Wage Dispersion in the Search and Matching Model with Intra-Firm Bargaining

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May 24, 2009

Abstract

Matched employer-employee data exhibits both wage and productivity dispersion across firms and suggest that a linear relationship holds between the average wage paid and a firm productivity. The purpose of this paper is to demonstrate that these facts can be explained by a search and matching model when firms are heterogenous with respect to productivity, are composed of many workers, and face diminishing returns to labor given the wage paid to identical workers is the solution to the Stole-Zwiebel bilateral bargaining problem. Helpman and Iskhoki (2008) show that a unique single wage (degenerate) equilibrium solution to the model exists in this environment. In this paper, I demonstrate that another equilibrium exists that can be characterized by a non-degenerate distribution of wages in which more productive firms pay more if employed workers are able to search. Generically this dispersed wage equilibrium is unique and exists if and only if firms are heterogenous with respect to factor productivity. Finally, employment is lower in the dispersed wage equilibrium than in the single wage equilibrium but this fact does not imply that welfare is higher in the single wage equilibrium.

1 Introduction

The simplicity of the canonical search and matching model offers many advantages for the purpose of understanding the determinants of unemployment. Furthermore, the model explains wage dispersion and a positive cross firm correlation between wage and productivity in the heterogeneous firm case. However, the special assumption that a firm is composed of a single worker and employer and no search on-the-job are obviously limiting restrictions. For example, it does not speak to the well known correlation wage and firm size. The principal purpose of this paper is to demonstrate that the canonical model's implications for wage dispersion continue to hold and are consistent with these addition facts when the model is generalized to allow for many worker firms provided that search on-the-job is possible.

A model of strategic bilateral wage bargaining when firms employ many worker is formulated and solved by Stole and Zwiebel (1996). In their formulation, the employer bargains with each worker as though he or she were the marginal employee. Because the wage outcome decreases with the number of workers employed in the diminishing returns case, they show that there is an incentive to "over employ" as a means of reducing the wage paid to all workers. Cahuc, Marque, and Wasmer (2007) study the implications of the Stole-Zwiebel model of wage determination in a search and matching framework when factors of production include many worker types and capital. Recently, Helpman and Iskhoki (2008) demonstrate that Stole-Zwiebel wages paid to identical workers are equalized across employers in a steady state search equilibrium when the production technology exhibits diminishing returns with respect to labor, even when firm factor productivities differ. This results would seem to contradict the claim of this paper. However, the Helpman-Iskhoki analysis does not consider the possibility of search on-the-job. If allowed, a dispersed wage equilibrium also exists.

In this paper, the wage paid at every point in time is the solution to the Stole-Zwiebel bargaining problem. I start by verifying the Helpman-Iskhoki conclusion that a single wage equilibrium always exists. Of course, were all employers paid the same wage, employed workers have no incentive to search. I then demonstrate that a second equilibrium exists characterized by wage dispersion if firms differ with respect to productivity and search by employed workers is not ruled out by assumption. Furthermore, generically there is only one disperse wage equilibrium in which more productive employers over a higher wage. Finally, I establish that employment is lower in the dispersed wage equilibrium than in the single wage equilibrium. But, because aggregate employment in the single wage equilibrium exceeds the social optimum, social welfare can be higher in the dispersed wage equilibrium.

There is clearly a relationship between the dispersed wage equilibrium in the search and matching model and the equilibrium wage distribution implied by the dynamic monopsony models of Burdett and Judd (1983) and Burdett and Mortensen (1998); particularly with the reinterpreted version of the Burdett-Mortensen model found in Mortensen (2003). Although the relationship is close, the results are not isomorphic. Wage dispersion in the both models arises because a continuum of wage-employment combinations exist that yield the same expected return to posting a vacancy. When the cost of posting vacancies is linear and the same for all firms, the equilibrium wage-employment relationship lies along this locus for each level of productivity. When firms differ with respect to productivity, the wages paid by more productive employers are higher at any level of employment. The search and matching model with diminishing returns to labor shares this feature with the Burdett-Mortensen dynamic monopsony model. However, forward looking strategic wage setting behavior by employers in the dynamic monopsony model generates dispersion even when there are no differences in firm productivity for strategic reasons that are not present when the wage is the outcome of a bargaining problem over current output, as assumed in this paper. For this reason, there is generally is no disperse wage equilibrium in the search and matching model when firms are homogenous.

2 The Model

The labor market is composed of a unit measure of employers and a continuum of identical workers of size ℓ. Labor is the only input and production is generally subject to diminishing returns. Let $p(x)f(n)$ denote the production function of firm $x \in [0, 1]$ where the variable n is a measure of employment and $p(x)$ is total factor productivity. The base line production function $f(n)$ is increasing and concave. Without loss of generality, firms with a higher index x are (weakly) more productive in the weak sense that $x > x'$ implies $p(x) \geq p(x')$. Given this convention, x is equal to the firm's percentile rank in the distribution of productivity.

Workers are identical and can be either employed or not. While unemployed they receive an income b, which I interpreted as the flow value of home production. While employed, they earn a wage w which is determined as the outcome of a bilateral bargaining problem specified later. A worker can choose to search or not at a unit intensity in the both unemployment and employment states. Finally, all agents are risk neutral.

2.1 Worker Search

The workers' common value of unemployment solves the Bellman equation

$$rU = b + \max_{\phi \in \{0,1\}} \left\{ \lambda\phi \int \max \langle W(w) - U, 0 \rangle \, dF(w) - \varepsilon\phi \right\} \tag{1}$$

where ϕ is an indicator that takes the value 1 if the worker searches and 0 otherwise, ε is a small fixed cost of search, b represents the value of home production, r is the interest rate, λ is the job finding rate, δ is the job destruction rate, $W(w)$ is the value of employment at wage w and $F(w)$ is the distribution of wage offers. The value of employment is the expected present value of the worker future income when employed. For a firm paying wage w, the state contingent value of employment satisfies the Bellman equation

$$rW(w) = \begin{array}{c} w + \delta(U - W(w)) \\ + \max_{\phi \in \{0,1\}} \left\{ \lambda\phi \int \max \langle W(z) - W(w), 0 \rangle \, dF(z) - \varepsilon\phi \right\}. \end{array}. \tag{2}$$

Of course, the worker can choose whether or not to search when employed as well as when unemployed and does so to maximize expected future income. Although the "search technology" as represented by the offer arrival rate may be different in the two states, we assume they are same in the paper for simplicity. For the same reason we also assume that the out of pocket cost of search $\varepsilon > 0$ is trivial. Hence, the worker's optimal search strategy is

$$\phi(w) = \begin{cases} 1 \text{ if unemployed and } 1 - F(b) > 0 \\ 1 \text{ if employed at value } W \text{ and and } 1 - F(w) > 0 \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

In words, a worker searches if and only if the expected gain in value is positive. By implication, trade requires that the highest wage offered in the market exceed the value of home production. Furthermore, all but those employed at the highest offer, $\overline{w}$, search if wages are disperse. Below I only consider the case in which trade takes place in equilibrium so that $\phi(b) = 1$ and assume that $\phi(w) = 1$ if and only if $1 - F(w) > 0$.

Given that employed workers do search, they move if and only if offered a strictly higher employment value. Hence, for a firm that pays w, the separation rate is

$$s(w) = \delta + \lambda\phi(w)(1 - F(w)). \tag{4}$$

Because an unemployed worker accepts all offers in equilibrium and an employed worker accepts only if offered a higher wage, the expected yield per vacancy advertised is

$$h(w) = \eta \left[\frac{u + (1-u)\phi(w)G(w)}{u + (1-u)\phi(w)} \right] \tag{5}$$

where u is the unemployment rate, $G(w)$ is the fraction of employed workers at firms that pay wage w or less, and η is the rate at which employers contact searching workers per vacancy posted. Because $s(w)$ is decreasing and $h(w)$ is increasing, an increase in the wage reduces turnover if and only if employed workers search.

Note that equation (4) and (2) imply $W'(w) = 1/(r + s(w))$. Hence, by rearranging terms and integrating by parts, equation (2) can be rewritten as

$$\begin{aligned}(r+\delta)\left(W(w) - U\right) &= w - rU + \lambda\phi(w)\int_{w}^{\overline{w}} \left[W(z) - W(w)\right] dF(z) \\ &= w - rU + \lambda\phi(w)\int_{w}^{\overline{w}} \left(\frac{1-F(z)}{r+s(z)}\right) dz.\end{aligned}$$

Therefore equation (1) and the definition of the reservation wage $U = W(\widehat{w})$ imply that an unemployed worker accepts a job offer if and only if the wage exceeds the solution to

$$\widehat{w} = b + \lambda[1 - \phi(\widehat{w})]\int_{\widehat{w}}^{\overline{w}} \left(\frac{1-F(z)}{r+s(z)}\right) dz. \tag{6}$$

In other words, if employed workers search, then the reservation wage is the value of home production. But, if there is no incentive to search on-the-job, then the reservation wage is equal to the value of home production plus the option value of continued search.

2.2 Wage Bargaining

A firm cannot commit to long term contracts and the employer bargains with each worker individually, Stole and Zwiebel (1996) show that the solution to a strategic non-cooperative bilateral bargaining in this environment yields a wage-employment profile that reflect Shapley values. Specifically, Stole and Zwiebel assume that the current wage is the outcome of a sequence of bilateral negotiations with its employees where each is regarded as the marginal worker. Costless renegotiation takes place whenever a worker leaves or is hired. In this paper, we adopt this wage determination model.

To justify their assumption, Stole and Zwiebel design a extensive form strategic bilateral bargaining game of a general Rubinstein (1982) kind between the employer and marginal worker. The game is composed of a sequence of negotiation rounds in which the role of the proposer alternates between worker and employer while the other party either accepts or rejects the proposed split of the surplus value attributable to employing the marginal worker. The game continues until either an agreement is reached or negotiations break down. The default position of the employer is no revenue and for the worker is a wage equivalent $\widehat{w}$.

As the firm's gross profit flow is

$$\pi(n,p) = pf(n) - wn, \tag{7}$$

renegotiation with the marginal worker under complete information leads immediately to an agreement that can be characterized by the following surplus sharing rule

$$\beta \pi_n(n,p) = \beta \left(pf'(n) - w - \frac{\partial w}{\partial n} n \right) = (1-\beta)\left(w - \widehat{w}\right). \tag{8}$$

where β, the worker's share of surplus, represents the bargaining power of the marginal workers. It can be interpreted as the probability that the worker makes the offer in any bargaining round. As Stole and Zwiebel show, the solution to this differential equation, the wage bargaining outcome function, is the generalized Shapley value

$$w(n,p) = (1-\beta)\widehat{w} + p \int_0^1 z^{\frac{1-\beta}{\beta}} f'(zn) dz \tag{9}$$

which is a decreasing function of employment given diminishing returns.

Stole and Zwiebel assume that the worker can instantaneously find employment elsewhere at the wage $\widehat{w}$ if bargaining breaks down. However, given search friction, we follow Cahuc et al. (2007) and Helpman and Iskhoki (2008) by supposing that unemployed search is the workers outside option in the bargaining problem. Of course, in the linear production function case, $f(n) = n$, the wage reduces to that in the canonical model, the weighted average of the reservation wage and labor productivity. If the production function is Cobb-Douglas of the form $f(n) = n^{\alpha}$,$\alpha \in (0, 1)$ then the wage equation is also linear in average product per worker,

$$w(n,p) = (1-\beta)\widehat{w} + \left(\frac{\beta\alpha}{1-\beta+\beta\alpha}\right)\frac{pf(n)}{n}. \tag{10}$$

However, this equation does not imply that the wage varies with p in equilibrium because employment is endogenous.

Were firms composed of a single worker, the employer clearly has an incentive to renegotiation if that employee receives an outside offer to the extent that doing so might affect retention. Cahuc, Postel-Vinay and Robin (2006) allow outside offer matching in such an environment. However, there are also good reasons for the employer to adopt a policy of not to respond to outside offers. As Mortensen (1978) points out, offer matching encourages inefficient rent seeking search effort. As a consequence, Postel-Vinay and Robin (2004) show that conditions exist for which an employer would commit exante not to match. In fact, except in jobs where worker productivity is general and observable to third parties, anecdotal evidence suggests that this policy dominate in the labor market.

In the case of multi-worker firms, interpersonal comparison of earning of observably identical workers provide another reason for a "non-response" policy. Offer matching induces intra-firm wage dispersion, which is can be costly to the extent that equity norms are important. Also dispersion of this form violates antidiscrimination laws when workers are interchangeable in the firm. For these reasons, I require equal treatment for observably equivalent workers, which rules out offer matching in the environment under study. Instead, a worker stays if doing so yields a higher value under the current and expected future wage agreements.

2.3 Job Vacancies

A firm chooses to post v vacancies. The hire flow yield per vacancy is $h(w)$ and the separate rate is $s(w)$ as defined respectively by equations (4) and (5). Hence, the law of motion for a firm's labor force is

$$\dot{n} = vh(w(n,p)) - s(w(n,p))n \tag{11}$$

where $w(n,p)$ denotes the Stole-Zwiebel wage outcome function defined in equation (9). The value of the firm satisfies the following continuous time

Bellman equation

$$rV(n,p) = \max_{v\geq 0}\left\{\begin{array}{c}\pi(n,p) - cv \\ +J(n,p)\left[h(w(n,p)v - s(w(n,p))n\right]\end{array}\right\} \tag{12}$$

where c is the cost of posting a vacancy and $J(n,p) = \partial V/\partial N$ is the value of the marginal worker to the firm. By the envelope theorem, the value of the marginal worker satisfies

$$\begin{aligned} rJ(n,p) &= \pi_n(n,p) - s(w(n,p))J(n,p) \\ &\quad +J_n(n,p)\left[h(w)v(n,p) - s(w)n\right] \\ &\quad +J(n,p)\left(h'(w)v(n,p) - s'(w)\,n\right)w_n(n,p), \end{aligned} \tag{13}$$

where

$$v(n,p) = \arg\max_{v\geq 0}\left\{h(w(n,p))vJ(n,p) - cv\right\} \tag{14}$$

is the optimal number of vacancies posted. The first term on the RHS of (13) is the profit earned on the marginal worker as defined in equation (7), the second term is the cost of separations per worker, and the last two terms account for the total effect of an additional worker on the capital gain or loss associated with any rate of change in the size of the labor force.

There are two effects of adding a worker to the firm's labor force that are not present in the canonical model search and matching model. However, the first of the two, the impact of a change in size on the value of the marginal match, is not unfamiliar when labor productivity diminishes. The second effect, represented by the last term on the RHS of (13), is more novel. This negative effect on the marginal value represents the fact than an increase in the number of employees drives down the wage thereby decreasing the yield on vacancies, $h(w)$, and increasing the separation rate, $s(w)$. *Of course, equations (9) and (4) imply that this effect is present if and only if employed worker search or the wage paid by the firm is not the lowest in the market.* These facts provide the rational for all of the original results found in the paper.

Because the vacancy posting cost, c, is constant by assumption, the firm posts no vacancies if the vacancy posting cost exceeds the expected return, $h(w(n_0,p)J(n_0,p)$, evaluated at its initial labor force size n_0. In this case, the firm either allows its labor force to fall through attrition until equality holds or it immediately lays off the redundant workers if doing so is costless. Once its steady state size is achieved, the firm posts the vacancies needed to replace those that quit. On the other hand, if the return exceeds the cost, the firm instantaneously hires the workers required to achieve an equality between the marginal cost and expected return to posting a vacancy. In sum, the firm's labor force size quickly adjusts to its steady state, which is defined by the requirement that the cost of posting a vacancy is equal to its expected return

$$c = h(w(n,p))J(n,p), \tag{15}$$

where the number of vacancies posted,

$$v(n,p) = \frac{s(w(n,p))n}{h(w(n,p))}, \tag{16}$$

is that required to keep the labor force at the desired steady state size. In the job search literature, equation (15) is referred to as the "free entry" condition for vacancy creation.

2.4 Labor Market Matching

The aggregate flow of matches that form per period is determined by an increasing, concave, and homogenous of degree one matching function of the aggregate number of vacancies and searching worker, denoted $M(v,z)$. The rates at which workers are randomly matched with vacant jobs and vacancies with workers are respectively

$$\lambda = M(v,z)/z = m(\theta) \tag{17}$$

and

$$\eta = M(v,z)/v = m(\theta)/\theta \tag{18}$$

where $m(\theta) = M(\theta,\ell)$,

$$z = [u + (1-u)\phi(w)]\,\ell \tag{19}$$

is the measure of search workers, and $\theta = v/z$ is market tightness. Note that $\theta = v/\ell$ when employed workers search.

3 Steady State Equilibria

3.1 Steady State Conditions

In a market steady state, the unemployment rate and the distribution of employment over firms are stationary by assumption. As the steady state unemployment rate equates the flows in and out of employment, it satisfies

$$\frac{u}{1-u} = \frac{\delta}{m(\theta)}. \tag{20}$$

The analogous requirement that the flow into employment with a firm that offers value w or less is equal to the flow out determines the employment weighted distribution of wages paid by employers, given any distribution of values over vacancies offered. Formally,

$$\lambda F(w)u\ell = (\delta + m(\theta)\phi(w)[1-F(w)])\, G(w)(1-u)\ell$$

where the left side is the flow of workers into employment paying wage w or less and the right side is the flow out. Hence, the steady state relationship between the distribution of wages paid employed workers and the distribution of wages offered over vacancies is given by

$$G(w) = \frac{\delta F(w)}{\delta + m(\theta)\phi(w)[1-F(w)]}. \tag{21}$$

These steady state conditions equations (5) and (18) imply that vacancy yield and the separation rate are inversely related in a steady state. That is,

$$h(w) = \eta \left[\frac{u + (1-u)\phi(w)G(w)}{u + (1-u)\phi(w)} \right] = \frac{\delta m(\theta)}{\theta s(w)} \tag{22}$$

where

$$s(w) = \delta + m(\theta)\phi(w)[1-F(w)] \tag{23}$$

from (4) and (17).

To maintain employment at its steady state level n, an employer must post $v = s(w)n/h(w)$ vacancies. This fact and the steady state relationship between the vacancy yield and the separation rate function given by (22), (23), imply that the effect of adding a worker to future turnover costs can be expressed as

$$J(n,p)\,(h'(w)v(n,p) - s'(w)\,n)w_n(n,p) = -2J(n,p)m(\theta)\phi(w)F'(w)g(w,p) \tag{24}$$

where from equation (9)

$$g(w,p) \equiv -w_n(w,p)n(w,p) = -n(w,p)p\int_0^1 z^{\frac{1}{\beta}} f''(zn(w,p))dz > 0 \tag{25}$$

and $n(w,p)$ is the inverse of the Stole-Zwiebel wage function defined in equation (9).

At this point, I remind the reader that a reduction in the wage has no effect on either the vacancy yield or the separation rate if the wage paid is the lowest in the market. Namely, because $\lim_{w\uparrow\underline{w}}\left\{\frac{F(\underline{w})-F(w)}{\underline{w}-w}\right\}=0$ and the sharing rule implies $\beta\pi_n=(1-\beta)(w-\widehat{w})$ from (8), equation (13) can be rewritten as

$$J(n(w,p),p)=\begin{cases}\frac{(1-\beta)(\underline{w}-\widehat{w})/\beta}{r+s(\underline{w})} \text{ if } w=\underline{w}\\ \\ \frac{(1-\beta)(w-\widehat{w})/\beta}{(r+s(w)+2m(\theta)g(w,p)\phi(w)F'(w)} \text{ if } w>\underline{w}\end{cases} \tag{26}$$

where $\underline{w}$ is the lower support of $F(w)$.

The function $g(w,p)$ plays a crucial role in the analysis in the paper. It represents the absolute amount by which the wage bill paid to existing employees falls when a new worker is hired or rises when an existing employee leaves the firm. Because

$$w=(1-\beta)\widehat{w}+\left(\frac{\beta}{1-\beta+\beta\alpha}\right)pn^{\alpha-1}$$

in the log linear production function case from (10),

$$\begin{aligned} g(n,p) &= -p\int_0^1 nz^{\frac{1}{\beta}}f''(zn)dz \qquad (27)\\ &= (1-\alpha)\alpha pn^{\alpha-1}\int_0^1 z^{\frac{1-\beta-\alpha\beta}{\beta}}dz=(1-\alpha)\alpha\left(w-(1-\beta)\widehat{w}\right). \end{aligned}$$

in the Cobb-Douglas case. Note that the function happens to be independent of firm productivity.

3.2 Degenerate Wage

I begin by verifying the Helpman-Iskhoki finding that a steady state equilibrium exists with the property that the "law of one price" holds. Since employed workers do not search is such an equilibrium ($\phi(w)\equiv 0$), the FONC for an optimal choice of vacancies, $c=h(w)J(n(w,p))$ and equation (26) imply that the common wage paid is

$$w=\widehat{w}+\left(\frac{\beta}{1-\beta}\right)\frac{(r+\delta)c\theta}{m(\theta)}\ .$$

Since $c/h(w)$ is the expected cost of filling a vacancy, the difference between the wage paid and the reservation wage demand is the amortized expected cost to the employer of finding a replacement for the marginal worker. This fact generalized the Stole-Zwiebel finding that wage paid must equal the worker's outside option in a model without matching friction.

Of course, equations (22) and (23) require that $s(w) = \delta$ and $h(w) = m(\theta)/\theta$ and

$$\widehat{w} = b + m(\theta)\left(\frac{w-\widehat{w}}{r+\delta}\right) = b + \left(\frac{\beta}{1-\beta}\right)c\theta$$

from equation (6). Hence, the solution to the system for the wage is

$$w = b + \left(\frac{\beta}{1-\beta}\right)(r+\delta+m(\theta))\frac{c\theta}{m(\theta)} \tag{28}$$

Finally, the steady state value of labor market tightness satisfies the employment identify

$$\frac{m(\theta)\ell}{\delta+m(\theta)} = (1-u)\,\ell = \int_0^1 n(w,p(x))dx. \tag{29}$$

Definition 1: A degenerate steady state equilibrium is a wage w and a market tightness parameter θ that satisfy (28), and (29) .

The following conditions are maintained:

Assumption 1: The job finding rate $m(\theta)$ is increasing and strictly concave function of θ and $m(0) = 0$, and $\lim_{\theta\to 0}\{\theta/m(\theta)\} = 0$.

Assumption 2: The baseline production function $f(n)$ is increasing, strictly concave thrice differentiable, and satisfies the Inada conditions, $\lim_{n\to 0} f'(n) = \infty$ and $\lim_{n\to\infty} f'(n) = 0$.

Proposition 1 *A unique degenerate steady state equilibrium with positive market tightness exists.*

Proof. Since (28) defines a continuous and strictly increasing functional relationship between w and θ, the right side of (29) can be represented as a continuous decreasing function of θ. As the LHS is continuos and increasing in θ, there is at most one solution for θ. Because the LHS is zero when $\theta = 0$, but the RHS is strictly positive, the equilibrium tightness is unique and strictly positive. ■

3.3 Equilibrium Wage Dispersion

Next, I turn to the principal focus of the paper, the implications of search-on-the job for wage dispersion in the canonical matching model. If wage are disperse and employed workers search ($\phi(w) = 1$) as a consequence, the reservation wage is $\widehat{w} = b$ by equation (6), the value of home production. Therefore, the free entry condition for vacancy creation (15) and equation (26) imply that the lowest wage paid in a market steady state is

$$\begin{aligned} \underline{w} &= b + \left(\frac{\beta}{1-\beta}\right)(r + \delta + m(\theta))\frac{c}{h(\underline{w})} \\ &= b + \left(\frac{\beta}{1-\beta}\right)(r + \delta + m(\theta))\frac{c\delta m(\theta)}{(\delta + m(\theta))\theta} \end{aligned} \tag{30}$$

where the second equality follows from (22). Equation (26) also implies that any equilibrium wage offer density must satisfy

$$F'(w) = \frac{h(w)\,(1-\beta)\,(w-b)/\beta - c(r + s(w))}{2cm(\theta)g(w,p)}, w \in (\underline{w}, \overline{w}). \tag{31}$$

where

$$h(w) = \frac{m(\theta)\delta}{\theta s(w)}, \tag{32}$$

and

$$s(w) = \delta + m(\theta)[1 - F(w)]. \tag{33}$$

Whether $g(w,p)$ is invariant with respect to p is important for our analysis. Indeed, in the special case in which $g(w,p)$ is independent of p, (31) is an ordinary differential equation that can be unique solved for the offer distribution for any specification of an initial condition, the value of $F(\underline{w})$. *This case is unique to the Cobb-Douglas production function specification.*

Lemma 2 (1) *The only baseline production functions in the set that satisfy Assumption 2 and $\partial g(w,p)/\partial p \equiv 0$ are members of the log linear family.*

Proof. From equation (9), the function $n(w,p)$ is the unique solution to

$$\frac{w - (1-\beta)\widehat{w}}{p} = \int_0^1 z^{\frac{1-\beta}{\beta}} f'(zn)dz.$$

By implication, $n(w,p)$ is a function of the LHS of this expression and so is any function of $n(w,p)$. Hence, $g_p(w,p) \equiv 0$ together with the definition (25) require

$$a\left(\frac{w-(1-\beta)\widehat{w}}{p}\right) = -n\int_0^1 z^{\frac{1}{\beta}} f''(zn)dz$$

for some constant $a > 0$. Given thee fact,

$$\begin{aligned} 0 &= \int_0^1 z[af'(zn) + z^{\frac{1}{\beta}} nf''(zn)]dz = 0 \\ &\Updownarrow \\ af'(n) &= -nf''(n) \forall n \in [0,1]. \end{aligned}$$

together with $f(0) = 0$ and $f'(n) > 0$ imply $f(n) = n^{1-a}/(1-a)$, $0 < a < 1$. Sufficiency is implied by equation (27). ■

When the effect of adding a worker to turnover costs, the denominator of (31), is not invariant to productivity, then the wage offer density depends on the productivity of any firm paying w. In this case, another differential equation is needed to fully describe an equilibrium wage offer distribution.

Let $\omega(x)$, $x \in [0,1]$, represent a function that assign a steady state wage rate to firm x and let $\digamma(x)$ denote the fraction of vacancies posted by those with productivity less than or equal to that the firm of productivity rank x. We seek a steady state equilibrium in which more productive employers pay more, at least weakly. Given $\omega'(x) \geq 0$, $\digamma(x) = F(\omega(x))$ and, consequently, $\digamma'(x) = F'(\omega(x))\omega'(x)$. Hence,

$$\omega'(x) = \frac{2cm(\theta)g(n(\omega(x),p(x)))\digamma'(x)}{h(\omega(x))\left(1-\beta)\right)(\omega(x)-b)/\beta-(r+s(\omega(x)))c}, x \in [0,1) \tag{34}$$

which provides one of the two differential equations for the system. The other is supplied by the employment identity

$$\begin{aligned} (1-u)\ell G(\omega(x)) &= \frac{m(\theta)\ell}{\delta+m(\theta)}\left(\frac{\delta\digamma(x)}{\delta+m(\theta)[1-\digamma(x)]}\right) \\ &= \int_0^x n(\omega(z),p(z))dz, , x \in [0,1] \end{aligned}$$

By differentiating both side and solving the result for $\digamma'(x)$, one obtains

$$\digamma'(x) = \left(\frac{(\delta+m(\theta))\left(\delta+m(\theta)[1-\digamma(x)]\right)^2}{\delta m(\theta)\ell}\right) n(\omega(x),p(x)), x \in (0,1). \tag{35}$$

Of course, in addition

$$\frac{m(\theta)\ell}{\delta + m(\theta)} = \int_0^1 n(\omega(z), p(z))dz. \tag{36}$$

3.4 Existence and Uniqueness

Definition 3 *A monotone increasing steady state dispersed wage equilibrium is an increasing wage assignment function* $\omega : [0,1] \to [\omega(0), \omega(1)]$ *, a vacancy c.d.f.* $F : [0,1] \to [0,1]$*, and a market tightness parameter* θ *that satisfy equations (34), (35), and (36).*

The following fact rules out multiple disperse wage equilibria except in the case of a Cobb-Douglas production function.

Lemma 4 (2) *If employed worker search and* $\partial g(w,p)/\partial p \neq 0$*, then firms that pay the same wage* w *must have the same productivity.*

Proof. Because the free entry condition, equation (26), can be written as

$$c = \frac{h(w)(w-b)}{r + s(w) + 2m(\theta)g(w,p)F'(w)},$$

$\omega(x_1) = \omega(x_2) = w$ implies $g(w, p(x_2)) - g(w, p(x_1)) = 0$ for all x_1 and x_2. ■

For a given value of the tightness parameter θ, one can use the phase diagram implied by the ODE system to characterize the set of solutions to the equations that could serve as possible wage offer distributions when $p(x)$ is continuous. Although the phase space generally has three dimensions with each point representing a particular (F, ω, x) combination, a slice of the space at any given value of x has the qualitative properties illustrated in Figure 1. The curve in the space with end points labeled w_0 and w_1 is the locus along which the denominator of the expression on the right side of equation (34) is zero. In other words, the curve is defined by

$$\left(\frac{m(\theta)\delta}{\theta(\delta + m(\theta)[1 - F(x)])}\right)(1-\beta)(\omega - b)/\beta - (r + \delta + m(\theta)[1 - F(x)])c = 0 \tag{37}$$

where w_0 is the solution for ω when $F(0) = 0$ and w_1 is the solution when $F(1) = 1$. Because the curve is independent of x, it is in the same position in Figure 1 for all x.

The curves in the phase diagram represented by arrows are the solution trajectories of the system where the points of each indicates the "direction of motion" as x increases. Note that there are two solution trajectories that initiate from any point on the curve defined by (37). Local multiplicity arises because $\omega'(0)$ converges to infinity as any such point is approached. In other words, the standard sufficient condition for a unique local solution to the ODE system, Lipschitz continuity, does not hold. However, because the RHS of (35) is always strictly positive, all solution trajectory that initiate above and to the right of the curve are unique and tend toward the north east in the diagram while only those to the left of it move northwest. Obviously, only the former can represent a possible offer distribution density function since $F'(w) = \mathcal{F}'(x)/\omega'(x) > 0$ implies that $\mathcal{F}'(x) > 0$ must hold. Finally, for each candidate, the upper support of the distribution, $\overline{w}$, is the value of ω at the point where the trajectory intersects the line $\mathcal{F} = 1$.

Still, we need two initial conditions to determine a unique particular solution. Of course, equation (30) provides one; namely, $\omega(0) = \underline{w} = w_0$ from (30) and (37) given the equilibrium value of θ. As an implication of Lemma (2), one can generally infer the second boundary condition. In the absence of a mass point in the distribution of productivity at it's lower support, equivalently $p(x)$ is strictly increasing at $x = 0$, then $F(\underline{w}) = 0$. In other words, the candidate solution for the wage offer distribution function in this case is the trajectory that originates at the point $(\omega(0), \mathcal{F}(0)) = (w_0, 0)$. An equilibrium value of the market tightness solves equation (36) when its RHS is evaluated using the offer distribution candidate uniquely associated with the solution value of θ. Obviously, equation (30) implies that the lowest wage of the candidate distribution increases with market tightness. Since $s(w)$ increases and $h(w)$ decreases with θ from (32) and (33), equation (31) and $F'(w) = \mathcal{F}'(\omega(x))/\omega'(x) > 0$ imply that the slopes of the all the solution trajectories fall continuous with θ. As a consequence, the wage distribution is stochastically increasing in θ which implies that the RHS of (36) is positive, continuous and decreasing function of θ.

To sum up, the following results holds.

Proposition 5 *If the productivity distribution is atomless ($p(x)$ is continuous and strictly increasing), a montone steady state equilibrium exists. Except in the case of the log linear family of production functions, there is only one and it is atomless.*

Corollary 6 *If the production function is Cobb-Douglas, then a continuum* *of equilibria exists even if all firms are equally productive.*

3.5 The Cobb-Douglas Case

The fact that a continuum of equilibria exist in the Cobb-Douglas case is somewhat disturbing, since that specification is particularly convenient in empirical work. Of course, no actual production function is exactly log linear. Hence, we can regard the disperse wage equilibrium of interest as that associated with the requirement that there is no mass point at the lowest wage, at least when there is no mass point of firms with the lowest productivity.

Moreover, the assumption that the cost of posting vacancies is a constant if also a simplification. Since adjustment to steady states is not instantaneous in fact, a convex cost of vacancy posting is a more realistic specification assumption. In this case, the increasing marginal cost function $c'(v)$ would replace c in all the equations above. It is a straight foreword but time consuming to show that all the other equations of the model as well as the arguments made hold in this generalization, at least in steady state. Furthermore, multiplicity also vanishes in the Cobb-Douglas case.

Proposition 7 *If the cost of posting vacancies is increasing and strictly convex, then employers pay the same wage if and only if they are of the same type.*

Proof. In this specification, the free entry can be written as

$$c'\left(\frac{s(w)n(w,p)}{h(w)}\right) = \frac{h(w)(w-b)}{r+s(w)+2m(\theta)g(w,p)F'(w)}, \tag{38}$$

in any steady state. As $c''(\cdot) > 0$ and $n_p > 0$ both hold, $\omega(x_1) = \omega(x_2)$ if and only if $p(x_1) = p(x_2)$. ■

4 Comparing the Equilibria

As $h(w) < m(\theta)/\theta$ from (32) when employed workers search, the lower support of the wage distribution in the disperse equilibrium is greater than the equilibrium single wage when evaluated at the same level of market tightness by equations (28) and (30). Workers receive more rent when they search while employed because the cost of replacing the marginal worker is higher given the same arrival rates. Because higher wages lower the incentive to post vacancies, the level of market tightness will be lower and unemployment will be higher in the disperse wage equilibrium. Indeed, an application of the argument used to prove existence and uniqueness of the dispersed wage equilibrium implies that the following relationship between wages and employment in the two equilibria.

Proposition 8 *Market tightness is lower in a dispersed wage equilibrium than in the single wage equilibrium and the equilibrium single wage is less than the upper support of the equilibrium wage offer distribution.*

Proof. Suppose that employed worker search and for the moment consider the case in which value of market tightness in the degenerate equilibrium; call it θ^*. When $\theta = \theta^*$, In this case, $w_0 > w^*$ in Figure 1 which implies that all the wages in the support of the candidate distribution exceed w^*. As a consequence, the LHS of (36) is strictly greater than the RHS when $\theta = \theta^*$ which in turn implies that the unique value of θ in the disperse wage equilibrium is strictly less than θ^*. Obviously, an analogous argument implies that the value of θ for which the upper support of the wage distribution is equal to w^* is such that the LHS of (36) is strictly greater than the RHS. Hence, the support of the equilibrium wage distribution includes the single wage equilibrium in its interior. ■

Employment is lower in the disperse wage equilibrium because employed workers have an incentive to search for a higher paying job. As a consequence of search on-the-job, the separation rate is higher than it would be for all but the highest paying firm and, consequently, turnover cost per worker are higher. In addition, the act of hiring another worker reduces the wage paid which adversely affect the net change in employment attributable to posting a vacancy. Both effects reduced the incentive to hire which results in lower employment in steady state.

Does lower employment imply less welfare in the dispersed wage equilibrium? Not necessarily. As Stole and Zwiebel (1996) point out in a particular equilibrium setting, their bilateral intra-firm bargaining solution provides an incentive for employers to "over employ" as a means of driving down wage costs. Cahuc, Marque, and Wasmer (2007) suggest that this conclusion continues to hold in search equilibrium when firms are composed of many workers and firms are equally productive. Below I extend the inefficiency result to include the case of heterogenous firms. Hence, wage dispersion and the additional search it induces off sets to some extent the incentive to over employ.

Suppose that the planner chooses vacancies for every firm to maximize the expected present value of aggregate income including home production. As is well known, search externalities generally exist in a matching model. However, as both Pissarides (1984) and Hosios (1990) have shown, these are internalized in the canonical search and matching model if the employers' share of match rent is equal to the elasticity of the matching function with respect to vacancies. In this section, I demonstrate that employment in the single wage equilibrium exceeds the solution to the planner's problem when the Hosios condition holds and production exhibits diminishing returns. In addition to excessive employment, workers are generally misallocated across firms as well when firm productivities differ.

The law of motion for employment in firm x is

$$\dot{n}(x) = v(x)q(\theta) - \delta n(x), x \in [0,1]$$

where $v(x)$ represents vacancies posted and θ is market tightness, the ratio of aggregate vacancies to the number of unemployed workers. Letting Ψ represent the maximal expected value of future aggregate income, the continuous time Bellman equation is

$$r\Psi = \max_{v(x)} \left\{ \begin{array}{c} \int_0^1 p(x)f(n(x))dx \\ +b\left(1-\int_0^1 n(x)dx\right) - c\int_0^1 v(x)dx \\ +\lambda\left[\theta\left(1-\int_0^1 n(x)dx\right) - \int_0^1 v(x)dx\right] \\ +\int_0^1 \Psi_{n(x)}\left[v(x)q\left(\theta\right) - \delta n(x)\right]dx \end{array} \right\} \tag{39}$$

where Ψ is the value of the optimal program, $\Psi_{n(x)}$ is its partial derivative with respect to $n(x)$, and λ is the shadow price of market tightness. The FONCs for interior vacancy choices are for every $x \in [0,1]$

$$\begin{aligned} \Psi_{n(x)}q(\theta) - \lambda - c &= 0 \\ \lambda\left(1-\int_0^1 n(x)dx\right) + \int_0^1 \Psi_{n(x)}v(x)q'\left(\theta\right)dx &= 0 \\ \theta\left(1-\int_0^1 n(x)dx\right) - \int_0^1 v(x)dx &= 0 \end{aligned}$$

where

$$(r+\delta)\,\Psi_{n(x)} = p(x)f'(n(x)) - b - \lambda\theta + \dot{\Psi}_{n(x)}$$

by the envelope theorem.

In steady state, the marginal value of a worker is the same across firms. Letting $\Psi_{n(x)} = \Psi_n$ for all x, market tightness satisfies

$$\Psi_n\left[q(\theta) + \theta q'(\theta)\right] - c = 0 \tag{40}$$

where the common marginal value of a worker solves

$$(r+\delta)\,\Psi_n = p(x)f'(n(x)) - b - \left(\frac{\theta q'(\theta)}{q(\theta)+\theta q'(\theta)}\right)c\theta, x \in [0,1] \tag{41}$$

and total employment satisfies

$$\frac{m(\theta)}{\delta + m(\theta)} = \frac{\theta q(\theta)\ell}{\delta + \theta q(\theta)} = \int_0^1 n(w, p(x))dx. \tag{42}$$

However, in a degenerate wage market equilibrium, equations (8) and (28) require

$$\beta\left(p(x)f'(n(x)) + g(p(x), w) - w\right) = (1-\beta)\left(w - b - \frac{c\beta\theta}{1-\beta}\right)$$

where

$$pf'(n(w,p(x))) + g(w,p(x)) - w = \frac{(r+\delta)c}{q(\theta)}$$

which together imply

$$w = b + \frac{\beta}{1-\beta}\left[\frac{(r+\delta)\,c}{q(\theta)} + c\theta\right]$$

In combination, these equations imply

$$p(x)f'(n(x))) + g(w,p(x)) - b - \frac{\beta}{1-\beta}c\theta = \frac{(r+\delta)c}{(1-\beta)q(\theta)}, x \in [0,1] \tag{43}$$

while equation (40) and (41) require

$$p(x)f'(n(x)) - b - \left(\frac{\theta q'(\theta)}{q(\theta)+\theta q'(\theta)}\right)c\theta = \left(\frac{q(\theta)}{q(\theta)+\theta q'(\theta)}\right)\frac{(r+\delta)c}{q(\theta)}, x \in [0,1] \tag{44}$$

Given the Hosios condition

$$1 - \beta = 1 + \frac{\theta q'(\theta)}{q(\theta)} = \frac{\theta m'(\theta)}{m(\theta)}, \tag{45}$$

intra-firm bargaining generates two distortions. First, because the marginal contribution to profit, $p(x)f'(n(x)) + g(w, p(x))$, is equalized across firms in the degenerate wage equilibrium rather than the marginal products, the allocation of workers across firms is not output maximizing except in the Cobb-Douglas production function case where $g(w, p(x))$ is independent of $p(x)$. Second, because $g(w, p(x)) > 0$, employment in every firm is too high given tightness.

Proposition 9 *If the Hosios condition holds, then the value of market tightness in the single wage equilibrium exceeds that in the planner's solution.*

Proof. By comparison of (43) and (44), the solution for $n(x)$ is larger to the first equation is larger than that of the second for all w and $p(x)$ given that $g(w, p(x)) > 0$ given the same value of θ. Hence, if the value of θ is chosen to be the solution to the planner's problem, the RHS of (42) exceeds the left in the single wage equilibrium. Hence, the equilibrium value of θ, that which equates to side, must be larger. ■

Although I did not allow for search on-the-job in formulating the social planner's problem, this restriction is not binding when the vacancy posting cost is linear. In steady state at least, the planner has no incentive to reallocated workers across firms because total output is already maximized given aggregate employment. However, because employment is too high in the single wage equilibrium and the allocation of employment is sub-optimal except in the case of a Cobb-Douglas production function, it is possible that the dispersed equilibrium with search-on-the job yields higher welfare than the single wage equilibrium.

5 Concluding Remarks

The purpose of this paper is to show that a dispersed wage steady state equilibrium exists in a version of the search and matching model in which firms have many employees, face diminishing returns in production, and wages are the outcome of intra-firm bargaining as modeled by Stole and Zwiebel (1996).

Helpman and Iskhoki (2008) establish that a unique single wage equilibrium exists in this environment. In this paper, I prove that a generically unique dispersed wage equilibrium also exists with the property that more productive firms pay higher wages if productivity differs across firms and employed workers are allowed to search. Finally, employment is lower in the dispersed wage equilibrium but welfare need not be because employment exceeds the social optimum in the single wage equilibrium. Excessive employment arises because employing another worker reduces the wage paid to all employees when the wage is the outcome of the Stole-Zwibel bilateral bargaining game. Employment is higher in the dispersed wage equilibrium because search by employed workers labor turnover and wage competition between firms and effects reduce the incentive to create jobs.

The obvious unanswered question has to do with stability of equilibrium. Will the market converge to the single wage or to the disperse wage steady state equilibrium? When the market is not in steady state, the nature of the wage equation implies that wages will differ and, consequently, employed worker will search. Is this fact sufficient to guarantee continued search in the transition and hence convergence of the market distribution to the non-degenerate steady state? Answering this question is a subject for future theoretical research.

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