Intermediary Leverage Cycles and Financial Stability

Tobias Adrian and Nina Boyarchenko

The views presented here are the authors’ and are not representative of the views of the Federal Reserve Bank of New York or of the Federal Reserve System.
Introduction

Questions about Financial Stability Policy

- Systemic distress of financial intermediaries raises questions about financial stability policies:
  - How does capital regulation affect the trade-off between the pricing of credit and the amount of systemic risk?
  - How does macroprudential policy function in equilibrium?
  - What are the welfare implications of capital regulation?
- We develop a theoretical framework to address these questions
Our Approach

- We use a standard macro model with a financial sector and add one key assumption:
  - Funding constraints of financial intermediaries are risk based, so intermediaries have to hold more capital when the riskiness of their assets increases
- This assumption is empirically motivated from risk management practices and regulatory constraints

- Equilibrium dynamics capture stylized facts:
  - Procyclical leverage of intermediary balance sheets
  - Procyclical share of intermediated credit
  - Relationship between asset risk premia and intermediary leverage
The Model

Economy Structure

**Producers**
random dividend stream, $A_t$, per unit of project financed by direct borrowing from intermediaries and households

**Intermediaries**
financed by households against capital investments

**Households**
solve portfolio choice problem between holding intermediary debt, physical capital and risk-free borrowing/lending

$A_t k_{ht}$

$i_t$

$A_t k_t$

$C_{bt} b_{ht}$
Production

- Aggregate amount of capital $K_t$ evolves as
  \[
dK_t = (I_t - \lambda_k)K_t dt
  \]
- Total output evolves as
  \[
  Y_t = A_t K_t
  \]
- Stochastic productivity of capital \(\{A_t = e^{at}\}_{t \geq 0}\)
  \[
da_t = \bar{a} dt + \sigma_a dZ_{at}
  \]
- $p_{kt}A_t$ denotes the price of one unit of capital in terms of the consumption good
Households

- Household preferences are:
  \[
  \mathbb{E} \left[ \int_0^{+\infty} e^{- (\xi_t + \rho_h t)} \log c_t \, dt \right]
  \]

- Liquidity preference shocks (as in Allen and Gale (1994) and Diamond and Dybvig (1983)) are \( \exp (-\xi_t) \)

  \[
  d\xi_t = \sigma_{\xi \rho_{\xi,a}} dZ_{at} + \sigma_\xi \sqrt{1 - \rho^2_{\xi,a}} dZ_{\xi t}
  \]

- Households do not have access to the investment technology

  \[
  dk_{ht} = -\lambda_k k_{ht} \, dt
  \]
Households’ Optimization

\[
\max_{\{c_t, k_{ht}, b_{ht}\}} \mathbb{E} \left[ \int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t \, dt \right]
\]

subject to

\[
dw_{ht} = r_{ft} w_{ht} \, dt + p_{kt} A_t k_{ht} (dR_{kt} - r_{ft} \, dt) + p_{bt} A_t b_{ht} (dR_{bt} - r_{ft} \, dt) - c_t \, dt
\]

and no-shorting constraints

\[
k_{ht} \geq 0 \\
b_{ht} \geq 0
\]
Intermediaries

- Financial intermediaries create new capital
  
  \[ dk_t = (\Phi(i_t) - \lambda_k) k_t dt \]

- Investment carries quadratic adjustment costs (Brunnermeier and Sannikov (2012))
  
  \[ \Phi(i_t) = \phi_0 \left( \sqrt{1 + \phi_1 i_t} - 1 \right) \]

- Intermediaries finance investment projects through inside equity and outside risky debt giving the budget constraint
  
  \[ p_{kt} A_t k_t = p_{bt} A_t b_t + w_t \]
Intermediaries’ Risk Based Capital Constraint

- Risk based capital constraint (Danielsson, Shin, and Zigrand (2011))

\[ \alpha \sqrt{ \frac{1}{dt} \left\langle k_t d \left( p_{kt} A_t \right) \right\rangle^2 } = w_t \]

- Implies a time-varying leverage constraint

\[ \theta_t = \frac{p_{kt} A_t k_t}{w_t} = \frac{1}{\alpha \sqrt{ \frac{1}{dt} \left\langle \frac{d(p_{kt} A_t)}{p_{kt} A_t} \right\rangle^2 } } \]

- Note that the constraint is such that intermediary equity is proportional to the Value-at-Risk of total assets

- This will imply that default probabilities vary over time

- Microfoundation of the risk based capital constraint is provided in Adrian and Shin (2010)
Risk-based Capital Constraints

*VaR is the potential loss in value of inventory positions due to adverse market movements over a defined time horizon with a specified confidence level. We typically employ a one-day time horizon with a 95% confidence level.*

### Average Daily VaR

<table>
<thead>
<tr>
<th>Risk Categories</th>
<th>2011</th>
<th>2010</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rates</td>
<td>$94</td>
<td>$93</td>
<td>$176</td>
</tr>
<tr>
<td>Equity prices</td>
<td>33</td>
<td>68</td>
<td>66</td>
</tr>
<tr>
<td>Currency rates</td>
<td>20</td>
<td>32</td>
<td>36</td>
</tr>
<tr>
<td>Commodity prices</td>
<td>32</td>
<td>33</td>
<td>36</td>
</tr>
<tr>
<td>Diversification effect $^1$</td>
<td>(66)</td>
<td>(92)</td>
<td>(96)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$113</strong></td>
<td><strong>$134</strong></td>
<td><strong>$218</strong></td>
</tr>
</tbody>
</table>

---

1. Equals the difference between total VaR and the sum of the VaRs for the four risk categories. This effect arises because the four market risk categories are not perfectly correlated.

Source: Goldman Sachs 2011 Annual Report
The Model

Commercial Bank Tightening Standards

\[ \rho = 0.68013 \]

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The Model

Systemic Distress

- Solvency risk defined by
  \[ \tau_D = \inf_{t \geq 0} \{ w_t \leq \bar{\omega} p_{kt} A_t K_t \} \]

- Term structure of systemic distress
  \[ \delta_t (T) = \mathbb{P} (\tau_D \leq T | (w_t, \theta_t)) \]

In distress

- Management changes
- Intermediary leverage reduced to \( \theta \approx 1 \) by defaulting on debt
- Intermediary instantaneously restarts with wealth
  \[ w_{\tau_D^+} = \frac{\theta_{\tau_D}}{\theta} w_{\tau_D} \]
Intermediaries’ Optimization

- Intermediary maximizes equity holder value to solve

\[
\max_{\{k_t, \beta_t, i_t\}} \mathbb{E} \left[ \int_0^{\tau_D} e^{-\rho t} w_t dt \right]
\]

subject to the dynamic intermediary budget constraint

\[
dw_t = k_t p_{kt} A_t (dR_{kt} + (\Phi (i_t) - i_t/p_{kt}) dt) - b_t p_{bt} A_t dR_{bt}
\]

and the risk based capital constraint

\[
\alpha \sqrt{\frac{1}{dt} \langle k_t d(p_{kt} A_t) \rangle^2} = w_t
\]
Equilibrium

An equilibrium in this economy is a set of price processes \( \{p_{kt}, p_{bt}, C_{bt}\}_{t \geq 0} \), a set of household decisions \( \{k_{ht}, b_{ht}, c_{t}\}_{t \geq 0} \), and a set of intermediary decisions \( \{k_{t}, \beta_{t}, i_{t}, \theta_{t}\}_{t \geq 0} \) such that:

1. Taking the price processes and the intermediary decisions as given, the household’s choices solve the household’s optimization problem, subject to the household budget constraint.
2. Taking the price processes and the household decisions as given, the intermediary’s choices solve the intermediary optimization problem, subject to the intermediary wealth evolution and the risk based capital constraint.
3. The capital market clears:
   \[ K_{t} = k_{t} + k_{ht}. \]
4. The risky bond market clears:
   \[ b_{t} = b_{ht}. \]
5. The risk-free debt market clears:
   \[ w_{ht} = p_{kt} A_{t} k_{ht} + p_{bt} A_{t} b_{ht}. \]
6. The goods market clears:
   \[ c_{t} = A_{t} \left( K_{t} - i_{t} k_{t} \right). \]
Related Literature

- **Leverage Cycles:** Geanakoplos (2003), Fostel and Geanakoplos (2008), Brunnermeier and Pedersen (2009)

- **Amplification in Macroeconomy:** Bernanke and Gertler (1989), Kiyotaki and Moore (1997)

Solution Strategy

- Equilibrium is characterized by two state variables, leverage $\theta_t$ and relative intermediary net worth $\omega_t$

$$\omega_t = \frac{w_t}{w_t + w_{ht}} = \frac{w_t}{p_{kt}A_tK_t}$$

- Represent state dynamics as

$$\frac{d\omega_t}{\omega_t} = \mu_{\omega t} dt + \sigma_{\omega a,t} dZ_{at} + \sigma_{\omega \xi, t} dZ_{\xi t}$$

$$\frac{d\theta_t}{\theta_t} = \mu_{\theta t} dt + \sigma_{\theta a,t} dZ_{at} + \sigma_{\theta \xi, t} dZ_{\xi t}$$

- Risk-based capital constraint implies

$$\alpha^{-2} \theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2$$
Volatility Risk

\[ y = 0.00074 - 0.12x \]
\[ R^2 = 0.013 \]

\[ y = 0.014 - 0.21x \]
\[ R^2 = 0.053 \]

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Mean</th>
<th>5%</th>
<th>Median</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_0 )</td>
<td>0.014</td>
<td>0.000</td>
<td>-0.003</td>
<td>0.000</td>
<td>0.003</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>-0.208</td>
<td>-0.105</td>
<td>-0.187</td>
<td>-0.104</td>
<td>-0.025</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.053</td>
<td>0.013</td>
<td>0.001</td>
<td>0.011</td>
<td>0.035</td>
</tr>
</tbody>
</table>
Solution

Intermediary Balance Sheets I

\[ y = 0.0086 + 0.56x \]
\[ R^2 = 0.056 \]

\[ y = -0.071 + 0.76x \]
\[ R^2 = 0.46 \]
## Intermediary Balance Sheets II

### Table: Procyclicality of Intermediated Credit

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Mean</th>
<th>5%</th>
<th>Median</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>-0.071</td>
<td>-0.112</td>
<td>-0.203</td>
<td>-0.108</td>
<td>-0.040</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.756</td>
<td>0.434</td>
<td>0.190</td>
<td>0.433</td>
<td>0.680</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.460</td>
<td>0.048</td>
<td>0.009</td>
<td>0.045</td>
<td>0.101</td>
</tr>
</tbody>
</table>
# Excess Returns

\[ y = 0.026 - 0.018x \]
\[ R^2 = 0.052 \]

\[ y = 0.12 - 0.31x \]
\[ R^2 = 0.17 \]

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Mean</th>
<th>5%</th>
<th>Median</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_0 )</td>
<td>0.118</td>
<td>0.076</td>
<td>0.068</td>
<td>0.076</td>
<td>0.084</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>-0.310</td>
<td>-0.031</td>
<td>-0.038</td>
<td>-0.031</td>
<td>-0.024</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.167</td>
<td>0.100</td>
<td>0.064</td>
<td>0.100</td>
<td>0.143</td>
</tr>
</tbody>
</table>
Equilibrium Prices of Risk I

Shocks

\[
d\hat{y}_t = \sigma_a^{-1} (d \log Y_t - \mathbb{E}_t [d \log Y_t]) = dZ_{at}
\]

\[
d\hat{\theta}_t = (\sigma_{\theta a, t}^2 + \sigma_{\theta \xi, t}^2)^{-\frac{1}{2}} \left( \frac{d\theta_t}{\theta_t} - \mathbb{E}_t \left[ \frac{d\theta_t}{\theta_t} \right] \right)
\]

\[
= \frac{\sigma_{\theta a, t}}{\sqrt{\sigma_{\theta a, t}^2 + \sigma_{\theta \xi, t}^2}} dZ_{at} + \frac{\sigma_{\theta \xi, t}}{\sqrt{\sigma_{\theta a, t}^2 + \sigma_{\theta \xi, t}^2}} dZ_{\xi t}.
\]
Equilibrium Prices of Risk II

Price of risk of leverage

\[ \eta_{\theta t} = \sqrt{1 + \frac{(\sigma_{ka,t} - \sigma_a)^2}{\sigma_{k\xi,t}^2}} \left( -\frac{2\theta_t \omega_t p_{kt}}{\beta (1 - \omega_t)} \sigma_{k\xi,t} + \sigma_{\xi} \sqrt{1 - \rho_{\xi,a}^2} \right) \]

- Price of risk of leverage is always positive (Adrian, Etula, and Muir (2013)), and depends on leverage growth in a nonmonotonic fashion (Adrian, Moench, and Shin (2010) find a negative relationship)
Equilibrium Prices of Risk III

Figure: Source: Adrian, Etula, and Muir (2013)
Equilibrium Prices of Risk IV

Price of risk of output

\[ \eta_{yt} = \sigma_a + \sigma_\xi \left( \rho_{\xi,a} - \frac{\sigma_{ka,t} - \sigma_a}{\sigma_{k\xi,t}} \sqrt{1 - \rho_{\xi,a}^2} \right) \]

- Switches sign, consistent with insignificant estimates of price of output risk
- Usually becomes negative when exposure to liquidity shock is small
Term Structure of Systemic Risk

Distress probability

Horizon

$\alpha = 2$
$\alpha = 4$

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Volatility Paradox
Constant Leverage Benchmark

- Constant expected output and consumption growth
- But lower level of output and consumption growth
- Constant investment and price of capital
- Liquidity shocks have no impact on real activity
A Sample Path of the Economy

Distortions and Amplification

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Household Welfare

![Graph showing welfare and distress probability over different time periods (6 months, 1 year, 5 years).](image)

- Welfare on the y-axis, with values ranging from 0 to 1.
- Distress probability on the y-axis, with values ranging from 0 to 1.
- Time periods are represented on the x-axis, with increments of 2 units (2, 4, 6, 8, 10).
- The graph compares welfare and distress probability for different time periods: 6 months, 1 year, and 5 years, indicated by different line styles and colors.

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Conclusion

- Dynamic, general equilibrium theory of financial intermediaries’ leverage cycle with endogenous amplification and endogenous systemic risk
- Conceptual basis for policies towards financial stability
- Systemic risk return trade-off: tighter intermediary capital requirements tend to shift the term structure of systemic downward, at the cost of increased prices of risk today

- Model captures important stylized facts:
  1. Procyclical intermediary leverage
  2. Procyclicality of intermediated credit
  3. Financial sector equity return and intermediary leverage growth
  4. Exposure to intermediary leverage shocks as pricing factor


Broker-Dealer Balance Sheets: Levels

Security Broker-Dealer: Assets, Liabilities, Equity, Leverage

Dollars (trillions)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0

Source: Flow of Funds

Leverage
(right axis)

Total Liabilities
(left axis)

Total Assets
(left axis)
Broker-Dealer Balance Sheets: Annual Growth

Security Broker-Dealer: Assets, Liabilities, Equity, Leverage

Source: Flow of Funds

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Broker-Dealer Balance Sheets: Adjustments

Security Broker Dealer Change of Assets as a Function of Change in Equity and in Liabilities (Annual)

Source: Federal Reserve Flow of Funds. Equity is at book value.
Balance Sheet Adjustments

Empirical Evidence

Investment Banks

Commercial Banks (FDIC)
Broker-Dealers and Banks

8 Largest Broker-Dealers and Investment Banks (1994Q1-2012Q2)

- Change in Equity & Change in Debt (Billions)
- Change in Assets (Billions)

Equation 1: $y = 0.002x + 9.45$
Equation 2: $y = 0.998x - 9.45$
Broker-Dealer VaR

[Graph showing the relationship between implied volatility and unit VaR over time, highlighting significant spikes in 2008 and 2009.]
Broker-Dealer VaR

![Chart showing the trend of Broker-Dealer VaR with time, indicating periods of increased leverage and variation in the pre-crisis standard deviations.]
### Simulation Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{a}$</td>
<td>0.0651</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>0.388</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.06</td>
</tr>
<tr>
<td>$\rho_h - \sigma^2/2$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>20</td>
</tr>
<tr>
<td>$\lambda_k$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\rho_{\xi,a}$</td>
<td>0</td>
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<tr>
<td>$\sigma_{\xi}$</td>
<td>0.0388</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2.5</td>
</tr>
</tbody>
</table>

- Ref.: Brunnermeier and Sannikov (2012)
- Monthly simulation frequency
- 10000 paths; 70 years
Equilibrium

Lemma 1

\[
\begin{align*}
\mu_{Rk,t} &= K_0(\omega_t, \theta_t) + K_a(\omega_t, \theta_t) \sigma_{ka,t} + \sigma_k \sqrt{1 - \rho_{\xi,a}^2} \\
\mu_{Rb,t} &= B_0(\omega_t, \theta_t) + B_a(\omega_t, \theta_t) \sigma_{ka,t} + B_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
\mu_{\omega t} &= O_0(\omega_t, \theta_t) + O_a(\omega_t, \theta_t) \sigma_{ka,t} + O_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
\mu_{\theta t} &= S_0(\omega_t, \theta_t) + S_a(\omega_t, \theta_t) \sigma_{ka,t} - O_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
r_{ft} &= R_0(\omega_t, \theta_t) + R_a(\omega_t, \theta_t) \sigma_{ka,t} \\
\sigma_{ba,t} &= \frac{2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_a - \frac{2\theta_t \omega_t p_{kt} + \beta (1 - \theta_t \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_{ka,t} \\
\sigma_{b\xi,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta (1 - \theta_t \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_{k\xi,t} \\
\sigma_{\theta a,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)}{\beta \omega_t} (\sigma_{ka,t} - \sigma_a) \\
\sigma_{\theta \xi,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)}{\beta \omega_t} \sigma_{k\xi,t} \\
\sigma_{k\xi,t} &= -\sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \sigma_{ka,t}^2} \\
\sigma_{ka,t} &= \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \left(1 + \frac{1 - \omega_t}{\omega_t (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t))}\right). 
\end{align*}
\]
Risk-free Rate

- **Household Euler equation**

\[ r_{ft} = \left( \rho_h - \frac{\sigma^2}{2} \right) + \frac{1}{dt} \mathbb{E} \left[ \frac{dc_t}{c_t} \right] - \frac{1}{dt} \mathbb{E} \left[ \frac{\langle dc_t \rangle^2}{c_t^2} + \frac{\langle dc_t, d\xi_t \rangle^2}{c_t} \right] \]

- **Goods market clearing implies**

\[ dc_t = d \left( K_t A_t - i_t k_t A_t \right) = A_t dK_t + \left( K_t - i_t k_t \right) dA_t - A_t k_t di_t - A_t i_t dk_t - k_t \langle di_t, dA_t \rangle \]
Stress Tests

*Inherent limitations to VaR include [...] VaR does not estimate potential losses over longer time horizons where moves may be extreme.*
Stress Tests

- Could consider a forward-looking capital constraint

\[
\theta_t^{-1} \geq \phi \sqrt{\mathbb{E}_t \left[ \int_t^T \left( \sigma_{ka,s}^2 + \sigma_{k\xi,s}^2 \right) ds \right]}.
\]

- Looks like a robust-control constraint

- Rewrite intermediary optimization as

\[
V_t(\psi) = \max_{\{i, \beta, k, \alpha_s\}} \mathbb{E}_t \left[ \int_t^{\tau_D} e^{-\rho(s-t)} w_t (i, \beta, k) ds \right]
\]

s.t.

\[
\frac{\theta_s^{-1}}{\alpha_s} \geq \sqrt{\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2}
\]

\[
\theta_t^{-1} = \phi \sqrt{\mathbb{E}_t \left[ \int_t^T \frac{\theta_s^{-2}}{\alpha_s^2} ds \right]}.
\]

- “Choose optimal capital plan”
Alternative Specification

- Two financial sectors: banks and funds

- Leveraged intermediaries have VaR constraint (as in the current paper) while funds have skin in the game constraint (as in He and Krishnamurthy (2012, 2013))

- Bank managers, fund managers, and households have log utility

- VaR constraint sometimes binds
Extensions

**Households**
Invest in risk-free debt, non-bank financial sector and bank financial sector

**Banks**
Create new capital; financed by debt issuance to the households

**Funds**
Hold existing capital; financed by profit sharing contracts with households

**Producers**
Productivity $A_t$ per unit of project; financed by financial sector
Intermediaries

- Two types of intermediaries: non-bank ("fund") and bank
- Unit mass of specialists manage funds; unit mass of bankers manage banks
- Future work: interactions between different intermediary types
Fund Sector

- Modeled as in He and Krishnamurthy (2012, 2013)
- Fund is formed each period $t$ as a random match between a specialist and a household
- Specialist contributes all of his wealth $w_{ft}$ to the fund
- Household contributes up to $mw_{ft}$ to the fund
- $m$: tightness of the specialists’ capital constraint
- Specialists control the allocation of fund capital to holding capital projects and risk-free debt

Notice:
- No new capital project creation
- No risky debt
Specialists’ Optimization I

Specialists’ maximize expected consumption

$$\max_{\{c_{ft}, \theta_{ft}\}} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} \log c_{ft} \, dt \right],$$

subject to the dynamic budget constraint

$$\frac{dW_{ft}}{W_{ft}} = \theta_{ft} (dR_{kt} - r_{ft} \, dt) + r_{ft} \, dt - \frac{c_{ft}}{W_{ft}} \, dt$$
Specialists’ Optimization II

Lemma 2

The specialists consume a constant fraction of their wealth

\[ c_{ft} = \rho w_{ft}, \]

and allocate the fund’s capital as a mean-variance investor

\[ \theta_{ft} = \frac{\mu_{Rk,t} - r_{ft}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}. \]
Banking Sector

- Banks create new capital

\[ dk_t = (\Phi(i_t) - \lambda_k) k_t dt \]

- Investment carries quadratic adjustment costs (Brunnermeier and Sannikov (2012))

\[ \Phi(i_t) = \phi_0 \left( \sqrt{1 + \phi_1 i_t} - 1 \right) \]

- Banks finance investment projects through inside equity and outside risky debt giving the budget constraint

\[ p_{kt} A_t k_t = p_{bt} A_t b_t + w_t \]
Bankers’ Optimization I

The representative banker solves

$$\max_{\theta_t, i_t, c_{bt}} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} \log c_{bt} dt \right]$$

subject to the dynamic budget constraint

$$\frac{dw_t}{w_t} = \theta_t \left( dR_{kt} - r_{ft} dt + \left( \Phi \left( i_t \right) - \frac{i_t}{\rho_{kt}} \right) dt \right)$$

$$- \left( \theta - 1 \right) \left( dR_{bt} - r_{ft} dt \right) + r_{ft} dt - \frac{c_{bt}}{w_t} dt,$$

and the risk-based capital constraint

$$\theta_t \leq \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}}.$$
Bankers’ Optimization II

Lemma 3

The representative banker optimally consumes at rate

\[ c_{bt} = \rho w_t \]

and invests in new projects at rate

\[ i_t = \frac{1}{\phi_1} \left( \frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right). \]

While the capital constraint in not binding, the banking system leverage is

\[ \theta_t = \frac{\sigma_{ba,t}^2 - \sigma_{ka,t} \sigma_{ba,t} + \sigma_{b\xi,t}^2 - \sigma_{k\xi,t} \sigma_{b\xi,t} - (\mu_{Rb,t} - r_{ft})}{\left( (\sigma_{ba,t} - \sigma_{ka,t})^2 + (\sigma_{b\xi,t} - \sigma_{k\xi,t})^2 \right)} \]

\[ + \frac{\left( \mu_{Rk,t} + \Phi(i_t) - \frac{i_t}{\phi_1 p_{kt}} - r_{ft} \right)}{\left( (\sigma_{ba,t} - \sigma_{ka,t})^2 + (\sigma_{b\xi,t} - \sigma_{k\xi,t})^2 \right)}. \]
Households

- Household preferences are:

\[
\mathbb{E} \left[ \int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t \, dt \right]
\]

- Liquidity preference shocks (as in Allen and Gale (1994) and Diamond and Dybvig (1983)) are \( \exp(-\xi_t) \)

\[
d\xi_t = \sigma_\xi dZ_{\xi_t}
\]

- Households allocate wealth between risky bank debt and contributions to funds
Households’ Optimization I

The representative household solves

$$\max_{\pi_{kt}, \pi_{bt}, c_t} \mathbb{E} \left[ \int_0^{+\infty} e^{-\xi_t - \rho_h t} \log c_t \, dt \right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ht}}{w_{ht}} = \pi_{kt} \theta_{ft} \left( dR_{kt} - r_{ft} \, dt \right) + \pi_{bt} \left( dR_{bt} - r_{ft} \, dt \right) + r_{ft} \, dt - \frac{c_t}{w_{ht}} \, dt,$$

the skin-in-the-game constraint

$$\pi_{kt} w_{ht} \leq m w_{ft},$$

and no shorting constraints

$$\pi_{kt} \geq 0$$

$$b_{ht} \geq 0.$$
Households’ Optimization II

Lemma 4

The households’ optimal consumption choice satisfies

\[ c_t = \left( \rho_h - \frac{\sigma^2_{\xi}}{2} \right) w_{ht}. \]

While the households are unconstrained in their wealth allocation, the households’ optimal portfolio choice is given by

\[
\begin{bmatrix}
\pi_{kt} \\
\pi_{bt}
\end{bmatrix} = \left( \begin{bmatrix}
\theta_{ft}\sigma_{ka,t} & \theta_{ft}\sigma_{k\xi,t} \\
\sigma_{ba,t} & \sigma_{b\xi,t}
\end{bmatrix}
\begin{bmatrix}
\theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\
\theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t}
\end{bmatrix} \right)^{-1} \begin{bmatrix}
\theta_{ft} (\mu_{Rk,t} - r_{ft}) \\
\mu_{Rb,t} - r_{ft}
\end{bmatrix} - \left( \begin{bmatrix}
\theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\
\theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t}
\end{bmatrix} \right)^{-1} \begin{bmatrix}
0 \\
\sigma_{\xi}
\end{bmatrix}.
\]
Equilibrium

An equilibrium in the economy is a set of price processes \( \{p_{kt}, \rho_{bt}, \rho_{ft}\}_{t \geq 0} \), a set of household decisions \( \{\pi_{kt}, b_{ht}, c_{t}\}_{t \geq 0} \), a set of specialist decisions \( \{k_{ft}, c_{ft}\}_{t \geq 0} \), and a set of intermediary decisions \( \{k_{t}, i_{t}, b_{t}, c_{bt}\}_{t \geq 0} \) such that the following apply:

1. Taking the price processes, the specialist decisions and the intermediary decisions as given, the household’s choices solve the household’s optimization problem, subject to the household budget constraint, the no shorting constraints and the skin-in-the-game constraint for the funds.
2. Taking the price processes, the specialist decisions and the household decisions as given, the intermediary’s choices solve the intermediary’s optimization problem, subject to the intermediary budget constraint, and the regulatory constraint.
3. Taking the price processes, the household decisions and the intermediary decisions as given, the specialist’s choices solve the specialist’s optimization problem, subject to the specialist budget constraint.
4. The capital market clears at all dates
   \[ k_{t} + k_{ft} = K_{t}. \]
5. The risky bond market clears
   \[ b_{t} = b_{ht}. \]
6. The risk-free debt market clears
   \[ w_{t} + w_{ft} + w_{ht} = p_{kt}A_{t}K_{t}. \]
7. The goods market clears
   \[ c_{t} + c_{bt} + c_{ft} + A_{t}k_{t}i_{t} = K_{t}A_{t}. \]
Additional Research

- Tradeoff between capital and liquidity regulation
- Stress tests
- Intermediation chains