t = 1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, I showed that given $1+r_{t-1}$, there is a uniquely determined interest rate at time t, that is, $1+r_t$. Therefore, by induction, I have shown that given an initial condition $1 + r_{-1}$, there is a unique path of interest rates for all $t \geq 0$.

In the third and final step, I show that the unique equilibrium path of the interest rates defined in step two, converges to the unique steady state characterized in Lemma 2, for given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and an initial condition $1 + r_{-1}$. Consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ first. Note that if $1 + r_{t-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, using 4 and 5 implies $1 + r_t = \theta_2 R_2(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, if $1 + r_{-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the path of interest rates is defined as $1 + r_t = \theta_2 R_2(2 + r_{t-1})$ for all $t \geq 0$. This path is clearly convergent to $1 + r_\ell^{ss} = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Now suppose $1 + r_{-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ which implies $1 + r_{-1} > \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Define the series $\{1 + \bar{r}_t\}_{t=-1}^{\infty}$ as $1 + \bar{r}_t = \theta_2 R_2(2 + \bar{r}_{t-1})$ for all $t \geq 0$ and $1 + \bar{r}_{-1} = 1 + r_{-1}$. If $1 + r_{t-1} \leq 1 + \bar{r}_{t-1}$, 5 implies $1 + r_t \leq \theta_2 R_2(2 + r_{t-1}) \leq \theta_2 R_2(2 + \bar{r}_{t-1}) = 1 + \bar{r}_t$. Hence by induction one must have $1 + r_t \leq 1 + \bar{r}_t$ for all $t \geq 0$. Since by Assumption 1, $\theta_2 R_2 < 1$, this immediately implies that, there is a finite t_0 for which $1 + r_{t_0} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, this case is similar to the previous part of the proof and so convergence is established.

Now consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ where Definition 2 implies that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Without loss of generality suppose $1 + r_{-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Define the series $\{1 + \underline{r}_t\}_{t=-1}^{\infty}$ as $1 + \underline{r}_t = \theta_1 R_1(2 + \underline{r}_{t-1})$ for all $t \geq 0$ and $1 + \underline{r}_{-1} = 1 + r_{-1}$. It is easy to see that there is a finite and unique $t_0 \geq 0$ such that $1 + \underline{r}_{t_0} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1 + \underline{r}_{t_0-1}$. Now one notes that if $1 + r_{t-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ for some $t \geq 0$, one must have $\theta_2 R_2(2 + r_{t-1}) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and therefore 4 and 5 give $1 + r_t = \max(\theta_1 R_1(2 + r_{t-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$. Using this observation and by induction, for $-1 \leq t \leq t_0 - 1$ one must have $1 + r_t = 1 + \underline{r}_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ and so $\theta_2 R_2(2 + r_t) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Using 4 and the definition of t_0, this implies that $1 + r_{t_0} = \max(\theta_1 R_1(2 + r_{t_0-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, the path of interest rates converges to the steady state interest rate, $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, in finite periods. The proof for the case $1 + r_{-1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar. Finally, if $1 + r_{-1} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the economy is already in the steady state and all future interest rates will be the same.

The proof for the illiquid region is very similar to the case of liquid region and so I do not provide it here. ∎

Proof of Proposition 2. First, I compute the boundaries of the illiquid and liquid regions as functions of θ_1. For the illiquid region the defining boundary is characterized by:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_1 R_1}{1 - \theta_1 R_1}\right).$$

Using 1 and solving the above as a function of θ_1 I get:

$$\theta_2^i(\theta_1) = \left(\frac{\theta_1 R_1(1 - \theta_1(1 + R_2))}{R_2(1 - \theta_1(1 + R_1))}\right).$$

This function is strictly increasing in θ_1 since $\theta_1 R_1$ is increasing and:

$$\frac{d}{d\theta_1}\left(\frac{(1-\theta_1(1+R_2))}{(1-\theta_1(1+R_1))}\right) = \frac{R_1 - R_2}{(1-\theta_1(1+R_1))^2} > 0\,.$$

Also observe that $\theta_2^i(0) = 0$ and so no matter how close to the origin, there are illiquid equilibria in any neighborhood of the $\boldsymbol{\theta} = \mathbf{0}$. Now the characterizing equation for the liquid region is:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\,.$$

Collecting terms involving θ_1 or θ_2 on different sides I obtain two distinct curves:

$$\begin{aligned}
\overline{\theta}_2^\ell(\theta_1) &= \left(\frac{(\theta_1 R_1(1+R_2)+R_2) + \sqrt{(\theta_1 R_1(1+R_2)+R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)}\right)\,,\\
\underline{\theta}_2^\ell(\theta_1) &= \left(\frac{(\theta_1 R_1(1+R_2)+R_2) - \sqrt{(\theta_1 R_1(1+R_2)+R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)}\right)\,.
\end{aligned}$$

Note that obviously $\underline{\theta}_2^\ell(\theta_1) \leq \overline{\theta}_2^\ell(\theta_1)$ and $\underline{\theta}_2^\ell(0) = 0$ and so the lower boundary characterizing the liquid region passes through the origin. This means that there are liquid steady state equilibria at any neighborhood of the origin.

Now let $\Delta(\theta_1) \equiv (\theta_1 R_1(1+R_2)+R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)$. Then the two curves $\underline{\theta}_2^\ell(\theta_1)$ and $\overline{\theta}_2^\ell(\theta_1)$ touch each other when $\Delta(\theta_1) = 0$. This equation has two roots:

$$\begin{aligned}
\overline{\theta}_1 &= \left(\frac{R_2\left((2(1+R_1)-(1+R_2)) + \sqrt{4(1+R_1)(R_1-R_2)}\right)}{R_1(1+R_2)^2}\right)\,,\\
\underline{\theta}_1 &= \left(\frac{R_2\left((2(1+R_1)-(1+R_2)) - \sqrt{4(1+R_1)(R_1-R_2)}\right)}{R_1(1+R_2)^2}\right)\,.
\end{aligned}$$

The smaller root is less than $\frac{1}{1+R_1}$ since:

$$
\begin{aligned}
\underline{\theta}_1 < \frac{1}{1+R_1} \Leftrightarrow (1+R_1)R_2\left((2(1+R_1)-(1+R_2)) - \sqrt{4(1+R_1)(R_1-R_2)}\right) \\
< R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1)-(1+R_2)) - R_1(1+R_2)^2 \\
< (1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)} \Leftrightarrow (R_1-R_2)(2R_1R_2+R_2-1) < \\
(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)}\,.
\end{aligned}
$$

If I square both sides and cancel $R_1 - R_2$ and collect the terms, I get:

$$
\Leftrightarrow R_1 < 4R_1^2R_2^3 + 8R_1^2R_2^2 + 4R_1R_2^3 + 7R_1R_2^2 + R_2^3 + 4R_1^2R_2 + 2R_2^2 + 2R_1R_2 + R_2\,.
$$

This is obviously the case given Assumption 1. The bigger root is greater than $\frac{1}{1+R_1}$ since:

$$
\begin{aligned}
\overline{\theta}_1 > \frac{1}{1+R_1} \Leftrightarrow (1+R_1)R_2\left((2(1+R_1)-(1+R_2)) + \sqrt{4(1+R_1)(R_1-R_2)}\right) \\
> R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1)-(1+R_2)) - R_1(1+R_2)^2 \\
> -(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)} \Leftrightarrow (R_1-R_2)(2R_1R_2+R_2-1) > \\
-(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)}\,.
\end{aligned}
$$

The last inequality is obvious given that one term is positive and the other is negative. Therefore the point at which the two curves $\overline{\theta}_2^{\ell}(\theta_1)$ and $\underline{\theta}_2^{\ell}(\theta_1)$ touch each other inside F is $\underline{\theta}_1$ and also the fact that $\underline{\theta}_1 < \frac{1}{1+R_1}$ proves that for high θ_1 there is no liquid steady state.

Next, I prove that $\overline{\theta}_2^{\ell}(\theta_1)$ is strictly decreasing and $\underline{\theta}_2^{\ell}(\theta_1)$ is strictly increasing.The derivatives are:

$$
\begin{aligned}
\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2) + \left(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2) - 2R_1R_2(1+R_1)\right)\Delta(\theta_1)^{-\frac{1}{2}}\right)\,,\\
\frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2) - \left(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2) - 2R_1R_2(1+R_1)\right)\Delta(\theta_1)^{-\frac{1}{2}}\right)\,.
\end{aligned}
$$

C_0 is just a constant. It is easy to see that the term in parenthesis just before $\Delta(\theta_1)^{-\frac{1}{2}}$ is always negative for $\theta_1 \leq \underline{\theta}_1$. Hence, $\frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ should be strictly positive. Now for the other case:

$$
\begin{aligned}
&\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} < 0 \\
\Leftrightarrow\ & R_1^2(1+R_2)^2\Delta(\theta_1) < \left(R_1^2(1+R_2)^2\theta_1 - R_1R_2(2(1+R_1)-(1+R_2))\right)^2 \\
\Leftrightarrow\ & (1+R_2)^2\Delta(\theta_1) < \left(R_1(1+R_2)^2\theta_1 - R_2(2(1+R_1)-(1+R_2))\right)^2 \\
\Leftrightarrow\ & (1+R_2) < 2(1+R_1)-(1+R_2)\,.
\end{aligned}
$$

The last statement is correct given Assumption 1. In the last step I used the definition of $\Delta(\theta_1)$ to cancel out all terms. What I proved show that for any $\boldsymbol{\theta} \in F_\ell$ one must have $\boldsymbol{\theta} \le (\frac{1}{1+R_1}, \frac{1}{1+R_1})$. This is because I showed that $\underline{\theta}_1 < \frac{1}{1+R_1}$ and that $\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ is strictly decreasing while $\overline{\theta}_2^{\ell}(\theta_1)$ stays above $\underline{\theta}_2^{\ell}(\theta_1)$ and intersects with $\theta_1 = 0$ at $\frac{1}{1+R_1}$.

In the next step I want to prove that the liquid region lies above the illiquid region. First, I observe the following:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial\theta_2} = \frac{(1-\theta_1)(R_1-R_2)R_1R_2}{((1-\theta_1)R_1-(1-\theta_2)R_2)^2} > 0\,.$$

Now suppose that $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \ge \frac{\theta_2R_2}{1-\theta_2R_2}$ and $r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \le \frac{\theta_1R_1}{1-\theta_1R_1}$ where $(\theta_1, \theta_2, \boldsymbol{R})$ and $(\theta_1, \theta_2', \boldsymbol{R})$ are in F. Then if $\theta_2 \le \theta_2'$, by the derivation above $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \le r_\Lambda(\theta_1, \theta_2', \boldsymbol{R})$ and hence:

$$\frac{\theta_2R_2}{1-\theta_2R_2} \le r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \le r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \le \frac{\theta_1R_1}{1-\theta_1R_1}\,.$$

This is not possible since it implies that $\theta_2R_2 \le \theta_1R_1$ and hence $(\theta_1, \theta_2, \boldsymbol{R})$ cannot be in F. In the last step of the proof, I show that $(\frac{1}{R_2}, \frac{1}{1+R_1}) \in F_i$. First, note that:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial\theta_1} = \frac{(1-\theta_2)(R_2-R_1)R_1R_2}{((1-\theta_1)R_1-(1-\theta_2)R_2)^2} < 0\,.$$

Second, observe that $\theta_2^i(\theta_1)$ is strictly increasing, passes through the origin and also converges to infinity when θ_1 gets close to $\frac{1}{1+R_1}$. This means that $\theta_2^i(\theta_1)$ cuts the horizontal border of F that is $\theta_2 = \frac{1}{R_2}$ at an interior point, say, $(\bar{\theta}_1, \frac{1}{R_2})$ where $\bar{\theta}_1 < \frac{1}{1+R_1}$. At this point

$\theta_2^i(\bar\theta_1) = \frac{\bar\theta_1 R_1}{1-\bar\theta_1 R_1}$. But since I proved above that $\frac{\partial r_\Lambda(\boldsymbol\theta,\boldsymbol R)}{\partial\theta_1} < 0$, for any $\theta_1 \in (\bar\theta_1, \frac{1}{1+R_1})$ I obtain:

$$\theta_2^i(\theta_1) < \theta_2^i(\bar\theta_1) = \frac{\bar\theta_1 R_1}{1-\bar\theta_1 R_1} < \frac{\theta_1 R_1}{1-\theta_1 R_1}\,.$$

This means that $\theta_1 \in F_i$ for $\theta_1 \in (\bar\theta_1, \frac{1}{1+R_1})$.

■

Proof of Proposition 3. When $R_1 > R_2 > 1$ and $1 > \theta_1 R_1 > \theta_2 R_2$, entrepreneurs only invest in type one since type two is dominated both in terms of liquidity and return. Hence, this economy collapses to the economy in Farhi and Tirole (2010a) with only one investment type, (θ_1, R_1), and no bubbles or outside liquidity. Farhi and Tirole (2010a) show in their Proposition 5 that under the assumption that $R_1 > 1$, all competitive equilibria are Pareto efficient and hence constrained Pareto efficient as well. ■

Proof of Proposition 4. By Proposition 2, the competitive equilibrium converges to a unique steady state corresponding to $(\boldsymbol\theta, \boldsymbol R) \in F_\ell \cup F_m$. This implies that there exist $T \geq 0$ and $\varepsilon > 0$ such that $x_{2t} \geq \varepsilon$ for $t \geq T$. If one reduces x_{2t} for $t \geq T$ by $\delta + \epsilon$ and increases x_{1t} for $t \geq T$ by ϵ where $\delta > 0$, $\epsilon > 0$ are such that $\epsilon + \delta < \varepsilon$ and:

$$\begin{aligned}
\delta &= (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta\,,\\
\epsilon &= \frac{1-\theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\,\delta\,.
\end{aligned}$$

Similar to what is shown in the text, the above reallocation leaves all middle aged at or after T strictly better off when $r_\Lambda(\boldsymbol\theta, \boldsymbol R) < 0$. If $r_\Lambda(\boldsymbol\theta, \boldsymbol R) = 0$ the reallocation does not affect the utility of middle aged after T but increases utility of middle aged at T. This proves that the competitive equilibrium is constrained Pareto inefficient whenever $r_\Lambda(\boldsymbol\theta, \boldsymbol R) \leq 0$.

If $(\boldsymbol\theta, \boldsymbol R) \in F_m$ then by definition $r_\Lambda(\boldsymbol\theta, \boldsymbol R) < 0$ implies a strictly negative interest rate at the steady state. If $(\boldsymbol\theta, \boldsymbol R) \in F_\ell$, by Lemma 6, $r_\Lambda(\boldsymbol\theta, \boldsymbol R) < 0$ implies:

$$\frac{\theta_2 R_2}{1-\theta_2 R_2} < \frac{\theta_1 R_1}{1-\theta_1 R_1} \leq 1 + r_\Lambda(\boldsymbol\theta, \boldsymbol R) < 1\,.$$

Hence by Lemma 2, the steady state interest rate is strictly negative which completes the proof. ■

Proof of Lemma 4. **(To Be Written)**. ■

Proof of Lemma 5. The straight line corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ is $\theta_2^\Lambda(\theta_1) = \left(\frac{R_1(R_2-1)}{R_2(R_1-1)}\right)\theta_1 + \left(\frac{R_1-R_2}{R_2(R_1-1)}\right)$. This line intersects horizontal lines $\theta_1 = 0$ and $\theta_1 = \frac{1}{R_1}$ at $\theta_2^\Lambda(0) = \frac{R_1-R_2}{R_2(R_1-1)}$ and $\theta_2^\Lambda(\frac{1}{R_1}) = \frac{1}{R_2}$. This implies $\theta_2^\Lambda(0) > 0$ and $\theta_2^\Lambda(\frac{1}{1+R_1}) < \theta_2^\Lambda(\frac{1}{R_1}) = \frac{1}{R_2}$. Hence, Proposition 2 and Proposition ?? imply that an strictly positive neighborhood of the origin, i.e., $\boldsymbol{\theta} = 0$, corresponds to inefficiently liquid equilibria and the top right corner of F and an strictly positive neighborhood of it correspond to inefficiently illiquid equilibria. Since by Proposition 2, any neighborhood of the origin contains liquid equilibria, by Proposition ??, it follows that there are inefficiently liquid equilibria in any small enough neighborhood of the origin. By Proposition 2, any neighborhood of the origin contains illiquid equilibria and therefore, close enough to the origin, no illiquid equilibrium is inefficiently illiquid since it lies below $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$. Note that by Proposition 2, the boundary of F_ℓ cuts the vertical axis $\theta_1 = 0$ at the origin and $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$ and also the upper part of the boundary is negatively sloped in (θ_1, θ_2) plane. Therefore, it follows that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ line passes through F_ℓ if and only if its intersection with $\theta_1 = 0$, that is $(0, \theta_2^\Lambda(0)$, lies below or at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$. This is the case whenever $\frac{R_1-R_2}{R_2-1} \leq 1$.

For the last part, let S_ℓ and S_i denote the unique intersections of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ with the boundaries of F_ℓ and F_i respectively. Since $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ is positively sloped, S_i lies above and to the right of S_ℓ. Note that S_i has the lowest θ_1 and θ_2 in the inefficiently illiquid part of F_i and so it is enough to show that there is no inefficiently liquid equilibria to the right of S_i. Suppose to the contrary that $\boldsymbol{\theta} = (\theta_1, \theta_2)$ corresponds to an inefficiently liquid equilibrium on the border of F_ℓ that lies on the right side of S_i. As argued above, both $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and S_ℓ lie below S_i and that S_ℓ lies to the left of S_i. Hence, there exists a point on the curve linking S_ℓ and $\boldsymbol{\theta} = (\theta_1, \theta_2)$, say $\boldsymbol{\theta} = (\widetilde{\theta}_1, \widetilde{\theta}_2)$ such that $\theta_1^{S_\ell} < \widetilde{\theta}_1 = \theta_1^{S_i} < \theta_1$ and $\widetilde{\theta}_2 < \theta_2^{S_i}$ where $S_\ell = (\theta_1^{S_\ell}, \theta_2^{S_\ell})$ and $S_i = (\theta_1^{S_i}, \theta_2^{S_i})$. This is a contradiction given that F_ℓ lies above F_i by Proposition 2 and so the proof is complete. ■

Proof of Proposition 5. For part of the proof, I use some of the results in Ghate and Smith (2005), specially their Theorem 2.6. This theorem shows that complementary slackness conditions are sufficient for optimality in a linear programming with infinite variables and infinite number of constraints, when feasible points, constraints and objective functions of both primal and dual problems are elements of appropriate spaces. A necessary condition for this result is that the feasible points of the primal problem, i.e. feasible allocations $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$, lie in ℓ_∞. To see this note that by 9 and Assumption 1:

$$x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \leq \theta_1 R_1 (x_{1t-1} + x_{2t-1}) + e$$

This together with 9 give:

$$\begin{cases} x_{1t} + x_{2t} \leq i_{-1} + \frac{e}{1-\theta_1 R_1}\,, \\ \\ c_t \leq R_1 (i_{-1} + \frac{e}{1-\theta_1 R_1}) + e\,. \end{cases}$$

i_{-1} is total investment at $t = -1$ which is an initial condition to the problem. The above proves that $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty} \in \ell_\infty$ for any feasible allocation.

Now let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and consider an allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ that satisfies 9 with equality for all $t \geq 0$. If there exists a series of strictly positive weights $\{\lambda_t\}_{t=0}^{\infty} \in \ell_1$ such that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ solves:

$$\begin{aligned} \max_{\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \lambda_t c_t \\ s.t. \quad & c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e \\ & x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \\ & c_t \geq 0\,,\ x_{1t} \geq 0\,,\ x_{2t} \geq 0 \end{aligned}$$

then $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient. Let $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ be the Lagrange multipliers for resource constraint, pledgeability constraint and non negativity constraints on x_{1t}, x_{2t} and c_t respectively. As discussed above any feasible allocation is bounded. Hence the sufficient conditions for $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ to be a maximum are:

$$\begin{cases} \lambda_t - \eta_t + \delta_{ct} = 0\,, \\ (R_1\eta_{t+1} - \eta_t) + (\theta_1 R_1 \gamma_{t+1} - \gamma_t) + \delta_{1t} = 0\,, \\ (R_2\eta_{t+1} - \eta_t) + (\theta_2 R_2 \gamma_{t+1} - \gamma_t) + \delta_{2t} = 0\,, \\ \eta_t \geq 0\,,\ \gamma_t \geq 0\,,\ \delta_{1t} \geq 0\,,\ \delta_{2t} \geq 0\,,\ \delta_{ct} \geq 0\,, \\ \delta_{1t}x_{1t} = 0\,,\ \delta_{2t}x_{2t} = 0\,,\ \delta_{ct}c_t = 0\,. \end{cases} \tag{SC}$$

for $t \geq 0$, provided that $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty} \in \ell_1$. First consider the case where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. In this case, if I set $\{\delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ to zero, solving the first three series of equations in SC, I obtain the following for $t \geq 0$:

$$\eta_t = \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1}\,,$$

$$\lambda_{t+2} = \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\lambda_{t+1}\,,$$

$$\lambda_1 = \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2 - \theta_1)R_1 R_2}\,(\lambda_0 + \gamma_0)\,.$$

The coefficient in the second difference equation above is $(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1}$. Therefore for any positive λ_0 and γ_0, λ_1 is given by the above and:

$$\lambda_t = \big(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\big)^{-(t-1)}\,\lambda_1\,.$$

Since $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, the resulting $\{\lambda_t\}_{t=0}^{\infty}$ and consequently all $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1. Therefore all the conditions above which are sufficient for optimality are satisfied and $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient.

Now, let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. If one sets $\{\delta_{1t}, \delta_{ct}\}_{t=0}^{\infty}$ to zero, the first three set of sufficient conditions in SC give the following for $t \geq 0$:

$$\eta_t = \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1} - \frac{1}{\theta_2 R_2 - \theta_1 R_1}\,\delta_{2t}\,,$$

$$\lambda_{t+2} = \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2-\theta_1)R_1R_2}\,\lambda_{t+1} + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\delta_{2t+1} - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\delta_{2t}\,,$$

$$\lambda_1 = \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2-\theta_1)R_1R_2}\,(\lambda_0 + \gamma_0 - \delta_{20})\,.$$

Given any positive λ_0 and γ_0, suppose one sets $\delta_{2t} = 0$ for $0 \le t \le T-1$. This implies $\lambda_t = \rho^{t-1}\lambda_1$ for $1 \le t \le T$, where $\rho = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1} > 1$. Moreover, let $\delta_{2T} = \alpha'\lambda_T$ and $\delta_{2t} = \alpha\lambda_t$ for $t \ge T+1$ where α and α' are positive constants to be determined. For $t \ge T+2$, the above equations lead to the following difference equation:

$$\lambda_{t+1} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha\big)\,\lambda_t - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha\lambda_{t-1}\,.$$

This difference equation has a solution of the form $\lambda_{t+1} = m\lambda_t$ where m is the smallest root of the characteristic equation:

$$m = \frac{1}{2}\Big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2} - \sqrt{\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\big)^2 - \frac{4\alpha}{(\theta_2-\theta_1)R_1R_2}}\;\Big)\,.$$

It is easy to see that:

$$m < 1 \quad \Leftrightarrow \quad \alpha < \frac{(\theta_2-\theta_1)R_1R_2(\rho-1)}{1-\theta_2 R_2}\,.$$

Hence if α is small enough and given the appropriate initial condition, i.e. $\lambda_{T+2} = m\lambda_{T+1}$, one can generate $\{\lambda_t\}_{t=0}^{\infty} \in \ell_1$. For time $T+1$ and $T+2$ the difference equation becomes:

$$\lambda_{T+1} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\big)\,\lambda_T\,,$$

$$\lambda_{T+2} = \big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha\big)\,\lambda_{T+1} - \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\lambda_T\,.$$

and therefore $\lambda_{T+2} = m\lambda_{T+1}$ if and only if:

$$\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha - m\big)\big(\rho + \frac{\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\big) = \frac{1}{(\theta_2-\theta_1)R_1R_2}\,\alpha'\,.$$

The above equation is linear in α'. Note that one always has $\theta_2 R_2 \rho < 1$ and hence for small enough α there is a strictly positive solution for α'. Therefore a small enough $\alpha > 0$ defines unique values of $0 < m < 1$ and $\alpha' > 0$ such that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ are in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient.

Finally, let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. Setting $\{\delta_{1t}, \delta_{ct}\}_{t=0}^{\infty}$ and $\{\delta_{2t}\}_{t=0}^{T-1}$ to zero implies $\lambda_t = \lambda_1$ for $1 \leq t \leq T$. Using SC for $k \geq 1$ one can obtain:

$$\lambda_{T+k} = \lambda_T - \zeta\Big(\sum_{j=0}^{k-2} \delta_{2T+j}\Big) + \nu\delta_{2T+k-1}\,.$$

where $\zeta = \frac{1-\theta_2 R_2}{(\theta_2-\theta_1)R_1R_2}$ and $\nu = \frac{1}{(\theta_2-\theta_1)R_1R_2}$. To satisfy the above condition, for any $j \geq 0$ define:

$$\delta_{2T+j} = \Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j+1},$$

$$\lambda_{T+j} = (\lambda_T + \zeta + \nu)\Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j}.$$

It is easy to see that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient and completes the proof. ■

Proof of Proposition 6. Using the values of steady state interest rates in Lemma 2, market clearings and the objective function in III, deriving V_ℓ^{ss}, V_i^{ss} and V_m^{ss} is straight forward. For the regulated economy remind that by IV the objective function when the social planner sets $\alpha_\ell = \alpha$ will be:

$$\begin{aligned} V_\alpha^{ss} &= (((1-\alpha)R_1 + \alpha R_2)(2 + r_\alpha^{ss}) - (1 + r_\alpha^{ss}))e \\ &= (R_\alpha(1 + \frac{\gamma_\alpha}{1-\gamma_\alpha}) - \frac{\gamma_\alpha}{1-\gamma_\alpha})e \\ &= \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right) e\,. \end{aligned}$$

The first part above uses 13. For last part of the proposition observe that 6 gives:

$$\widetilde{\alpha}_\ell = \frac{(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}\,.$$

The interest rate is:

$$
\begin{aligned}
&1 + r^{ss}_{\widetilde{\alpha}_\ell} = \frac{\gamma_{\widetilde{\alpha}_\ell}}{1-\gamma_{\widetilde{\alpha}_\ell}} = \\
&\frac{((1+r_\Lambda) - \theta_1 R_1(2+r_\Lambda))\theta_2 R_2 + (\theta_2 R_2(2+r_\Lambda) - (1+r_\Lambda))\theta_1 R_1}{((1+r_\Lambda) - \theta_1 R_1(2+r_\Lambda))(1-\theta_2 R_2) + (\theta_2 R_2(2+r_\Lambda) - (1+r_\Lambda))(1-\theta_1 R_1)} = \\
&\frac{(\theta_2 R_2 - \theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,.
\end{aligned}
$$

Therefore the utility levels at the steady state should be the same. Note that $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \leq R_{\widetilde{\alpha}_\ell}$ by Assumption 1 and so the proof is complete. ∎

Proof of Lemma 6. First, consider $V^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R}) - V^{ss}_i(\boldsymbol{\theta}, \boldsymbol{R})$. Note that:

$$
\begin{aligned}
V^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R}) > V^{ss}_i(\boldsymbol{\theta}, \boldsymbol{R}) &\Leftrightarrow \frac{(1-\theta_2)R_2}{1-\theta_2 R_2} > \frac{(1-\theta_1)R_1}{1-\theta_1 R_1} \Leftrightarrow \\
\frac{R_2 - 1}{1-\theta_2 R_2} > \frac{R_1 - 1}{1-\theta_1 R_1} &\Leftrightarrow (R_2 - 1)(1-\theta_1 R_1) > (R_1 - 1)(1-\theta_2 R_2) \Leftrightarrow \\
(R_2 - \theta_1 R_1 R_2 + \theta_1 R_1) &> (R_1 - \theta_2 R_1 R_2 + \theta_2 R_2) \Leftrightarrow \\
(\theta_2 - \theta_1) R_1 R_2 &> (1-\theta_1)R1 - (1-\theta_2)R_2 \Leftrightarrow \\
1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) &= \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{(1-\theta_1)R1 - (1-\theta_2)R_2}\right) > 1\,.
\end{aligned}
$$

Now define the following terms:

$$
\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{(\theta_2 - \theta_1)R_1 R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2)}{(\theta_2 R_2 - \theta_1 R_1)}\right) e\,,
$$

$$
\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2 R_2((1-\theta_1)R1 - (1-\theta_2)R_2) - (1-\theta_2 R_2)(\theta_2 - \theta_1)R_1 R_2}{(1-\theta_2 R_2)((1-\theta_1)R1 - (1-\theta_2)R_2)}\,.
$$

Notice that the denominators of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ are strictly positive. Moreover it is easily seen that the numerator of $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 1$ and the numerator of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2} = 1 + r^{ss}_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ or equivalently $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the welfare gains per unit of reduction in x_1 of investing the freed resources in x_2 and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the maximum amount of reduction in x_1

that can possibly occur (see Section 3.1). By Proposition 6:

$$V_m^{ss} = \left(\frac{(\theta_2 - \theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)} \right) e \,.$$

Now I want to compute and simplify $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{DEN}{NUM}$. The common denominator and the numerator are:

$$\begin{aligned}
DEN &= (1-\theta_2 R_2)((1-\theta_1)R1 - (1-\theta_2)R_2)(\theta_2 R_2 - \theta_1 R_1)\,,\\
NUM &= (1-\theta_2 R_2)(\theta_2-\theta_1)R_1R_2\Big((\theta_2-\theta_1)R_1R_2\\
&\quad - \Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2)\Big)\Big)\\
&\quad + \theta_2 R_2\left((1-\theta_1)R1 - (1-\theta_2)R_2\right)\Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2)\Big)\,,\\
&= ((1-\theta_1)R1 - (1-\theta_2)R_2)\Big((\theta_2-\theta_1)R_1R_2 + \theta_2 R_2^2 - \theta_2 R_1 R_2 - \theta_2 R_2(\theta_2 R_2 - \theta_1 R_1)\Big)\,,\\
&= ((1-\theta_1)R1 - (1-\theta_2)R_2)(1-\theta_2)R_2(\theta_2 R_2 - \theta_1 R_1)\,.
\end{aligned}$$

Therefore:

$$\begin{aligned}
V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) &= \frac{\theta_2 R_2}{1-\theta_2 R_2} = V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R})\,, \Rightarrow\\
V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) &= \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})\,.
\end{aligned}$$

By last equation it is obvious that the sign of $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. For the last case define:

$$\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{((1-\theta_1)R1 - (1-\theta_2)R_2) - (\theta_2-\theta_1)R_1R_2}{(\theta_2 R_2 - \theta_1 R_1)} \right) e\,,$$

$$\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(1-\theta_1 R_1)(\theta_2-\theta_1)R_1R_2 - \theta_1 R_1((1-\theta_1)R1 - (1-\theta_2)R_2)}{(1-\theta_1 R_1)((1-\theta_1)R1 - (1-\theta_2)R_2)}\,.$$

Note that $\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) = -\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$. Similar simplifications lead to:

$$V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = \Omega_i(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R})\,.$$

Hence $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$. ■

Proof of Proposition 7. (To Be Revised)

Throughout I use the fact that if $\frac{X}{Y} > \frac{Z}{T}$ for strictly positive values of X, Y, Z, T, then for any pair $\alpha', \alpha \in (0,1)$ one must have $\frac{\alpha' X+(1-\alpha')Z}{\alpha' Y+(1-\alpha')T} > \frac{\alpha X+(1-\alpha)Z}{\alpha Y+(1-\alpha)T}$ if and only if $\alpha' > \alpha$.

First, suppose that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ for $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and so according to Lemma 6 one must have $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$. Let $\widetilde{\alpha}_\ell$ be the ratio of liquid investment to the total funds invested for the particular $(\boldsymbol{\theta}, \boldsymbol{R})$. Now if the planner sets the fraction of liquid investment to the total equal to $\widetilde{\alpha}_\ell - \epsilon$ where $\epsilon > 0$, then Proposition 6 and the fact that $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ imply:

$$V^{ss}_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(1-(\widetilde{\alpha}_\ell-\epsilon))(1-\theta_1)R_1 + (\widetilde{\alpha}_\ell-\epsilon)(1-\theta_2)R_2}{(1-(\widetilde{\alpha}_\ell-\epsilon)(1-\theta_1 R_1) + (\widetilde{\alpha}_\ell-\epsilon)(1-\theta_2 R_2)}\right) > \left(\frac{(1-\widetilde{\alpha}_\ell)(1-\theta_1)R_1 + \widetilde{\alpha}_\ell(1-\theta_2)R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) = V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})\,.$$

Note also, that the interest rate would be lower in the regulated economy (which itself is less than zero):

$$1 + r^{ss}_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(1-(\widetilde{\alpha}_\ell-\epsilon))\theta_1 R_1 + (\widetilde{\alpha}_\ell-\epsilon)\theta_2 R_2}{(1-(\widetilde{\alpha}_\ell-\epsilon)(1-\theta_1 R_1) + (\widetilde{\alpha}_\ell-\epsilon)(1-\theta_2 R_2)}\right) < \left(\frac{(1-\widetilde{\alpha}_\ell)\theta_1 R_1 + \widetilde{\alpha}_\ell\theta_2 R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1\,.$$

Note that I have used the fact that by Assumption 1, $\frac{\theta_1 R_1}{1-\theta_1 R_1} < \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Since $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2)$ and $\min(R_1, R_2) \leq R_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})$, one must have $1 + r_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})^{ss} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})$ which ensures that the steady state of the regulated economy exists and it conforms with Definition 1. Note that $R_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})$ is defined as before:

$$R_{(\widetilde{\alpha}_\ell-\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) = (1-(\widetilde{\alpha}_\ell-\epsilon))R_1 + (\widetilde{\alpha}_\ell-\epsilon)R_2\,.$$

Now suppose that $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and so Lemma 6 implies $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$. In this case the planner raise the fraction of liquid investments to $\widetilde{\alpha}_\ell + \epsilon$ to improve the steady state welfare. Hence, I obtain:

$$V^{ss}_{(\widetilde{\alpha}_\ell+\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(1-(\widetilde{\alpha}_\ell+\epsilon))(1-\theta_1)R_1 + (\widetilde{\alpha}_\ell+\epsilon)(1-\theta_2)R_2}{(1-(\widetilde{\alpha}_\ell+\epsilon)(1-\theta_1 R_1) + (\widetilde{\alpha}_\ell+\epsilon)(1-\theta_2 R_2)}\right) > \left(\frac{(1-\widetilde{\alpha}_\ell)(1-\theta_1)R_1 + \widetilde{\alpha}_\ell(1-\theta_2)R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) = V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})\,.$$

The interest rate would be higher in the regulated economy in this case (which itself is greater than zero):

$$1 + r^{ss}_{(\widetilde{\alpha}_\ell+\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(1-(\widetilde{\alpha}_\ell+\epsilon))\theta_1 R_1 + (\widetilde{\alpha}_\ell+\epsilon)\theta_2 R_2}{(1-(\widetilde{\alpha}_\ell+\epsilon)(1-\theta_1 R_1) + (\widetilde{\alpha}_\ell+\epsilon)(1-\theta_2 R_2)} \right) > \left(\frac{(1-\widetilde{\alpha}_\ell)\theta_1 R_1 + \widetilde{\alpha}_\ell \theta_2 R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)} \right) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 1\,.$$

As in the previous case it only remains to show that the regulated interest rate is in conformance with the optimality condition of problem IV and Definition 1. Moreover using Assumption 1 I observe that $1+r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \leq R_{(\widetilde{\alpha}_\ell+\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})$. Last part of Proposition 6 gives $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 1 + r^{ss}_{\widetilde{\alpha}_\ell}(\boldsymbol{\theta}, \boldsymbol{R})$, and by continuity if the social planner chooses an small enough $\epsilon > 0$, one must have $1 + r^{ss}_{(\widetilde{\alpha}_\ell+\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \leq R_{(\widetilde{\alpha}_\ell+\epsilon)}(\boldsymbol{\theta}, \boldsymbol{R})$. This ensures that the regulated steady state exists and satisfies Definition 1. ■

Proof of Proposition ??. **(To Be Written)**. ■

Proof of Lemma 7. **(To Be Written)**. ■

Proof of Proposition 8. Using problem III_b and conditions 14 and 15 for $z \in \{\ell, m, i\}$, one observes that:

$$V^{ss}_z(\sigma) = \big((1-\sigma)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r^{ss}_z(\sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r^{ss}_z(\sigma)) - \sigma r^{ss}_z(\sigma)\big)e\,.$$

using the definitions of Λ and Φ, I can obtain a more explicit form of the objective function:

$$\begin{aligned} V^{ss}_z(\sigma) &= \left(\frac{(1-\sigma)\big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r^{ss}_z(\sigma))\big)}{\theta_2 R_2 - \theta_1 R_1} \right) e \\ &+ \left(\frac{(1+r^{ss}_z(\sigma))(\theta_2-\theta_1)R_1R_2 - r^{ss}_z(\sigma)(\sigma)(\theta_2R_2 - \theta_1R_1)}{\theta_2R_2 - \theta_1R_1} \right) e\,. \end{aligned}$$

Now using the expressions for $r_z^{ss}(\sigma)$ for $z \in \{\ell, m, i\}$ in Lemma 7 I can take the derivative for each $z \in \{\ell, m, i\}$. For $z = \ell$:

$$\frac{d(1+r_\ell^{ss}(\sigma))}{d\sigma} = \left(\frac{1}{1-\sigma}\right)^2 (1+r_\ell^{ss}(\sigma)) = \left(\frac{\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)^2 .$$

Now using above I get:

$$\begin{aligned}\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} &= \left(\frac{\big(-(\theta_2-\theta_1)R_1R_2 + ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\big)}{\theta_2 R_2 - \theta_1 R_1}\right) e \\ &+ \left(\frac{\Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1 - (1-\theta_2)R_2)\Big)(1+r_\ell^{ss})^2}{(\theta_2 R_2 - \theta_1 R_1)}\right) e \\ &- r_\ell^{ss} e .\end{aligned}$$

I simplify to:

$$\begin{aligned}\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} &= \left(\frac{\big((\theta_2-\theta_1)R_1R_2(2+r_\ell^{ss}) - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\big)}{\theta_2 R_2 - \theta_1 R_1}\right) r_\ell^{ss} e \\ &- r_\ell^{ss} e .\end{aligned}$$

I note that $r_\ell^{ss} = \frac{\theta_2 R_2}{1-\theta_2 R_2}$ and so I can simplify to get:

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{R_2 - 1}{1-\theta_2 R_2}\right) r_\ell^{ss} e .$$

The proof for $z = i$ is very similar. For $z = m$, one observes that $r_m^{ss}(\sigma) = r_m^{ss}$ for any (small enough) σ and so:

$$\begin{aligned}\frac{d(1+r_m^{ss}(\sigma))}{d\sigma} &= 0 , \\ \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_m^{ss}(\sigma)) &= 0 .\end{aligned}$$

Hence:

$$\frac{dV_m^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = -r_m^{ss} e .$$

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