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Maximum Likelihood in the Frequency Domain: A Time To Build Example

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Abstract

A well known result is that the Gaussian log-likelihood can be expressed as the sum over different frequency components. This implies that the likelihood ratio statistic has a similar linear decomposition. We exploit these observations to devise diagnostic methods that are useful for interpreting maximum likelihood parameter estimates and likelihood ratio tests. We apply the methods to the estimation and testing of two real business cycle models. The standard real business cycle model is rejected in favor of an alternative in which capital investment requires a planning period.

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1 Introduction

Full information econometric methods in empirical macroeconomics flourished in the early 1980s, stimulated in large part by the work of Hansen and Sargent (1980). Subsequently, limited information methods became more popular. A prominent example of such methods is the calibration methodology advocated by Kydland and Prescott (1982).[1] This shift in interest reflected increased concern with the notion that, since models are abstractions, any model is necessarily misspecified on some dimensions. A key perceived shortcoming of full information methods is that these specification errors have unpredictable and hard to diagnose implications for the parameter values and for model fit.[2] More recently, it has been emphasized that limited information methods have their own problems. For example, their small sample properties may be poor compared with those of full information methods.[3] Considerations such as these have helped to renew interest in full information methods in empirical macroeconomics.[4]

Our objective is to draw attention to the potential value of the frequency domain for diagnosing estimation and testing results based on full information, Gaussian maximum likelihood methods.[5] In the process of illustrating these methods, we provide evidence in favor of a particular class of business cycle models.

We propose a set of tools for evaluating the impact on parameter estimation and model fit of different frequency components of the data. We exploit the well known fact that the log, Gaussian density function has a linear decomposition in the frequency domain. This decomposition has two implications. First, the likelihood ratio statistic for testing a model can be represented as the sum of likelihood ratios in the frequency domain. As a result, if a model is rejected because of a large likelihood ratio statistic, then it is possible to determine,

[1] Other, related methods include those based on Hansen and Singleton's (1982) generalized method of moments (GMM) framework. These include the exactly identified GMM methodology of Christiano and Eichenbaum (1992), and the overidentified GMM known as simulated method of moments (Duffie and Singleton 1993). In addition, there are the diagnostic methods proposed by Watson (1993) and Diebold, Ohanian and Berkowitz (1998).

[2] A recent example by Hansen and Sargent (1993) illustrates the principle. They show how misspecification of the seasonal component of a model can, using maximum likelihood methods, lead to distortions in the estimated values of all model parameters.

[3] The 1996 issue of the Journal of Business and Economic Statistics reports evidence on the small sample properties of limited information methods based on generalized method of moment estimators. For a particular empirical application, Fuhrer, Moore, and Schuh (1995) made the case that the small sample problems are so severe that maximum likelihood performs better than limited information methods, even in the presence of plausible forms of specification error. Cogley (1998), however, displays an example in which GMM performs better than maximum likelihood when the technology shock is misspecified.

[4] Recent examples include Altug (1989); Christiano (1988); Christiano, Eichenbaum, and Marshal (1991); McGratten (1994); Hall (1996); Ireland (1997); Kim (1998); Leeper and Sims (1994); and McGrattan, Rogerson, and Wright (1997).

[5] Another paper which does this is Altug (1989). Her methods complement ours.

arithmetically, which frequencies of the data are responsible for the poor model fit. Second, if parameter estimates look 'strange', then it is possible to determine which frequencies are responsible.

We illustrate the method by estimating and testing simple real business cycle models using data on aggregate, quarterly, US output growth. We start with a standard real business cycle model, in which the technology shock is a geometric random walk. We first work with a version of the model in which the only free parameter is the variance of the technology shock. All other parameters are fixed at the estimated values reported in Christiano and Eichenbaum (1992). The likelihood ratio statistic, testing this model against an unrestricted alternative, rejects the model. When we examine the likelihood ratio statistic in frequency domain, the reason for the rejection is clear. The model fit is very poor in two frequency bands of the data: those corresponding to periods of oscillation in the range of 2.5 - 8 years and those corresponding to periods of oscillation in the range of 7 - 7.5 months. When we free up some of the other model parameters, our Gaussian estimation criterion drives them into regions that cause the model to conform better to the data over all frequency bands. However, the estimated parameter values appear implausible on other grounds. Overall, our results are consistent with the findings reported in Christiano (1988, p. 274), Cogley and Nason (1995), and Watson (1993). The poor fit in the 2.5 - 8 year range reflects the difficulty the standard real business cycle model has in generating output persistence.

We next consider a version of the real business cycle model where capital investment requires four periods to build. We estimate the fraction of overall resources that must be put into place in each of the first, second, third, and fourth periods of construction. The parameter estimates are plausible from the perspective of microeconomic studies of investment projects. They imply that the amount of resources allocated in the early part of an investment project is relatively small. For reasons explained in Christiano and Todd (1996), incorporating this feature of investment projects into the time to build model allows that model to generate persistence in output growth. This in turn helps the model to match the 2.5 - 8 year component of the data. In addition, the estimated model also does well in matching the 7 ¡ 7.5 month component of the data. As a result, our time to build model is not rejected by the data.

We now consider the relationship of our paper to the existing literature. Several other papers exploit the fact that the Gaussian density function can be decomposed in the frequency domain. For example, Altug (1989) demonstrates its value for estimating models with measurement error. Other papers emphasize its value in the estimation of time-aggregated models.[6] Christiano and Eichenbaum (1987) and Hansen and Sargent (1993) exploit the de-

[6] See, for example, Hansen and Sargent (1980a), Christiano (1985), Christiano and Eichenbaum (1987)

composition to evaluate the consequences for maximum likelihood estimates of certain types of model specification error.[7]

The value of comparing model and data spectra has also been emphasized in the recent contributions of Watson (1993) and Diebold, Ohanian and Berkowitz (1998). Watson (1993)'s objective is to provide descriptive tools only, and so his approach is not designed for conducting statistical inference. Ours is, since our methods are simply designed to help interpret the results of standard statistical estimation and testing procedures.

Our approach is most closely related to that of Diebold, Ohanian and Berkowitz (1998). They also do estimation using the frequency domain decomposition of the Gaussian density function. Their paper differs from ours in three respects. First, they use the frequency domain decomposition as a convenient way to exclude frequency bands from the analysis. We incorporate all frequency bands into our analysis, and use the frequency domain decomposition as a device for gaining insight into the results of analysis based on all frequencies. Second, their approach to testing is different from ours. We focus on the likelihood ratio statistic and the value of the frequency domain for diagnosing its magnitude. Third, the application we use to illustrate the method differs from theirs.

The following section presents our econometric framework. Section 3 presents the results. Section 4 concludes.

2 Econometric Framework

This section describes the econometric framework of our analysis. First, we display the frequency domain decomposition of the Gaussian density function. Second, we derive the log-likelihood function of the unrestricted representation of the data. Third, we display the likelihood of the representation restricted by the various real business cycle models that we consider. Finally, we display the linear, frequency domain decomposition of the likelihood ratio statistic.

2.1 Spectral Decomposition of the Gaussian Likelihood

The logarithm of the Gaussian density function for a T dimensional vector of observations, $y_1, ..., y_T$, is:

$$L(y) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\log|V| - \frac{1}{2}y'V^{-1}y$$

and Christiano, Eichenbaum and Marshall (1991).

[7] These approaches to specification error analysis are similar in spirit to the early approach taken in Sims (1972).

where V is the T by T covariance matrix of $y = [y_1, ..., y_T]'$. It is well known (Harvey, 1989, p. 193) that for T large, this expression is, approximately,

$$L(y) = -\frac{1}{2}\sum_{j=0}^{T-1}\left[2\log 2\pi + \log f(\omega_j) + \frac{I(\omega_j)}{f(\omega_j)}\right] \tag{1}$$

where $I(\omega)$ is the periodogram of the data:

$$I(\omega) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} y_t \exp(-i\omega t)\right|^2 \tag{2}$$

and

$$\omega_j = \frac{2\pi j}{T},\ j = 0, 1, ..., T-1.$$

Finally, $f(\omega)$ is the spectral density of y at frequency ω implied by V for large T.[8]

We find it convenient, for later purposes, to express the likelihood function as a weighted likelihood, as in Diebold, Ohanian and Berkowitz (1998):

$$L(y) = -\frac{1}{2}\sum_{j=0}^{T-1} w_j\left[2\log 2\pi + \log f(\omega_j) + \frac{I(\omega_j)}{f(\omega_j)}\right]. \tag{3}$$

In our analysis, we will consider $w_j \in \{0, 1\}$.

2.2 Likelihood Function for The Structural Model

This subsection derives the restricted reduced form representation for output growth implied by two structural models, and their associated log likelihood functions.

2.2.1 Real Business Cycle Model

The representative agent in our model has preferences, $E_0 \sum_{t=0}^{\infty} \beta^t [\log(C_t) + \psi \log(1 - n_t)]$, where C_t denotes consumption and n_t denotes hours worked. The time endowment is normalized to unity and the parameters β and ψ satisfy $0 < \beta < 1$, $\psi > 0$. The resource constraint is $C_t + I_t \leq Y_t$, where

$$Y_t = K_t^{\theta} (z_t n_t)^{(1-\theta)},\ 0 < \theta < 1,$$

with a technology shock z_t

[8] Let V_{lj} denote the j^{th} element of the l^{th} row of V. Then,

$$f(\omega) = V_{ll} + 2\sum_{j=l+1}^{\infty} V_{lj}\cos(\omega(j-l)),$$

for any l.

$$\log(z_t) = \log(z_{t-1}) + \eta_t$$

where η_t is i.i.d. Normal with mean μ and variance σ^2. In the real business cycle (RBC) version of this model,

$$K_{t+1} - (1 - \delta)K_t = I_t, \; 0 < \delta < 1.$$

We denote the unknown parameter values of the RBC model by the vector Φ. In the next section's estimation exercise, we consider two cases. In one, $\theta = 0.344$, $\psi = 3.92$, $\delta = 0.021, \beta = 1.03^{-0.25}$ and $\Phi = \sigma_\eta$. In the other, $\psi = 3.92$, $\beta = 1.03^{-0.25}$ and $\Phi = (\sigma_\eta, \delta, \theta)$. These choices are made to enhance the illustrative value of the application studied in Section Three.

2.2.2 Time to Build Model

The time to build model differs from the RBC model only in the investment technology. Period t investment is:

$$I_t = \phi_1 x_t + \phi_2 x_{t-1} + \phi_3 x_{t-2} + \phi_4 x_{t-3},$$

where $\phi_i \geq 0$ for $i = 1, 2, 3, 4$, and

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 \leq 1.$$

The investment technology requires that if x_t units of net investment are to occur during period $t + 3$, i.e.,

$$K_{t+4} - (1 - \delta)K_{t+3} = x_t,$$

then, resources in the amount $\phi_1 x_t$ must be applied in period t, $\phi_2 x_t$ must be applied in period $t+1$, $\phi_3 x_t$ must be applied in period $t+2$, and finally $\phi_4 x_t$ must be applied in period $t + 3$. Once initiated, an investment project's scale cannot be expanded or contracted. As in the RBC model, Φ denotes the vector of parameters to be estimated. In our analysis, $\Phi = (\sigma_\eta, \phi_1, \phi_2, \phi_3)$.

2.2.3 Reduced Form Representation and Likelihood Function

We used the undetermined coefficient method described in Christiano (1998) to approximate the policy rules for employment and capital that solve the planning problem associated with the above two model economies. We manipulated these approximate policy rules to obtain a reduced form representation for $y_t = \log(Y_t/Y_{t-1})$:

$$y_t = \alpha(L; \Phi)\eta_t = \alpha_0(\Phi)\eta_t + \alpha_1(\Phi)\eta_{t-1} + \alpha_2(\Phi)\eta_{t-2} + ... \tag{4}$$

This representation is a restricted $ARMA(4, 8)$ model[9]. That is, $\alpha(L; \Phi)$ is the ratio of an 8^{th} order polynomial in the lag operator, L, and a 4^{th} order polynomial in L. We restrict Φ so that

$$\sum_{i=0}^{\infty} \alpha_i(\Phi)^2 < \infty\ ,$$

guaranteeing that the spectral density of y_t exists. We also restrict Φ so that $\alpha(z; \Phi) = 0$ implies $|z| \geq 1$.

The spectral density of y_t at frequency ω is

$$f^r(\omega; \Phi) = \frac{\sigma_\eta^2}{2\pi} \alpha(e^{-i\omega}; \Phi)\alpha(e^{i\omega}; \Phi),$$

where the superscript, r, indicates the restricted model for y_t. The frequency domain approximation to the restricted likelihood function is (1) with $f(\omega)$ replaced by $f^r(\omega; \Phi)$.

2.3 Unrestricted Reduced Form Likelihood

In order to test our model, we need to estimate an unrestricted version of (4):

$$y_t = \alpha^u(L)\varepsilon_t, \tag{5}$$

where

$$\alpha^u(L) = 1 + \alpha_1^u L + \alpha_2^u L^2 + \ldots.$$

Also,

$$\sum_{i=0}^{\infty} (\alpha_i^u)^2 < \infty\ ,$$

and $\alpha^u(z) = 0$ implies $|z| \geq 1$. These correspond to the analogous restrictions imposed on the restricted reduced form. The polynomial in L, $\alpha^u(L)$, is the ratio of an 8^{th} order polynomial and a 4^{th} order polynomial, with constant terms normalized to unity. This specification nests the real business cycle model and the time to build model. It has 13 free parameters: the 12 parameters of $\alpha^u(L)$, and σ_ε. We denote these by the 13 dimensional vector, γ. Let $f^u(\omega; \gamma)$ denote the spectral density of y_t:

$$f^u(\omega; \gamma) = \frac{\alpha^u(e^{-i\omega})\alpha^u(e^{i\omega})}{2\pi} \sigma_\varepsilon^2,$$

The frequency domain approximation to the unrestricted likelihood function is (1) with $f(\omega)$ replaced by $f^u(\omega; \gamma)$.

[9] The appendix presents the derivation of this $ARMA$ representation.

2.4 Cumulative Likelihood Ratio

The likelihood ratio statistic is

$$\lambda = 2(L^u - L^r),$$

where L^r and L^u are the maximized values of the restricted and unrestricted log likelihoods, respectively. Under the null hypothesis that the restricted model is true, this statistic has a chi-square distribution with degrees of freedom equal to the difference between the number of parameters in the restricted and unrestricted models (Harvey, 1989, p. 235). Define

$$\lambda(\omega) = \log \frac{f^r(\omega; \widehat{\Phi})}{f^u(\omega; \widehat{\gamma})} + I(\omega) \left[\frac{1}{f^r(\omega; \widehat{\Phi})} - \frac{1}{f^u(\omega; \widehat{\gamma})} \right], \tag{6}$$

where a hat over a variable indicates its estimated value. Then, it is easily confirmed that,

$$\lambda = \sum_{j=0}^{T-1} \lambda(\omega_j).$$

This expression can be simplified because of the symmetry properties of $\lambda(\omega)$:

$$\lambda(\omega_{\frac{T}{2}-l}) = \lambda(\omega_{\frac{T}{2}+l}),\ l = 1, 2, ..., \frac{T}{2} - 1.$$

These imply that λ can be written:

$$\lambda = \lambda(0) + 2 \sum_{j=1}^{\frac{T}{2}-1} \lambda(\omega_j) + \lambda(\pi). \tag{7}$$

This is our linear, frequency domain decomposition of the likelihood ratio statistic.

If λ is large, then we should be able to determine which ω_j's are responsible for this. To assist in this, we define the cumulative likelihood ratio:

$$\begin{aligned} \Lambda(\omega) &= \lambda(0) + 2 \sum_{\omega_j \leq \omega} \lambda(\omega_j),\ 0 < \omega < \pi \\ \Lambda(0) &= \lambda(0), \\ \Lambda(\pi) &= \lambda. \end{aligned} \tag{8}$$

A sharp increase in $\Lambda(\omega)$ in some region of ω's signals a frequency band where the model fits poorly.

3 Results

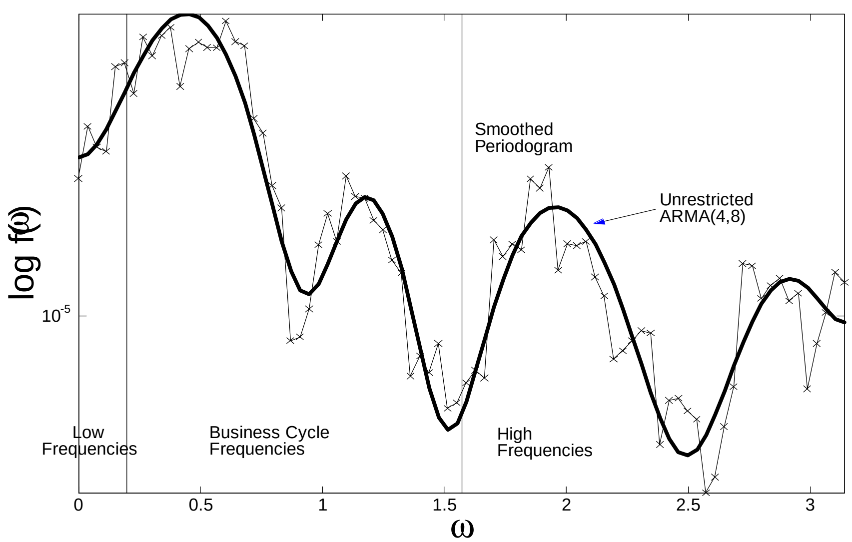
This section presents our results for estimating and testing the RBC and time to build models. The periodogram of the data, (2), and the spectral density of the unrestricted reduced form are important ingredients in the analysis, and so we begin by presenting these. The following two subsections present the analysis of the RBC and the time to build models, respectively.

3.1 Periodogram and Spectrum of Unrestricted Reduced Form

Figure 1 presents a smoothed version of $I(\omega)$ for $\omega \in (0, 2\pi)$, based on (2).[10] The thick solid line in Figure 1 is the spectrum of our unrestricted $ARMA(4,8)$ representation of US GDP growth. Note how similar these are. This is to be expected, since both represent consistent estimates of the spectrum of the data.

Vertical bars draw attention to three frequency bands, the low frequencies (those corresponding to period 8 years to infinity), the business cycle frequencies (period 1 year to 8 years) and the high frequencies (period 2 quarters to 1 year). Note that the low and business cycle frequencies have high power. In addition, the spectrum has pronounced dips in the $7 - 7.5$ months (near $\omega = 2.5$) range and in the higher frequency component of the business cycle (near $\omega = 1.5$).

Figure 1: Estimated Spectral Density



3.2 Estimation and Testing of RBC Model

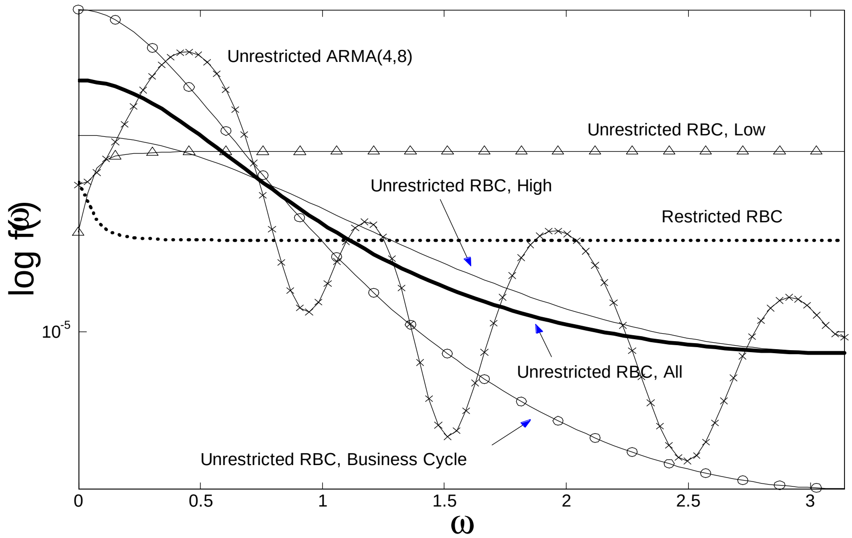
We begin by estimating the version of the RBC model in which only the innovation variance of the technology shock, σ_η, is free. We call this the 'restricted RBC' model. We then turn to the version (the 'unrestricted RBC' model) in which δ and θ are also free.

The spectrum of the estimated restricted RBC model is displayed in Figure 2. For convenience, Figure 2 reproduces the spectrum of the unrestricted $ARMA(4,8)$ representation of the data from Figure 1. As emphasized in Watson (1993), the spectrum of the RBC model

[10] The data are seasonally adjusted, cover the period 1955Q3 to 1997Q1, and are from the Citibase database. The sample mean of y_t is subtracted from the data, so that $I(0)$ is zero. We present the smoothed version of the periodogram because, as is well known, the unsmoothed periodogram is quite volatile. The smoothed periodogram at frequency ω_j is a centered, equally weighted average, $\sum_{i=-3}^{3} I(\omega_{j+i})/7$.

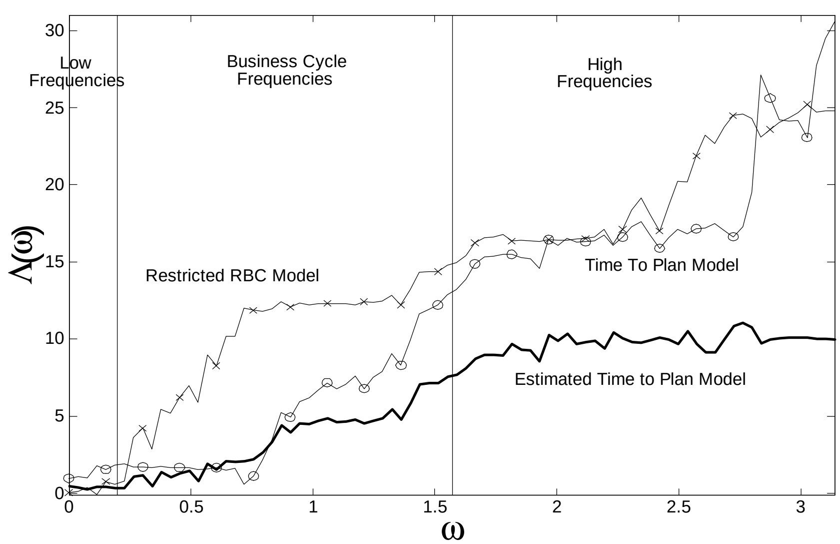
is essentially flat. To a first approximation, the model implies that aggregate output inherits the persistence properties of the technology shock, which is a random walk by assumption.

Figure 2 Spectra Relevant to the Analysis of the RBC Model



For a formal evaluation of model fit, consider Figure 3 which displays the cumulative likelihood ratio, (8). Note that λ is just under 25 (see the cumulative likelihood ratio for $\omega = \pi$). Under the null hypothesis that the restricted RBC model is true, λ is the realization of a chi-square statistic with 12 degrees of freedom. The statistic has a p-value of 1.5 percent and hence the model is rejected at the five percent significance level. To see why the model is rejected, note that the cumulative likelihood ratio displays sharp increases in the low frequency component of the business cycle, and in the frequencies corresponding to periods 7-7.5 months.

Figure 3: Cumulative Likelihood Ratio



We now turn to the unrestricted RBC model. The estimated parameter values are $\hat{\theta} = 0.37$ and $\hat{\delta} = 0.73$. Although the estimated value of capital's share is reasonable, the estimated value of δ is much larger than seems plausible in light of data on investment and the stock of capital (Christiano and Eichenbaum,1992). To see what frequency component of the data drives this result, we recomputed $\hat{\theta}$, $\hat{\delta}$ several times using alternative weights in the weighted likelihood function, (3). The estimation results are displayed in Table 1 and Figure 2.

Table 1: Weighted Likelihood Estimation Results

Frequencies	θ	δ	σ_η	λ	λ_w	Observations Used
High	0.25	0.99	0.0126	9.6	3.7	50%
Business Cycle	0.51	0.99	0.0170	26.1	2.3	43%
Low	0.15	0	0.0100	37.3	-0.2	7%
All	0.37	0.73	0.0144	8.5	8.5	100%

Notes: These are the results of estimating the unrestricted RBC model by weighted maximum likelihood (i.e., by maximizing (3)). Low frequencies: w_j equals 1 only for w_j's that belong to frequencies corresponding to periods 8 years and up. Business cycle frequencies: w_j equals 1 only for w_j's that belong to frequencies corresponding to periods 1 to 8 years. High frequencies: w_j equals 1 only for w_j's that belong to frequencies corresponding to periods 2 quarters to 1 year; All frequencies: w_j equals 1 for all j. Percent of observations used: fraction of $j \in \{0, 1, ..., T-1\}$ equal to unity in the weighted likelihood estimation. λ: likelihood ratio statistic. λ_w: likelihood ratio statistic based only on the subinterval for the weighted likelihood function.

The business cycle and high frequency components of the data drive δ to nearly unity. With δ near one, the model reduces to the scalar version of the model in Long and Plosser (1983), in which output growth is a first order autoregression with autoregressive parameter θ. With the spectrum of this process, proportional to $1/(1+\theta^2 - 2\theta\cos(\omega))$, the model is able to match the shape of the data spectrum in the business cycle and high frequencies (see 'Unrestricted RBC, Business Cycle' and 'Unrestricted RBC, High' in Figure 2). However, different values of θ work better in the two frequency ranges.

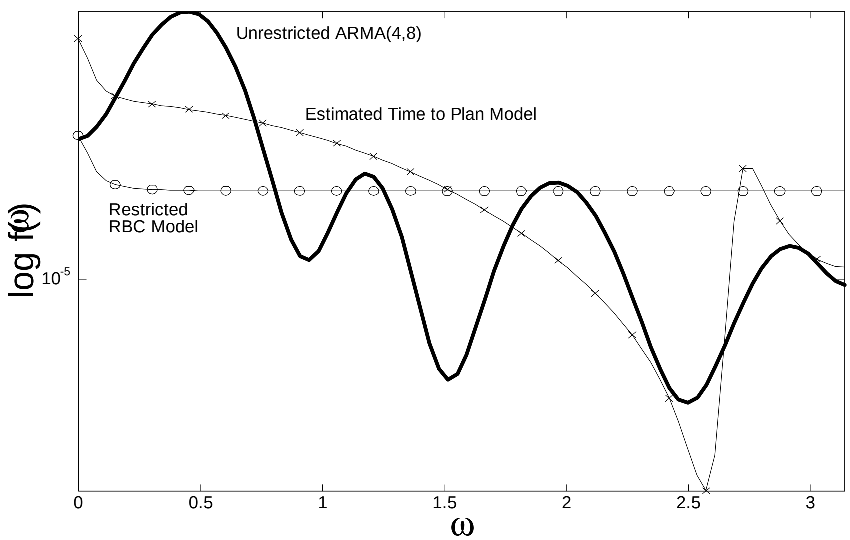
To match the low frequencies, a very different parameterization is needed, with δ nearly 0 and θ small (see 'Unrestricted RBC, Low' in Figure 2). The parameter estimates based on all frequencies are roughly an average of the results over the various frequencies.

3.3 Time To Build Model

Results for estimating the time to build model are displayed in Figure 4. For convenience, Figure 4 displays the spectrum of the restricted RBC model, and of the data. Both of these are taken directly from Figure 2. Our estimates of the weights are: $\phi_1 = 0.01$, $\phi_2 = 0.28$, $\phi_3 = 0.48$, and $\phi_4 = 0.23$. Note that the first weight is almost zero. This implies that in the first period of an investment project, essentially no resources are used. This motivates referring to this first period as a *planning period*, one in which plans are drawn up, permits are secured, etc.[11] We refer to this as the estimated time to plan model.

Note how well the spectrum of the time to plan model conforms with the spectrum of the data. The time to plan model even matches the dip in the spectrum in the 7-7.5 month range. This is reflected in the good performance of the model's cumulative likelihood ratio (see Figure 3). The cumulative likelihood ratio rises slowly with frequency and achieves a maximum value just under 10. Under the null hypothesis that the model is true, this is the realization of a chi-square distribution with 9 degrees of freedom. Under these conditions, the p-value is 35 percent. As a result, the model is not rejected at conventional levels.

Figure 4 Estimation Results for the Time to Build Model



We compare the estimated time to build model with two others: the time to build model suggested in Kydland and Prescott (1982), where $\phi_i = 0.25$, $i = 1, 2, 3, 4$; and the version of the time to plan model analyzed in Christiano and Todd (1996), where ϕ_1 ¼ 0, $\phi_i = 1/3$, $i = 2, 3, 4$. We do not display the spectrum implied by Kydland and Prescott's model, because that essentially coincides with the spectrum of the restricted RBC model (King, 1995). As a result, Kydland and Prescott's model is rejected like the restricted RBC model. For a detailed discussion of the similarity of these models, see Christiano and Todd (1996) and Rouwenhorst (1991). The Christiano and Todd (1996) parameterization of the time to plan model is an improvement over the restricted RBC model in the business cycle components of the data (Figure 3). Over all frequencies, the two models, however, have a comparable fit.

[11] See Christiano and Todd (1996), who argue that the notion of a planning period conforms well with studies of investment projects.

4 Conclusions

We have described some advantages, for diagnosing model estimates and fit, of using the frequency domain decomposition of the likelihood function. We illustrate the approach with an empirical analysis of the standard RBC model and a version with a time to build technology. We reject the former in favor of the latter. The time to build technology that fits the data best appears to be one in which investment projects begin with a planning period, during which relatively few resources are expended. Christiano and Todd (1996) emphasize that this specification conforms well with microeconomic studies of investment projects, and discuss other advantages to this model for business cycle analysis.

5 Appendix: Showing that y is an ARMA(4,8)

The policy rules that solve the time-to-build model are linear equations in the log of capital $\ln K$ and of hours-worked $\ln n$ and the technology shocks (where $\ln z_t$ equals $\ln z_{t-1} + \eta_t$).

$$\begin{aligned} \ln K_t &= (1 - A(1)) \ln K + A(L) \ln K_t + (1 - A(L)) \ln z_{t-4} + +B(L) (\eta_{t-4} - \mu) \\ \ln n_t &= \ln n - C(1) \ln k + C(L) \ln K_{t+4} - C(L) \ln z_t + D(L) (\eta_t - \mu) \end{aligned}$$

The terms $A(L)$ and $B(L)$ are polynomials of degree four and $C(L)$ and $D(L)$ are polynomials of degree three in the lag operator. Capital is a function of the past capital and the shocks to technology from four to eight periods ago. Hours worked is a function of future capital (since you have to work for the investment that you have already committed to making) and the current and lagged shocks. The variables without the time subscript are the variables at steady state.

Taking the first difference of the above two equations eliminates the steady state values.

$$\begin{aligned} \Delta \ln K_t &= A(L) \Delta \ln K_t + (1 - A(L)) \Delta \ln z_{t-4} + B(L) \Delta (\eta_{t-4} - \mu) \\ &= \frac{(1 - A(L) + B(L)(1 - L)) L^4}{1 - A(L)} \eta_t \end{aligned}$$

$$\begin{aligned} \Delta \ln n_t &= C(L) \Delta \ln K_{t+4} - C(L) \Delta \ln z_t + D(L) \Delta (\eta_t - \mu) \\ &= \left(\frac{C(L)B(L)(1 - L)}{1 - A(L)} + D(L)(1 - L) \right) \eta_t \end{aligned}$$

The next step is to derive an equation for y. Output is produced using a Cobb-Douglas production function. Hence, output can be written as

$$\ln Y_t = \theta \ln K_t + (1 - \theta) \ln n_t + (1 - \theta) \ln z_t$$

Taking the first difference

$$y_t = \Delta \ln Y_t = \theta \Delta \ln K_t + (1 - \theta) \Delta \ln n_t + (1 - \theta) \eta_t$$

Substituting in the values for $\Delta \ln K_t$ and $\Delta \ln n_t$ we have

$$(1 - A(L))\, y_t = \left(\begin{array}{c} \theta\,(1 - A(L) + B(L)\,(1 - L))\, L^4 + \\ (1 - \theta)\,(C(L)B(L)\,(1 - L) + (1 - A(L))\, D(L)(1 - L) + 1) \end{array} \right) \eta_t$$

The polynomials A and C are fourth order and the polynomials D and B are third order. As the first difference operator is also present, the moving average component is an eighth order polynomial. The autoregressive term is the same order as A. The time-to-build model, therefore, can be characterized as a restricted version of an ARMA(4,8) model. The RBC model nests inside this specification.

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