Uncertainty shocks, asset supply and pricing over the business cycle*

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Abstract

This paper studies a DSGE model with endogenous financial asset supply and ambiguity averse investors. An increase in uncertainty about financial conditions leads firms to substitute away from debt and reduce shareholder payout in bad times when measured risk premia are high. Regime shifts in volatility generate large low frequency movements in asset prices due to uncertainty premia that are disconnected from the business cycle.

1 Introduction

This paper studies a DSGE model with endogenous financial asset supply and ambiguity averse investors. Firms face frictions in debt and equity markets and decide on capital structure and net payout. Investors perceive time varying uncertainty about real and financial technology. Uncertainty shocks lead firms to reoptimize capital structure as relative asset prices such as risk premia change. In an estimated model that allows for both smooth changes in ambiguity and regime shifts in volatility, concerns about financial conditions generates low frequency movements in asset prices that are disconnected from the business cycle.

We model ambiguity aversion by recursive multiple priors utility. When agents evaluate an uncertain consumption plan, they use a worst case conditional probability drawn from a set of beliefs. A larger set indicates higher uncertainty. In our DSGE context, beliefs are parameterized by the conditional means of innovations to real or financial technology. Conditional means are drawn from intervals centered around zero. The width of the interval measures the amount of ambiguity. It can change either smoothly with the arrival of intangible information or it can jump discretely across regimes with different stochastic volatility. Both types of change in uncertainty work like a drop in the conditional mean and hence have first order effects on decisions.

Time variation in ambiguity leads econometricians to measure time varying premia in asset markets. Indeed, when investors evaluate an asset *as if* the mean payoff is low, then

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they are willing to pay only a low price for it. To an econometrician, the return on the asset – actual payoff minus price – will then look unusually high. The more ambiguity investors perceive, the lower is the price and the higher is the subsequent return. An econometrician who runs a regression of return on price (normalized by dividends) will thus find a positive coefficient. If interest rates are stable – say because bonds are less ambiguous than stocks – then the price-dividend ratio helps forecast *excess* returns on stocks, that is, there are time varying risk premia on stocks. A convenient feature of our model is that asset premia are due to perceptions of low mean, and therefore appear in a standard loglinear approximation to the equilibrium.

The real side of our model is that of an RBC model with adjustment costs and variable capacity utilization. To focus on financial frictions, we abstract from nominal features and labor market frictions. However, the supply of equity and corporate debt, and hence leverage, is endogenously determined. Firms face an upward sloping marginal cost curve for debt: debt is cheaper than equity at low levels of debt, but becomes eventually more expensive as debt increases. Firms also have a preference for dividend smoothing. To maximize shareholder value, they find interior optima for leverage and net shareholder payout. Firm decisions are sensitive to ambiguity since shareholder value incorporates uncertainty premia. In particular, an increase in ambiguity about real or financial technology leads firms to substitute away from debt and reduce leverage.

We estimate the model with postwar US data on eight observables. The macro quantities are GDP growth, investment growth and consumption growth. The financial quantities are net nonfinancial corporate debt, net nonfinancial corporate payout and the value of nonfinancial corporate equity, all measured as ratios relative to GDP. Including the latter two variables implies that we match the corporate price/payout ratio, which behaves similarly to the price-dividend ratio. Finally, we include measures of the real short term interest rate and the slope of the yield curve. In sum, we ask our model to account for the price and quantity dynamics of equity and debt, as well as standard macro aggregates.

Estimation delivers two main results. First, regime shifts in volatility help understand jointly the heteroskedasticity of macro quantities and the low frequency movements in asset prices. When we allow for two regimes for stochastic volatilities with symmetric priors, we identify a low and a high volatility regime. The latter dominates a prolonged period of time from the early '70s to the second half of the '80s, when quantities were volatile and the price/payout ratio was low. A switch from the low volatility regime to the high volatility regime determines a drop in stock prices of around 20% on impact that is followed by a further drawn out decline that can last for decades. This is because higher volatility increases ambiguity and generates a substantial price discount.[1]

The second result is that financial quantities depend relatively more on uncertainty shocks than real variables. In particular, changes in uncertainty about future financing costs are important for understanding the positive comovement of debt and net payout to shareholders. Those changes also help our model account for the excess volatility of stock prices. Indeed, since financing costs affect corporate cash flow relatively more than consumption, uncertainty about financing costs moves stock prices more than bond prices. Moreover, the model can

[1]If the economy happens to revert to the low volatility regime, a symmetric pattern occurs, with a stock market boom followed by a slow return to the low volatility conditional steady state.

generate movements in stock prices that are somewhat decoupled from the business cycles. Importantly, both dividends and prices are endogenously determined in our model as optimal responses to uncertainty shocks.

Relative to the literature, the paper makes three contributions. First it introduces a class of linear DSGE models that accommodate both endogenous asset supply and time varying uncertainty premia. Second, it shows how to extend that class of models to allow for first order effects of stochastic volatility. Finally, the results suggest a prominent role for uncertainty shocks in driving jointly asset prices and firm financing decisions. [details to be written]

The paper is structured as follows. Section 2 presents a simplified version of the model with exogenous output and a simpler dividend smoothing motive. Since that model can be worked out in closed form it serves to illustrate the mechanics of endogenous asset supply and asset valuation with ambiguity aversion. Section 3 then describes the quantitative model, our solution and estimation strategy, and then discusses the estimation results.

2 A simple model

In this section we consider a simple model of asset pricing with endogenous supply. The key simplification relative to our estimated model below is that output is exogenous – firms' only decisions are debt and net payout to shareholders. We also simplify the firms dividend smoothing motive by assuming that payout has to be fixed one period in advance.

The section has two goals. The first is to illustrate the tradeoffs faced by the firm using closed form solutions. The main point here is that if a firm optimizes payout and capital structure in a world with uncertainty shocks, then uncertainty shocks tend to make net debt and net payout go together. The second goal is to illustrate how asset pricing works in our linearized multiple priors model. In particular, we derive unconditional asste premia, such as the equity premium and the yield curve, from deterministic steady state conditions, and we derive time varying premia from linearized first order conditions.

2.1 Setup

A representative household invests in equity and debt issues by financing constrained firms.
Technology: production and financing

Production is exogenous. It consists of a certain component L and a random component F_t. Here L contains labor income as well as income generated by firms that are not publicly traded, whereas F_t is the cash flow of publicly traded firms, net of fixed financing costs. In the simple model of this section, this division is exogenous. In the richer model estimated below, both components are variable and imperfectly correlated. An important feature of both this simple model and our estimated model is that there are shocks to the profitability of publicly traded firms that do not affect other components of GDP.

Publicly traded firms decide on net payout to shareholders D_t as well as the face value of short term debt B_t. Payout to shareholders has to be chosen one period in advance. Debt can be changed at short notice, but the marginal cost of debt slopes upward. The cost of debt between $t-1$ and t also depends on financial conditions, captured by a time varying parameter Ψ_t The firm budget constraint for date t is

$$D_t = F_t + Q_t B_t - B_{t-1} - \kappa\left(B_{t-1}; \Psi_t\right) \tag{1}$$

where the cost function κ is convex in debt B and its derivatives satisfy $\kappa_1(0; \Psi) < 0$ and $\kappa_{12}(B, \Psi) > 0$. The first condition says that the first dollar of debt is cheaper than outside equity. The second condition says that Ψ determines the marginal cost of a dollar of debt.

The resource constraint is

$$C_t = L + F_t - \kappa\left(B_{t-1}; \Psi_t\right) \tag{2}$$

While consumption depends on the choice of debt and is thus endogenous, we will work with a cost function that makes the effect of debt on total resources second order. To first order, the model can be thought of as an endowment economy. Our assumption of a negative marginal cost of debt at zero implies that the firm issues positive debt in steady state and realizes a gain $-\kappa$. We view F as incorporating a fixed financing cost so the effective total financing cost is positive.

Households own the equity of the firm which trades at a price P_t. The representative household budget constraint is

$$C_t + Q_t B_t + P_t \theta_t = L + (P_t + D_t)\theta_{t-1} + B_{t-1}$$

Households enter the period with equity – on which they receive net payout D_t – and debt. They decide to consume or save in the form of debt and equity. While total savings in a NIPA accounting sense are zero in this economy, firms leverage up and pay out cash flow in form of dividends and interest. Household financial wealth consists of positive positions in equity and debt.

Uncertainty

There are two sources of technological uncertainty in the economy: firm cash flow F_t and the financing cost parameter Ψ_t. We define a vector $\hat{\tau}_t = (\hat{f}_t, -\hat{\psi}_t)$ to collect the log deviations of technology from its steady state value. Our sign convention is that high τ for both components means a "good" technology realization (in the sense that consumption increases, as will become clear below). The data generating process for τ is

$$\hat{\tau}_{t+1} = \phi_\tau \hat{\tau}_t + \mu_t^* + \sigma_\tau \varepsilon_{t+1} \tag{3}$$

where ε is an iid vector of shocks and μ_t^* is a deterministic sequence. The decomposition of the innovation to τ into two components μ and $\sigma\varepsilon$ serves to distinguish between ambiguity and risk, respectively.

Consider the ambiguous component μ. We assume that agents know the long run properties of the sequence μ_t^*. In particular, they know that the long run empirical distribution of μ_t^* is iid with mean zero and variance $\tilde{\sigma}_\tau \tilde{\sigma}_\tau' - \sigma_\tau \sigma_\tau'$. However, agents do not know the exact sequence μ^* and are thus uncertain about the conditional mean relevant for forecasting technology one period ahead. They receive intangible information about the mean next period, which allows them to narrow down their range of forecasts. For example, they reduce the range of forecasts about $\hat{f}_{t+1}$ to a range $[-a_{t,f}, a_{t,f}]$ centered around zero, and similarly for

$-\hat{\psi}_{t+1}$. Agents are not confident enough to further integrate over alternative forecasts (and so in particular they do not use a single forecast).

The vector $a_t = (a_{t,f}, a_{t,\psi})'$ summarizes the ambiguity agents perceive about the different components of technology. We think of $-a_{t,i}$ as an indicator of information quality about technology component i: if $a_{t,i}$ is low, then agents find it relatively easy to forecast $\hat{\tau}_{t,i}$ and and their behavior will be relatively close to that of expected utility maximizers (who use a single probability when making decisions). In contrast, when $a_{t,i}$ is high, then agents do not feel confident about forecasting. Information quality itself evolves as an AR(1)

$$a_{t+1} - \bar{a} = \phi_a (a_t - \bar{a}) + \sigma_a \varepsilon_{t+1}$$

This specification allows for persistence in the quality of information. It also allows for correlation between innovations in τ and the quality of information about τ. In the richer model below, ambiguity a is also allowed to depend on the volatility of τ.

Preferences

The representative household has recursive multiple priors utility. Every period households observe a vector of shocks ε_t. Let ε^t denote the history of shocks up to date t. A consumption plan is a family of functions $c_t(\varepsilon^t)$. Conditional utilities from some consumption plan c are defined recursively by

$$U\left(c; \varepsilon^t\right) = \log c_t\left(\varepsilon^t\right) + \beta \min_{\mu_{t,i} \in \times_i [-a_{t,i}, a_{t,i}]} E^{\mu}\left[U\left(c; \varepsilon^t, \varepsilon_{t+1}\right)\right], \tag{4}$$

where the conditional distribution over ε_{t+1} uses the means $\mu_{t,i}$ that minimize expected continuation utility. If $a_t = 0$, we obtain have standard separable log utility with those conditional beliefs. If $a_t > 0$, then lack of information prevents agents from narrowing down their belief set to a singleton. In response, households take a cautious approach to decision making – they act as if the worst case mean is relevant.[2]

In what follows, we consider equilibria with positive debt. It is then easy to solve the minimization step in (4) at the equilibrium consumption plan: the worst case expected cash flow is low and the worst case expected marginal financing cost is high. Indeed, consumption depends positively on cash flow and, since debt is positive, it depends negatively on the marginal financing cost. It follows that agents act throughout as if forecasting under the worst case mean $\mu_t = -a_t$. This property pins down the representative household's worst case belief after every history and thereby a worst case belief over entire sequences of data. We can thus also compute worst case expectations many periods ahead, which we denote by stars. For example $E^* D_{t+k}$ is the worst case expected dividend k periods in the future.

[2]In the expected utility case, time t conditional utility can be represented as as $E_t\left[\sum_{\tau=0}^{\infty} \log c_{t+\tau}\right]$ where the expectation is taken under a conditional probability measure over sequences that is updated by Bayes' rule from a measure that describes time zero beliefs. An analogous representation exists under ambiguity: time t utility can be written as $\min_{\pi \in \mathcal{P}} E_t^{\pi}\left[\sum_{\tau=0}^{\infty} \log c_{t+\tau}\right]$. The time zero set of beliefs $\mathcal{P}$ can be derived from the one step ahead conditionals $\mathcal{P}_t$ as in the Bayesian case, see Epstein and Schneider (2003) for details.

2.2 Characterizing equilibrium

To describe t-period ahead contingent claims prices, we define random variables M_0^t that represent prices normalized by conditional *worst case* probabilities. This particular normalization is convenient for summarizing the properties of prices, which are derived from households' and firms' first order conditions. We also define a one-period-ahead pricing kernel as $M_{t+1} = M_0^{t+1}/M_0^t$. From household utility maximization, we have the familiar equations

$$\begin{aligned} M_{t+1} &= \beta C_t / C_{t+1} \\ Q_t &= E_t^* \left[M_{t+1} \right] \\ P_t &= E_t^* \left[M_{t+1} \left(P_{t+1} + D_{t+1} \right) \right] \end{aligned}$$

The only difference to a standard model is that expectations are taken under the worst case belief, indicated by a star.

The firm maximizes shareholder value

$$E_0^* \sum M_0^t D_t$$

Shareholder value depends on worst case expectations. This is because state prices determined in financial markets reflect households' attitudes to uncertainty, as illustrated by the household Euler equations above.

Let λ_t denote the shadow value of funds inside the firm at date t, normalized by the contingent claims price M_0^t. The firm's first order equations for debt and dividends, respectively, are

$$\begin{aligned} \lambda_t Q_t &= E_t^* \left[\lambda_{t+1} M_{t+1} \left(1 + \kappa_1 \left(B_t; \Psi_{t+1} \right) \right) \right] \\ E_t^* M_{t+1} &= E_t^* \left[\lambda_{t+1} M_{t+1} \right], \end{aligned}$$

When choosing debt, the firm equates the marginal benefit of a bond issued (which contributes Q dollars, or $Q\lambda$ dollars within the firm) to the marginal cost of repaying the debt. The latter consists of the value of debt next period (at the firm's own shadow prices) and the financing cost. When choosing net shareholder payout one period ahead, the firm equates the expected shadow value of a dollar to the expected value of a dollar outside the firm. It follows that the shadow value of funds within the firm will typically be different from one. Indeed, since short run adjustment is costly for debt and impossible for equity, a dollar within the firm differs in value from a dollar outside.

An equilibrium is characterized by (1), (2), the household and firm Euler equations, as well as the dynamics of the exogenous variables. We compute an approximate solution in three steps. First, we find the "worst case steady state", that is, the state to which the model were to converge if the data were generated by the worst case probability belief. The worst case steady state used in steps 1 and 2 should be viewed a computational tool that helps describe agents' optimal choices. Agents choose conservative policies in the face of uncertainty, and this looks *as if* the economy were converging to the worst case steady state.Second, we linearize the model around the worst case steady state. Finally, we derive the true dynamics of the system, taking into account that the exogenous variables follow the true data generating process.

In the third step, we make use of the fact that the true deterministic sequence μ_t^* behaves like a realization of an iid normal stochastic process. This means that we can compute model implied moments using the moments of the iid normal process. By construction, those moments do not depend on the particular sequence μ_t^*, only on its long run properties. We think of the combined variance of the risky and ambiguous component – introduced above as $\tilde{\sigma}_\tau \tilde{\sigma}_\tau'$ – as the model moment that is to be matched to the variance of τ in the data. An explicit decomposition into a true μ_t^* and risky shocks is thus not needed for the quantitative assessment of the model. The point of the decomposition is to clarify that agents cannot learn certain aspects of the data even in a stationary environment, and are thus fruitfully modeled as perceiving ambiguity.

Worst case steady state

Denote the steady values of cash flow and the financing cost parameter by (F, Ψ). Households faced with ambiguity act *as if* the economy converges to a state with worse technology. This induces cautious behavior and asset premia. At the worst case steady state, conditional forecasts of technology $\hat{\tau}_t = (\hat{f}_t, -\hat{\psi}_t)$ are constant at $-\bar{a} = (-\bar{a}_f, -\bar{a}_\psi)$. In other words, households behave as if long cash flow is lower by $\bar{a}_f$ percent, at $F^* = F \exp(-\bar{a}_f)$ and the long run financing cost is higher by $\bar{a}_\psi$ percent, at $\Psi^* = \Psi \exp(\bar{a}_\psi)$. We work with the cost function $\kappa(B, \Psi) = -\psi B + \frac{1}{2}\Psi B^2$, with $\psi > 0$. While a quadratic cost function is not globally sensible because it penalizes positive bond holdings, it works well for a local approximation around a steady state with positive debt. We further choose parameters so that $D > 0$ in steady state, that is, the firm makes a positive net payout.

In the worst case steady state, the pricing kernel and the riskless bond price are simply the household's discount factor: $M^* = Q^* = \beta$. Interest rates do not depend on $\bar{a}$ and are thus the same in the worst case steady state and in a steady state with rational expectations. However, the long run debt, dividend and consumption levels all depend on the amount of ambiguity:

$$
\begin{aligned}
B^* &= (\psi/\Psi)\exp(-\bar{a}_\psi) \\
D^* &= F\exp(-\bar{a}_f) - (1-\beta)B^* + \psi B - \frac{1}{2}\Psi\exp(\bar{a}_\psi)\, B^{*2} \\
C^* &= L + F\exp(-\bar{a}_f) + \frac{1}{2}(\psi^2/\Psi)\exp(-\bar{a}_\psi)
\end{aligned}
\tag{5}
$$

Here the last term in the consumption equation reflects the gain from debt financing realized in steady state. More ambiguity about financing conditions (higher $\bar{a}_\psi$) shrinks this gain and leads firms to behave as if debt needs to be lower in the long run. Moreover agents act as if cash flow and consumption are lower. The rational expectations steady state levels are obtained by setting $\bar{a} = 0$.

Loglinear approximation

We now loglinearize the model around the worst case steady state. We use hats to indicate log deviations and stars to signal that we are expanding around the worst case steady state. We start with the resource constraint:

$$
\hat{c}_t^* = \omega_\tau \hat{\tau}_t^*, \tag{6}
$$

where the vector $\omega_\tau = (\omega_f, \omega_\psi) = (F^*/C^*, \frac{\psi}{2}B^*/C^*)$ collects the steady state GDP shares of cash flow and financing costs. Variations in debt have only a second order effect on consumption and do not appear in a first order approximation. To first order, the model is thus an endowment economy in which technology alone determines consumption. In a sensibly quantified model, the coefficient on $\hat{\psi}$ is an order of magnitude smaller than the coefficient on $\hat{f}$.[3] In what follows, it is thus helpful to think about changes in technology $\omega_\tau \hat{\tau}_t$ as mostly driven by changes in cash flow. The main effect of financing costs will come through the firm's marginal cost of debt, rather than through the actual resource cost spent..

The loglinearized pricing kernel and the household Euler equation for debt are

$$\hat{m}^*_{t+1} = \hat{c}^*_t - \hat{c}^*_{t+1}$$
$$\hat{q}^*_t = E^*_t[\hat{m}_{t+1}]$$

State prices vary across states of nature with both firm cash flow f and financing cost ψ. However, in our loglinear framework this variation is not important for pricing – what matters are conditional means under the worst case belief. The short term interest rate is $\hat{r}^*_t = -\hat{q}^*_t = -E^*_t \hat{m}^*_{t+1}$.

The firm's problem can be written using a single endogenous state variable, namely the funds the firm plans to pay to outsiders – shareholders or bondholders – in the next period. We write the log deviation from steady state of "planned payout" as

$$\hat{w}^*_t =: (\omega_d/\omega_b)\hat{d}^*_{t+1} + \beta^{-1}\hat{b}^*_t.$$

Here $\omega_b = Q^*B^*/C^*$ is the GDP share of (the market value of) corporate debt and $\omega_d = D^*/C^*$ is the GDP share of dividends. Both components of planned payout $\hat{w}_t$ are selected at date t but the actual payments – redemption of debt and payout to shareholders – are made at date $t+1$.

The loglinearized budget constraint of the firm is

$$\hat{b}^*_t = -(\omega_\tau/\omega_b)\hat{\tau}^*_t - \hat{q}^*_t + \hat{w}_{t-1},$$

The firms issues debt in response to current technology and bond prices so as to satisfy the firm budget constraint. At the same time, it must respect the planned payout $\hat{w}_{t-1}$ from the previous period. The presence of $\hat{w}_{t-1}$ indicates that there is some (short run) propagation in the model. Indeed, if a shock prompted the firm to increase planned payout at date $t-1$, then it also issues more debt at date t.

From the Euler equation for shareholder payout D, the firm wants to keep the shadow value of funds at its steady state level in expectation, or $E^*_t[\hat{\lambda}^*_{t+1}] = 0$. This is accomplished by setting planned payout as a function of expected future technology and interest rates. Combining household and firm Euler equations, equilibrium planned payout is [4]

$$\hat{w}^*_t = E^*_t \left[(\omega_\tau/\omega_b)\, \hat{\tau}^*_{t+1} + \hat{q}^*_{t+1} - \hat{\psi}^*_{t+2} \right] \tag{7}$$

[3] The coefficient ω_ψ is the corporate debt/GDP ratio multiplied by the parameter ψ which determines the subsidy, per dollar of debt for issuing debt. We think of ψ as a few percentage points at most – for example, if the subsidy is the tax advantage of debt, then it corresponds to a tax rate multiplied by an interest rate.

[4] Loglinearizing the firm's first order condition for debt delivers

$$\hat{\lambda}_t + \hat{q}_t = E^*_t \left[\hat{\lambda}_{t+1} + \hat{m}_{t+1} \right] + \psi \left(E^*_t \hat{\psi}_{t+1} + \hat{b}_t \right).$$

Under the worst case belief, firms plan to pay out more if cash flow is higher, interest rates are lower (that is, bond prices are higher) or the marginal cost of debt is lower.

The firm's decision rules for debt and dividends now follow from the budget constraint and the definition of $\hat{w}$:

$$\hat{b}_t^* = -\left(\omega_\tau/\omega_b\right)\left(\hat{\tau}_t^* - E_{t-1}^*\hat{\tau}_t^*\right) - \left(\hat{q}_t^* - E_{t-1}^*\hat{q}_t\right) - E_{t-1}^*\hat{\psi}_{t+1}^*$$
$$\hat{d}_{t+1}^* = \left(\omega_b/\omega_d\right)\left(\hat{w}_t - \beta^{-1}\hat{b}_t\right)$$

The firm can immediately react to shocks only by adjusting debt. This adjustment corrects "forecast errors" about technology or interest rates that were not taken into account when planning payout the period before. In addition, the firm wants to stabilize the shadow value of funds. As a result, debt on average reflects expected financing costs, that is, $E_t^*\hat{b}_{t+1} = -E_t^*\psi_{t+2}$.

Dividends are then set to implement the forward looking payout rule, taking into account the adjustment of debt. Dividends thus make the connection between the forward looking choice of planned payout $\hat{w}_t$ and the adjustment of debt, which is mostly backward looking (although it also responds to the bond price, itself a forward looking variable). A key implication is that current shocks to technology will have opposite effects on new debt and planned dividends: if the firm has more internal funds today because of higher cash flow or lower financing costs, then it will reduce debt and plan to pay out more dividends.

Closed form solution

We now derive the equilibrium law of motion for all relevant variables. This solution describes how the model responds to shocks. Indeed, while we have linearized around the worst case, we can approximate the dynamics of the model around its actual "zero risk" steady state using the same coefficients. As we have seen above, ambiguity $\bar{a}$ affects levels, while the coefficients that depend on $\bar{a}$ involve ratios such as ω_d/ω_b. As a result, as long as $\bar{a}$ is not too large, it has a minor effect on the coefficients in the loglinearized system.

The solution for the bond price is

$$\hat{q}_t = \eta_{q\tau}\hat{\tau}_t^* + \eta_{qa}\hat{a}_t^*; \qquad \eta_{q\tau} = \omega_\tau\left(I - \phi_\tau\right), \quad \eta_{qa} = \omega_\tau$$

An increase in technology increases the bond price if it lowers expected consumption growth. This is the relevant case – it obtains for example if cash flow and financing costs are persistent and do not help forecast each other (ϕ_τ diagonal with positive elements). An increase in ambiguity always increases bond prices – a precautionary savings effect.

Planned payout can also be written as a function of current technology and ambiguity only. Let e_ψ denote a unit vector that selects financial technology $-\psi$ out of the technology vector τ. We can then write

$$\hat{w}_t^* = \left(\omega_\tau/\omega_b + \eta_{q\tau}\right)\left(\phi_\tau\hat{\tau}_t^* - \hat{a}_t^*\right) + e_\psi\phi_\tau(\phi_\tau\hat{\tau}_t^* - \hat{a}_t^*) + (\eta_{qa} - e_\psi)\phi_a\hat{a}_t^* \tag{8}$$

Using the equation for the price of bonds and the fact that the expected shadow value of the firm is zero, we obtain

$$\hat{\lambda}_t = \psi\left(E_t^*\hat{\psi}_{t+1} + \hat{b}_t\right)$$

Internal funds are more valuable for the firm in periods when debt is high and it is costly to borrow.

Since planned payout is a purely forward looking variable, it is affected by technology shocks only if technology is persistent ($\phi_\tau \neq 0$). With iid technology, ambiguity alone drives planned payout. The precise effects reflect forecasts and uncertainty about future internal funds, prices and financing costs.

The first term summarizes the effect of technology shocks on the firm's internal funds. The firm gains if technology is better or bond prices are higher – here the direct and the price effect go in the same direction. The anticipation of changes in internal funds translates into two responses to current shocks. On the one hand, if there is a positive technology shock, then planned payout is increased if (and only if) technology is persistent ($\phi_\tau \neq 0$). On the other hand, if there is an increase in ambiguity, then firms are concerned about future internal funds and cautiously reduce planned payout. The second term describes the firm's response to changes in expected financing costs two periods ahead – again there is an expectations and ambiguity effect. Finally, an ambiguity shock has a knock-on effect if ambiguity is persistent ($\phi_a \neq 0$): firms then anticipate higher bond prices and possibly higher borrowing costs next period.

Debt and dividends can be written as a function of current shocks as well as the endogenous state variable $\hat{w}_{t-1}^*$:

$$
\begin{aligned}
\hat{b}_t^* &= \hat{w}_{t-1} - \left(\omega_\tau/\omega_b + \eta_{q\tau}\right)\hat{\tau}_t^* - \eta_{qa}\hat{a}_t^* \\
\hat{d}_{t+1}^* &= -\left(\omega_b/\omega_d\right)\beta^{-1}\hat{w}_{t-1} + \eta_{d\tau}\hat{\tau}_t^* + \eta_{da}\hat{a}_t^* \\
\eta_{d\tau} &= \left(\omega_\tau/\omega_d + (\omega_b/\omega_d)\eta_{q\tau}\right)\left(\beta^{-1}I + \phi_\tau\right) + \left(\omega_b/\omega_d\right)e_\psi\phi_\tau^2 \\
\eta_{da} &= -\left(\omega_\tau/\omega_d + (\omega_b/\omega_d)\eta_{q\tau}\right) - \left(\omega_b/\omega_d\right)e_\psi(\phi_\tau + \phi_a) + \eta_{qa}\left(\phi_a + \beta^{-1}I\right)
\end{aligned}
$$

The solution reflects the backward and forward looking effects discussed above. If the firm inherits large payment obligations, it rolls them over by issuing debt and then pays them off by lowering dividends next period. Similarly, a bad technology shock is addressed first by borrowing, followed by lower dividends. Technology shocks thus move debt and net shareholder payout in opposite directions. Ambiguity shocks lower debt. They also lower planned dividends provided that the price effect of ambiguity (the last term in η_{da}) is small enough. This will be true as long as steady state debt is sufficiently large. Ambiguity shocks then generate positive comovement between debt and net shareholder payout.

The detailed formulas again reflect the internal funds, prices and financing cost channels. The first term in the elasticity $\eta_{d\tau}$ shows how technology shocks affect dividends both through the budget constraint and through expectations. Better technology means more internal funds, which the firm uses immediately to pay down debt (cf. the first equation). The firm then plans to pay out the resulting savings to shareholders one period later – the coefficient β^{-1} enters because of saved interest on the debt. If technology is persistent, then dividends are increased even further in anticipation of higher internal funds next period as well as possibly lower financing costs two periods ahead. The first two terms in η_{da} show how an increase in ambiguity affects dividends as firms become uncertain about internal funds next period as well as financing costs two period ahead, respectively. There is a counteracting effect as ambiguity increases bond prices which leads firms to increase dividends.

2.3 Steady state and unconditional asset premia

Suppose all shocks are equal to zero, but agents use decision rules that reflect their aversion to ambiguity. In particular, agents perceive constant ambiguity, as in the worst case steady state. We study the zero risk steady state using the decision rules derived above by linearization around the worst case steady state. From this perspective, the true steady state cash flow $(F, -\Psi)$ looks like a positive deviation summarized by the vector $\bar{a}$. Mechanically, we now need to find the steady state of a system in which technology is always at $\bar{a}$, but in which agents act as if the economy is on an impulse response towards the worst case steady state. The latter impulse response can be computed from the closed form solution for the equilibrium derived above.

Consumption and short term interest rate

The zero risk steady state captures the effect of the average amount of ambiguity on decisions and prices, as well as unconditional asset premia. Under the worst case belief, agents expect consumption to revert from its temporarily high level towards the worst case steady state according to $\hat{c}_t^* = \omega_\tau \phi_\tau^t \bar{a}$. Asset prices follow from the anticipated sequence of pricing kernels $\hat{c}_t^* - \hat{c}_{t+1}^*$. As we have seen above, the worst case steady state bond price is the same as the bond price in the absence of ambiguity. The average log price of a short bond predicted by the model is therefore

$$\bar{q} = \log \beta + \omega_\tau \left(I - \phi_\tau\right) \bar{a}$$

Ambiguity unconditionally increases bond prices and lowers interest rates, due to precautionary savings. The uncertainty premium is smaller if technology is more persistent. Intuitively, agents worry about bad technology, but they also observe current technology. If technology is more persistent, then agents also know this and hence worry less about what happens in the near term. As a result, they demand less compensation on short term bonds. As we will see below, more persistent technology implies that agents demand relatively more compensation on long term assets.

Payout and capital structure

Consider the firm's planned payout, expressed as a deviation from the worst case steady state. It follows from substituting $\hat{\tau}_t^* = \bar{a}$ and $\hat{a}_t^* = 0$ in the decision rule (8):

$$\hat{w}^* = \left(\omega_\tau / \omega_b + \eta_{q\tau} + e_\psi \phi_\tau\right) \phi_\tau \bar{a}$$

If technology is serially independent ($\phi_\tau = 0$), then the firm always keeps planned payout at its worst case steady state level. With persistent technology, shareholders worry less about the near term and commits to more payout. Mechanically, the firm acts as if the current unusually high cash flow or low financing cost spills over to next period. It also expects low interest rates to continue next period, which further increases planned payout.

We can now compute firms' steady state capital structure and shareholder payout. Denote log debt and shareholder payout in the rational expectations steady state by $\bar{b}^{RE}$ and $\bar{d}^{RE}$, respectively. With ambiguity, steady state debt and shareholder payout are[5]

[5]From (5) we have $\bar{b}^{RE} = \log(\psi/\Psi)$ and $\bar{d}^{RE} = \log(F - (1-\beta)\psi/\Psi + \frac{1}{2}\psi^2/\Psi)$. We can write both in terms of percentage deviations from the worst case steady state and substituting $\hat{w}_{t-1}^* = \hat{w}^*$, $\hat{\tau}_t^* = \bar{a}$ and $\hat{a}_t^* = 0$ into the decision rules for debt and dividends.

$$\bar{b} = \bar{b}^{REE} - \omega_\tau/\omega_b \left(I - \phi_\tau\right) \bar{a} - \eta_{q\tau} \left(I - \phi_\tau\right) \bar{a} - e_\psi \left(I - \phi_\tau^2\right) \bar{a}$$
$$\bar{d} = \bar{d}^{REE} + \left(\beta^{-1} - 1\right) \left(\bar{b}^{REE} - \bar{b}\right) + (\omega_b/\omega_d)\eta_{q\tau}\bar{a}$$

The first line says that ambiguity lowers the average level of debt – when firms are uncertain about the future, they cautiously plan lower borrowing. The second line says that ambiguity increases average shareholder payout. Indeed, lower debt means lower interest cost, which is directly paid out to shareholders. This effect would be there even if the interest rate was unchanged at β^{-1} – the first term in the second line. There is an additional effect since interest rates decline with higher ambiguity.

The formula shows three separate channels at work. First, firms worry about future internal funds which depend on both cash flow and financing costs. Here the effect of financing costs scales with the second entry in ω_τ/ω_b, namely $\psi/2\beta$ and is large only if ψ is sufficiently large. Second, firms worry about bond prices. Finally, firms worry about financing costs directly. All channels are weaker if technology is more persistent. Intuitively, with persistent technology firms worry less about near term cash flow and financing costs and hence leverage and pay out more.

Stock price discount and equity premium

The loglinearized household Euler equation for stocks can be written as

$$\hat{p}_t^* - \hat{d}_t^* = E_t^* \left[\beta(\hat{p}_{t+1}^* - \hat{d}_{t+1}^*) + (\hat{d}_{t+1}^* - \hat{c}_{t+1}^*) - \left(\hat{d}_t^* - \hat{c}_t^*\right)\right] \tag{9}$$

Here the left hand side is the log price dividend ratio, or more precisely the price payout ratio. Its worst case steady state value is equal to $\beta/\left(1-\beta\right)$, the same as in the rational expectations steady state. A key property of stock market data is that prices are much more volatile than scaled measures of payout. In other words, the log price dividend ratio moves around over time.

We can solve forward to express the price dividend ratio as the present value of future growth rates in the dividend-consumption ratio

$$\hat{p}_0^* \ - \hat{d}_0^* = E_t^* \sum_{t=0}^{\infty} \beta^t \left((\hat{d}_{t+1}^* - \hat{c}_{t+1}^*) - \left(\hat{d}_t^* - \hat{c}_t^*\right)\right) \tag{10}$$

As is familiar from asset pricing with separable utility under risk, the price dividend ratio is driven by counteracting cash flow and interest rate effects. For example, bad news about dividends decrease expected dividend stream and thereby the present value of dividends. At the same time, since dividends are part of consumption, bad news decreases interest rates, thus lowering the present value of dividends. If dividends are equal to consumption, then the price dividend ratio is constant – with log utility, income and substitution effects cancel. Inη contrast, if dividends are a small share of consumption (as in the data), then the cash flow effect dominates and bad news decrease the price dividend ratio. In our model, changes in uncertainty work like changes in means and so this intuition carries over directly. Ambiguity about dividends that does not affect consumption much can generate excess volatility of stock prices.

At the zero risk steady state, stocks are priced according to (10) with expectations reflecting the impulse response from the zero risk steady state to the worst case steady state. In the first period, this impulse response reflects only the adjustment of consumption, since shareholder payout is predetermined:

$$\hat{d}_0^* - \hat{c}_0^* = -\frac{\omega_b}{\omega_d}\beta^{-1}\hat{w}^* + (\eta_{d\tau} - \omega_\tau)\,\bar{a}$$
$$\hat{d}_1^* - \hat{c}_1^* = -\frac{\omega_b}{\omega_d}\beta^{-1}\hat{w}^* + (\eta_{d\tau} - \omega_\tau\phi_\tau)\,\bar{a}$$

The first element in the sum (10) is therefore the ambiguity premium in the short term bond price $\bar{q} - \log\beta$

Along the impulse response for $t > 1$, the dividend consumption ratio evolves as

$$\begin{aligned}\hat{d}_t^* - \hat{c}_t^* &= -\left(\omega_b/\omega_d\right)\beta^{-1}\hat{w}_{t-2}^* + \eta_{d\tau}\tau_{t-1}^* - \omega_\tau\tau_\tau^* \\ &= \left((\omega_\tau/\omega_d)(1-\omega_d) + (\omega_b/\omega_d)\eta_{q\tau}\right)\phi_\tau^t\bar{a} - \left(\omega_b/\omega_d\right)e_\psi\left(\beta^{-1}I - \phi_\tau\right)\phi_\tau^t\bar{a}\end{aligned}$$

Consider first ambiguity about cash flow only. When firms worry about cash flow, they cautiously act as if cash flow declines towards the worst case steady state. Lower cash flow leads endogenously to lower dividends. Investors thus price stocks as if there is a declining path of dividends. Indeed, with ambiguity about cash flow only, the last term vanishes and the dividend consumption ratio declines geometrically to the worst case steady state from above. Since $\omega_d < 1$, ambiguity about cash flow is not offset by the effect of ambiguity on interest rates (cf the first term). Comparing coefficients, it follows that the sum over growth rates in (10) is negative and the steady state price dividend ratio $\bar{p} - \bar{d}$ is below the rational expectations steady state (which coincides with the worst case steady state). Ambiguity about cash flow thus leads to a steady state price discount, works because cash flow uncertainty makes investors fear low a payoff of stocks relative to bonds.

Consider now ambiguity about financing costs and focus first on the case where the resource cost effect is negligible (ω_ψ small). The effect is then described only by the last term. When firms worry about the marginal cost of debt, they act cautiously as if costs will increase towards the worst case steady state. Higher financing costs leads endogenously to lower debt and higher payout to shareholders. Investors thus price stocks as if there is an increasing path of dividends. This creates a force that makes the price dividend ratio higher in steady state. Since the resource cost effect works like a decrease in cash flow, there is also an offsetting force, but we would expect its effect to be quantitatively small. The discussion here this illustrates the importance of taking firm decisions into account.

The equity premium at the zero risk steady state is[6]

$$\log\left(\bar{p} + \bar{d}\right) - \log\bar{p} + \bar{q} = (1-\beta)\left(\bar{d} - \bar{p}\right) + \omega_\tau\left(I - \phi_\tau\right)\bar{a}$$

Ambiguity can make the steady state equity premium positive for two reasons. First, the average stock return is higher than under RE if the price dividend ratio is lower. This is

[6]The log stock return at the zero risk steady state is

$$\log\left(\bar{p} + \bar{d}\right) - \log\bar{p} \approx (1-\beta)\left(\bar{d} - \bar{p}\right) - \log\beta$$

where we are using the fact that all asset returns are equal to $-\log\beta$ at the worst case steady state.

the first term. Second, the interest rate is lower. The second effect is small if dividends are a small share of consumption, that is, both components of ω_τ are relatively small. We emphasize the role of the first effect: it says that average equity returns themselves are higher than in the rational expectations steady state. Ambiguity does not simply work through low real interest rates.

Term structure of interest rates

From the household Euler equations, we can compute the price of any asset, including long term bonds that are in zero net supply. Let $\hat{q}_t^{*(n)}$ denote the log deviation from the worst case steady state for an n period zero coupon bond. The linearized Euler equation for that bond is

$$\hat{q}_t^{*(n)} = E_t^* \left[\hat{m}_{t+1}^* + \hat{q}_{t+1}^{*(n-1)} \right] = E_t^* \left[\hat{c}_t^* - \hat{c}_{t+n}^* \right]$$

This Euler equation must hold also for every n along the impulse response from the zero risk steady state, under the deterministic belief that $c_t = \omega_\tau \phi_\tau^t \bar{a}$.

The steady state n period interest rate (quoted as a continuously compounded yield to maturity) is therefore

$$\bar{\imath}^{(n)} = -\log \bar{q}^{(n)}/n = -\log \beta - \frac{1}{n} \omega_\tau \left(I - \phi_\tau^n \right) \bar{a}$$

For persistent technology, consumption slopes down away from the zero risk steady state towards the worst case steady state. This implies that short rates will be lower than long rates. In particular the short rate $\log \delta - \omega_\tau \left(I - \phi_\tau \right)$ is smaller then the infinite maturity rate $\lim_{n \to \infty} i^{*n} = -\log \beta$. With a geometrically declining impulse response we expect a geometrically upward sloping average yield curve. The slope depends on the persistence of technology.

2.4 Predictability of excess returns

A standard measure of uncertainty premia in asset markets is the expected excess return on an asset computed from a regression on a set of predictor variables. The log excess stock return implied by our model can be approximated as

$$\begin{aligned} x_{t+1}^e &= \log(p_{t+1} + d_{t+1}) - \log p_t - \log(i_t) \\ &\approx \beta \hat{p}_{t+1}^* + (1 - \beta) \hat{d}_{t+1}^* - \hat{p}_t^* + \hat{q}_t^* \\ &= \beta \left(\hat{p}_{t+1}^* - \hat{d}_{t+1}^* - E_t^* [\hat{p}_{t+1}^* - \hat{d}_{t+1}^*] \right) + \hat{d}_{t+1}^* - E_t^* \hat{d}_{t+1} \end{aligned}$$

Here the second line is due to loglinearization of the return around the worst case steady state. The third line follows from the household Euler equation for stocks.

Consider now an econometrician who attempts to predict excess stock returns in the model economy. Suppose for concreteness that he has enough predictor variables to actually recover theoretical conditional expectation of payoff next period given the state variables of the model. With a large enough sample, he will measure the expected excess return

$E_t x^e_{t+1}$, where the expectation is taken with the conditional mean $\mu^*_t = 0$.[7] Using the above expression, we can write the measured risk premium as

$$E_t x^e_{t+1} = \beta(E_t - E^*_t)[\hat{p}^*_{t+1} - \hat{d}^*_{t+1}] + (E_t - E^*_t)[\hat{d}^*_{t+1} - E^*_t \hat{d}_{t+1}],$$

where $(E_t - E^*_t)$ represents the difference between the expectation under $\mu^*_t = 0$ and the worst case expectation. This is a term that is proportional to ambiguity a_t. This expression suggests an interesting approach to quantify ambiguity in a linear model. Since risk premia must be due to ambiguity, it is possible to learn about ambiguity parameters up front from simple linear regressions without solving the DSGE model fully.

3 An estimated model

This section describes the model that we use to quantify the role of uncertainty in driving US business cycles and asset prices. The basic structure is the same as in the previous section – a representative household invests in debt and equity issued by a financing constrained firm. However, there are three key changes.

First, on the real side, there is endogenous production and capital accumulation. The production technology resembles that in many recent DSGE models; in particular, we allow for investment rate adjustment costs and endogenous capacity utilization. Labor supply is endogenous and the labor market is competitive. Technology shocks affect firm profits and household wages. We also introduce a government subject to spending shocks.

Second, on the financial side, we make more explicit the sources of shocks. In particular, there is a fixed cost of accessing credit markets, in additional to a marginal cost shock. In a model with exogenous cash flow, the fixed cost is simply a negative cash flow shock. However, firm cash flow is now endogenous and affected by all shocks. In particular, it depends on technology shocks that also move around labor income. A fixed cost shock is then special because it is a shock to profits that does not affect labor income at the same time. As a result, ambiguity about the fixed cost makes investors worry about stocks more than about bonds.

Third we now allow ambiguity to be affected by regime switching volatility. This not only allows for an explicit connection between ambiguity and volatility and for first order effects of volatility, but it also introduce correlation across fundamental shocks.

3.1 Model

3.1.1 Uncertainty

The "fundamental" shocks of the economy consist of real and financial technology – as in section 2 – as well as government spending shocks. The real shock is the stochastic growth rate of labor augmenting technical process ξ_t. The financial shocks are a fixed cost

[7] Indeed, since all unconditional empirical moments converge to those of a process with $\mu^*_t = 0$ by construction, the same is true for conditional moments

of accessing debt market f_t and a marginal cost shifter Ψ_t. For simplicity, we assume that these shocks are orthogonal. We follow the notation of section 2 and denote by τ_t the log deviations of the shocks $[\xi_t, \ -f_t, -\Psi_t, -g_t]'$ from their corresponding steady state values $[\xi, -f, -\Psi, -g]'$. The last three shocks – including government spending g_t– have a negative sign to facilitate the interpretation of positive innovations to τ_t as 'good' realizations, in the sense of increasing equilibrium consumption.

In contrast to section 2, we now allow for heteroskedasticity. To describe the true data generating process for τ_t, we modify (3) to get

$$\tau_{t+1} = P\tau_t + \mu_t^* + \Sigma_t \varepsilon_{t+1} \tag{11}$$

where P is a diagonal matrix with entries $\rho_\xi, \rho_f, \rho_\Psi, \rho_g$ and the rest of the elements equal to zero. The vector $\varepsilon_t \sim N(0, I)$ contains the exogenous Gaussian shocks, and the matrix Σ_t contains the stochastic volatilities, with elements denoted by $\sigma_\xi, \sigma_f, \sigma_\Psi, \sigma_g$.

Volatility Σ_t is known one period in advance and follows a regime-switching process. We work with a two-state Markov chain that we write as a VAR

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix} = H^{vo} \begin{bmatrix} e_{1,t-1} \\ e_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} \tag{12}$$

Here $e_{j,t} = 1_{s_t=j}$ is an indicator operator if the volatility regime s_t is in place, and the shock v_t is defined such that $E_{t-1}[v_t] = 0$. This representation is useful to derive a loglinear approximation to equilibrium in the presence of stochastic volatility. We denote the transition matrix of the Markov chain by H^{vo}.

The decomposition of the innovation to τ into two components μ^* and $\Sigma\varepsilon$ again serves to distinguish between ambiguity and risk, respectively. Agents know all long run empirical moments of the sequence μ^*, but they do not know the number μ_t^* when they make decisions at date t. Based on date t information, the agent contemplates an interval of conditional means $\mu_{t,i} \in [-a_{t,i}, a_{t,i}]$ for each component τ_i. The vector a_t summarizes ambiguity perceived about fundamentals and can be thought of as an inverse measure of confidence.

A key new element in this section is that ambiguity a_t depends on volatility Σ_t. The idea is that agents are less confident about the future when there is more "turbulence" in fundamentals in the sense of larger realized shocks. Formally, we assume

$$a_{t,i} = \eta_{t,i}\Sigma_{t,i} \tag{13}$$

$$\eta_{t,i} = \rho_{\eta,i}\overline{\eta}_i + (1-\rho_{\eta,i})\eta_{t-1,i} + \sigma_{\eta,i}\varepsilon_t \tag{14}$$

There are now two sources of variation in ambiguity. Within a regime, volatility is fixed and ambiguity changes linearly with the arrival of intangible information about fundamentals, as in section 2. This within regime dynamics are described by the process η_t which we specify below such that it is negative only with negligible probability. In addition, volatility changes across regimes also affect ambiguity.

We can interpret η as an inverse measure of information quality conditional on the regime. Indeed we have that $\mu_{t,i} \in [-a_{t,i}, a_{t,i}]$ if and only if

$$\frac{\mu_{t,i}^2}{2\Sigma_{t,i}^2} \leq \frac{1}{2}\eta_{t,i}^2$$

The left hand side is the relative entropy between two normal distributions that share the same standard deviation $\Sigma_{t,i}$ but have different means $\mu_{t,i}$ and zero, respectively. The agent thus contemplates only those conditional means that are sufficiently close to the long run average of zero in the sense of conditional relative entropy. The relative entropy distance captures that intuition through the fact that when $\Sigma_{t,i}$ increases it is harder to distinguish different models.

3.1.2 Production

Firms can produce numeraire goods using capital services $\overline{K}_t$ and labor L_{t-1} and they can invest in trade physical capital K_t.subject to adjustment costs

$$\begin{aligned} Y_t &= \overline{K}_t^{\alpha} \left(\epsilon_t L_{t-1}\right)^{1-\alpha} \\ K_t &= (1-\delta)K_{t-1} + \left[1 - \frac{S''}{2}\left(\frac{I_t}{I_{t-1}} - \overline{\xi}\right)^2\right] I_t, \end{aligned} \tag{15}$$

Numeraire production depends on the technology shock ϵ_t, whose growth rate $\xi_t \equiv \Delta \log \epsilon_t$ is stochastic. The process for ξ_t is described in (11), such that ξ and $\overline{\xi}$ are the steady states under the true DGP and the worst-case belief, respectively.[8] Physical capital depreciates and is produced from numeraire. Adjustment costs are convex in the growth rate of investment. As detailed in section 3.1.6 below, we solve the model by loglinearizing around the worst-case steady state. It is then helpful to define the adjustment costs in (15) so that the level and the marginal adjustment cost are zero along the balanced growth path of the worst-case steady state. Thus, in the loglinear approximation to equilibrium, only S'' matters for dynamics.

Production of capital services from capital is subcontracted to short-lived firms who rent capital and select a capital utilization rate u_t that applies to the beginning of period t stock of physical capital $K_{t-1} = \bar{K}_t/u_t$. Increased utilization requires increased maintenance costs in terms of investment goods per unit of physical capital measured by

$$a(u_t) = \frac{1}{2} r^k \vartheta u_t^2 + r^k(1-\vartheta)u_t + r^k\left(\frac{1}{2}\vartheta - 1\right).$$

The function $a(.)$ is increasing and convex with $a\,(1) = 0$. It is normalized such that, in the nonstochastic steady state, $u = 1$ and $a''\,(u) = \vartheta r^k$, where r^k is the steady state rental price of capital. As a result, $a''\,(u)\,/a'\,(u) = \vartheta > 0$ is a parameter that controls the degree of convexity of utilization costs. In the loglinear approximation to equilibrium, only ϑ matters for dynamics.

3.1.3 Financing

As in section 2, the firm maximizes shareholder value evaluated under the worst-case belief. We model the benefit of debt explicitly as a tax advantage. Let τ_k denote the corporate income tax rate. The firm's budget constraint is

[8] We further discuss the stochastic properties of the shocks in section 3.1.6 below. The worst-case belief here is that productivity is low. Thus, as detailed in the appendix 4.1 and formula (20), $\overline{\xi} = \xi - \frac{\overline{\eta}_\xi \overline{\sigma}_\xi}{1-\rho_\xi}$, where $\overline{\sigma}_\xi$ is the ergodic mean of the volatility of the growth rate shock that evolves as in (12).

$$D_t = (1-\tau_k)\left[Y_t - W_t L_t - K_{t-1}a(u_t) - B^f_{t-1}\left(1-Q^b_{t-1}\right) - \phi\left(D_t\right) - \delta Q^k_{t-1}K_{t-1}\right] -$$
$$- \kappa\left(B^f_{t-1}\right) + \delta Q^k_{t-1}K_{t-1} - I_t - B^f_{t-1}Q_{t-1} + B^f_t Q_t,$$

where W_t denotes wages, B^f_{t-1} represents the face value of the debt that the firm enters period t, Q^k_t is the price of capital, I_t is investment, and δ is the capital depreciation rate.

Dividends D_t are chosen in the current period and are subject to an adjustment cost

$$\phi\left(D_t\right) = \frac{\phi''}{2}\frac{1}{\epsilon_t}(D_t - \overline{D})^2$$

This specification, which appears also in Jermann and Quadrini (2012), penalizes deviations of dividends from some long-run payout target $\overline{D}$. We assume that $\overline{D}$ equals the steady state dividends under the worst-case belief. The function ϕ is convex with steady state values of $\phi = \phi' = 0$ and $\phi'' > 0$. At the balanced growth path, this means that the level and the marginal adjustment cost are zero. In the short run, the effects are similar as when dividends must be set one period in advance, as in the previous section.

When issuing debt the firm has to pay a fixed cost as well as a marginal cost that slopes upward. Given the debt B^f_{t-1}, the time t costs associated with debt are

$$\kappa\left(B^f_{t-1}\right) = f_t\epsilon_t + \frac{\Psi_t}{2}\frac{1}{\epsilon_t}\left(B^f_{t-1}\right)^2$$

where f_t and Ψ_t are shocks to the two components of the costs.[9]

3.1.4 Households

Utility is now defined over uncertain streams of consumption bundles $\overrightarrow{C} = \left(\overrightarrow{C}_t\right)_{t=0}^{\infty}$. The date t consumption bundle $\overrightarrow{C}_t\left(\varepsilon^t, v^t\right)$ contains consumption and leisure. It depends on histories of both the normal shocks ε_t and the innovations to the Markov chain v_t. The agent's felicity function is:

$$u\left(\overrightarrow{C}_t\right) = \log C_t - \frac{\chi_L}{1+\sigma_L}L_t^{1+\sigma_L}, \tag{16}$$

where C_t denotes consumption of the final good, L_t denotes working hours supplied by the household, χ_L is a labor disutility parameter and σ_L^{-1} is the Frisch elasticity of labor supply.

The household budget constraint is:

$$(1+\tau_c)C_t + Q^e_t\theta_t = (1-\tau_l)\left[W_t L_t + \Pi_t + D_t\theta_{t-1} - B^h_{t-1}\left(1-Q^b_{t-1}\right)\right] +$$
$$+ Q^e_t\theta_{t-1} - B^h_{t-1}Q^b_{t-1} + B^h_t Q^b_t + T_t$$

[9]Technically we assume that the fixed cost has a shochastic short-run component and a deterministic long-run component. Ambiguity is over the former since it is the only one uncertain. Appendix 4.2.1 details this decomposition.

where C_t is consumption, τ_c is the consumption tax rate, θ_t are shares of the firm, Π_t is an endowment (from sources not showing up in the production function of the firms), Q_t^e is the price of a share, B_{t-1}^h represents the face value of the debt that the household enters period t, W_t is wages and T_t are government lump sum transfers. We assume that the endowment $\Pi_t = \pi\epsilon_t$, where π is a parameter.

State prices can again be read off household first order conditions. Let stars indicate worst case conditional beliefs that support household choices. The only difference to the simple model above is that the pricing kernel must take into account taxation. To simplify this adjustment, we assume that capital gains are taxed immediately when they occur and that the capital gains tax rate is the same as the labor income tax rate τ_l. We then obtain

$$M_{t+1} = \beta \frac{C_t}{C_{t+1}} \frac{1-\tau_l}{1-\tau_l \beta E_t^* \left[C_t/C_{t+1}\right]},$$

where C_t is the representative agent's consumption of the final good.

3.1.5 Government

Government is subject to exogenous spending G_t, financed by debt B_t^g and distortionary taxes τ_l, τ_c, τ_k so that the budget constraint holds as:

$$\begin{aligned} G_t = {} & \tau_l \left[W_t L_t + \Pi_t + D_t \theta_{t-1} - B_{t-1}^h \left(1 - Q_{t-1}^b\right)\right] + \\ & + \tau_k \left[Y_t - W_t L_t - K_{t-1} a(u_t) - B_{t-1}^f \left(1 - Q_{t-1}^b\right) - \phi\left(D_t\right) - \delta Q_{t-1}^k K_{t-1}\right] + \\ & + \tau_c C_t - T_t - B_{t-1}^g + B_t^g Q_t^b. \end{aligned}$$

Lump sump transfers T_t follow the process:

$$T_t = T_o - \varkappa \left(B_{t-1}^g - B^g\right)$$

where T_o, B^g are the steady states of transfers and government debt and $\varkappa$ is a parameter such to insure that we have stability of the government debt, i.e. $\varkappa > 1 - \beta$. We model government expenditures as $G_t = g_t \epsilon_t$, where g_t is a stochastic process.

3.1.6 Solution strategy, worst case belief and pricing

In the richer model of this section, it is less obvious what the worst case belief is. To solve the model, we extend the guess and verify procedure in Ilut and Schneider (2011). The idea there is to first guess a worst case belief. A natural candidate here is that the conditional means of the technology shocks are always at their lower bounds while those of the financing costs and government spending are at their upper bounds. Given a guess, we can solve the model as in section 2 by linearizing around the worst case steady state. Given a solution of the model, we can evaluate the value function (using a second order approximation) and verify the guess. The above candidate "works" because the value function ends up being monotonic in all components of the fundamentals vector τ.

A key detail is how we deal with volatility. The volatility chain is stationary and ergodic and its dynamics are the same under the true and worst case dynamics. One component of the worst case steady state is thus the matrix $\overline{\Sigma}$ that contains the ergodic volatilities. The fact that volatility is in fact changing with respect to this ergodic value will imply "shocks" to volatility and therefore shifts in the constant with respect to the worst case steady state. Given the VAR representation of the Markov chain (12), let Σ_j denote the matrix of values of Σ_t if $e_{j,t} = 1$. For each element $\Sigma_{t,i}$ we can define $\widehat{\Sigma}_{t,i} = \Sigma_{t,i} - \overline{\Sigma}_i$, where $\overline{\Sigma}_i$ is the corresponding ergodic volatility. This means that if we augment our DSGE state vector with the vector e_t we can control for the first order effects of the shifts in volatility. For example, when the volatility regime 1 is in place, $e_{1,t} = 1$, $e_{2,t} = 0$ and our system of equations will load Σ_1 and put zero zero weight on Σ_2. The volatility regimes imply that a shift in regime simultaneously changes all values of the standard deviations in Σ, thereby implying correlated ambiguity shocks.

Choosing ambiguity parameters

Time variation in ambiguity is governed by equation (13). The entropy bound $\eta_{t,i}$ evolves linearly according to (14) and its process is a function of three parameters $\overline{\eta}_i, \rho_{\eta_i}$ and σ_{η_i}. Two considerations matter for selecting a prior over these parameters. The first is technical: we would like the interval for $\mu_{t,i}$ to remain centered around zero which is true only if $\eta_{t,i}$ remains nonnegative. Unfortunately, nonnegativity is incompatible with a linear law of motion for η_i. We thus require parameters such that the unconditional mean $\overline{\eta}_i$ is more than three unconditional standard deviations away from zero:

$$\overline{\eta}_i \geq 3 \frac{\sigma_{\eta_i}}{\sqrt{1 - \rho_{\eta_i}^2}}. \tag{17}$$

As a result, the probability that $\eta_{t,i}$ becomes negative is .13%, and any negative $\eta_{t,i}$ will be small. Any $\eta_{t,i}$ close to zero will thus represent a small set of belief that is close to having a single mean close to zero - a very confident agent.

The second consideration is that we want to bound the lack of confidence by the measured variance of the shock that agents perceive as ambiguous. As detailed in Ilut and Schneider (2011), we argue that a reasonable upper bound for a is given by $2\sqrt{\rho}\Sigma_i$, where $\rho \in [0, 1]$ is the share of the variability in the data that agents attribute to ambiguity. This means that the bound on η_i is given by $2\sqrt{\rho}$. When $\rho = 1$, which means that agents attribute all the observed variation in the shock to the ambiguous component, we obtain the largest upper bound, i.e. $\eta_{t,i} \leq 2$. Again we cannot enforce the bound exactly, but assume that it is violated with probability .13%:

$$\overline{\eta}_i + 3 \frac{\sigma_{\eta_i}}{\sqrt{1 - \rho_{\eta_i}^2}} \leq 2. \tag{18}$$

In preliminary estimations of the model, we find that when the three ambiguity parameters $\overline{\eta}_i, \rho_{\eta_i}$ and σ_{η_i} are separately estimated the implied unconditional volatility of the $\eta_{t,i}$ process is so large that it implies very frequent negative realizations of $\eta_{t,i}$. We thus restrict attention to the subset of the parameter space in which (17) is binding. It then implies that $\overline{\eta}_i \in [0, 1]$ because of (18). We then estimate two ambiguity parameters $\overline{\eta}_i$ and ρ_{η_i} for each shock i, together with the other parameters of the model. We can then infer σ_{η_i} from (17).

3.2 Estimates

We include eight observables based on US data: GDP growth, investment growth, consumption growth, dividend to GDP ratio, equity price to GDP ratio, real interest rate, 10 year spread, and firm debt to GDP ratio. The time period is 1959Q2 to 2011Q3. The variables are reported in Figure 1. We estimate the model using Bayesian methods. Specifically, we first compute the posterior mode and then we make 500,000 draws from the posterior using a Metropolis-Hastings algorithm. Please refer to Bianchi (2012) for details about the estimation strategy.

3.2.1 Parameter estimates and regime probabilities

Table 1 contains the parameter estimates Regime 2 is associated with higher volatility for all shocks. We will label this regime the High Volatility regime. Figure 2 reports the smoothed probabilities of the High Volatility regime. The regime turns out to dominate a prolonged period of time starting from the early '70s until 1987. After that, we observe only a brief spike around 1992.

3.3 Stock prices and ambiguity

The top panel of Figure 3 reports the evolution of the price-to-GDP ratio and a counterfactual series constructed setting all shocks to zero, but the ambiguity shock about fixed costs. The lower panel contains the smoothed series for ambiguity about fixed costs at the posterior mode. The figure provides a visual characterization of the importance of ambiguity about fixed costs in determining fluctuations in the price-to-GDP ratio.

3.4 Financial variables and fixed costs

Figure 4 reports impulse responses to shocks to fixed costs and ambiguity about fixed costs for the dividend-to-GDP and the price-to-GDP ratios. We assume that the low volatility regime has been in place for a prolonged period of time, implying that the starting point is given by its conditional steady state. Furthermore, the low volatility regime is assumed to be in place over the relevant horizon of ten years. Notice that the dynamics under the high volatility regime are identical, but shifted because of the different conditional steady state.

An increase in the fixed cost or an increase in ambiguity about fixed costs determine a fall in both the dividend-to-GDP ratio and in the price-to-GDP ratio. The drop following the increase in ambiguity is more pronounced, but it lasts substantially less. Instead, following an increase in fixed costs we observe a prolonged decline in both variables that even after ten years is far from being re-absorbed. These results are in line with what shown for the simple model of Section 2, but with a hump shape since the dividend adjustment cost draws out the response longer than when keeping dividends fixed one period ahead.

3.5 Asset Prices and Risk

Figure 5 reports the evolution of the price-to-GDP ratio induced by the typical path for the regimes as implied by the posterior mode. We assume that the economy starts from the low volatility regime conditional steady state. The first switch to the high volatility regime determines a large drop in the price-to-GDP ratio and then a further decline, as the economy moves closer to the High volatility conditional steady state. In a similar way, the return to the low volatility regime generates an initial boom in the stock price, followed by a prolonged and slow moving increase as the variable keeps moving closer to the conditional steady state associated with the low volatility regime.

It is interesting to compare Figure 5 with the impulse responses reported in Figure 4. It is immediate to see that a change in volatility determines a more pronounced swing in the dynamics of the price-to-GDP ratio than an increase in fixed costs. The order of magnitude is very different: Following an increase in volatility, the price-to-GDP ratio falls by a value around .2, i.e. around 20%, while following an increase in the fixed cost the fall is .015 ($\cong$ 1.5%). An increase in ambiguity about fixed costs determines a fall of around 11% on impact and it is therefore closer to what implied by the increase in volatility. However, this drop is short lasting, while the increase in volatility is followed by further declines in the price-to-GDP ratio as the economy gets closer to the High Volatility conditional steady state.

3.6 Spectral decomposition

Figure 6 reports the normalized spectrum conditioning on the two regimes. The red vertical bars mark the business cycle frequencies (from 6 quarters to 32 quarters). It is immediate to see that while the macroeconomic variables present a variability concentrated at business cycle and high frequencies, financial variables turn out to be very persistent, i.e., the largest fraction of their variability is associated with low frequencies. Furthermore, we notice important differences between the two regimes for the macroeconomic variables, while the spectrum for the financial variables appears much more similar.

Figures 7 and 8 report the spectral decomposition conditioning on each of the two regimes. Two aspects are worth pointing out. First, the shocks to the cost of financing and to ambiguity to the costs of financing are important for the financial variables at all frequencies, but they do not substantially affect consumption growth. Instead, shocks to technology and to ambiguity about the growth rate combined explain more than 50% of the variability at all frequencies. Second, across the two regimes, the spectral decomposition is similar for the financial variables, while it is somehow different for consumption growth. Specifically, shocks to the growth rate of technology explain a larger fraction of uncertainty at business cycle frequencies when under the high volatility regime, while at low frequencies the decomposition is substantially unaffected.

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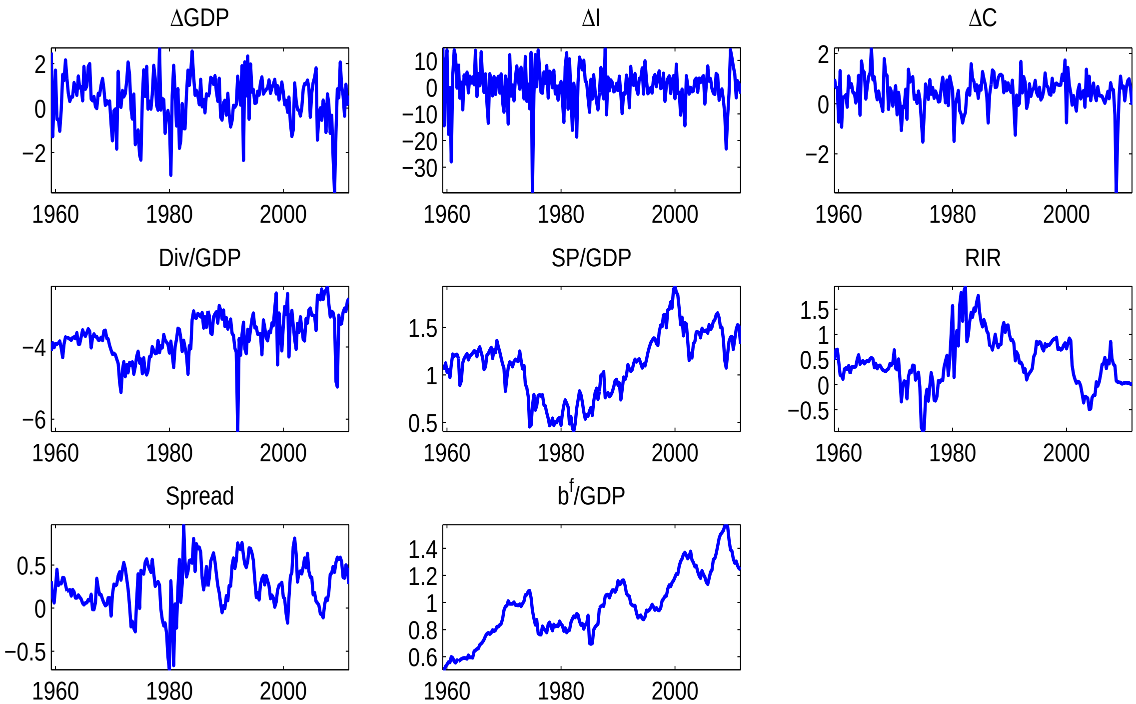

Figure 1: Variables used for the estimation of the model.

4 Appendix

4.1 Solution method for a model with ambiguity and MS volatility

Here we describe our approach to solve a general model with ambiguity and Markov switching volatility. The steps of the solution are the following:

1. Describe the law of motion for the shocks

   (a) Write the perceived law of motion for the continuous shocks as in (11):

   $$\tau_{t+1} = P\tau_t + \mu_t + \Sigma_t \varepsilon_{t+1}$$

   where by formula (13) each element i in the vector μ_t belongs to a set

   $$\mu_{t,i} \in [-\sqrt{2\psi_{t,i}}\Sigma_{t,i}, \sqrt{2\psi_{t,i}}\Sigma_{t,i}] \tag{19}$$

   (b) Suppose there are two Markov-switching regimes in Σ_t. Write the MS process as in (12)

   (c) Suppose the relative entropy bound evolves linearly as in (14).

2. Guess and verify the worst-case scenario. As discussed in section 2 and in detail in Ilut and Schneider (2011), the solution to the equilibrium dynamics of the model can be found through a guess-and-verify approach. To solve for the worst-case belief that minimizes expected continuation utility over the i sets in (19), we propose the following procedure:

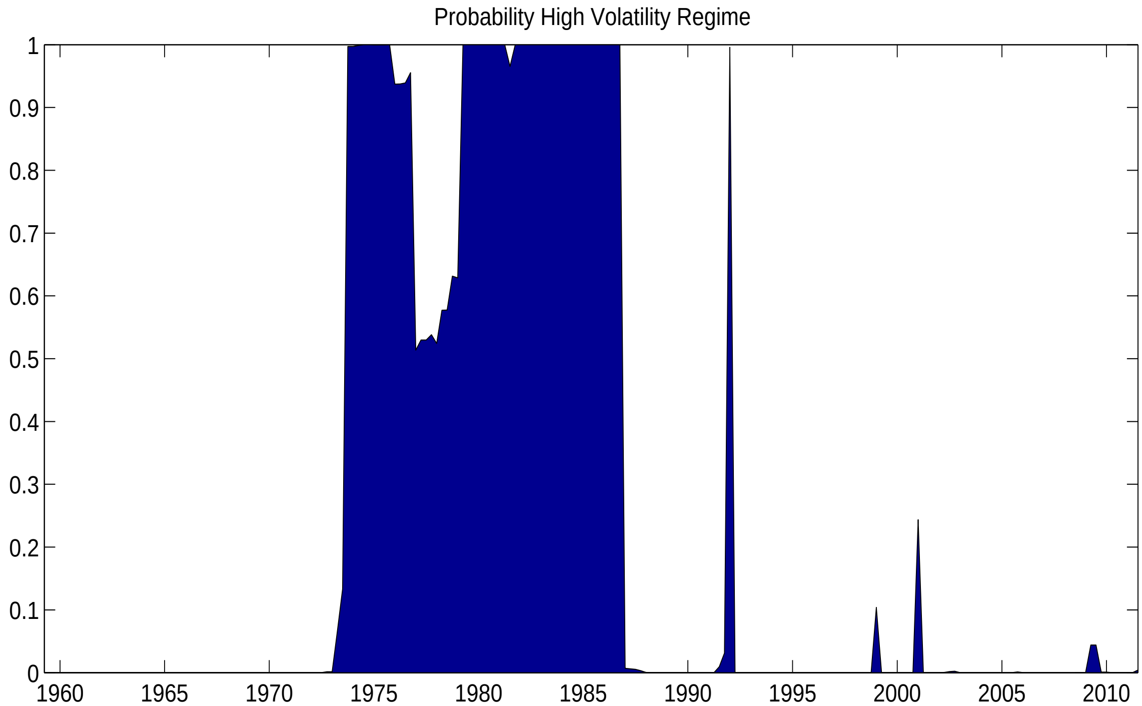

Figure 2: Smoothed probability of Regime 2, the high volatility regime, at the posterior mode.

(a) guess the worst case belief p^0

(b) solve the model assuming that agents have expected utility and beliefs p^0.

(c) compute the agent's value function V

(d) verify that the guess p^0 indeed achieves the minima.

The following steps detail the point 2.b) above. Here we use an observational equivalence result saying that our economy can be solved *as if* the agent maximizes expected utility under the belief p^0. Given this equivalence, we can use standard perturbation techniques that are a good approximation of the nonlinear decision rules under expected utility. In particular, we will use linearization.

3. Compute worst-case steady states

(a) Compute the ergodic mean $\overline{\Sigma}_i$ for the stochastic volatility based on (12).

(b) Suppose that the shocks are normalized so that the guess above involves setting $\mu^*_{t,i} = -a_{t,i}$ for each shock i. Then, denoting by τ_i the mean of the shock of the true DGP process in (11), the worst-case steady state is

$$\overline{\tau}_i = \tau_i - \frac{\overline{\eta}_i \overline{\Sigma}_i}{1-\rho_i}, \tag{20}$$

where ρ_i is the AR(1) coefficient in the P matrix corresponding to shock i.

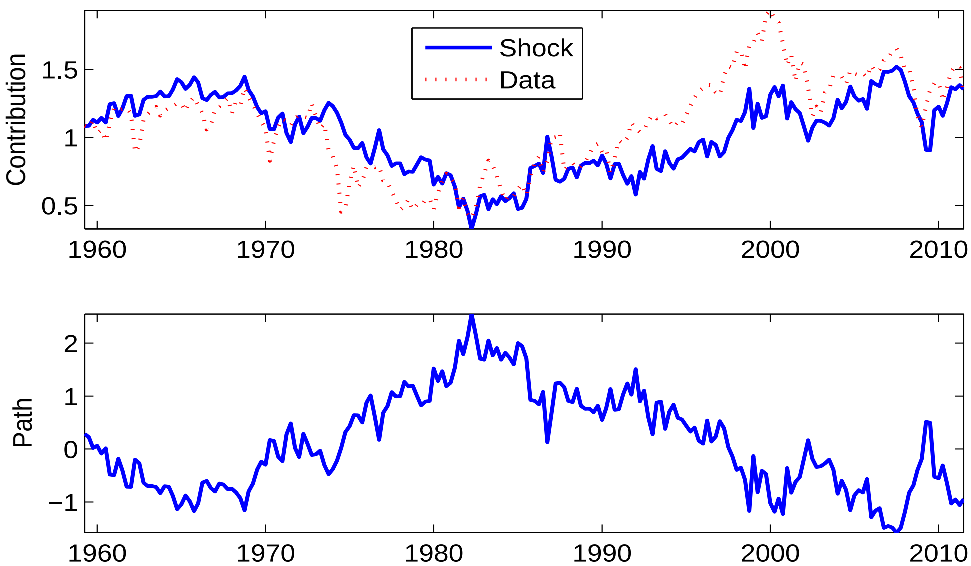

Figure 3: Stock prices and ambiguity. The first panel reports the evolution of the Price-to-GDP ratio (red sahed line) and a counterfactual series obtained setting all shocks to zero, but the shocks to ambiguity about the fixed costs of financing. The lower panel reports the smoothed series for ambiguity about the fixed costs of financing.

(c) Compute the worst-case steady state $\overline{Y}$ of the endogenous variables. For this, use the FOCs of the economy based on their deterministic version in which the one step ahead expectations are computed under $\mu_i^* = -a_i$. Denote the solution as $\overline{Y} = f(\overline{\tau})$.

4. Dynamics:

(a) Linearize around $\overline{Y}, \overline{\tau}, \overline{\eta}, \overline{\Sigma}$:

$$\widetilde{Y}_t \equiv Y_t - \overline{Y} \ , \ \widetilde{\tau}_{t,i} \equiv \tau_{t,i} - \overline{\tau}_i$$
$$\widetilde{\eta}_{t,i} \equiv \eta_{t,i} - \overline{\eta}_i \ , \ \widetilde{\Sigma}_{t,i} = \Sigma_{t,i} - \overline{\Sigma}_i.$$

by finding the coefficient matrices from linearizing the FOCs. The linearized FOCs can be written in the canonical form of solving rational expectations models as:

$$\widetilde{\Gamma}_0 \widetilde{S}_t = \widetilde{\Gamma}_1 \widetilde{S}_{t-1} + \widetilde{\Psi} \widetilde{\Sigma}_t \left[\varepsilon_t', v_t'\right]' + \Pi \eta_t$$

where S_t is the DSGE state.

(b) Given that the shock v_t is defined such that $E_{t-1}\left[v_t\right] = 0$, a standard solution method to solve rational expectations general equilibrium models can be employed. The solution can then be rewritten as a MS-VAR in which the constant is also time-varying:

$$\widetilde{S}_t = C_t + T\widetilde{S}_{t-1} + R\Sigma_{t-1}\varepsilon_t \tag{21}$$

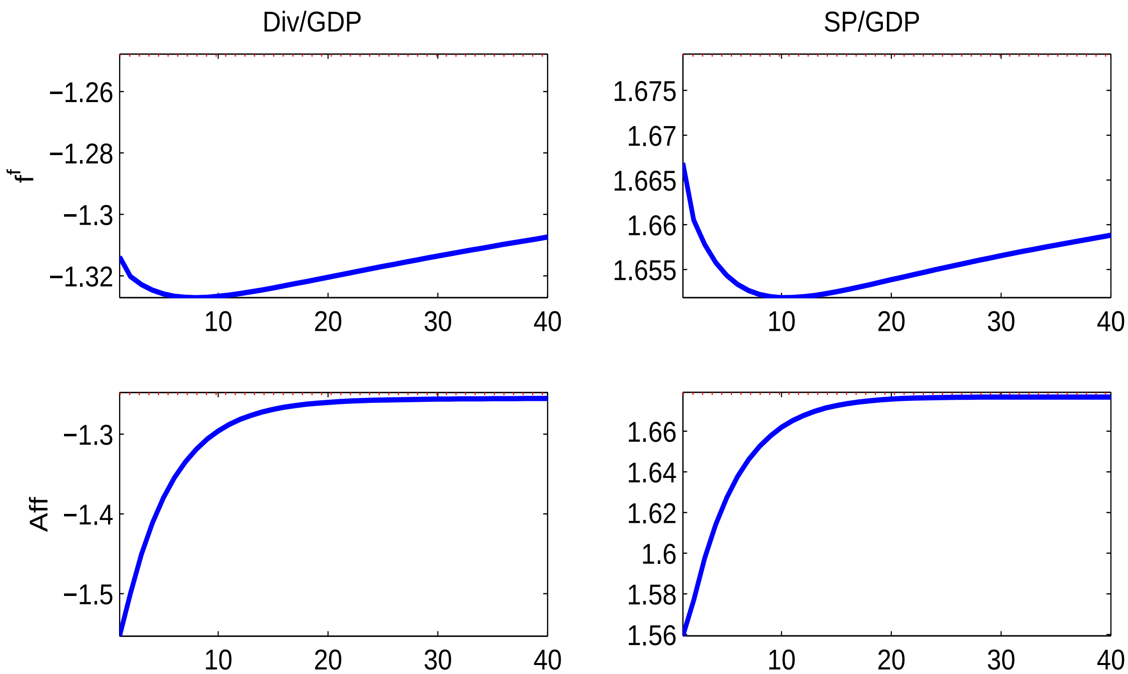

Figure 4: Impulse responses to shocks to fixed costs and ambiguity about fixed costs for the dividend-to-GDP and the price-to-GDP rations

The changes in the constant control the first order effects of stochastic volatility that arise because of ambiguity. Notice that the solution is expressed in terms of the original DSGE state variables as the variables $e_{1,t}$ and $e_{2,t}$ have been replaced with the MS constant C.

(c) Verify that the guess p^0 indeed achieves the minima of the time t expected continuation utility over the sets (19).

5. Equilibrium dynamics under the true DGP. The above equilibrium was derived under the worst-case beliefs. We need to characterize the economy under the econometrician's law of motion. There are two objects of interest: the zero-risk steady state of our economy and the dynamics around that steady state.

   (a) The zero-risk steady state, denoted by Y^*. This is characterized by shocks, including the volatility regimes, being set to their ergodic values under the true DGP. Y^* can then be found by looking directly at the linearized solution and adding $R\overline{\eta}\overline{\Sigma}$:

$$Y^* - \overline{Y} = T\left(Y^* - \overline{Y}\right) + R\overline{\eta}\overline{\Sigma} \tag{22}$$

   where the latter uses that the worst-case scenario is the minus of a_t.

   (b) Dynamics. The law of motion in (21) needs to take into account that expectations are under the worst-case beliefs which differ from the true DGP. Then, defining $\widehat{S}_t \equiv S_t - S^*$ and using (21) together with (22) we have:

$$\widehat{S}_t = C_t + T\widehat{S}_{t-1} + R\Sigma_{t-1}\varepsilon_t + R\left(\overline{\eta}\widetilde{\Sigma}_{t-1} + \overline{\Sigma}\widetilde{\eta}_{t-1}\right)$$

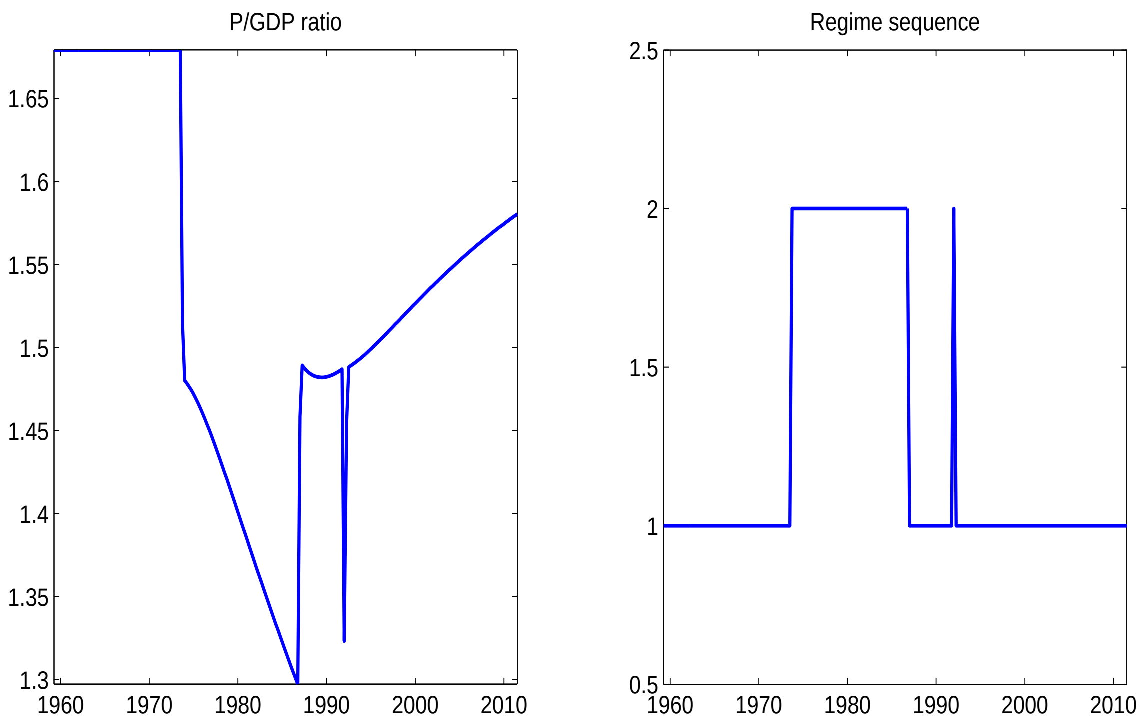

Figure 5: Change in the price-to-GDP ratio (left panel) induced by the typical regime sequence identified at the posterior mode (right panael).

4.2 Equilibrium conditions for the estimated model

Here we describe the equations that characterize the equilibrium of the estimated model in Section 3. To solve the model, we first scale the variables in order to induce stationarity. The variables are scaled as follows:

$$c_t = \frac{C_t}{\epsilon_t}, y_t = \frac{Y_t}{\epsilon_t}; g_t = \frac{G_t}{\epsilon_t}; t_t = \frac{T_t}{\epsilon_t}; k_t = \frac{K_t}{\epsilon_t}, i_t = \frac{I_t}{\epsilon_t}$$

Prices:

$$w_t = \frac{W_t}{\epsilon_t}; q_t^e = \frac{Q_t^e}{\epsilon_t}$$

Financial variables:

$$d_t = \frac{D_t}{\epsilon_t}, b_t^i = \frac{B_t^i}{\epsilon_t}; i = f, h, g;$$

The borrowing costs:

$$\frac{\kappa\left(B_{t-1}^f\right)}{\epsilon_t} = f_t + \frac{\Psi_t}{2}\left(\frac{b_{t-1}^f}{\xi_t}\right)^2;\ \phi\left(D_t\right)\frac{1}{\epsilon_t} = \frac{\phi''}{2}\left(d_t - \overline{d}\right)^2$$

We now present the nonlinear equilibrium conditions characterizing the model, in scaled form. The expectation operator in these equations, denoted by E_t^*, is the one-step ahead conditional expectation under the worst case belief p^0. According to our model, the worst

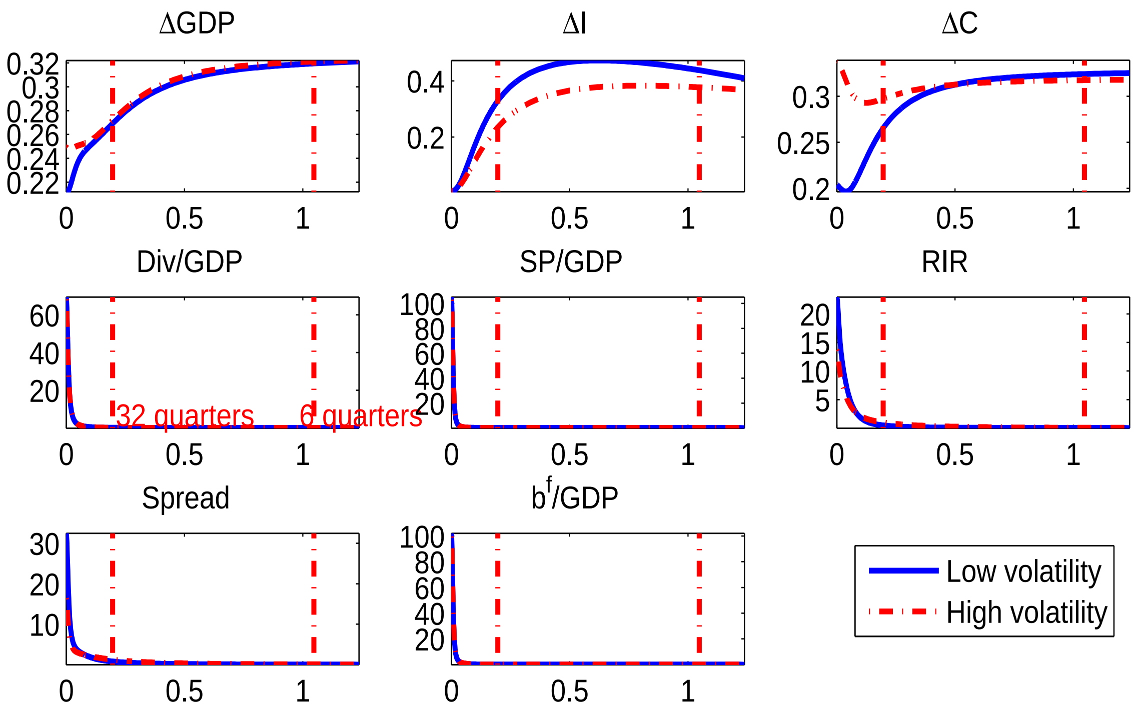

Figure 6: Normalized spectrum as implied by the posterior mode estimates.

case is that future productivity is low, and that financing costs and government spending are high. Thus, according to p^0 τ_{t+1} evolves as

$$\tau_{t+1} = P\tau_t - a_t + \Sigma_t \varepsilon_{t+1} \tag{23}$$

where a_t is the corresponding vector of $a_{t,i}$ that evolve as in (13).

The firm problem is

$$\max E_0^* \sum M_{0.t}^f D_t$$

subject to the budget constraint

$$\begin{aligned} d_t = (1-\tau_k)\left[y_t - w_t L_t - k_{t-1}\frac{a(u_t)}{\xi_t} - \frac{b_{t-1}^f}{\xi_t}\left(1 - Q_{t-1}^b\right) - \frac{\phi''}{2}\left(d_t - \overline{d}\right)^2\right] - \\ - f_t - \frac{\Psi_t}{2}\left(\frac{b_{t-1}^f}{\xi_t}\right)^2 + \delta\tau_k q_{t-1}^k \frac{k_{t-1}}{\xi_t} - i_t - \frac{b_{t-1}^f}{\xi_t} Q_{t-1}^b + b_t^f Q_t^b \end{aligned} \tag{24}$$

and the capital accumulation equation

$$k_t = \frac{(1-\delta)k_{t-1}}{\xi_t} + \left[1 - \left(\frac{S''}{2}\frac{i_t \xi_t}{i_{t-1}} - \overline{\xi}\right)^2\right] i_t \tag{25}$$

Let the LM on the budget constraint be $\lambda_t M_{0.t}^f \varepsilon_t^*$ and on the capital accumulation be $\mu_t M_{0.t}^f \varepsilon_t^*$. Then the scaled pricing kernel is

$$m_{t+1}^f \equiv M_{t+1}\frac{\varepsilon_{t+1}^*}{\varepsilon_t^*} = \beta \frac{c_t}{c_{t+1}} \frac{1-\tau_l}{1 - \tau_l \beta E_t^* \left[c_t/\xi_{t+1} c_{t+1}\right]} \tag{26}$$

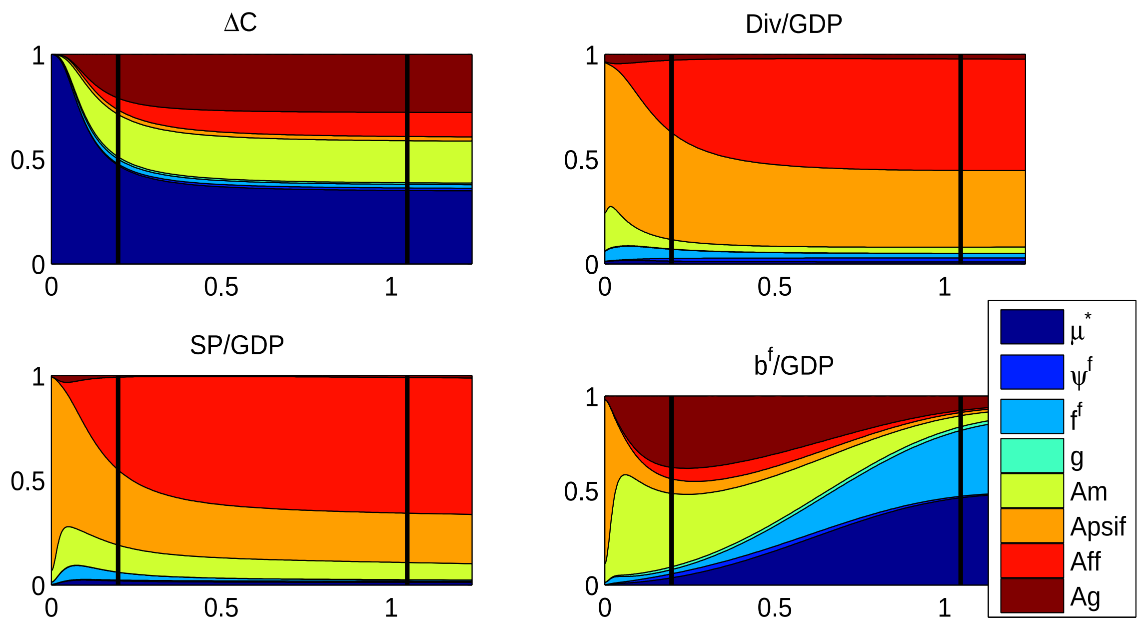

Figure 7: Spectral decomposition conditioning on Regime 1 for consumption growth, dividend-to-GDP ratio, price-to-GDP ratio, and debt-to-GDP ratio. The vertical bars mark business cycle frequencies. The shocks in the legend, from top to bottom are: growth rate, marginal cost, fixed cost, government spending and ambiguity about them

The FOCs associated with the firm problem are then:

1. Labor demand:

$$w_t = (1-\alpha)E_t^* \left(\frac{u_{t+1}k_t}{\xi_{t+1}}\right)^{\alpha} L_t^{-\alpha} \tag{27}$$

2. Dividends:

$$1 = \lambda_t \left[1 + (1-\tau_k)\phi''\left(d_t - \overline{d}\right)\right] \tag{28}$$

3. Bonds:

$$Q_t^b \lambda_t = E_t^* m_{t+1}^f \lambda_{t+1} \frac{1}{\xi_{t+1}} \left[1 - \tau_k\left(1 - Q_t^b\right) + \Psi_{t+1}\left(\frac{b_t^f}{\xi_{t+1}}\right)\right] \tag{29}$$

4. Investment:

$$1 = q_t^k \left[1 - \frac{S''}{2}\left(\frac{i_t\xi_t}{i_{t-1}} - \overline{\xi}\right)^2 - S''\left(\frac{i_t\xi_t}{i_{t-1}} - \overline{\xi}\right)\frac{\xi_t}{i_{t-1}}\right] + \tag{30}$$
$$+ E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^k S'' \frac{i_{t+1}^2 \xi_{t+1}}{i_t^2}\left(\frac{i_{t+1}\xi_{t+1}}{i_t} - \overline{\xi}\right)$$

where

$$q_t^k \equiv \frac{\mu_t}{\lambda_t}$$

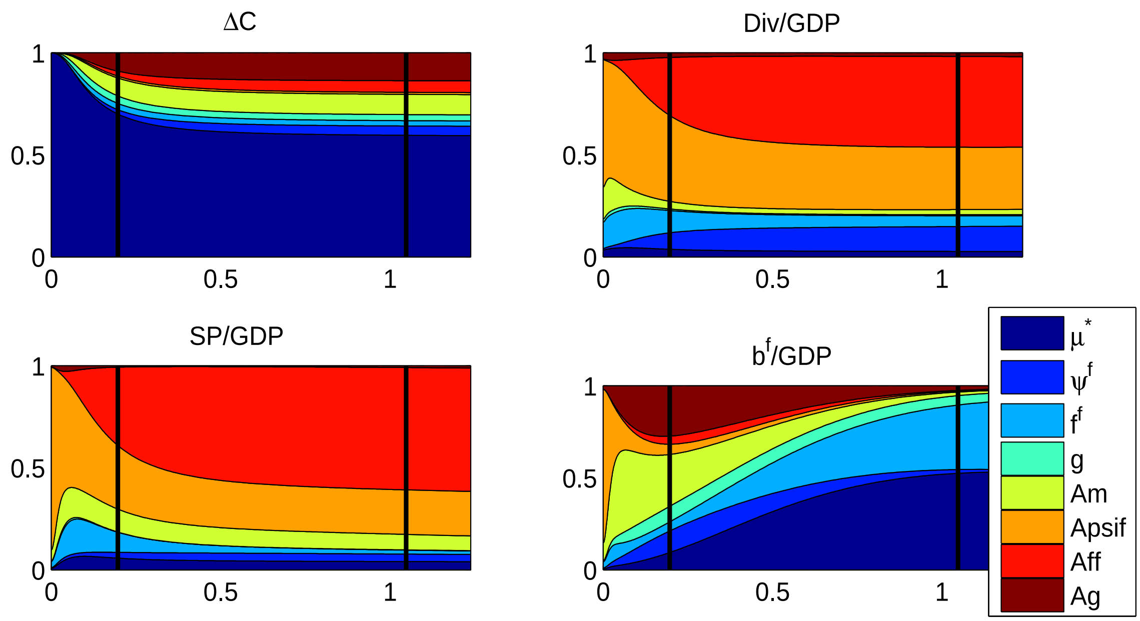

Figure 8: Spectral decomposition conditioning on Regime 2 for consumption growth, dividend-to-GDP ratio, price-to-GDP ratio, and debt-to-GDP ratio. The vertical bars mark business cycle frequencies. The shocks in the legend, from top to bottom are: growth rate, marginal cost, fixed cost, government spending and ambiguity about them

5. Capital:

$$1 = E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\lambda_t} \frac{R_{t+1}^K}{\xi_{t+1}} \tag{31}$$

$$R_{t+1}^k \equiv \frac{(1-\tau_k)\left[\alpha u_{t+1}^\alpha \left(\frac{k_t}{\xi_{t+1}}\right)^{\alpha-1} L_t^{1-\alpha} - a(u_{t+1})\right] + (1-\delta) q_{t+1}^k}{q_t^k} + \delta\tau_k$$

6. Utilization rate:

$$\alpha\left(\frac{u_t k_{t-1}}{\xi_t}\right)^{\alpha-1} L_{t-1}^{1-\alpha} = r^k \vartheta u_t + r^k(1-\vartheta) \tag{32}$$

The household problem is as follows:

$$\max E_0^* \sum \beta^t \left[\log C_t - \frac{\chi_L}{1+\sigma_L} L_t^{1+\sigma_L}\right]$$

$$(1+\tau_c)c_t + q_t^e\theta_t = (1-\tau_l)\left[w_t L_t + \pi + d_t\theta_{t-1} - \frac{b_{t-1}^h}{\xi_t}\left(1 - Q_{t-1}^b\right)\right] + \tag{33}$$

$$+ q_t^e \theta_{t-1} - \frac{b_{t-1}^h}{\xi_{t+1}} Q_{t-1}^b + b_t^h Q_t^b + t_t$$

Thus, the FOCs associated to the household problem are:

1. Labor supply

$$\frac{(1-\tau_l)w_t}{(1+\tau_c)c_t} = \chi_L L_t^{\sigma_L} \tag{34}$$

2. Bond demand:

$$Q_t^b = \beta E_t^* \frac{c_t}{c_{t+1}} \frac{1}{\xi_{t+1}} \left[1 - \tau_l \left(1 - Q_t^b\right)\right] \tag{35}$$

3. Equity holding:

$$q_t^e = \beta E_t^* \frac{c_t}{c_{t+1}} \left(q_{t+1}^e + (1-\tau_l)d_{t+1}\right) \tag{36}$$

The market clearing conditions characterizing this economy are:

$$b_t^h + b_t^f + b_t^g = 0 \tag{37}$$

$$c_t + i_t + g_t + \frac{\phi''}{2}\left(d_t - \overline{d}\right)^2 + f_t + \frac{\Psi_t}{2}\left(\frac{b_{t-1}^f}{\xi_t}\right)^2 = y_t + \pi \tag{38}$$

$$\theta_t = 1$$

corresponding to the market for bonds, goods and equity shares, respectively.

The government budget constraint is:

$$g_t + t_t = \tau_k \left[y_t - w_t L_t - k_{t-1}\frac{a(u_t)}{\xi_t} - \frac{b_{t-1}^f}{\xi_t}\left(1 - Q_{t-1}^b\right) - \frac{\phi''}{2}\left(d_t - \overline{d}\right)^2 - \delta q_{t-1}^k \frac{k_{t-1}}{\xi_t}\right] + \tag{39}$$

$$+ \tau_l(w_t L_t + \pi + d_t\theta_{t-1}) + \tau_c c_t - \frac{b_{t-1}^g}{\xi_t} + b_t^g Q_t^b$$

and the lump sump transfers follow the process:

$$t_t = t_o - \varkappa\left(b_{t-1}^g \frac{1}{\xi_t} - b^g \frac{1}{\xi}\right) \tag{40}$$

Thus, we have the following 16 unknowns:

$$k_t, u_t, i_t, L_t, w_t, b_t^f, b_t^h, b_t^g, Q_t^b, q_t^e, c_t, d_t, q_t^k, t_t, \lambda_t, m_t^f$$

The equations (25), (26), (27), (28), (29), (30), (31), (32), (34), (35), (36), (37), (38) and (40) give us 14 equations. By Walras' law, we can then use two out of the three budget constraints in (24), (33) (using $\theta_t = 1$) and (39). This gives us a total of 16 equations.

4.2.1 Parametrization

Short and long run fixed cost

Motivated by the observed firm debt to GDP ratio evolution that trends upward, we allow for the possibility that the financing fixed cost f_t has a deterministic trend. To model this, consider the following separation of the shock into a short run and long run component:

$$\log\left(f_t/f\right) = \log\left(\frac{f_t}{f_t^L}\right) + \log\left(\frac{f_t^L}{f}\right)$$

Then defining the deviation $f_t^s \equiv f_t / f_t^L$ and assuming a deterministic trend, we have

$$\log\left(f_t^s\right) = \rho_f \log\left(f_{t-1}^s\right) + \sigma_f \varepsilon_{f,t}$$
$$\log\left(f_t^L / f\right) = \rho_{f^L} \log\left(f_{t-1}^L / f\right)$$

The filtering algorithm would then determine the initial long run distance from the steady state and all the remaining fluctuations would be explained by the short run component. Since we assume a deterministic trend, then ambiguity is only over the short run component.

Rescaling and calibrating parameters

For the steady state calculation of the model it is helpful to rescale some parameters. Specifically, denote by $\overline{y}^{gdp}$ the worst-case steady state measured GDP, i.e. total goods $y + \pi$ minus financing costs. Then, define the following ratios:

$$f_y = \frac{f}{\overline{y}^{gdp}}; \Psi_y = \frac{\Psi}{\overline{y}^{gdp}}; \pi_y = \frac{\pi}{\overline{y}^{gdp}}, t_{o,y} = \frac{t_o}{\overline{y}^{gdp}}$$

The results reported in section 3.2 are based on some parameters that are estimated, with values reported in Table 1, and some that are calibrated. The latter are reported in Table 2.

The Frisch labor elasticity is set to a relatively standard value in the literature, while the disutility parameter χ_L only scales the economy. The government transfer parameter $\varkappa$ is set to a reasonably high value to guarantee stability.

The other parameters are calibrated to match some key ratios from the NIPA accounts. First, total measured GDP in our model, denoted here by y^{gdp}, corresponds to the non-financial corporate sector (NFB) output plus goods produced by the other productive sectors- financial, non-corporate and household. We associate the firm in our model with the NFB sector and thus π_y equals goods produced by other productive sectors divided by y^{gdp}. The tax parameters are computed as follows: τ_l equals total personal taxes and social security contributions divided by total income, where the latter is defined as total wages plus dividends. τ_k equals NFB taxes divided by NFB profits and τ_c equals NFB sales taxes divided by NFB output. The government spending ratio g equals government net purchases from other sectors plus net exports divided by y^{gdp}. The ratio $t_{o,y}$ equals government transfers (including social security and medicare) plus after-tax government wages divided by y^{gdp}.

	$Mode$	$Mean$	5%	95%	$Type$	$Mean$	Std
ρ_ξ	0.0068	0.0064	0.0057	0.0069	B	0.50	0.15
ρ_ψ	0.8039	0.8056	0.7743	0.8420	B	0.50	0.15
ρ_f	0.7726	0.7586	0.7331	0.7922	B	0.50	0.15
ρ_g	0.8857	0.8856	0.8653	0.9049	B	0.50	0.15
ρ_{f^L}	0.9841	0.9778	0.9636	0.9880	B	0.95	0.025
$100f_y$	0.0208	0.0208	0.0201	0.0216	B	0.30	0.20
ϕ''	9.1411	8.5334	7.9619	8.9935	G	60.00	30.00
Ψ_y	0.0022	0.0023	0.0022	0.0025	G	0.005	0.001
$100(\xi - 1)$	1.4180	1.4210	1.3773	1.4927	G	0.30	0.05
δ	0.0381	0.0382	0.0368	0.0398	B	0.0250	0.0030
α	0.2215	0.2199	0.2132	0.2255	B	0.35	0.05
S"	0.0002	0.0002	0.0001	0.0002	G	10.00	5.00
ϑ	3.1488	3.2347	3.1302	3.3804	G	4.00	2.00
$100(\beta^{-1} - 1)$	0.5001	0.5242	0.5045	0.5501	G	0.30	0.10
$\overline{\eta}_\xi$	0.1923	0.1965	0.1861	0.2058	B	0.50	0.25
$\overline{\eta}_\psi$	0.3142	0.2712	0.2335	0.3158	B	0.50	0.25
$\overline{\eta}_f$	0.6906	0.6492	0.5575	0.7281	B	0.50	0.25
$\overline{\eta}_g$	0.9884	0.9619	0.9265	0.9878	B	0.50	0.25
ρ_{η_ξ}	0.9494	0.9482	0.9358	0.9599	B	0.50	0.25
ρ_{η_ψ}	0.9900	0.9880	0.9823	0.9925	B	0.50	0.25
ρ_{η_f}	0.7981	0.8047	0.7625	0.8273	B	0.50	0.25
ρ_{η_g}	0.8769	0.8770	0.8570	0.8995	B	0.50	0.25
$\sigma_\xi\,(1)$	0.0355	0.0357	0.0325	0.0394	IG	0.05	0.05
$\sigma_\psi\,(1)$	1.4563	1.4214	1.2892	1.5785	IG	0.05	0.05
$\sigma_f\,(1)$	4.6368	4.7600	4.3803	5.1324	IG	0.05	0.05
$\sigma_g\,(1)$	0.0705	0.0707	0.0631	0.0790	IG	0.05	0.05
$\sigma_\xi\,(2)$	0.0658	0.0657	0.0591	0.0740	IG	0.05	0.05
$\sigma_\psi\,(2)$	4.0379	4.3355	3.5977	5.1608	IG	0.05	0.05
$\sigma_f\,(2)$	8.0898	9.0933	8.0466	10.0037	IG	0.05	0.05
$\sigma_g\,(2)$	0.1981	0.2092	0.1835	0.2421	IG	0.05	0.05
H_{11}	0.9340	0.9300	0.9088	0.9511	D	0.9048	0.0626
H_{22}	0.9869	0.9808	0.9597	0.9941	D	0.9048	0.0626

Table 1: Parameter estimates. Left hand side: Mode, mean, and 90% error bands. Right hand side: Priors.

τ_l	τ_k	τ_c	π_y	$t_{o,y}$	σ_L	χ_L	g	$\varkappa$
0.189	0.193	0.09	0.3	0.21	1	1	0.05	0.03

Table 2: Calibrated parameters