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Abstract

This paper examines whether theoretical models of bubbles based on the notion that the price of an asset can deviate from its fundamental value are useful for understanding historical episodes that are often described as bubbles, and which are distinguished by features such as asset price booms and busts, speculative trading, and seemingly easy credit terms. In particular, I focus on risk-shifting models similar to those developed in Allen and Gorton (1993) and Allen and Gale (2000). I show that such models could give rise to these phenomena, and discuss under what conditions price booms and speculative trading would emerge. In addition, I show that these models imply that speculative bubbles can be associated with low spreads between borrowing rates and the risk free rate, in accordance with observations on credit conditions during historical episodes often suspected to be bubbles.

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1 Introduction

The large fluctuations in U.S. equity and housing prices over the past decade and a half have led to renewed interest in the phenomenon of asset bubbles. Among non-economists, the term "bubble" has come to refer to any historical episode in which asset prices rise and fall significantly over a relatively short period of time.[1] By contrast, economists usually use the term "bubble" to mean that an asset trades at a price that differs from its fundamental value, i.e. the expected discounted value of the dividends it generates. The latter notion is meant to capture the idea that asset prices convey distorted signals as to the true value of the underlying assets. The two notions are certainly compatible: The price of an asset can rise above fundamentals and then collapse. But economists working on models of bubbles have largely ignored the question of whether these models can account for the key features of the historical episodes suspected to be bubbles, focusing instead on whether assets can trade at a price that differs from fundamentals. The distinguishing features of the historical episodes include not just a boom and bust in asset prices, but a high incidence of speculative trading in which agents buy assets with the explicit aim of profiting from selling them later on rather than accruing dividends and low spreads on loans taken out against these assets.

This paper examines whether one particular class of models that can generate a gap between the price of an asset and its fundamental value – the risk-shifting theory of bubbles developed by Allen and Gorton (1993) and Allen and Gale (2000) – can also generate the qualitative features common to the historical episodes often described as bubbles. In risk-shifting models, traders purchase risky assets with funds obtained from others. The financial contracts traders use to secure these funds are assumed to involve limited liability. The latter feature implies traders would be willing to pay more for assets than the expected dividends they yield, since traders can shift any losses they incur on to their financiers. In other words, agents value assets above the dividends these assets generate because of the option to default on loans issued against the asset. This intuition can be illustrated in a purely static model where assets trade hands exactly once. Thus, overvaluation can occur independently of boom-bust dynamics or speculative trade.

While both Allen and Gorton (1993) and Allen and Gale (2000) consider dynamic models of risk-shifting, the various assumptions they impose to maintain tractability make it difficult to gauge whether and when these models give rise to the key features that characterize the historical episodes often described as "bubbles." For example, Allen and Gorton (1993) assume bilateral trades rather than a market for assets. This implies asset prices in their model are not uniquely determined, and so their model has little to say on when price booms and busts will arise. In addition, both papers effectively require agents who buy an overvalued asset to sell it after some exogenously determined holding period. As such, they cannot address whether

[1] For example, Merriam-Webster.com defines a bubble as "a state of booming economic activity (as in a stock market) that often ends in a sudden collapse," while the New York Times Online Guide to Essential Knowledge defines it as a "market in which the price of an asset continues to rise because speculators believe it will continue to rise even further, until prices reach a level that is not sustainable; panic selling begins and the price falls precipitously."

these models give rise to "speculative" bubbles in which traders buy an asset with the aim of profiting from selling it, or to what Hong and Sraer (2011) dub "quiet" bubbles in which assets are overvalued but trade only infrequently. To address these questions requires a dynamic model in which agents trade strategically.

There has also been little work on whether risk-shifting models of bubbles are consistent with the lax credit conditions that accompany historical episodes identified as bubbles. At a first glance, these models seem to imply that if anything, bubbles should be associated with higher borrowing costs. In particular, traders who buy overvalued assets are only able to raise funds by pooling with other borrowers whom financiers would like to finance. If creditors understand that some of those they fund will buy risky assets, they would presumably charge a spread over the risk-free rate to cover the expected losses on speculators. One of the points of this paper is to show why this intuition can be misleading, and to outline reasons why the spread between the borrowing rate and the risk-free rate may in fact be lower in speculative episodes.

In what follows, I develop a dynamic model of risk-shifting where agents borrow to buy risky assets and then choose if and when to sell them. If agents fear that future traders may not always buy the asset in order to gamble at the expense of creditors, assets can exhibit rapid price appreciation as long as they continue to trade. The intuition is related to the one Blanchard and Watson (1982) derive for rational bubbles: If an asset might cease to become overvalued in the future, rational agents must be compensated for holding the asset now while it is still overvalued rather than sell it. This compensation accrues as capital gains if the asset remains overvalued. However, an important difference between the Blanchard and Watson (1982) framework and mine is that their model is silent on whether assets trade, since agents in their model are always indifferent between holding and selling an asset. This is not true in my model, where agents sometimes sell the asset and sometimes hold it to maturity. This feature of my model reveals that the price dynamics in Blanchard and Watson (1982) should be seen as a bound on the rate of asset price appreciation rather than the rate at which rational bubbles always grow. It also reveals when bubbles will be "noisy" rather than quiet and feature repeated trade: When assets are more overvalued, when asset price appreciation is high, and when assets returns are skewed towards a high upside potential relative to the mean. An additional insight is that noisy bubbles can be associated with lower borrowing spreads than quiet bubbles. This is because when traders sell bubble assets, the risk lenders are exposed to in lending against assets is partly shifted to future lenders. Thus, it may be cheap to borrow against assets precisely when assets trade hands repeatedly, a common feature of empirical episodes suspected to be bubbles.

The second modification I consider is to allow lenders to design contracts optimally. This modification reveals another reason why speculation can be associated with low borrowing spreads, especially for speculators. Specifically, lenders will want to minimize the losses they incur from speculators, e.g. by restricting loan size or providing only short-term financing. At the same time, lenders would not want to impose these features on profitable borrowers. To induce speculators to accept more restricted contracts, lenders must make them more attractive on some other dimension, which can include lower rates.

While this paper only considers risk-shifting models of bubbles, other models have been developed in which assets can trade above their fundamental value. One example are models in which there is a shortage of assets that perform some essential function such as a store of value or liquidity. This scarcity can lead people to value whatever assets are available above and beyond the dividends they yield. Examples of such models include overlapping generations models such as Samuelson (1958), Diamond (1965), and Tirole (1985).[2] Another example are the so-called "greater-fool" models of bubbles, where agents buy assets they view as overvalued because they expect to profitably resell these assets to other agents who value the asset differently. These models include Allen, Morris, and Postlewaite (1993) and Conlon (2004). Without denying the importance of these models, there does seem to be value in studying risk-shifting models of bubbles in particular. One reason is that in these models credit markets play a central role in allowing bubbles to arise, which accords with the observation that in many historical episodes traders would often borrow against the assets they purchase. The central role of credit markets also implies these models can be used to explore one potential concern about bubbles, namely that the collapse of a bubble can be especially consequential if it results in default by those who borrowed against these assets. Finally, risk-shifting models of bubbles can lead to different policy implications than alternative models that give rise to bubbles. For example, since bubbles do not arise in these models because assets play a valuable role such as a store of value of liquidity, assets that trade above fundamentals may be inefficiently oversupplied. At the same time, unlike in greater fool models, risk-shifting models allow for it to be common knowledge that some assets are overpriced, so a policymaker with no more knowledge than private agents might want to intervene.

The paper is structured as follows. Section 2 lays out the basic features of my model economy. Section 3 studies the implications of the model for price dynamics and speculation when the set of contracts agents can enter is exogenously restricted to simple debt contracts. Section 4 allows for endogenous contracting, and examines what type of contracts will be offered when speculative bubbles arise. Section 5 concludes.

2 Setup

I begin by describing the key features of the environment I study. The model is meant to generalize the Allen and Gale (2000) model to allow for strategic dynamic trades and, later on, for endogenous contracting. To keep the analysis tractable, I restrict attention to a two-period model, where periods are indexed by $t \in \{1, 2\}$. I first describe the assets that agents can trade. I then describe the agents who populate the

[2] There are also examples of such models where agents have infinite horizons, e.g. Kocherlakota (1992) and Santos and Woodford (1997). Both offer examples of bubbles in models inspired by Bewley (1980) where there are finitely many agents with infinite horizons but limits on what agents can trade. Both papers go on to show that if agents face borrowing limits, bubbles can also emerge on assets available in zero net supply. The latter seems less relevant for understanding historical episodes suspected to be bubbles, where the relevant assets were in positive net supply. While early models focused on the case where assets served the role as a store of value, more recent work, including Caballero and Krishnamurthy (2006), Farhi and Tirole (2011), and Rocheteau and Wright (2011) instead focus on the case where assets provide a liquidity role.

economy. Finally, I describe the operation of credit markets in my economy.

As in Allen and Gale (2000), I assume a continuum of assets that are available in fixed supply and cannot be sold short. Restricting supply in some manner is necessary for assets to trade above their fundamental value. More generally, I could have allowed for an upward sloping supply schedule within a period, although this would have been more cumbersome. The fixed supply of the asset can be viewed as a technological constraint on the production of additional assets. The restrictions on short sales can be motivated by similar informational frictions to those I consider, e.g. agents who sell short cannot be trusted to deliver the assets they sell or to replicate its payoffs. However, in what follows I take short sales restrictions as given rather than derive them. Since the supply of assets is assumed to be fixed, I can normalize its mass to 1.

The payouts on assets are risky. For simplicity, suppose the assets pay a single common dividend d at the end of date $t = 2$ that can take on just two values:

$$d = \begin{cases} D > 0 & \text{with probability } \epsilon \\ 0 & \text{with probability } 1 - \epsilon \end{cases} \tag{1}$$

For example, an asset can represent a claim to the profits of a firm with a patent that may or may not pan out, and D represents the value of profits to the firm if the patent is successful.

At the beginning of date 1, agents know only that d is distributed according to (1). For reasons I explain shortly, I allow d to be revealed with some probability between dates 1 and 2. That is, before agents trade the asset at date 2, d will be revealed with probability $q \in [0, 1]$. For example, sticking to the patent example, a technological discovery may occur at the end of date 1 that reveals whether the patent is viable. Formally, let $\mathcal{I}_t$ denote the information traders observe at date t concerning dividends. Then $\mathcal{I}_1 = \varnothing$ and

$$\mathcal{I}_2 = \begin{cases} d & \text{with probability } q \\ \varnothing & \text{with probability } 1 - q \end{cases}$$

When $0 < q < 1$, agents who buy assets at date 1 are uncertain whether these assets will remain risky at date 2. This turns out to be important, since the riskiness of the asset affects demand for it, and so demand for the assets at date 2 is uncertain as of date 1. There are other reasons why demand for the asset might be uncertain, e.g. the number of agents who show up at date 2 might be random, or there might be uncertainty as to other assets agents may buy at date 2. I focus on early revelation of d only for convenience.

For simplicity, I will assume agents are all risk neutral and do not discount. As such, the expected utility value of the dividends the asset generates for any agent is just $E\left[d|\mathcal{I}_t\right]$. I will refer to this expectation as the *fundamental value* of the asset. It represents the value to society of creating an additional unit of the asset at date t. That is, if additional units of the asset could be produced at some cost at date t, the price of the asset would have to equal $E\left[d|\mathcal{I}_t\right]$ to provide proper incentives for creating additional units.

I now turn to the agents that populate the economy. Since there are several types of agents in the model, differing in their endowments and opportunity sets, it will help to begin with a brief overview. At the core of the model are two types who stand to gain by trading with each other, although this trade does not involve the assets just described. Rather, some agents, whom I call *creditors*, are endowed with resources but can earn only low returns on their savings, while other agents, whom I call *entrepreneurs,* lack resources but have access to a production technology that yields a high rate of return. Creditors can thus benefit from lending to entrepreneurs in exchange for a higher return. The reason these two types have any bearing on the assets just described is that there is a third group of agents, whom I call *non-entrepreneurs*, that lack both resources and access to a production technology, but who can buy the aforementioned risky assets. If creditors cannot distinguish entrepreneurs from non-entrepreneurs, in trying to trade with entrepreneurs they may end up lending to non-entrepreneurs who buy risky assets. The desire by creditors to trade with entrepreneurs allows resources to flow into the asset market and influence asset prices.

Closing the model requires two additional groups: The first group, whom I call *original owners*, is endowed with the risky assets at the beginning of date 1. The second group, whom I call *non-participants*, lacks resources, lacks access to a productive technology, and cannot trade assets. This group will only become relevant when I consider endogenous contracts in the next section, where their presence serves to limit the type of contracts creditors offer. In particular, their presence prevents creditors from paying non-entrepreneurs not to buy assets, since this would also draw in non-participants who cannot buy assets.

Formally, the endowments and opportunity sets of the five types can be characterized as follows:

1. **Creditors:** Creditors are endowed with a large amount of resources at date 1 that they can store until date 2 at a gross return of 1. They can also buy risky assets, although they will never strictly prefer to do so in equilibrium. The mass of creditors is assumed to be large, more than enough to supply the demand of potential borrowers whom I discuss next.

2. **Entrepreneurs**: Entrepreneurs are endowed with neither resources nor assets. However, they have access to a productive technology that allows them to produce $R > 1$ units of output per unit of input invested, up to a capacity of 1 unit of input. I assume they cannot buy risky assets. This avoids having to verify that they prefer production over buying assets, although there are parameters that guarantee this will be the case. Entrepreneurs arrive at exogenously set times, either at $t = 1$ or $t = 2$, and if they want to invest they must contact creditors the period they arrive. Regardless of when they invest, the output from their investment accrues at the end of date 2.

3. **Non-Entrepreneurs:** Non-entrepreneurs are also endowed with neither resources nor assets. In contrast to entrepreneurs, they do not have access to a productive technology. But they can buy risky assets. Like entrepreneurs, they arrive at exogenously set times, either at $t = 1$ or $t = 2$, and if

they wish to buy risky assets they must both contact creditors and buy assets the period they arrive. Let $n_t < \infty$ denote the mass of non-entrepreneurs who arrive at date t, and let $m_t < \infty$ denote the combined mass of entrepreneurs and non-entrepreneurs who arrive at date t. To cut down on the number of parameters, I assume the ratio of non-entrepreneurs to entrepreneurs is the same in both periods. That is, regardless of t, the number of non-entrepreneurs $n_t = \phi m_t$ for some $\phi \in (0, 1)$.

4. **Original Owners:** Original owners are endowed with one unit of the asset each and a large amount of resources, which they can store at a gross return of 1. For ease of exposition, I assume that unlike creditors, they do not participate in the credit market, either as lenders or borrowers. This assumption is not restrictive given that in equilibrium they could not profit from trading in the credit market.

5. **Non-Participants**: Non-participants are endowed with neither resources, assets, nor a productive technology. In addition, they face a prohibitive cost of entering the market for risky assets. The mass of such agents is large, in a sense that will be clarified in Section 4 when they become relevant.

Many of the assumptions above only serve to simplify the analysis, and can be considerably relaxed without affecting some of the key results. In particular, the result that assets can trade above their fundamental value $E[d|\mathcal{I}_t]$ arises because creditors can profitably finance entrepreneurs, but only on terms that would make it profitable for non-entrepreneurs to borrow and buy risky assets. The existence of bubbles thus hinges on there being at least some entrepreneurs with profitable investment opportunities but limited resources, allowing non-entrepreneurs to borrow and then buy risky assets with little of their own at stake. The remaining assumptions I impose – such as the exogenous arrival dates of agents, the finite number of agents who arrive at each period, and the finite capacity of entrepreneurs – are unnecessary for this result.

At the same time, my assumptions do matter for price dynamics and trade volume. Specifically, if there were no limit on how much agents could borrow at date 1, asset prices would be bid up immediately, assets would trade hands only once, and asset prices would not grow. To generate price appreciation and repeated trading, we need total borrowing at date 1 to be finite, so the finite capacity of entrepreneurs is important. Similarly, assuming that agents arrive at exogenous dates is not entirely innocuous. If entrepreneurs and non-entrepreneurs all arrived in period 1 and chose when to trade, the number of agents who trade in each period would be endogenous. Since my results depend on the number of traders that arrive at each period, it is not obvious that endogenous timing would accommodate all of the phenomena I emphasize. However, as will become clear below, for certain parameter values such as a small ϵ, price appreciation and repeated turnover would probably arise even if I allowed agents to time their actions.

To focus on the key features of the model I am after, I impose two parameter restrictions. First, I restrict the return on production R to an intermediate range of values:

$$1 + \frac{\phi(1-\epsilon)}{1-\phi(1-\epsilon)} < R < \frac{1}{\epsilon} \tag{2}$$

It is easy to verify that this range is nonempty for $\phi \in (0,1)$ and $\epsilon \in (0,1)$. The reason R cannot be too low is that the earnings of entrepreneurs must cover the expected losses creditors incur on non-entrepreneurs who use the funds they borrow to buy risky assets. At low values of R, lending would simply shut down. But high values of R can also be problematic. In particular, if R exceeded $1/\epsilon$, entrepreneurs would earn a higher return than non-entrepreneurs could ever earn from buying risky assets. This implies non-entrepreneurs might not be able to profit from buying risky assets, since borrowers may be charged a high rate that still attracts entrepreneurs but would make it unprofitable to buy and hold risky assets. In addition, once I allow more general contracts, if entrepreneurs earn more than non-entrepreneurs ever could they could prove their type to creditors by showing their earnings. Creditors could then avoid lending to non-entrepreneurs.

Second, I restrict the total number of non-entrepreneurs over the two periods to not be too large:

$$n_1 + n_2 < (1-q)\, D/R + q\epsilon D \tag{3}$$

Since each agent will be able to borrow at most one unit of resources given the finite capacity of entrepreneurs, assumption (3) restricts the total amount of resources agents can borrow to spend on risky assets. As we shall see below, this will impose an upper bound on the price of the asset. (3) rules out the case where the price of the asset is high enough that non-entrepreneurs who buy risky assets earn zero expected profits, rendering them indifferent between buying and not buying the asset. The latter case introduces an additional variable – the fraction of non-entrepreneurs who buy assets – and is thus more tedious to analyze.

Remark 1: Note that the second inequality in (2) implies $D/R > \epsilon D$. Hence, the upper bound in (3) is strictly greater than ϵD for $q < 1$. In the next section I show that $n_1 + n_2 > \epsilon D$ is a necessary and sufficient condition for a bubble. My parametric restrictions are thus compatible with the possibility of a bubble.

Finally, I need to describe the functioning of credit markets where creditors can trade with entrepreneurs, non-entrepreneurs, and non-participants. Creditors cannot distinguish the different types that seek to borrow, nor can they monitor what agents do with the funds they receive. To motivate this assumption, we can think of entrepreneurs as earning the return R by purchasing some type of asset, e.g. purchasing an asset they can manage better than its current owner, or buying an undervalued asset based on private information as in Allen and Gorton (1993). If creditors cannot tell apart different types of assets, entrepreneurs and non-entrepreneurs will look indistinguishable. Alternatively, creditors may not even get to observe the underlying assets, as is sometimes the case with hedge funds that don't divulge their trading strategies.

Credit markets are run as follows: First, creditors post contracts. Agents then arrive and choose among contracts. Specifically, agents flow in according to some pre-arranged order, where the fraction of non-entrepreneurs within each arriving cohort is ϕ. The reason I require sequential arrivals is that, as we shall see below, it is possible that more than one type of contract will be offered in equilibrium. In this case, more attractive contracts must be rationed, and sequential arrival rations them to those who arrive first.

In terms of the types of contracts creditors can offer, for now I restrict lenders to offering only a limited set of contracts, retaining comparability with Allen and Gale (2000) who also focus on a particular set of contracts. I will allow a more general class of contracts in Section 4. The set of contracts I initially study are fixed-size, full recourse, simple debt contracts. Specifically, at each date t, lenders can offer to lend one unit of resources to borrowers who show up at that date, the most creditors would ever agree to offer given the finite capacity of entrepreneurs. The borrower is required to pay back a pre-specified amount $1 + r_t$ at the end of date 2, which is when entrepreneurs would first be able to make a payment. The only dimension along which lenders can compete is the rate r_t they charge borrowers. If the borrower fails to repay his obligation in full, the lender has full recourse to go after the borrower's remaining resources, up to the amount of the obligation. Hence, entrepreneurs cannot escape repaying their debt, and wealthy agents will not find it profitable to borrow and buy risky assets given they will always be liable for losses they incur.[3]

Beyond the threat of recourse, an important reason borrowers repay their debt in the real world is that default tends to be costly, e.g. it may be associated with a loss of access to future credit. One can crudely capture this intuition in my model by assuming that if the borrower pays $z < 1 + r_t$, he will incur a cost $k(1 + r_t - z)$ proportional to his shortfall. I want to allow for this possibility, but to avoid keeping track of another parameter I focus on the limiting case where $k \to 0$. All of my results extend to the case where k is positive but small. At the same time, taking the limit as $k \to 0$ is not equivalent to setting $k = 0$. For example, when $k > 0$, agents will not borrow if they expect to default with certainty, a property that will be preserved in the limit as $k \to 0$. But agents would be willing to borrow and default when $k = 0$. Looking at the limit as $k \to 0$ thus rules out equilibria that are not robust to the introduction of small default costs.

3 Equilibrium

I now proceed to analyze the equilibrium of this economy. Intuitively, an equilibrium consists of state-contingent paths for the price of the asset $\{p_t(\mathcal{I}_t)\}_{t=1}^2$ and the borrowing rate $\{r_t(\mathcal{I}_t)\}_{t=1}^2$ that ensure both the asset market and credit markets clear. Specifically,

1. At each date t, for each information set $\mathcal{I}_t$, demand for the asset by potential buyers at price p_t is equal to the amount those who already own the asset are willing to sell at price p_t
2. At each date t, for each information set $\mathcal{I}_t$, creditors earn zero expected profits when they offer a loan at rate r_t, and creditors cannot expect to earn positive profits by offering an alternative contract

[3] One could interpret these loans as collateralized by the assets the borrower purchases: If the borrower fails to repay, full recourse allows the creditor to seize any dividends that accrue to the asset. However, as will become clear once I allow creditors to design their contracts, creditors must have very limited information about the assets used as collateral, or else the contract would naturally make use of such information in a way that would make it different from a debt contract.

As anticipated in the previous section, there may be situations in which condition (2) will require creditors to offer more than one interest rate at date $t = 1$. Thus, a proper definition of equilibrium needs to be modified to allow for multiple interest rates. With this caveat in mind, I now proceed to characterize the conditions that ensure the asset market clears and that creditors not expect to earn positive profits.

First, though, I introduce some terminology that will help in describing equilibrium in the asset market. I will refer to the risky asset as a *bubble* if at any date t, its price p_t differs from the fundamental value $E[d|\mathcal{I}_t]$. Note that my definition for fundamental value only reflects the value of dividends and not the option value to default on loans borrowed against an asset, even though non-entrepreneurs value the asset in part because they can default on loans against it. The justification for doing so is that society as a whole is no better off from this option, which merely redistributes resources from creditors to borrowers. Thus, this option value should not be viewed as something intrinsic that makes the asset more valuable.

Next, following Harrison and Kreps (1978), I define *speculation* to mean that agents assign a positive value to the right to resell the asset when they purchase it. This definition is meant to capture the notion that agents who buy an asset intend to profit by selling the asset rather than merely waiting to collect all of its dividends.[4] Consistent with these definitions, I will refer to a *speculative bubble* as a bubble where the agents who buy it at date 1 would strictly prefer to sell it at date 2 in some state of the world.[5] A speculative bubble thus implies that the same assets trade hands multiple times in some states of the world. However, a bubble asset may trade hands multiple times even if it does not meet the definition of a speculative bubble, since agents who buy the asset at date 1 may in fact be indifferent about selling it at date 2. To distinguish this case from the one in which the asset is a bubble but agents who buy at date 1 hold on to their assets until d is realized, I borrow the terminology of Hong and Sraer (2011) and refer to the case where an asset is a bubble that trades hands at most once as a *quiet bubble*. Likewise, I refer to a bubble that trades hands more than once with some probability as a *noisy bubble*.[6] A speculative bubble will always be noisy, but not all noisy bubbles are speculative. As I show below, depending on parameter values, my model admits

[4] Note that per this definition, finitely-lived agents who buy infinitely-lived assets are engaging in speculation. But this is not because traders view selling the asset as inherently more profitable than waiting to collect all of its dividends, which is the notion Harrison and Kreps claimed to be after. Rather, it is because finitely-lived agents cannot collect all dividend payments. However, in both the Harrison and Kreps (1978) model and my model, agents live long enough to collect all dividends. In that case, this definition for speculation does seem to capture the notion of trying to profit by selling the asset.

[5] At first glance, the notions of a bubble and speculation may seem identical. On the one hand, if asset prices were always equal to fundamentals, agents should be indifferent between selling an asset and holding it to maturity and the right to sell the asset would be worthless. Thus, speculation would seem to imply a bubble. But this intuition breaks down if agents value assets differently, as occurs here where agents value the asset differently because they borrow different amounts against it. Although defining fundamentals is tricky when agents value the asset differently, Barlevy and Fisher (2012) provide an explicit example where prices can arguably be said to equal fundamentals but leveraged agents are engaged in speculation. In the opposite direction, it might seem intuitive that assets can only trade above their fundamental value if agent expect to sell their assets. However, the fact that bubbles can arise in static risk-shifting models where there is only one round of trading such as Allen and Gale (2000) suggests an asset can be overvalued even without speculation.

[6] Allen and Gorton (1993) use the term "churning" to refer to the same phenomenon.

the possibility of no bubbles, quiet bubbles in which assets are overvalued but trade hands no more than once, and noisy and speculative bubbles in which assets trade hands multiple times.

To solve for equilibrium, I work backwards from date 2. At this point, $\mathcal{I}_2 \in \{\varnothing, d\}$. Consider first the case where $\mathcal{I}_2 = d$, i.e. where the dividend is revealed before agents trade. In this case, the equilibrium price for the asset must be d. For suppose the price exceeded d. In that case, supply would be strictly positive: Agents who own the asset earn more from selling the asset than holding it.[7] But demand for the asset would be zero, since agents with resources would prefer storage to buying the asset, while agents without resources would have to default with certainty if they had to compensate creditors for their opportunity cost. We can similarly rule out a price below d. In that case, demand for the asset would be strictly positive since agents with resources could earn a higher return than from storage or from lending. At the same time, supply would be zero, since agents would earn more holding the asset than selling it. This leaves $p_2(d) = d$. Since $p_2(d)$ is trivial to characterize, I will henceforth use p_2 to refer to $p_2(\varnothing)$, the price if d is not revealed.

The formal analysis for the case where $\mathcal{I}_2 = \varnothing$ is carried out in an Appendix. Here, I only sketch the argument. Assumption (2) ensures that the return R on entrepreneurial activity is high enough to make it profitable to extend credit even when a fraction ϕ of borrowers are non-entrepreneurs who are expected to incur losses to their lenders. Thus, creditors will provide loans in equilibrium. Assumption (3) ensures that the equilibrium price of the asset at date 2 is low enough that borrowing to buy risky assets and defaulting if $d = 0$ is profitable for non-entrepreneurs. Specifically, I derive the following result in the Appendix:

Lemma 1: Let p_t denote the equilibrium price of the asset at state $\mathcal{I}_t = \varnothing$. If (2) and (3) hold, then $p_t < D/R$ for $t \in \{1, 2\}$.

Lemma 1 implies $D/p_t > R$, i.e. holding the asset to maturity will yield more income if $d = D$ than an entrepreneur could be asked to pay. Pretending to be an entrepreneur and buying risky assets will thus guarantee positive expected profits. Since my assumptions on the various types ensure only non-entrepreneurs ever buy risky assets, computing demand is straightforward: Each of the n_2 non-entrepreneurs will borrow one unit of resources to buy assets, and so the amount of assets demanded at price p_2 is n_2/p_2.

Turning to asset supply, those who own the asset at date 2 can include original owners who did not sell their holdings at date 1 and non-entrepreneurs who showed up at date 1 and bought assets. In fact, using assumptions (2) and (3), we can deduce that all n_1 non-entrepreneurs who could have borrowed at date 1 would have done so, and hence n_1/p_1 shares will be held at the beginning of date 2 by agents who bought them at date 1, while $1 - n_1/p_1$ shares will be held by original owners who held on to their asset at date

[7] Here the fact that I focus on the limiting case where the cost of default $k \to 0$ is important, since it implies that even agents who intend to default will strictly prefer to sell their asset holdings.

1. At date 2, the two groups have different reservation prices at which they would agree to sell the asset. Original owners would sell the asset at any price p_2 above ϵD, their expected payoff from holding the asset. Agents who borrowed one unit of resources at rate r_1 at date 1 to buy assets have a different reservation price. To see this, note that if they held on to their assets, their expected profits would equal

$$\epsilon\left(\frac{D}{p_1}-(1+r_1)\right) \tag{4}$$

If they sold their assets at date 2 instead, they would earn

$$\frac{p_2}{p_1}-(1+r_1) \tag{5}$$

Comparing the two, they should be willing to sell their assets if

$$p_2 \geq \epsilon D+(1-\epsilon)(1+r_1)\,p_1 \tag{6}$$

Traders who bought assets at date 1 with borrowed funds have a higher reservation price than original owners. Moreover, their reservation price is increasing in the rate r_1 they are charged. Given the interest rates charged at date 1, we can readily derive the supply schedule for the asset at date 2.

A market clearing price p_2 is one at which supply and demand are equal. Figure 1 plots the supply and demand curves assuming all borrowers are charged a single interest rate r_1 at date 1. Under this assumption, the supply schedule is a two-step function. The demand curve is a hyperbola that depends on n_2. Figure 1 shows the different ways supply and demand could intersect. The figure suggests four cases are possible:

a. At least some original owners hold on to assets until the end of date 2. In this case, $p_2 = \epsilon D$.

b. All original owners sell by date 2, but no non-entrepreneur who bought at date 1 sells at date 2.

c. All original owners sell by date 2, and some non-entrepreneurs who bought at date 1 sell at date 2.

d. All original owners and all non-entrepreneurs who bought at date 1 sell at date 2.

Note that in cases (b)-(d), the price p_2 exceeds $\epsilon D = E[d|\varnothing]$, i.e. there is a bubble. Although Figure 1 is suggestive, two important caveats are in order. First, it is only meant to illustrate the different ways in which the demand curve could intersect a step-function. It does not correspond to the effects of increasing n_2, the number of non-entrepreneurs arriving at date 2. This is because changing n_2 will in general affect the price of the asset at date 1, and will therefore affect both demand and supply for the asset at date 2.

Second, the supply curve in Figure 1 is drawn assuming all borrowers at date 1 are charged the same rate r_1 in equilibrium. For cases (a), (b), and (d), this will indeed be the case. But in case (c), there will in fact be two different rates offered at date 1. To see why, suppose all borrowers were charged the same rate r_1 at date 1. Since in case (c) only some of the non-entrepreneurs who buy assets at date 1 sell them at date 2, there must be some creditor who lends at date 1 and assigns probability less than 1 that a non-entrepreneur who borrows from him would sell at date 2. Suppose this creditor charged a slightly lower rate. From (6), we know that the reservation price is increasing in r_1. Hence, the creditor could induce a discrete jump in the probability that a non-entrepreneur who borrows from him sells the asset. Inducing the borrower to sell the asset and pay back his loan with certainty rather than hold on to the asset and pay back his loan if $d = D$ leads to a discrete rise in the creditors ex-ante expected profits from the non-entrepreneur that can more than offset the lower interest rate. Case (c) is thus incompatible with a single interest rate r_1.

Instead, in case (c), equilibrium requires two different interest rates at date 1, a low rate r_{1*} and a high rate r_1^*. Non-entrepreneurs who are charged r_{1*} will sell the asset at date 2, while those charged r_1^* will hold on to it. The supply curve in this case will be a three-step function, as shown in Figure 2. The number of contracts offered with each rate depends on the volume of trade between non-entrepreneurs who buy at date 1 and non-entrepreneurs who buy at date 2. If we increase n_2, specifically if we increase the number of borrowers m_2 but keep the fraction of non-entrepreneurs ϕ fixed, more non-entrepreneurs who buy at date 1 will have to sell at date 2, and so a larger fraction of the loans at date 1 will charge the lower rate r_{1*}.

In all four cases (a)-(d), supply and demand for the asset intersect exactly once, so the market clearing price p_2 is unique. Moving back to date 1, we can similarly derive supply and demand for the asset. Appealing to assumptions (2) and (3), all non-entrepreneurs will borrow one unit of resources and use it to buy assets. Demand for the asset at date 1 is thus n_1/p_1. As for supply, original owners of the asset can either sell the asset for p_1 or hold the asset until date 2. Per my discussion above, original owners always weakly prefer to sell their asset at date 2 regardless of the realization of $\mathcal{I}_2$. Hence, waiting to sell the asset would yield an expected payoff of $q\epsilon D + (1 - q)\, p_2$. This expression corresponds to the reservation price of original owners at date 1. Given a value for p_2, there will be a unique equilibrium price p_1 at date 1.

To summarize, the requirement that p_t clear the market at each $\mathcal{I}_t$ yields a pair of conditions associated with market clearing at $\mathcal{I}_1 = \varnothing$ and $\mathcal{I}_2 = \varnothing$ respectively that uniquely determine p_1 and p_2. The market clearing at date 1 implies p_1 will depend on what agents believe about the price p_2, consistent with the usual notion that asset prices are forward looking. More interestingly, market clearing at date 2 implies p_2 will depend on the price p_1 that prevailed at date 1. This non-traditional backward-looking aspect arises because p_1 governs the reservation price of agents who bought the assets at date 1. Since these agents are leveraged, the value of their option to default will depend on the price of the asset at date 1.

The last step in solving for equilibrium is to use zero-profit conditions to pin down interest rates $r_t(\mathcal{I}_t)$. When $\mathcal{I}_2 = d$, risk-shifting opportunities disappear. Loans are thus riskless, and competition will drive the net interest rate on loans to 0, i.e. $r_2(d) = 0$. Once again, since interest rates are trivial to characterize in this case, I will use r_2 to refer to $r_2(\varnothing)$. If $\mathcal{I}_2 = \varnothing$, creditors who extend credit at date 2 will earn a return

r_2 from entrepreneurs and from non-entrepreneurs if $d = D$, but will recoup nothing from non-entrepreneurs if $d = 0$. Their expected profits will equal 0 if r_2 satisfies

$$(1-\phi)\, r_2 + \phi\left[\epsilon\,(1+r_2) - 1\right] = 0 \tag{7}$$

Note that r_2 does not depend on the price of the asset. In particular, we have

$$r_2 = \frac{\phi\,(1-\epsilon)}{1-\phi\,(1-\epsilon)} \tag{8}$$

Moving to date 1, let μ denote the probability that a non-entrepreneur who buys assets at date 1 will sell them at date 2 if $\mathcal{I}_2 = \varnothing$. As discussed above, equilibrium requires that $\mu = 0$ or $\mu = 1$, i.e. a creditor must not anticipate that a non-entrepreneur who borrows from him will randomize whether he sells the asset or not. For each μ, expected profits must equal 0. Hence, r_1 must satisfy

$$(1-\phi)\, r_1 + \phi\mu\left[(1-q+q\epsilon)\,(1+r_1) - 1\right] + \phi\,(1-\mu)\left[\epsilon\,(1+r_1) - 1\right] = 0 \tag{9}$$

A borrower who expects to sell his assets with certainty will be charged a lower rate, which I denote r_{1*}, while a borrower who expects to hold on to his assets will be charged a higher rate, which I denote r_1^*. Substituting in for $\mu \in \{0,1\}$ yields the following expressions

$$r_{1*} = \frac{\phi q\,(1-\epsilon)}{1-\phi q\,(1-\epsilon)}, \qquad r_1^* = \frac{\phi\,(1-\epsilon)}{1-\phi\,(1-\epsilon)} \tag{10}$$

Remark 2: Note that r_{1*}, r_1^*, and r_2 are all positive. Hence, non-participants will never benefit from taking out a loan, which is why I can ignore them for now in analyzing the credit market.

The discussion above can be summarized as follows:

Proposition 1: Given (2), for each $(n_1, n_2) \in \mathcal{N} \equiv \left\{(n_1, n_2) \in \mathbb{R}^2_{++} : n_1 + n_2 < (1-q)\, D/R + q\epsilon D\right\}$, there exists a unique price path $\{p\,(\mathcal{I}_t)\}_{t=1}^2$ and a unique distribution of interest rates offered at each $\mathcal{I}_t$ that ensure market clearing and no positive expected profits to lenders. $\mathcal{N}$ can thus be partitioned into regions $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{D}$ corresponding to the different types of asset market equilibria (a) - (d) when $\mathcal{I}_2 = \varnothing$.

The regions $\mathbb{A} - \mathbb{D}$ are illustrated graphically in Figure 3, and their boundaries are derived in the proof of Proposition 1. In region $\mathbb{A}$, asset prices equal fundamentals. Region $\mathbb{B}$ is associated with quiet bubbles in which the asset trades above fundamentals but each asset trades hands no more than once. Region $\mathbb{C}$ is associated with noisy bubbles but not with speculation, i.e. some traders who buy assets at date 1 will turn around and sell them at date 2, but they are indifferent between selling at date 2 and holding on to the asset. Lastly, region $\mathbb{D}$ corresponds to speculative (and thus noisy) bubbles: Traders who buy assets at date 1 will strictly prefer to sell them if $\mathcal{I}_2 = \varnothing$. The remainder of this section highlights several features of the equilibrium and provides an economic interpretation for when the different cases arise.

3.1 Asset Price Levels

As evident from Figure 3, whether or not a bubble arises depends on the total number of traders $n_1 + n_2$. When $n_1 + n_2$ is small, specifically when $n_1 + n_2 < \epsilon D$, the price of the asset will equal fundamentals in both periods, i.e. $p_t = E[d|\mathcal{I}_t]$. When $n_1 + n_2 > \epsilon D$, the price of the asset can exceed its fundamental value in date 1 and in date 2 if $\mathcal{I}_2 = \varnothing$. In this case, the degree to which the asset is overvalued – i.e. the size of the bubble component – is uniquely determined. This is in contrast to some other models of bubbles, e.g. overlapping generations models, where the size of the bubble is indeterminate.

In particular, if a bubble exists, the size of the bubble component $b_t = p_t - E[d|\mathcal{I}_t]$ depends on how many non-entrepreneurs trade in assets markets. Here, it is important to distinguish between the absolute number of such traders, n_t, and their relative share among all borrowers that creditors finance, ϕ. The share parameter ϕ determines interest rates r_t, but does not affect the price of the asset directly other than by affecting r_t. By contrast, the number of traders n_t affects asset prices, but does not affect the level of interest rates.[8] Consider increasing the total number of borrowers m_t while holding ϕ fixed, i.e. increasing both entrepreneurs and non-entrepreneurs while keeping their relative shares fixed. The proof of Proposition 1 shows that p_1 and p_2 are both weakly increasing in n_1 and n_2. Since $E[d|\mathcal{I}_t]$ is constant, this means b_1 and b_2 are also weakly increasing in n_1 and n_2. In other words, the bubble component in asset prices will be bigger the greater the aggregate amount that is borrowed against these assets.

The intuition behind this result is that the model gives rise to what Allen and Gale (1994) describe in a different context as "cash-in-the-market pricing" meaning asset prices depend on the ratio of the cash brought by asset buyers and the amount of assets up for sale.[9] To better appreciate this, consider the case where $q = 0$, so d will not be revealed until the end of date 2. Thus, there is no uncertainty about trade at date 2. The only possible equilibrium is one where the asset trades at the same price in both periods, and assets only trade hands once. This is because the price of the asset cannot rise between dates 1 and 2, or else all original owners will wait to sell at date 2, meaning no one will meet the demand for assets from date-1 non-entrepreneurs. But the price of the asset also cannot fall, since in that case only those who bought the asset at date 1 could sell it at date 2, yet their reservation price exceeds p_1. Hence, the unit supply of the asset held by the original owners will be sold off to non-entrepreneurs in exchange for the resources they can bring to the asset market. Since each can borrow one unit, this means $p_1 = p_2 = n_1 + n_2$.

[8] The number of traders n_t will, however, affect how many date 1 contracts are offered with rates r_{1*} and r_1^*, respectively. In particular, a higher n_2 will imply more contracts that charge a low rate r_{1*}. I will return to this result in Section 3.4.

[9] In Allen and Gale (1994), agents choose between cash and assets in advance, and then a random number of agents are hit with an immediate need for liquidity and must sell their assets for cash. If more agents are hit with liquidity shocks than expected, assets trade *below* their fundamental value, with the price equal to the ratio of cash held by liquid agents and assets held by illiquid agents. By contrast, here cash corresponds to the amount of resources agents can borrow and then use to buy risky assets, and assets trade *above* their fundamental value.

This intuition for why higher n_1 and n_2 increase the size of the bubble continues to hold with uncertainty.

3.2 Asset Price Growth

Next, I examine the rate at which asset prices grow. Since I assume no time discounting, the risk-free rate is zero. This implies that the price of the asset cannot rise unless the price of the asset at date 2 is itself uncertain, or else agents could earn above the risk-free rate with no risk. Indeed, when $q = 0$ so the only possible state at date 2 is $\mathcal{I}_2 = \varnothing$, I argued above that $p_1 = p_2$. But when $q > 0$, the price of the asset at date 2 will depend on whether d is revealed, and if it is revealed, on the value of d. In this case, asset prices can appreciate in some states of the world. Clearly, the price of the asset will rise if d is revealed to be D. The more relevant question for understanding historical episodes often taken to be bubbles is the rate at which asset prices grow if d is not revealed. That is, can assets become increasingly overvalued over time even when there is no commensurate growth in the fundamental value of the asset?

The rate at which the asset price appreciates if d remains hidden turns out to depend on whether in equilibrium all original owners sell their assets at date 1 or only some do. When only some of sell their assets at $t = 1$, they must be indifferent between selling the asset and waiting to sell at date 2. This implies

$$p_1 = q\epsilon D + (1 - q)\, p_2 \tag{11}$$

Let b_t denote the size of the bubble at date t, i.e. $b_t = p_t - E\left[d|\mathcal{I}_t\right]$. The price of the asset at date 1 can thus be expressed as the sum of the fundamental value and a bubble component, i.e.

$$p_1 = \epsilon D + b_1 \tag{12}$$

Substituting this expression for p_1 into (11) and solving for p_2 yields

$$p_2 = \epsilon D + \frac{b_1}{1 - q} \tag{13}$$

Thus, the bubble component $b_2 = p_2 - \epsilon D$ will be larger than the bubble component b_1 at date 1 as long as $q < 1$. That is, the asset will appreciate in price and becomes increasingly more overvalued. These dynamics are identical to those derived by Blanchard and Watson (1982), who showed that when traders are rational, bubbles grow at a risk-adjusted interest rate. Intuitively, the original owners of the asset require some compensation to hold the asset at date 1, since by holding the asset they risk giving up the opportunity to sell an overvalued asset. This compensation accrues in the form of capital gains if the bubble survives, allowing them to earn potentially higher profits if they wait. Previous risk-shifting models of bubbles such as Allen and Gorton (1993) and Allen and Gale (2000) do not give rise to these pricing dynamics because they require original owners to sell their assets rather than let them trade strategically.

When all original owners sell off their holdings at date 1, though, the growth rate of asset prices can deviate from the dynamics derived by Blanchard and Watson (1982). In this case, the original owners must weakly prefer to sell the asset for price p_1 at date 1 than to hold it and sell at date 2, and so $p_1 \geq q\epsilon D + (1-q)\,p_2$. Thus, $(1-q)^{-1}$ represents an upper bound on the rate at which the bubble can grow. But there is also a lower bound on the rate at which the bubble can grow. In particular, if all original owners sell their assets at date 1, non-entrepreneurs who show up at date 2 will have to buy assets from non-entrepreneurs who arrived at date 1. The latter will only agree to sell if the price is at least equal to their reservation price, $\epsilon D + (1-\epsilon)(1+r_{1*})\,p_1$. Hence, the price of the asset p_2 must satisfy

$$\epsilon D + (1-\epsilon)\,p_1\,(1+r_{1*}) \leq p_2 \leq \epsilon D + \frac{b_1}{1-q} \tag{14}$$

We know from Lemma 1 that $p_1\,(1+r_{1*}) < D$, and so the lower bound on p_2 implies that $p_2/p_1 \geq (1+r_{1*})$, i.e. the price of the asset must grow faster than the rate at which non-entrepreneurs must pay to borrow resources.[10] This is because non-entrepreneurs who sell the asset at date 1 must not only earn enough to pay their creditors, but must be compensated for the option value of default they give up and which they could have exercised if they held on to their assets. Intuitively, price appreciation in this case is driven by the fact that an increase in price allows for trade. In particular, if the price of the asset rises, agents who buy the asset borrow more against each asset than those from whom they buy the assets. As a result, the option to default is more valuable for new buyers than it is to its existing owners, meaning they will value the asset more than the original cohort of buyers. The implied rate of price appreciation may be slower than the growth rate due to the considerations first pointed out by Blanchard and Watson (1982). Since this lower rate of price appreciation occurs when the original owners sell all of their asset holdings at date 1, asset price appreciation will generally be lower when trade volume in the bubble asset is high early on.

In sum, the rate at which the bubble component grows depends on when traders show up and is bounded by $(1-q)^{-1}$, a bound which is increasing in the probability of the bubble bursting. This insight suggests why the phenomena that distinguish historical episodes suspected to be bubbles, e.g. rapid price appreciation, might be rare. When rapid appreciation is possible, i.e. when q is large, appreciation is also less likely to materialize since the bubble is likely to burst. However, the fact that we observe rapid asset price appreciation only infrequently does not imply that bubbles – instances in which assets are overvalued – are also inherently rare. In particular, quiet bubbles associated with low values of q can be quite common, although they are also harder to identify empirically since this requires estimating fundamentals.

3.3 Quiet Bubbles, Noisy Bubbles, and Speculation

I now turn to the predictions of the model for trading patterns. In particular, if the asset is overvalued, will it be a quiet or a noisy bubble? The answer to this question can be inferred from Figure 3. The figure shows

[10] Since the borrowing rate r_{1*} exceeds the risk free rate, the asset must appreciate faster than the risk-free rate. In models where bubbles cannot have a finite end-date, e.g. overlapping generation models, this is problematic: It requires the economy to grow faster than the risk-free rate to ensure agents can always afford the asset. This in turns implies the economy is dynamically inefficient. Here, the bubble ends in finite time, and so there is no need for high rates of economic growth.

the nature of the asset market equilibrium at date 2 for different values of n_1 and n_2. Recall that region $\mathbb{B}$ corresponds to quiet bubbles where assets trade hands only once, while regions $\mathbb{C}$ and $\mathbb{D}$ correspond to noisy bubbles where at least some of the agents who buy assets at date 1 will sell them at date 2. Figure 3 is drawn as if all four types of equilibria are possible. However, this will not be true in general. In particular, whether equilibria corresponding to regions $\mathbb{C}$ and $\mathbb{D}$ are possible depends on the location of the outer boundary for region $\mathbb{B}$ relative to the outer boundary of the set $\mathcal{N}$. If the outer boundary for region $\mathbb{B}$ – whose location depends on the parameters q, ϵ, and ϕ – is sufficiently far away from the origin, then the only type of equilibrium bubbles that can arise for $(n_1, n_2) \in \mathcal{N}$ are quiet bubbles. Whether speculation and noisy bubbles can arise depends on these parameters.

The formal analysis of how these parameters affect the types of equilibria is contained in the Appendix. Here, I only review the results. Consider first q, which I argued in Section 3.2 governs the rate at which the bubble can grow. As $q \to 0$, region $\mathbb{B}$ expands outwards to cover all pairs $(n_1, n_2) \in \mathcal{N}$ for which $n_1 + n_2 > \epsilon D$. In other words, for small q, only quiet bubbles are possible. Intuitively, the smaller the risk of the bubble collapsing, the less the bubble can grow over time without inducing agents to hold on to the asset. But if asset prices do not grow much over time, traders who borrow to buy the asset will not earn enough when they sell the asset to cover their interest obligation and the option value to default they give up by selling the asset. Hence, noisy bubbles will not arise for low values of q. Can they arise for high values? As $q \to 1$, region $\mathbb{B}$ shrinks towards the origin, meaning fewer values of (n_1, n_2) will be associated with quiet bubbles. However, the set $\mathcal{N}$ also shrinks. In the Appendix, I show that if R is close to its upper bound of $1/\epsilon$, noisy bubbles will exist for some $(n_1, n_2) \in \mathcal{N}$ when q is close to 1. When R is close to its lower bound in (2), noisy bubbles still exist for some $(n_1, n_2) \in \mathcal{N}$ when q is close to 1 and either ϵ or ϕ are large. Since q governs the rate at which asset prices can appreciate, it follows that noisy bubbles are more likely to arise when price appreciation is high. As in Section 3.2, it also follows that speculative trading will be rarely observed, since when it can take place the bubble is likely to burst before speculators can sell.

Turning to ϵ, in the limit as $\epsilon \to 1$, any $(n_1, n_2) \in \mathcal{N}$ for which $n_1 + n_2 > \epsilon D$ must fall in region $\mathbb{B}$. Thus, when ϵ is large, if there is a bubble it must be a quiet bubble. In the opposite direction, as $\epsilon \to 0$, then as long as $q < 1$, in the limit all $(n_1, n_2) \in \mathcal{N}$ lie in either region $\mathbb{C}$ or $\mathbb{D}$. That is, when ϵ is small, bubbles must arise and are inherently noisy.[11] To understand this result, note that for small values of ϵ, holding on to the asset is likely to be unprofitable, since d is likely to assume its low realization. Thus, a high ϵ will encourage those who buy the asset early to turn around and sell it rather than hold on to it. This suggests penny stocks, i.e. stocks that trade at low prices but have a skewed return distribution, should be particularly vulnerable to speculative bubbles. Interestingly, Eraker and Ready (2013) find that such

[11] Interestingly, Hong and Sraer (2011) also find that noisy bubbles are more likely for assets with a high upside potential, but for different reasons. In particular, a high upside in their model allows for more potential for disagreement as to the value of the asset, which encourages trade. This feature is absent from my setting.

stocks seem to be overvalued, since they tend to have a negative expected return that cannot be explained by asset characteristics. As for ϕ, region $\mathbb{B}$ becomes larger as ϕ rises, i.e. higher values for ϕ are more likely to be associated with quiet bubbles. Intuitively, a higher ϕ increases the cost of borrowing, which raises the reservation price of agents who borrowed to buy the asset. High values of ϕ thus discourage turnover.

While q, ϵ, and ϕ determine whether bubbles *can* be noisy, it is clear from Figure 3 that when noisy bubbles are possible, they occur only when either n_1 or n_2 are large. Intuitively, noisy bubbles require that either a lot of traders buy at date 1 who can later sell, or else a large number arrive and wish to buy assets at date 2. Recall from Subsection 3.1 that other things equal, higher values of n_1 and n_2 imply the asset will be more overvalued. This implies that noisy bubbles will tend to be associated with assets that are more overvalued, while assets that are only slightly overvalued will be associated with quiet bubbles. What is not evident from Figure 3 is that too much overvaluation discourages noisy bubbles. In particular, if $n_1 + n_2 > q\epsilon D + (1-\epsilon) D/R$, we can no longer use (3) to ensure that all non-entrepreneurs will buy the asset. Although I omit the formal analysis of this case, with a large number of traders the expected profits from buying the asset will turn to zero. This will reduce the volume of trade associated with the asset.

Although either n_1 or n_2 must be large for bubbles to turn noisy, the role of n_1 and n_2 in contributing to noisy bubbles is not symmetric. On the one hand, holding the total number of non-entrepreneurs $n_1 + n_2$ fixed, a noisy bubble is more likely to arise if most traders arrive early rather than late, i.e. when n_1 is large rather than n_2. Intuitively, if a lot of traders show up at date 1, they will have to be those who sell the asset at date 2 if there is additional demand. But if few traders show up at date 1, additional demand at date 2 might still be potentially met by original owners. At the same time, the features most associated with historical episodes suspected to be bubbles arise when n_2 is large rather than n_1. First, recall from Section 3.2 that asset price growth is likely to be higher when n_1 is small. In addition, Figure 3 reveals that region $\mathbb{D}$ where speculative bubbles arise requires that n_2 be large relative to n_1. Thus, scenarios with rapid asset price growth and traders who are keen to both buy and sell assets are more likely to be associated with a rising trade volume rather than a falling one.

3.4 Borrowing Rates in Credit Markets

Finally, many of the historical episodes suspected to be bubbles are associated with low interest rates charged to those who borrow against supposedly bubble assets. This may seem at odds with the risk-shifting view of bubbles: Since risk-shifting requires creditors to charge higher spreads to cover losses on speculators, spreads should presumably be higher with bubbles than when bubbles are absent. However, my model implies that noisy bubbles can be associated with lower borrowing rates than quiet bubbles. Whether historical episodes should be associated with high or low borrowing spreads depends on whether the tranquil times against which these episodes are judged are periods in which asset prices reflect fundamentals or in which the assets exhibit quiet bubbles. In principle, as noted by Diba and Grossman (1987), assets that can potentially exhibit bubbles in the future must contain a bubble component in the present, even if it is arbitrarily small. Since we argued in Section 3.3 that small overvaluation is likely to be associated with quiet bubbles, it seems reasonable to view tranquil times as periods in which the asset is slightly overvalued.

To see that noisy bubbles can be associated with lower borrowing spreads than quiet bubbles, consider the effect of increasing the absolute number of non-entrepreneurs n_2 who arrive at date 2 while holding their share among total borrowers ϕ fixed. Graphically, this implies moving up along Figure 3. The higher is n_2, the larger the fraction of date 1 buyers who will have to sell their assets. Thus, as long as we are in an equilibrium of type (c), more borrowers will be offered a low rate r_{1*} rather than a high rate r_1^*, until eventually we move into an equilibrium of type (d) in which all the agents who buy at date 1 will sell at date 2 if the payoff on the asset remains uncertain. The intuition for this is due to dynamic risk-shifting: When traders turn around and sell the overvalued assets they buy, their lenders will anticipate that part of the risk from lending against an overvalued asset will be borne by future lenders. This allows them to charge lower rates to their borrowers. However, since noisy bubbles are more likely at high values of q, the lower rate r_{1*} may not be substantially lower than r_1^* since much of the risk is still associated with early periods. In the next section, where I let creditors choose from a larger set on contracts, I discuss another reason for why speculative bubbles may be associated with low spreads.

3.5 Summary

The preceding analysis shows that risk-shifting models of bubbles in which assets can trade above their fundamental value may give rise to patterns consistent with historical episodes taken to be bubbles, e.g. rapid price appreciation, speculative trading, and low spreads charged to those who borrow against assets used for speculative trading. However, an overvalued asset will not always exhibit these patterns. Assets that are only slightly overvalued will tend to trade infrequently and can exhibit low rates of price appreciation. More overvalued assets, especially those with a high trade volume and against which a large amount is borrowed, are those that are prone to become speculative bubbles. But since rapid appreciation and speculation can only be high when bubbles are likely to burst early, such patterns should be observed relatively infrequently.

4 Endogenous Contracts

Up to now, I allowed creditors to offer only fixed-size, full-recourse, simple debt contracts. In this section, I allow lenders to choose from a broader set of contracts. This serves two purposes. First, it offers a robustness exercise: Can speculative bubbles still emerge when lenders can choose from a broader set of contracts? Second, it reveals what types of contracts will emerge when speculative bubbles arise. To preview my results, I find that endogenizing contracts makes it harder but not impossible for bubbles to occur: Although lenders would like to avoid funding agents who plan to buy overvalued assets, they will not always be able to avoid doing so. When lenders cannot avoid taking on non-entrepreneurs, they will design contracts to minimize losses on non-entrepreneurs, and the purchase of overvalued assets will be financed using certain types of contracts. In particular, my model gives rise to Spence-Miyazaki-Wilson contracts, along the lines of Spence (1977), Miyazaki (1977), and Wilson (1977), whereby creditors earn profits on entrepreneurs that cross-subsidize the minimal losses possible on non-entrepreneurs.

Analyzing contract choice requires being more explicit about what agents know and can condition their contacts on. In line with my focus on debt contracts, I consider a costly state verification model as in Townsend (1979) and Gale and Hellwig (1985) that gives rise to debt-like contracts. That is, creditors cannot immediately observe what assets borrowers purchased. However, at the end of date 2, creditors can learn what assets the borrower purchased as well as their cash holdings by paying a cost. The auditing cost is assumed to be a fraction $\lambda < 1$ of the borrower's cash holdings, meaning it is less costly to audit agents with fewer resources to hide. Assuming a purely proportional cost is analytically convenient, since it implies an agent with no cash can be audited at no cost. Adding a fixed cost of verification would not fundamentally change my results as long as the fixed cost component was small, but it would raise the issue of stochastic auditing as a way to minimize auditing costs. I wish to avoid randomization, not just because it is rarely employed in practice but because it raises some technical issues that I discuss below. In what follows, I assume λ is close to 1. As I discuss below, a sufficiently high value for λ will deter lenders from auditing borrowers who claim to be entrepreneurs unless they fail to pay the amount required of them.

Appealing to the revelation principle, we can replicate any contract with a direct mechanism in which borrowers report their private information, and these reports trigger transfers as well as an audit strategy. Transfers can only depend on the borrower's actual private information if the lender audits the borrower. Contracts can of course still depend on publicly observable information, e.g. payments can depend on $\mathcal{I}_2$ or d even if creditors do not observe whether a given borrower bought risky assets. However, this conditioning is possible only because I've assumed one type of risky asset. With more than one type of risk asset, such conditioning might not be possible. In particular, suppose that we allowed for a continuum of ex-ante identical risky assets, all of which pay a single dividend d distributed according to (1). However, ex-post, only a fraction ϵ pay out $d = D$, and for a fraction q of each type of asset the value of d is revealed before date 2. Non-entrepreneurs who secure funds would prefer to concentrate in one asset rather than diversify, and each period non-entrepreneurs would spread out uniformly across all assets whose dividend remains uncertain. As long as we adjust the number of traders in period 2 to account for the fact that fewer assets will trade at date 2 than at date 1, we can reinterpret the model in the previous section with only one type of asset as one with different assets but no aggregate uncertainty and thus nothing for lenders to condition on.[12] Motivated by this interpretation, I consider contracts that do not condition on asset market variables. This also leads to a contract that more closely resembles a standard debt contract.[13]

Formally, a contract amounts to a sequence of announcements by borrowers at the various stages where they have private information, and a sequence of transfers and actions triggered by these announcements. I use $\widehat{\theta}_j$ denote the reports at stage $j = 1, 2, 3, ...$ and $x_j(\widehat{\theta}_j)$ as the transfer from the borrower following each report. Figure 4 summarizes the contract for agents who arrive at date 1 as a flow chart. The contract for agents who arrive at date 2 can be constructed similarly.

A contract begins with an agent reporting upon his arrival whether he is an entrepreneur (denoted $\widehat{\theta}_1 = e$) or not ($\widehat{\theta}_1 = n$). While I could allow non-entrepreneurs and non-participants to distinguish themselves, this turns out to be unnecessary. The contract then stipulates a transfer $x_1(\widehat{\theta}_1)$ from the creditor to the borrower. Since agents own no resources, $x_1(\widehat{\theta}_1) \geq 0$.

The next stage at which the agent has private information is after he chooses what to do with the funds he received. Assumptions (2) and (3) ensure agents will not be indifferent between actions, and will allocate all of the funds they receive to a single use. This simplifies what agents can report. An agent who reported he was an entrepreneur can only truthfully report that he either initiated entrepreneurial activity ($\widehat{\theta}_2 = e$) or did nothing ($\widehat{\theta}_2 = \varnothing$), and so without loss of generality I restrict agents to these two reports. An agent who reported he was not an entrepreneur can only truthfully report that he bought assets ($\widehat{\theta}_2 = b$) or did nothing ($\widehat{\theta}_2 = \varnothing$), so I restrict him to these two reports. Let $x_2(\widehat{\theta}_1, \widehat{\theta}_2)$ denote the transfer from the lender to the borrower following this report. If the agent reports that he used his funds, i.e. $\widehat{\theta}_2 \in \{e, b\}$, he would have no resources to transfer if he were truthful, so $x_2(\widehat{\theta}_1, \widehat{\theta}_2) \geq 0$. If the agent reports he did nothing, the constraint is $x_1(\widehat{\theta}_1) + x_2(\widehat{\theta}_1, \widehat{\theta}_2) \geq 0$, i.e. he cannot transfer more resources than he originally received.

For agents who arrive at date 1, the next stage at which they may receive private information is at the beginning of date 2, after the asset market clears but before d and R are paid out. At this point, an agent who reported buying assets, i.e. $\widehat{\theta}_2 = b$, must report whether he sold his assets before d was revealed ($\widehat{\theta}_3 = p_2$), sold them after learning d is high ($\widehat{\theta}_3 = D$), sold them after learning d is low ($\widehat{\theta}_3 = 0$), or held on to them ($\widehat{\theta}_3 = h$). An agent who reports $\widehat{\theta}_2 \in \{e, \varnothing\}$ would have nothing to report at this stage if he were truthful. For completeness, I let them report $\widehat{\theta}_3 = \varnothing$. Let $x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3)$ denote the net transfer from the lender to the borrower after report $\widehat{\theta}_3$ is submitted.

[12] In particular, if we want the mass of traders at date 2 to equal n_2, the number of traders who arrive must equal $n_2/(1-q)$.

[13] Allowing contracts to condition on the asset market would not change my qualititative results. In particular, it would lead creditors to charge high payments in states where non-entrepreneurs earn profits, e.g. if $d = D$, and low (even negative) payments in other states. However, the most a contract can demand from entrepreneurs in any state is R, and assumption (3) implies the return to buying the risk asset and holding it to maturity exceeds R. Hence, non-enteperneurs would want to trade with creditors if contracts were contingent on asset market data.

At the end of date 2, an agent must issue a final report on his earnings. I denote this report $\widehat{y}$. If the initial transfer $x_1(\widehat{\theta}_1) > 0$, it will be convenient to let $\widehat{y}$ denote earnings divided by $x_1(\widehat{\theta}_1)$. If $x_1(\widehat{\theta}_1) = 0$, the agent cannot do anything anyway, so I can restrict him to reporting $\widehat{y} = 0$. When $x_1(\widehat{\theta}_1) > 0$, five reports are possible for $\widehat{y}$: $R - 1$, if the agent chose to produce; $p_2/p_1 - 1$, if the agent bought and sold assets before learning d; $D/p_1 - 1$, if $d = D$ and the agent did not sell the assets before d was revealed; -1, if $d = 0$ and the agent did not sell the asset before d was revealed; and 0, if the agent did nothing. I let the agent report any of these $\widehat{y}$ regardless of previous reports. In particular, I need to let an agent who bought assets but pretended to engage in entrepreneurial activity come clean if $d = 0$. After these reports, the lender can choose whether to audit the agent. This means that the contract must specify an auditing rule $\sigma_{\widehat{y}}(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y})$ equal to 1 if the agent is audited and 0 otherwise. I do not allow for stochastic audits where $0 < \sigma_{\widehat{y}} < 1$. This restriction prevents a lender from offering a contract with a slightly higher audit probability and a slightly lower interest rate that would attract entrepreneurs who have no reason to fear auditing. Allowing lenders to offer such contracts would imply no equilibrium exists, for the reason pointed out in Rothschild and Stiglitz (1976): Any potential equilibrium would be vulnerable to cherry-picking of good types. Restricting contracts to deterministic audits is one way to address this non-existence problem. Other approaches would allow for stochastic auditing, but are considerably more involved.[14]

Finally, the contract stipulates a transfer $x_{\widehat{y}}(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y})$ from the lender to the borrower if there is no audit, and a transfer $x_y(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y}, y)$ if there is an audit, where y denotes actual earnings.

Now that I've described contracts, I can proceed to characterize equilibrium contracts. Since the lender chooses contracts optimally, I begin with some observations about optimal contracts. First, suppose a borrower announces he has no cash at the end of date 2, i.e.

$$x_1(\widehat{\theta}_1) + x_2(\widehat{\theta}_1, \widehat{\theta}_2) + x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3) + \widehat{y}x_1(\widehat{\theta}_1) = 0 \tag{15}$$

If this report is truthful, the agent will not be able to make any transfers, and so $x_{\widehat{y}}(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y}) \geq 0$. As such, it will be optimal for the lender to audit the borrower and grab all of his resources if he is untruthful. The reason is that if the agent is telling the truth then the audit is costless, while if he is not telling the truth the agent can be punished. Since $\lambda < 1$, auditing and punishing an agent is never costly for the lender. At the same time, punishing false reports relaxes incentive constraints, so the lender should do it.

A second observation is that a lender can use transfers to limit the misreports agents can make. In particular, if $x_1(\widehat{\theta}_1) > 0$, the lender can confirm a report that the agent did nothing with his funds, i.e. $\widehat{\theta}_2 = \varnothing$, by asking the agent to transfer back his resources, i.e. $x_2(\widehat{\theta}_1, \widehat{\theta}_2) = -x_1(\widehat{\theta}_1)$. This would only

[14] Wilson (1977), Miyazaki (1977), and Spence (1977) suggested dealing with this non-existence by redefining equilibrium so that an equilibrium contract cannot be improved upon only by contracts that remain profitable when unprofitable contracts are withdrawn. Mimra and Wambach (2011) and Netzer and Scheuer (2013) provide game-theoretic foundations for this idea by allowing lenders withdraw contracts (possibly at a small cost) after observing the contracts offered by others.

be feasible if $\theta_2 = \varnothing$. Subsequent transfers can then be used to achieve any desired terminal payoff for this type. Likewise, the lender can confirm whether an agent who bought assets at date 1 sold them at date 2 for a positive price, i.e. $\widehat{\theta}_3 \in \{p_2, D\}$, by asking the agent to transfer more than $x_1(\widehat{\theta}_1)$ after this announcement, i.e. $x_2(\widehat{\theta}_1, \widehat{\theta}_2) < x_1(\widehat{\theta}_1)$. Again, this would only be feasible if the agent in fact bought and sold assets. Lastly, the lender can verify whether the agent earned D per asset by requiring an agent who reports $\widehat{y} = D/p_1 - 1$ to transfer D/p_1. The upper bound on R in (2), together with (3), puts a limit on the price p_2, implying that $D/p_2 > R$. Thus, a non-entrepreneur who buys assets has enough funds to pass himself off as an entrepreneur, but not vice-versa. Note that confirming an agent earned $y = D/p_1 - 1$ may require an additional transfer after $x_{\widehat{y}}(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y})$ to achieve a given payoff to the borrower, i.e. the agent and lender must transfer resources back and forth. These transfers greatly limit the type of misreports by agents and thus reduce the number of incentive constraints the contract is subject to.

The last observation is that it will be optimal for the lender to audit the agent and seize all of his cash if the agent is ever inconsistent in his reports, e.g. announcing that he bought assets but then reporting $\widehat{y} = R - 1$. This implies that in equilibrium, an agent who reports he chose to produce, i.e. $\widehat{\theta}_2 = e$, will have to pay some amount if he reports his income is $R - 1$, and will be audited and left with no cash otherwise. Likewise, an agent who reports he bought assets, i.e. $\widehat{\theta}_2 = b$, will have to pay some amount that may depend on what he says he did with the assets he bought, and will be audited and left with no cash otherwise.

We can use the observations above to determine when the equilibrium contract will call for an audit. First, without loss of generality, an agent who announces $\widehat{y} = -1$ without contradicting past reports will be audited. This is because we can always take the equilibrium contract and construct an equivalent contract with $x_2(\widehat{\theta}_1, \widehat{\theta}_2) = x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3) = 0$ for reports consistent with a report of $\widehat{y} = -1$, ensuring the agent will have no cash at the end of date 2 if the true $y = -1$. This allows the lender to audit at no cost if the agent is truthful while discouraging misreports. If we set $x_{\widehat{y}}(\cdot)$ to ensure the same terminal payoff to a truthful agent as in the original contract, this contract will do at least as well as the original equilibrium contract.

Next, suppose the agent reports $\widehat{y} = 0$ or $\widehat{y} = D/p_1 - 1$. Since the lender can directly confirm these reports using transfers, neither report will be audited in equilibrium. The lender cannot fully verify $\widehat{y} = p_2/p_1 - 1$, since an agent who sells his assets early at price D can pass himself as this type. Since lenders prefer non-entrepreneurs sell their assets if d is not revealed, they may want to audit such reports to reward agents. For now, I take no stand on whether this report will be audited, although I argue below that it will not.

Finally, suppose the agent reports $\widehat{y} = R - 1$. As anticipated above, it will be optimal not to audit the borrower following this report if λ is close to 1. For suppose the optimal contract did audit agents making this report. As $\lambda \to 1$, the rate that must be charged to entrepreneurs to ensure non-negative profits must tend to R, since auditing costs on these agents are equal to λR while the expected profit from lending to non-entrepreneurs is negative. At the same time, the first inequality in (2) implies there exists a rate strictly below R such that charging this rate to all borrowers and collecting it in full from entrepreneurs and with probability ϵ from non-entrepreneurs will yield positive profits. Since the latter is a lower bound on the profits of the lender, it follows that as $\lambda \uparrow 1$, not auditing a report of $\widehat{y} = R - 1$ will be optimal.

It follows that equilibrium contracts are essentially variable-size debt contracts with a repayment schedule: An agent who arrives at date 1 and reports $\widehat{\theta}_1 \in \{e, n\}$ receives $x_1^{\widehat{\theta}_1}$ resources together with a particular repayment schedule. An agent who reports $\widehat{\theta}_1 = e$ will be asked to pay back $(1 + r_1^e)\, x_1^e$ at the end of date 2 or else be audited. An agent who reports $\widehat{\theta}_1 = n$ will be asked to pay back an amount that depends on when he pays. If he waits until the end of period 2, meaning he reported holding on to his assets ($\widehat{\theta}_3 = h$), he will have to pay $x_1^n(1 + r_1^{n,\text{late}})$ and will be audited otherwise. If he pays at the beginning of date 2, meaning he sold assets for a positive price, he would owe $x_1^n(1+r_1^{n,D})$ if he sold at price D and a potentially different amount $x_1^n(1+r_1^{n,p_2})$ if he sold at price p_2. If he pays earlier still at date 1, meaning he did nothing ($\widehat{\theta}_2 = \varnothing$), he would owe $x_1^n(1 + r_1^{n,\varnothing})$. That is, the borrower either receives x_1^e with a requirement to repay at rate r_1^e at the end of date 2, or he receives x_1^n with a rate schedule $\left\{r_1^{n,\varnothing}, r_1^{n,D}, r_1^{n,p_2}, r_1^{n,\text{late}}\right\}$. This description abstracts from the back-and-forth transfers to verify the borrower's reports that are needed to sustain the contract, but it correctly capture the terminal payoffs to the lender.

By a similar argument, an agent who arrives at date 2 receives a variable-size debt contract depending on his report $\widehat{\theta}_1 \in \{e, n\}$ that specifies an initial transfer $x_2^{\widehat{\theta}_1}$ and an amount he must repay at the end of period 2, but which might charge a different rate if he shows he did nothing with the funds he received.

Given the optimal audit rules I derived above, agents who report $\widehat{\theta}_1 = n$ will at some point be either audited or forced to make a transfer that reveals their identity. As such, we need not worry about whether entrepreneurs will try to report $\widehat{\theta}_1 = n$. Instead, the relevant incentive constraints are that non-entrepreneurs not try to pass themselves as entrepreneurs or misreport when they sold assets or for how much. Alas for lenders, these incentive constraints have real bite, since non-entrepreneurs can secure themselves positive expected profits from pretending to be entrepreneurs. In particular, they can buy risky assets and hold them to the end of date 2, then pay back no more than $R < D/p$ if $d = D$, all without being audited. Creditors who want to to fund entrepreneurs thus have no choice but to also take on non-entrepreneurs and offer them rents. However, it does not follow from this that once creditors finance entrepreneurs, non-entrepreneurs must end up buying risky assets. Rather, creditors can provide rents to non-entrepreneurs by paying them off rather than lending to them. Formally, they can offer $r_1^{n,\varnothing} < 0$, essentially paying non-entrepreneurs not to buy assets. The next proposition establishes that if all agents without a production opportunity can buy risky assets – meaning all can participate in the asset market – it would indeed be optimal for creditors to bribe non-entrepreneurs not to buy the asset, and a bubble would not emerge.

Proposition 2: If there are no non-participants, allowing lenders to choose contracts optimally implies that the equilibrium price of the asset must be $p_1 = p_2 = \epsilon D$, i.e. bubbles will not emerge.

Intuitively, letting a non-entrepreneur buy an overvalued asset is a costly way for the creditor to provide him with rents: When the asset is overvalued, the creditor ends up paying some resources to whoever the non-entrepreneur buys the asset from beyond the rents he provides the non-entrepreneur. Thus, where there is a bubble, it is more efficient to pay agents directly than to reward them by having them buy assets.

Paying agents not to buy risky assets may not work well in practice, however. The reason is that it may draw in people who have no intention to speculate. Here is where non-participants finally play a role. While they do nothing in equilibrium, their presence precludes creditors from paying non-entrepreneurs not to speculate. As long as there are enough non-participants, the cost of having to pay them together with non-entrepreneurs will be enough to make contracts with $r_1^{n,\varnothing} < 0$ too costly, and creditors who need to deliver rents to non-entrepreneurs will choose to do so by letting the latter purchase risky assets.

Formally, if there is a large enough group of non-participants, any contract must satisfy the property that if the cumulative transfers from the creditor to the borrower are positive – i.e. if the lender ends up giving any gift to the borrower – it must be that the borrower has zero cash holdings at the end of date 2. Otherwise, agents with no intention to buy risky assets will also accept the contract. Since an agent with zero cash holdings will be audited in equilibrium, this restriction can be formalized as follows. First, if there is no audit, net transfers from the lender to the borrower must be non-positive:

$$x_1(\widehat{\theta}_1) + x_2(\widehat{\theta}_1, \widehat{\theta}_2) + x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3) + x_{\widehat{y}}(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{y}) \leq 0 \tag{16}$$

Second, if there is a positive transfer of resources from the creditor to the agent with auditing, then the terminal cash holdings of the agent must be 0, i.e.

$$x_1(\widehat{\theta}_1) + x_2(\widehat{\theta}_1, \widehat{\theta}_2) + x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3) + x_y(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, y) > 0 \tag{17}$$

implies

$$x_1(\widehat{\theta}_1) + x_2(\widehat{\theta}_1, \widehat{\theta}_2) + x_3(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3) + x_y(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, y, y) + y \cdot x_1(\widehat{\theta}_1) = 0 \tag{18}$$

Under these conditions, non-participants will never find it profitable to borrow. We can therefore ignore them for the remainder of our discussion: Creditors will never have to engage them, and their only role comes from imposing the additional constraints (17) and (18) on contracts.

Since the first inequality in (2) ensures it will be profitable to finance entrepreneurs, we can conclude that $x_1^e = x_2^e = 1$, i.e. entrepreneurs receive the maximum amount of funding they can employ productively. Otherwise, a lender could simply scale x_t^e and x_t^n up proportionally but charge entrepreneurs slightly more than under the original contract, a small enough increase that they would still prefer to borrow a larger amount at a slightly higher rate. Since entrepreneurs cannot pass themselves off as non-entrepreneurs, they will choose the contract designed for them. Non-entrepreneurs will prefer the contract offered to them given that under the original equilibrium they preferred the contract offered them when entrepreneurs were charged a lower rate. Since scaling up both loans and leaving interest rates fixed would leave profits at zero, charging entrepreneurs higher rates ensures positive profits. Hence, $x_t^e < 1$ cannot be an equilibrium.

The only remaining terms of the contracts offered to entrepreneurs are the rates r_1^e and r_2^e. Since lenders earn zero expected profits in equilibrium, the amount collected from entrepreneurs must equal the expected losses on non-entrepreneurs. This, in turn, depends on the terms offered to non-entrepreneurs. To solve for these, I begin with a preliminary result. Let $V_t(\theta_1, \widehat{\theta}_1)$ denote the expected utility to an agent of type θ_1 who announces his type as $\widehat{\theta}_1$ and then behaves optimally. The next lemma establishes that in equilibrium, a non-entrepreneur will be indifferent about what type he announces:

Lemma 2: In equilibrium, $V_t\,(n, n) = V_t\,(n, e)$

Lemma 2 implies that the equilibrium contract offered to non-entrepreneurs solves the problem of maximizing the expected revenue from non-entrepreneurs while leaving the non-entrepreneur with the same utility as if he reported $\widehat{\theta}_1 = e$. For suppose the equilibrium contract did not solve this problem, i.e. there exists a contract x' that offers identical terms to entrepreneurs and leaves non-entrepreneurs with the same utility $V_t\,(n, e)$ but collects more revenue from them. A lender could then offer a contract x'' equivalent to x' but which charges non-entrepreneurs slightly less so that their utility under x'' is still higher than $V_t\,(n, e)$. This allows the lender to also charge entrepreneurs slightly less than under the original equilibrium contract without inducing non-entrepreneurs to report $\widehat{\theta}_1 = e$. Since the original contract earned zero profits, x' would generate positive profits if it attracted both types since it collects more from non-entrepreneurs but the same amount from entrepreneurs. Since the profits from x'' can be made arbitrarily close to the profits from x' when it attracts both types, it follows that the original contract could not have been an equilibrium. Essentially, the creditor would like to grab as much resources from non-entrepreneurs to make his contract as attractive as possible to entrepreneurs.

Applying this insight, we can solve for the equilibrium contracts offered to agents who show up at date 2. In particular, the terms offered to non-entrepreneurs who arrive at this date can be summarized by a loan size x_2^n and a repayment rate r_2^n that satisfy

$$\max_{x_2^n, r_2^n} \; x_2^n \left[\epsilon \left(1 + r_2^n\right) - 1\right] \tag{19}$$

subject to

$$\begin{array}{ll} \text{i.} & \epsilon\left(\dfrac{D}{p_2} - \left(1 + r_2^n\right)\right) x_2^n = \epsilon\left(\dfrac{D}{p_2} - \left(1 + r_2^e\right)\right) \\ \text{ii.} & r_2^n \geq 0 \end{array}$$

Constraint (i) corresponds to the requirement that $V_2\,(n, n) = V_2\,(n, e)$. Constraint (ii) follows from the requirement that rates cannot be negative given (17) and (18). Solving (19) yields

Proposition 3: In an equilibrium where $p_2 > \epsilon D$, non-entrepreneurs will be offered a contract in which $x_2^n = \frac{D-(1+r_2^e)p_2}{D-p_2} < 1$ and $r_2^n = 0$, while entrepreneurs will be offered a contract where $x_2^e = 1$ and r_2^e solves

$$(1-\phi)\, r_2^e + x_2^n \phi\, (\epsilon r_2^e - (1-\epsilon)) = 0.$$

To understand this result, observe that creditors always want to lend less to non-entrepreneurs, but they need to charge a lower rate on such loans to ensure non-entrepreneurs choose these contracts which may not make them so profitable. When $p_2 > \epsilon D$, it turns out that lenders do not need to lower rates too much to attract non-entrepreneurs to smaller loans. In particular, lenders will find it optimal to lower interest rates and offer non-entrepreneurs smaller loans, up to the point where they hit the constraint that interest rates must be nonnegative. This reveals a distinct reason for why speculative bubbles might be associated with low spreads between the borrowing rate and the risk-free rate: In order to steer agents who intend to buy risky assets towards contracts that incur smaller losses for lenders, these loans must carry low rates.

Remark 5: The fact that lenders do not need to worry about entrepreneurs pretending to be non-entrepreneurs plays an important role behind Proposition 3. If entrepreneurs could pass themselves off as non-entrepreneurs, e.g. if they could engage in hidden borrowing or if we precluded back-and-forth transfers that would allow non-entrepreneurs to prove their types, we would need to add a constraint to (19) to insure the entrepreneur does not prefer to take the smaller loan at a low rate, i.e.

$$R - (1 + r_2^e) \geq (R - (1 + r_2^n))\, x_2^n$$

Comparing this constraint with (i) reveals that the nature of the equilibrium contract depends on how D/p_2 compares to R. When $D/p_2 > R$, the fact that non-entrepreneurs are just indifferent between a higher rate on a large loan and a zero rate on a small loan implies entrepreneurs would prefer the smaller loan. In that case, in equilibrium both types will be offered the same contract, i.e. $x_2^n = x_2^e = 1$ and $r_2^e = r_2^n = \dfrac{(1-\epsilon)\,\phi}{1-(1-\epsilon)\,\phi}$. That is, if lenders are precluded from using transfers to restrict what borrowers can report, the equilibrium contract terms may be the same for both types. ■

Finally, consider contracts offered to agents who arrive in period 1. A few observations allow me to simplify the characterization of these contracts. As in the previous section, in equilibrium lenders must believe the non-entrepreneurs they lend to will either hold or sell their assets at date 2 with certainty if d is not revealed. For lenders who expect a borrower to hold on to his assets, the equilibrium contract can be described as a simple *long-term* debt contract $\{x_1^n, r_1^{n,\text{late}}\}$ in which the agent receives x_1^n upon arrival and is asked to repay $x_1^n \left(1 + r_1^{n,\text{late}}\right)$ at the end of date 2 or else be audited. To see this, denote the original loan amount by $\widehat{x}_1^n$. Since rates must be nonnegative, agents will prefer buying assets to doing nothing, so we can ignore the rate for repayment at date 1. Denote the remaining rates by $\{\widehat{r}_1^{n,D}, \widehat{r}_1^{n,p_2}, \widehat{r}_1^{n,\text{late}}\}$. Construct a new contract $\{x_1^n, r_1^{n,\text{late}}\}$ where $x_1^n = \widehat{x}_1^n$ and $r_1^{n,\text{late}} = q\widehat{r}_1^{n,D} + (1-q)\,\widehat{r}_1^{n,\text{late}}$. If the borrower holds on to his assets, he will only repay if $d = D$. Under the original contract, he would have expected to pay $q\widehat{r}_1^{n,D} + (1-q)\,\widehat{r}_1^{n,\text{late}}$ with probability ϵ. The same is true for the new contract. If the agent prefers to sell his assets under the new contract if d is not revealed, the original contract could not have been optimal, since the new contract recoups the same expected amount but with a larger probability. Since all agents are indifferent between the new contract and the original equilibrium contract, and the trading strategy of non-entrepreneurs is unchanged, the contract $\{x_1^n, r_1^{n,\text{late}}\}$ is equivalent to the original equilibrium contract.

Next, consider a borrower who intends to sell his assets. Here the equilibrium contract can be represented using a simple *short-term* debt contract $\{x_1^n, r_1^{n,\text{early}}\}$ in which the agent receives x_1^n upon arrival and is asked to repay $x_1^n\left(1 + r_1^{n,\text{early}}\right)$ at the beginning of date 2 or else be audited and be stripped of all of his cash holdings. To see this, denote the terms of the equilibrium contract by $\widehat{x}_1^n$ and $\{\widehat{r}_1^{n,D}, \widehat{r}_1^{n,p_2}, \widehat{r}_1^{n,\text{late}}\}$. Construct a new contract $\{x_1^n, r_1^{n,\text{early}}\}$ where $x_1^n = \widehat{x}_1^n$ and $r_1^{n,\text{early}} = \frac{1-q}{q\epsilon+(1-q)}\widehat{r}_1^{n,p_2} + \frac{q\epsilon}{q\epsilon+(1-q)}\widehat{r}_1^{n,\text{late}}$. If the borrower intends to sell his assets, this contract would ensure the same expected repayment. But a borrower who accepts a short-term contract would necessarily sell the asset early at date 2. Hence, the short-term contract $\{x_1^n, r_1^{n,\text{short}}\}$ is equivalent to the original contract, since all agents are indifferent between this and the original equilibrium contract and it has no effect on the trading strategy of non-entrepreneurs.

We can therefore derive the terms for non-entrepreneurs using a problem analogous to (19) in which the contract involves a loan size x_1^n and a single interest rate, either $r_1^{n,\text{early}}$ or $r_1^{n,\text{late}}$ depending on what the lender believes the borrower will do with the asset at date 2. Solving these reveals the following:

Proposition 4: Non-entrepreneurs who arrive at period 1 will receive the following contracts:

1. Non-entrepreneurs who intend to keep their assets will receive a long-term contract with $x_1^n < 1$ and $r_1^{n,\text{late}} = 0$ in any equilibrium where $p_1 > \epsilon D$. If $p_1 = \epsilon D$, the value of x_1^n is not uniquely determined, but a pooling contract where $x_1^n = x_1^e = 1$ and $r_1^{n,\text{late}} = r_1^e$ is an equilibrium.
2. Non-entrepreneurs who intend to sell their assets will receive a short-term contract with $x_1^n < 1$ and $r_1^{n,\text{early}} = 0$ in any equilibrium where $p_1 > q\epsilon D + (1-q)\,p_2$. If $p_1 = q\epsilon D + (1-q)\,p_2$, the value of x_1^n is not uniquely determined, but a contract where $x_1^n = x_1^e = 1$ and $r_1^{n,\text{early}} = r_1^e$ is an equilibrium.

There are two differences between date-1 contracts and date-2 contracts. First, there is now another dimension that creditors can exploit to reduce their expected losses on non-entrepreneurs, namely shortening the maturity of the loan. Requiring lenders to repay early forces them to sell the asset, which the lender would prefer since it minimizes his exposure to losses if the purchase of the asset turns out to be unprofitable. Second, when non-entrepreneurs intend to sell the asset at date 2 if $\mathcal{I}_2 = \varnothing$, reducing the loan size may not be profitable if the non-entrepreneur is just indifferent about selling the asset. In this case, any benefit to the creditor from reducing the loan size x_1^n is fully offset by the reduction in interest rate r_1^n.

Remark 6: The fact that contracts are separating even though creditors incur losses on some contracts may seem surprising: Why don't creditors offer only the contracts that entrepreneurs accept? The reason is that from Lemma 2, non-entrepreneurs are indifferent between the two contracts. Thus, if a creditor unilaterally withdrew the contract offered to non-entrepreneurs, he could not be sure that he won't end up with a non-entrepreneur. A fully specified equilibrium with contracts must include the beliefs of creditors on whether they can avoid non-entrepreneurs by withdrawing contracts, and in equilibrium creditors cannot believe they can dodge non-entrepreneurs. Another question is why creditors don't exclusively offer non-entrepreneurs short-term contracts and force them to sell early. The answer is that not all non-entrepreneurs can sell the asset in equilibrium, and so again, in equilibrium creditors must believe that only offering short term contracts will lead non-entrepreneurs to choose the contract offered to entrepreneurs.

The analysis above only considers what type of contracts will be offered in equilibrium. A full analysis would need to solve for the equilibrium prices of the asset $p_t(\mathcal{I}_t)$. Here, there would be little innovation over the analysis of the previous section: Depending on parameter values, there will be equilibria in which non-entrepreneurs all sell their assets, all hold on to their assets, or some hold and some sell if d remains uncertain at the beginning of date 2. For brevity, I skip the analysis.

To conclude, allowing lenders to design contracts reveals that this makes it more difficult, but not impossible, for bubbles to arise. Creditors will try to avoid entering into financial arrangements with agents who intend to buy overvalued assets, and would try to dissuade borrowers from buying overvalued assets if possible. However, if lenders cannot just pay agents not to speculate, bubbles can emerge in equilibrium. In these cases, lenders would want to offer contracts that minimize the losses they incur from speculators. Specifically, the contracts that emerge involve lower rates on smaller, shorter-term loans. Barlevy and Fisher (2012) build on this last observation and show that in a risk-shifting model adapted to features of the housing market that my model here ignores, there is a similar tendency for speculative bubbles to be associated with contracts that encourage early repayment. They find that such contracts were indeed used heavily in cities with house price booms, but these contracts were used only sparingly elsewhere.

5 Conclusion

The distinguishing feature of equity and housing markets in recent years has been the rapid run-up in the prices of these assets followed by an equally sharp collapse in these prices. While these episodes are often described as "bubbles" in the popular press, economists who reserve the term "bubble" to mean a deviation between the price of an asset and its underlying value should find it less obvious whether these episodes were associated with overvaluation. In particular, an alternative potential explanation for volatile asset prices is that the underlying fundamentals are themselves highly volatile. This is particularly true in models where different agents value the same asset differently for whatever reason. In that case, if the price of the asset was equal to the value that the marginal trader assigns to the asset, shocks that change the identity of the marginal trader can result in especially violent swings in asset prices, a theme explored in recent work by Geanakopolos (2009), Burnside, Eichenbaum, and Rebelo (2011), and Albagli, Hellwig, and Tsyvinski (2011). Thus, one could potentially understand these episodes without having to appeal to the notion that asset prices are unhinged from the underlying valuation that some agent attaches to them. Yet the question of whether asset prices reflect fundamentals is important. If asset prices fail to reflect their underlying value, they signal to potential producers of the asset to produce a different amount of these assets than is socially optimal, i.e. than the value of providing the marginal trader with another asset. In that case, there may be a role for policy intervention that would not have a parallel when asset prices reflect volatile fundamentals. It is therefore important to gauge whether episodes often branded are bubbles correspond to mispricing.

This paper shows that a model in which the deviation between asset prices and fundamentals is due to risk-shifting concerns can generate some of the key aspects of these episodes – booms and busts, high turnover in asset positions, and low spreads charged to those who borrow against these assets. Thus, the analysis here suggests that overvaluation of assets remains a viable explanation for these episodes. Importantly, the model is potentially falsifiable: These patterns do not arise in general, but only under certain conditions. The model also implies that to the extent that a boom-bust pattern reflects a speculative bubble, agents who borrow to speculate would be offered certain types of contracts. Failure to observe these contracts during speculative episodes would also provide evidence against the model.

To be sure, since the model developed here is quite stylized, the contracts it predicts will be used when assets are overvalued may not match the type of contracts we observe in practice for reasons that need to be incorporated into the model. However, the intuition behind the results is likely to carry over to more realistic settings. For example, the model developed here assumes that creditors have very limited information on what borrowers do with the funds they secure. This assumption is clearly implausible in some settings, e.g. housing markets, where creditors can and typically do inspect the assets agents post as collateral. However, risk-shifting can arise in settings where imperfect information pertains to variables other than the actual assets themselves, e.g. creditors may lack information about the preferences or income flows of the agents they lend to. Barlevy and Fisher (2012) pursue this idea in a model of housing, where creditors can observe the assets agents buy but are uncertain as to how much a given borrower values owning a home. They find that the type of contracts one might expect if assets are overvalued, which are more elaborate than the type of contracts that emerge here, empirically did seem to be used more heavily in places where house price appreciation was unusually high. Further work is needed to relate dynamic risk-shifting models like the ones examined here to more realistic settings.

Appendix

Lemma A1: Let p_t denote the equilibrium price at date t if $\mathcal{I}_t = \varnothing$. Then $p_t \geq \epsilon D$.

Proof of Lemma A1: Suppose $p_t < \epsilon D$. None of the original owners would agree to sell assets. Thus, demand for assets must be zero for the market to clear, so the expected utility to a non-entrepreneur must be zero in equilibrium. However, suppose a lender at date t were to offer to lend out one unit of resources at rate $r_t = \dfrac{1}{\epsilon} - 1$. Since $p_t < \epsilon D$, it follows that

$$(1 + r_t)\, p_t < \frac{1}{\epsilon}\,(\epsilon D) = D$$

This implies a non-entrepreneur could earn positive profits, and since he earns zero in equilibrium, he would accept this contract. The expected profits to the lender on this loan would equal

$$\epsilon\left(\frac{1}{\epsilon}\right) - (1 - \epsilon) = \epsilon$$

If entrepreneurs also chose this contract, profits would be even greater, since they will repay $\min\left(\dfrac{1}{\epsilon}, R\right) > 1$. Since no lender can expect to earn positive expected profits in equilibrium, it follows that $p_t < \epsilon D$ cannot be an equilibrium.

Proof of Lemma 1: Suppose $n_1 + n_2 \leq \epsilon D$. We can verify that $p_1 = p_2 = \epsilon D$ is an equilibrium in this case: Original owners are indifferent between holding and selling and are willing to supply any amount. Since the second inequality in (2) implies $\epsilon D < D/R$, non-entrepreneurs strictly prefer to buy the asset at these prices, since they can ensure themselves positive expected profits by buying and holding the asset. Finally, since $\frac{n_1}{\epsilon D} + \frac{n_2}{\epsilon D} < 1$, the demand by non-entrepreneurs can be fully met by original owners selling their asset.

At the same time, no price $p_t > \epsilon D$ can be an equilibrium. For suppose this were the case. By Lemma A1, $p_t \geq \epsilon D$ for all t. Then either $p_2 > \epsilon D$ or $p_2 = \epsilon D$ and $p_1 > \epsilon D$. In either case, all original owners would have sold their asset holdings by date 2. But since $p_t \geq \epsilon D$, we have that $\frac{n_1}{p_1} + \frac{n_2}{p_2} < \frac{n_1}{\epsilon D} + \frac{n_2}{\epsilon D} \leq 1$, which is a contradiction.

Finally, since $p_1 = p_2 = \epsilon D < D/R$, the statement of the lemma holds for these values of (n_1, n_2)

Next, suppose $\epsilon D < n_1 + n_2 < (1 - q)\, D/R + q\epsilon D$. I first argue that $p_1 < n_1 + n_2$. Since $n_1 + n_2 < (1 - q)\, D/R + q\epsilon D < D/R$ given (2) and (3), this implies the lemma holds at date 1. First, if $p_1 = \epsilon D$, the statement follows. So suppose $p_1 > \epsilon D$. As argued above, in this case $\frac{n_1}{p_1} + \frac{n_2}{p_2} \geq 1$. There are two cases to consider. First, if the asset is traded at date 2, then it must be the case that $p_2 \geq p_1$. Otherwise, all the original owners would sell at date 1, and none of the non-entrepreneurs who bought the asset at date 1 would agree to sell it at date 2 if $p_2 < p_1$ since then they would have to default. But if $p_2 \geq p_1$, it follows that $\frac{n_1}{p_1} + \frac{n_2}{p_2} \leq \frac{n_1}{p_1} + \frac{n_2}{p_1}$. It follows that $1 \leq \frac{n_1}{p_1} + \frac{n_2}{p_1}$, i.e. $p_1 \leq n_1 + n_2$, as claimed. Next, suppose the asset is not traded at date 2. This requires that $p_2 \geq D/R$, or else non-entrepreneurs could earn positive profits from buying the asset at date 2. But since $p_1 > \epsilon D$, and there is no trade at date 2, this requires that all original owners sell their assets at date 1. Since demand is given by x/p_1 where $x \leq n_1$ is the amount non-entrepreneurs borrow at date 1, market clearing implies $x/p_1 = 1$, and so $p_1 = x \leq n_1 < n_1 + n_2$, as claimed.

The last step is to show that if $n_1 + n_2 > \epsilon D$, then $p_2 \leq D/R$. If $p_2 = \epsilon D$, the statement follows from (2) and (3). If $p_2 > \epsilon D$, then $p_1 > \epsilon D$, or else there would be excess demand for the asset at date 1 given the asset is worth at least ϵD regardless of whether d is revealed, so all agents would strictly prefer to buy it at a price of $p_1 = \epsilon D$. Moreover, since the original owners can wait to sell, the fact that some agents will sell at date 1 requires that $p_1 \geq q\epsilon D + (1-q)\,p_2$. But this implies

$$\begin{aligned} p_2 &\leq \frac{p_1 - q\epsilon D}{1-q} \\ &\leq \frac{n_1 + n_2 - q\epsilon D}{1-q} \\ &\leq \frac{(1-q)\,D/R}{1-q} = \frac{D}{R} \end{aligned}$$

The claim follows. ■

Proof of Proposition 1: I first argue that the restrictions (2) and (3) ensure that in equilibrium, entrepreneurs will produce in both periods and that non-entrepreneurs in period 1 buy risky assets. I then use these observations to set up and solve for the equilibrium for all (n_1, n_2) in the set

$$\mathcal{N} \equiv \left\{(n_1, n_2) \in \mathbb{R}^2_{++} : n_1 + n_2 < (1-q)\,D/R + q\epsilon D\right\} \tag{20}$$

As discussed in the text, the price of the asset if $\mathcal{I}_2 = d$ must equal d. Hence, the only relevant prices are at $\mathcal{I}_1 = \varnothing$ and $\mathcal{I}_2 = \varnothing$. Denote these by p_1 and p_2, respectively. Lemma A1 implies that $p_1 \geq \epsilon D$ and $p_2 \geq \epsilon D$.

I now argue that equilibrium interest rates are at most $\overline{r} = \dfrac{\phi\,(1-\epsilon)}{1-\phi\,(1-\epsilon)}$ in both periods. In period 2, if some lender charges above $\overline{r}$, another lender could charge $r_2 = \overline{r} + \varepsilon$. For ε sufficiently small, they would be assured positive profits. This is because when the equilibrium contract exceeds $\overline{r}$, there exists an ε that would attract both entrepreneurs and non-entrepreneurs away from the equilibrium contract. The expected payoffs to the creditor are given by

$$(1-\phi)\,r_2 + \phi\,[\epsilon\,(1+r_2) - 1] = [(1-\phi) + \phi\epsilon]\,\varepsilon > 0$$

Next, at date 1, if lenders charged more than $\overline{r}$, a lender could once again make positive expected profits by charging $r_1 = \overline{r}+\varepsilon$ and attracting both types. This contract would yield them a profit of $[(1-\phi)+\phi\epsilon]\,\varepsilon > 0$ if non-entrepreneurs held on to their assets at date 2, and even higher profits if non-entrepreneurs sold their assets at date 2. This is because in the limiting case where $k \to 0$, agents will not sell their assets unless they can repay their debt in full, so lenders recover the full debt obligation if agents sell the asset.

Assumption (2) tells us that $R > 1+\overline{r}$. From Lemma 1, it follows that

$$D/p_t > R > 1+\overline{r} \geq 1+r_t$$

Hence, all non-entrepreneurs will want to buy assets, since they can guarantee themselves positive expected profits by buying assets and holding them to maturity. Hence, demand for the assets at period t is equal to n_t/p_t: All non-entrepreneurs would wish to buy risky assets in equilibrium, and only non-entrepreneurs wish to buy.

Next, consider what happens when $\mathcal{I}_2 = \varnothing$. Non-entrepreneurs who arrive at date 2 could potentially earn information rents by buying the asset. They would have to buy these assets from either the original owners, who value the assets at ϵD, or non-entrepreneurs who bought the assets at date 1 and who value the assets at $\epsilon D + (1-\epsilon)(1+r_1)\,p_1$. Hence, four types of equilibria are possible in this state of the world: (a) at least some of the original owners hold on to them until maturity; (b) all original owners who own assets at date 2 sell them, but none of those who bought them back in period 1 sell them; (c) all original owners prefer to sell at date 2 and some but not all of those who bought the asset back in period 1 are are willing to sell; and (d) all those who own the asset at date 2 sell it. Each of these cases yield a system of equations that pins down p_1 and p_2.

Before laying out these equations and solving for the equilibrium values of p_1 and p_2, I derive some preliminary results for case (c) in which only some of those who buy assets in period 1 sell them in period 2, and which involves some important subtleties. I first argue that traders who buy the risky asset at date 1 must not randomize on whether to sell these assets at date 2. For suppose traders did randomize. This would mean such traders are indifferent between holding on to the assets for an expected profit of $\epsilon\left(\dfrac{D}{p_1}-(1+r_1)\right)$ and selling for an expected profit of $\dfrac{p_2}{p_1}-(1+r_1)$. Suppose a lender in period 1 were to offer a slightly lower interest rate than r_1 to the same trader. Since the trader was just indifferent about selling the asset at r_1, he would strictly prefer to sell at a lower rate. But this would allow the lender to earn higher profits, since the loss in interest rate can be infinitesimally small while the probability of recovering nearly $1+r_1$ jumps up discretely. It follows that in equilibrium there can only be two types of traders at date 1: Those who will definitely sell the asset at date 2 and those who will definitely hold on to it.

Next, I argue that in equilibrium the two types of traders cannot receive the same interest rate. For suppose they did. Let μ denote the fraction of agents who buy assets at date 1 that will definitely sell at date 2. Further, let π_0 and π_1 denote the expected profits from lending to those who will hold and those who will sell, respectively. Expected profits to the lender are given by

$$(1-\phi)\, r_1 + \phi\left((1-\mu)\,\pi_0 + \mu\pi_1\right)$$

If the interest rate is the same for both types, then $\pi_1 > \pi_0$. But then a lender can earn positive expected profits by offering a rate slightly below r_1: given some buyers are willing to sell the asset at date 2, at a slightly lower rate all buyers would prefer to sell, yielding the lender an expected profit that is arbitrarily close to

$$(1-\phi)\, r_1 + \phi\pi_1$$

which is strictly higher than the original equilibrium profits. So borrowers cannot all be charged a common rate in an equilibrium of type (c). Thus, in equilibrium, the two types must receive different rates. Moreover, they have to earn zero profits on each type. This implies that in equilibrium, non-entrepreneurs who sell the asset at date 2 receive a single rate r_{1*} that satisfies

$$(1-\phi)\, r_{1*} + \phi\left[(1-q+q\epsilon)\,(1+r_{1*}) - 1\right] = 0$$

or, rearranging,

$$1 + r_{1*} = \frac{1}{1 - q\phi\,(1-\epsilon)}$$

Likewise, non-entrepreneurs who hold the asset at date 2 must receive a single rate r_1^* that satisfies

$$(1-\phi)\, r_1^* + \phi\left[(1-\epsilon)\,(1+r_1^*) - 1\right] = 0$$

or, rearranging

$$1 + r_1^* = \frac{1}{1 - \phi\,(1-\epsilon)}$$

Since non-entrepreneurs would prefer the low rate contract, these have to be rationed. Given the assumption on sequential arrivals, the borrowers who show up first will be those who receive these contracts.

Next, I argue that r_{1*} must leave non-entrepreneurs just indifferent between holding it and selling it at date 2, i.e.

$$p_2 = \epsilon D + (1-\epsilon)\,(1+r_{1*})\, p_1$$

For suppose that the non-entrepreneurs who sell their holdings in period 2 strictly prefer to sell at r_{1*}. Then a lender could earn positive expected profits by offering all agents who in equilibrium receive a rate above r_{1*} a new contract with a rate that is just a little above r_{1*}. Since non-entrepreneurs would buy the asset and sell it, they would act in the same way as those receiving the rate r_{1*}. Entrepreneurs would continue to borrow and produce. Since expected profits were zero at r_{1*}, they would be strictly positive at a higher rate. The only way to ensure lenders who offer a high rate are unwilling to offer other contracts is if the rate at which traders are just indifferent about selling at date 2 yields zero expected profits.

I am now in a position to provide equations associated with the different cases (a) - (d) above. In case (a), the price in period 2 must be ϵD, or else all original owners would want to sell the asset. But then $p_1 = \epsilon D$, or else all of the original owners would have sold at date 1. But in that case, the only agents who can sell the asset in period 2 would have a reservation price that exceeds ϵD, making it impossible for the asset market to clear at $p_2 = \epsilon D$. Hence, case (a) implies

$$\begin{aligned} p_1 &= \epsilon D \\ p_2 &= \epsilon D \end{aligned} \tag{21}$$

In case (b), all original owners must sell the asset at either date 1 or date 2, so total demand must equal the original stock of the asset, 1. In addition, since demand is positive in both periods, the original owners must be indifferent between selling at date 1 and waiting to sell at date 2. Hence, the prices p_1 and p_2 must satisfy the system of equations

$$\begin{aligned} \frac{n_1}{p_1} + \frac{n_2}{p_2} &= 1 \\ (1-q)\,p_2 + q\epsilon D &= p_1 \end{aligned} \tag{22}$$

To ensure that none of those who bought the asset at date 1 wish to sell it, we must check that the solution to (22) satisfies $p_2 \leq \epsilon D + (1-\epsilon)(1+r_{1*})\,p_1$, i.e. p_2 does not exceed the reservation price of agents who bought the asset at date 1. The reservation price depends on r_{1*} since an agent who sells the asset will receive the rate r_{1*}.

In case (c), some but not all of those who bought the asset at date 1 sell it at date 2. From the analysis above, we know that the agents who buy the asset at date 1 and sell at date 2 are offered a rate r_{1*} at date 1 and are indifferent about selling at date 2. In addition, either all of the original owners sell their assets at date 1, implying $n_1/p_1 = 1$, or else the original owners must be indifferent between the two periods. These conditions can be summarized as follows:

$$\begin{aligned} p_2 &= \epsilon D + (1-\epsilon)(1+r_{1*})\,p_1 \\ p_1 &= \max\{(1-q)\,p_2 + q\epsilon D, n_1\} \end{aligned} \tag{23}$$

To ensure that not all of those who bought the assets at date 1 sell in date 2,we need to verify that at the prices that solve the system (23), $n_2/p_2 < 1$.

Finally, in case (d), all of those who own the asset at the beginning of date 2 sell it. In this case, buyers who show up at date 2 must buy up all existing shares of the asset. Once again, either some of the original owners sell in either period, requiring them to be indifferent between the two periods, or else the original owners buy up all of the assets at date 1, implying $n_1/p_1 = 1$. The two conditions are thus

$$\begin{aligned} p_2 &= n_2 \\ p_1 &= \max\{n_1, (1-q)\, n_2 + q\epsilon D\} \end{aligned} \tag{24}$$

The rest of the proof shows that we can partition the set $\mathcal{N}$ into distinct regions where the unique equilibrium in each region corresponds to a different case. These regions are illustrated graphically in Figure A1 and identified in the analysis below.

First, case (a) is the unique equilibrium whenever $n_1 + n_2 < \epsilon D$, which corresponds to region $\mathbb{A}$ in Figure A1. This is because at $p_1 = p_2 = \epsilon D$, the prices that must prevail in this equilibrium, all non-entrepreneurs in both periods would buy the asset. But if $n_1 + n_2 > \epsilon D$, the sum of demands in the two periods exceeds 1, and so we cannot have an equilibrium in which some of the original owners hold on to the asset to maturity. By contrast, it is easy to confirm that $p_1 = p_2 = \epsilon D$ is an equilibrium when $n_1 + n_2 < \epsilon D$. One can verify that none of the other cases can be supported as equilibria in this region: The solution to the system of equations (22) require that either $p_1 < \epsilon D$ or $p_2 < \epsilon D$ or both, which cannot occur in equilibrium, while the systems of equations (23) and (24) yield solutions for p_1 and p_2 at which total demand for the asset over the two periods $\frac{n_1}{p_1} + \frac{n_2}{p_2}$ is less than 1, implying not all original owners sell the asset as required in these two cases.

Next, consider case (b). We already ruled out that this can be an equilibrium in region $\mathbb{A}$. The two equilibrium conditions in (22) can be reduced to a single polynomial in p_2, specifically

$$(1-q)\, p_2^2 - (n_1 + (1-q)\, n_2 - q\epsilon D)\, p_2 - n_2 q\epsilon D = 0 \tag{25}$$

Only one of the roots exceeds ϵD. Denote this root by $p_2^*(n_1, n_2)$. Hence, there is a unique price market clearing price p_2, and thus a unique price $p_1 = q\epsilon D + p_2$, for each (n_1, n_2). Recall that for these prices are consistent with an equilibrium of type (b), we need to verify that the implied p_2 is not so high that agents who bought at date 1 wish to sell their assets, i.e. $p_2 \leq \epsilon D + (1-\epsilon)(1 + r_{1*})\, p_1$. Since $p_1 = (1-q)\, p_2 + q\epsilon D$, we can substitute for r_{1*} and p_1 to get

$$p_2 \leq \epsilon D + \frac{(1-\epsilon)\left[(1-q)\, p_2 + q\epsilon D\right]}{1 - (1-\epsilon)\, q\phi}$$

which, upon rearranging, yields

$$p_2 \leq \frac{1 + q\,(1-\epsilon)\,(1-\phi)}{1 - (1-\epsilon)\,(1 - q\,(1-\phi))}\epsilon D \equiv p^*$$

Using the equation $\frac{n_1}{q\epsilon D + (1-q)\, p_2} + \frac{n_2}{p_2} = 1$, it is immediate that $p_2^*(n_1, n_2)$ is strictly increasing in both n_1 and n_2. For $n_1 = 0$, the condition $p_2^*(0, n_2) \leq p^*$ implies $n_2 \leq \frac{1 + (1-\epsilon)\, q\,(1-\phi)}{1 - (1-\epsilon)\,(1 - q\,(1-\phi))}\epsilon D$. For

$n_2 = 0$, the condition the condition $p_2^*(n_1, 0) \leq p^*$ implies $n_1 \leq \dfrac{1-(1-\epsilon)\,q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D$. By totally differentiating (25), one can show that the boundary condition $p_2^*(n_1, n_2) = p^*$ defines a negative linear relationship between n_1 and n_2. Hence, the region in which there exists an equilibrium consistent with case (b) is bounded. This region is shown graphically as region $\mathbb{B}$ in Figure A1. Since values in the interior of region $\mathbb{B}$ satisfy $p_2 < \epsilon D + (1-\epsilon)(1+r_{1*})\,p_1$ by construction, there can be no equilibria associated with cases (c) or (d) inside region $\mathbb{B}$.

Next, we move to case (c). To determine the values of (n_1, n_2) that allow such equilibria, it will help to examine separately equilibria in which $(1-q)\,p_2 + q\epsilon D$ exceeds n_1 and equilibria in which it does not. If $(1-q)\,p_2 + q\epsilon D > n_1$, then we have two equations and two unknowns, which have a unique solution:

$$\begin{aligned} p_2 &= \frac{1+q\,(1-\epsilon)\,(1-\phi)}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D \\ p_1 &= \frac{1-(1-\epsilon)\,q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D \end{aligned}$$

One restriction that needs to be satisfied is that $n_1 < (1-q)\,p_2 + q\epsilon D = \dfrac{1-(1-\epsilon)\,q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D \equiv \overline{n}_1^*$. At the same time, we need to verify that $n_2 < p_2$, or else demand by non-entrepreneurs who arrive at date 2 would exceed the total supply of the asset. This implies $n_2 < \dfrac{1+q\,(1-\epsilon)\,(1-\phi)}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D \equiv \overline{n}_2^*$. Hence, an equilibrium consistent with case (c) in which $n_1 < (1-q)\,p_2 + q\epsilon D$ is confined to values $\{(n_1, n_2) : (n_1, n_2) \notin A \cup B,\ n_1 < \overline{n}_1^* \text{ and } n_2 < \overline{n}_2^*\}$. This set is illustrated graphically as region $\mathbb{C}_1$ in Figure A1. Note that the equilibrium prices p_1 and p_2 do not vary with n_1 and n_2 in this region.

Next, consider case (c) where $n_1 > (1-q)\,p_2 + q\epsilon D$. In this case, the system of equations is given by

$$\begin{aligned} p_2 &= \epsilon D + \frac{(1-\epsilon)\,n_1}{1-q\phi\,(1-\epsilon)} \\ p_1 &= n_1 \end{aligned}$$

As before, there are two restrictions that need to be satisfied to ensure these prices are indeed an equilibrium. First, we need to confirm that $n_1 > (1-q)\,p_2 + q\epsilon D$. Substituting in the value of p_2 yields $n_1 > \overline{n}_1^*$ as defined above. Second, we again need to make sure $n_2 < p_2$ so demand by non-entrepreneurs at date 2 would exceed the total supply of the asset. In this case, this condition implies

$$n_2 < \epsilon D + \frac{1-\epsilon}{1-q\phi\,(1-\epsilon)} n_1$$

The defines the region $\mathbb{C}_2 = \left\{(n_1, n_2) \in \mathbb{R}^2_{++} : n_1 > \overline{n}_1^*, n_2 < \epsilon D + \dfrac{1-\epsilon}{1-q\phi\,(1-\epsilon)} n_1\right\}$ depicted in Figure A1, which is the only region in which an equilibrium that accords with case (c) arises. Note that in region $\mathbb{C}_1$, the expressions for p_1 and p_2 are independent of n_1 and n_2, while in region $\mathbb{C}_2$, both are increasing in n_1 and independent of n_2.

Finally, we consider case (d). Observe that prices are weakly increasing in n_1 and n_2 in this case as well. Again, we separately consider the case where $(1-q)\,n_2 + q\epsilon D$ exceeds n_1 and the case where the opposite is true. If $(1-q)\,n_2 + q\epsilon D > n_1$, then we have $p_1 = (1-q)\,n_2 + q\epsilon D$ and $p_2 = n_2$. We need to confirm that $n_1 < (1-q)\,n_2 + q\epsilon D$. Since all the agents who buy the asset in period 1 must be willing to sell it in period 2, we need to verify that

$$p_2 \geq \epsilon D + (1-\epsilon)\,(1+r_{1*})\,p_1$$

Substituting in for p_1, p_2, and r_{1*} yields

$$n_2 \geq \epsilon D + \frac{(1-\epsilon)\,((1-q)\,n_2 + q\epsilon D)}{1-q\phi\,(1-\epsilon)}$$

which, upon rearranging, yields

$$n_2 \geq \frac{1+q\,(1-\epsilon)\,(1-\phi)}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D \equiv \overline{n}_2$$

This corresponds to region $\mathbb{D}_1$ in Figure A1, which has no overlap with region $\mathbb{C}_1 \cup \mathbb{C}_2$. Thus, this type of equilibrium cannot occur in these regions. Finally, we have case (d) where $n_1 > (1-q)\,n_2 + q\epsilon D$. Again, to ensure that all the agents who buy the asset in period 1 are willing to sell it in period 2 requires that $p_2 \geq \epsilon D + (1-\epsilon)\,(1+r_{1*})\,p_1$. Substituting in for p_1, p_2, and r_{1*} yields

$$n_2 \geq \epsilon D + \frac{(1-\epsilon)\,n_1}{1-q\phi\,(1-\epsilon)}$$

Hence, this region, denoted $\mathbb{D}_2$ in Figure A1, is defined by

$$\mathbb{D}_2 \equiv \left\{(n_1, n_2) \in \mathbb{R}^2_{++} : \epsilon D + \frac{(1-\epsilon)\,n_1}{1-q\phi\,(1-\epsilon)} \leq n_2 \leq \frac{n_1 - q\epsilon D}{1-q}\right\}$$

and does not overlap with $\mathbb{C}_1 \cup \mathbb{C}_2$. The uniqueness of equilibrium is thus established for all $(n_1, n_2) \in \mathcal{N}$.
■

Comparative Statics of Equilibrium in Proposition 1: In the text, I discuss some comparative static results regarding the asset market equilibrium at date 2. I present the formal derivation for these results here. Note that the outer boundary of region $\mathbb{B}$ is given by the line defined by two points,

$$(n_1, n_2) = \left(0, \frac{1+(1-\epsilon)\,q\,(1-\phi)}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D\right) \text{ and } \left(\frac{1-(1-\epsilon)\,q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D, 0\right)$$

In the limit as $q \to 0$, these points converge to $(0, D)$ and $(D, 0)$, respectively, implying the set $\mathbb{B}$ converges to $\left\{(n_1, n_2) \in \mathbb{R}^2_{++} : \epsilon D < n_1 + n_2 < D\right\}$. Conversely, when $q \to 1$, these points converge to $\left(0, \frac{1+(1-\epsilon)(1-\phi)}{1-\phi(1-\epsilon)}\epsilon D\right)$ and $(\epsilon D, 0)$, respectively.

As for the outer boundary of $\mathcal{N}$, for ease of comparison we can also define it by two points, namely

$$(n_1, n_2) = \left(0, (1-q)\,\frac{D}{R} + \epsilon D\right) \text{ and } \left((1-q)\,\frac{D}{R} + \epsilon D, 0\right)$$

In the limit as $q \to 0$, these points converge to $\left(0, \frac{D}{R}\right)$ and $\left(\frac{D}{R}, 0\right)$, respectively, implying the set $\mathcal{N}$ converges to $\left\{(n_1, n_2) \in \mathbb{R}^2_{++} : n_1 + n_2 < \frac{D}{R}\right\}$. Since $\frac{D}{R} < D$, it follows that $\mathcal{N} \setminus (\mathbb{A} \cup \mathbb{B}) = \varnothing$, i.e. the set $\mathcal{N}$ contains only regions $\mathbb{A}$ and $\mathbb{B}$. By continuity, the the same is true for q close to 0. Hence, for q close to 0, the only possible outcomes are either no bubbles or quiet bubbles.

Conversely, when $q \to 1$, the above points converge to $(0, \epsilon D)$ and $(\epsilon D, 0)$, respectively. In the limit when $q = 1$, then $\mathcal{N} \setminus (\mathbb{A} \cup \mathbb{B}) = \varnothing$. However, it is not clear what happens for q close to but strictly less than 1. Define

$$\chi = \frac{(1-q)\, D/R + \epsilon D}{\frac{1-(1-\epsilon)q\phi}{1-(1-\epsilon)(1-q(1-\phi))}\epsilon D}$$

as the ratio of the values of n_1 at the outer boundary of $\mathcal{N}$ and $\mathbb{B}$, respectively, when $n_2 = 0$. Differentiating χ with respect to q and evaluating at $q = 1$ reveals that

$$\lim_{R \to 1/\epsilon} \left. \frac{d\chi}{dq} \right|_{q=1} = -\frac{1-\epsilon}{1-\phi\,(1-\epsilon)} < 0$$

Since $\chi = 1$ at $q = 1$, it follows that for large values of R, $\mathcal{N} \setminus (\mathbb{A} \cup \mathbb{B}) \neq \varnothing$ in a neighborhood of $q = 1$. In other words, for q sufficiently close to but not equal 1, noisy bubbles will be possible when $n_1 > \frac{1-(1-\epsilon)\, q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D$ and $n_2 = 0$. For smaller values of R, we have

$$\lim_{R \to 1+\frac{\phi(1-\epsilon)}{1-\phi(1-\epsilon)}} \left. \frac{d\chi}{dq} \right|_{q=1} = \frac{(1-\epsilon)\left((1-\phi)^2 - \epsilon\left(1-\phi+\phi^2\right)\right)}{1-\phi\,(1-\epsilon)}$$

This expression will be negative if either ϵ or ϕ are sufficiently large to turn $(1-\phi)^2 - \epsilon\left(1-\phi+\phi^2\right)$ negative, provided both ϵ and ϕ are both strictly between 0 and 1. Hence, even for low values of R, there exist parameters such that noisy bubbles will be possible for q sufficiently close to 1.

Turning to ϵ, as $\epsilon \to 1$, the two points that define the outer boundary of $\mathbb{B}$ converge to $(0, D)$ and $(D, 0)$. By contrast, the two points that define the outer boundary of $\mathcal{N}$ converge to $\left(0, (1-q)\frac{D}{R} + qD\right)$ and $\left((1-q)\frac{D}{R} + qD, 0\right)$, respectively. Since $D/R < D$, it follows that $\mathcal{N} \setminus (\mathbb{A} \cup \mathbb{B}) = \varnothing$, i.e. the set $\mathcal{N}$ contains only regions $\mathbb{A}$ and $\mathbb{B}$. By continuity, the the same is true for ϵ close to 1. Hence, for ϵ close to 1, the only possible outcomes are either no bubbles or quiet bubbles.

Conversely, in the limit as $\epsilon \to 0$, the two points that define the outer boundary of $\mathbb{B}$ both converge to $(0, 0)$. By contrast, the two points that define the outer boundary of $\mathcal{N}$ converge to $\left(0, (1-q)\frac{D}{R}\right)$ and $\left((1-q)\frac{D}{R}, 0\right)$, respectively, which strictly exceed $(0, 0)$ as long as $q < 1$. Thus, for small values of ϵ, noisy bubbles and speculation both become possible, and in the limit these will be the only possible outcomes.

Finally, the parameter ϕ only affects the outer boundary of region $\mathbb{B}$. Differentiating the two values reveal that both are increasing in ϕ. Specifically,

$$\begin{aligned}\frac{d}{d\phi}\frac{1+(1-\epsilon)\,q\,(1-\phi)}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D &= \frac{q\,(1-\epsilon)^2}{(1-(1-\epsilon)\,(1-q\,(1-\phi)))^2}\epsilon D \\ \frac{d}{d\phi}\frac{1-(1-\epsilon)\,q\phi}{1-(1-\epsilon)\,(1-q\,(1-\phi))}\epsilon D &= \frac{(1-q)\,q\,(1-\epsilon)^2}{(1-(1-\epsilon)\,(1-q\,(1-\phi)))^2}\epsilon D\end{aligned}$$

Hence, higher values of ϕ increase the size of region $\mathbb{B}$, meaning a larger set of parameters associated with bubbles will correspond to quiet rather than noisy bubbles. ■

Proof of Proposition 2: The proof is by contradiction. Suppose there was an equilibrium where the price of the asset at date 2 exceeded ϵD, i.e. $p_2 > \epsilon D$. Let x_2^n denote the amount of resources non-entrepreneurs receive upon arrival at date 2 under the equilibrium contract, and denote the amount they would have to repay if $d = D$ by $(1 + r_2^n)\,x_2^n$. The expected payoff to a non-entrepreneur from buying the asset would then equal

$$x_2^n\epsilon\left(\frac{D}{p_2} - (1 + r_2^n)\right)$$

while the expected payoff to the lender in this case is

$$x_2^n\,(\epsilon\,(1 + r_2^n) - 1)$$

Summing together yields

$$x_2^n\left(\frac{\epsilon D}{p_2} - 1\right)$$

This expression is negative when $p_2 > \epsilon D$. Hence,

$$x_2^n\epsilon\left(\frac{D}{p_2} - (1 + r_2^n)\right) < x_2^n\,(1 - \epsilon\,(1 + r_2^n)) \tag{26}$$

Consider a creditor who offered a contract $\widehat{x}$ in which non-entrepreneurs receive nothing upon arrival, i.e. $\widehat{x}_2^n = 0$ and then paid the agent $\epsilon\left(\frac{D}{p_2} - (1 + r_2^n)\right)x_2^n$ at the end of date 2, which is just the expected payoff from buying the asset under the original contract. From (26), we know that lenders are strictly better off from this arrangement, since their incur smaller losses. Since (26) is a strict inequality, and since the first inequality in (**??**) ensures entrepreneurs strictly prefer production to buying risky assets, it will be possible to increase the payment to non-entrepreneurs a little to make them strictly better off and to lower the interest charged to entrepreneurs in a way that makes them better off but still induces them to choose the contract intended for them and to use the funds they borrow for production. Hence, if $p_2 > \epsilon D$, no agents buy the asset at date 2. Since any original owners who still hold the asset at date 1 would prefer to sell at a price of ϵD than hold on to the asset, the only possible equilibrium is one in which all the original owners sell the asset at date 1.

Next, we move to date 1. First, consider whether there can be an equilibrium in which the asset is purchased at date 1. Suppose $p_1 > \epsilon D$. In that case, we know that $(1 + r_1)\,p_1 \geq p_1 > \epsilon D$, so that any agents who buy the asset at date 1 would only agree to sell the asset at date 2 for a price above ϵD. But we just argued that if $p_2 > \epsilon D$, there cannot be an equilibrium in which agents buy the asset at date 2 in equilibrium. Hence, non-entrepreneurs who buy the asset at date 1 expect to hold on to the asset until d is revealed. But by a similar argument to before, if $p_1 > \epsilon D$, then creditors can earn positive expected profits by giving non-entrepreneurs zero resources up front and then paying them after date 2 a little above their expected profits from buying risky assets. Hence, the only possible equilibrium is one where $p_1 = \epsilon D$.

However, if $p_1 = \epsilon D$, there cannot be an equilibrium where $p_2 > \epsilon D$. If there were such an equilibrium, all original owners will hold on to their asset holds until date 2 and sell if $p_2 > \epsilon D$, i.e. if d was not revealed. But we know that if $p_2 > \epsilon D$, there will not be any buyers at date 2 in equilibrium. Hence, the market will not clear at date 2. It follows that not only is $p_1 = \epsilon D$ in equilibrium, but $p_2 = \epsilon D$. ■

Proof of Lemma 2: Suppose not, i.e. a non-entrepreneur strictly prefers to announce n than to announce e. It is easy to check that in equilibrium, $r_t^e > 0$ for both $t \in \{1, 2\}$: non-entrepreneurs will incur a cost on creditors that must be made up to ensure creditors earn zero expected profits. Consider a lender who only offers the following contract $\{\widetilde{x}_t^e, \widetilde{r}_t^e\}$ aimed at entrepreneurs:

$$\begin{aligned} \widetilde{x}_t^e &= x_t^e = 1 \\ \widetilde{r}_t^e &= r_t^e - \varepsilon \end{aligned}$$

where $\varepsilon \in (0, r_t^e)$ so the interest rate remains positive. Entrepreneurs will clearly prefer this contract to the equilibrium contract. Non-entrepreneurs originally strictly preferred the contract they received in equilibrium to the contract $\{1, r_t^e\}$, and hence for ε small enough will also prefer it to $\{1, \widetilde{r}_t^e\}$. Hence, this contract will only attract entrepreneurs, and for ε small enough the expected profits to the creditor who offers it will be strictly positive. Hence, the original contract could not have been an equilibrium. ■

Proof of Proposition 3: Consider the maximization problem

$$\max_{x_2^n, r_2^n} \; x_2^n \left[\epsilon \left(1 + r_2^n\right) - 1\right]$$

subject to

$$\begin{array}{ll} \text{i.} & \epsilon \left(\dfrac{D}{p_2} - \left(1 + r_2^n\right)\right) x_2^n = \epsilon \left(\dfrac{D}{p_2} - \left(1 + r_2^e\right)\right) \\ \text{ii.} & r_2^n \geq 0 \end{array}$$

Substituting (ii) into the objective function yields

$$\left(\frac{\epsilon D}{p_2} - 1\right) x_2^n - \frac{\epsilon D}{p_2} + \epsilon \left(1 + r_2^e\right)$$

When $p_2 > \epsilon D$, this expression is decreasing in x_2^n, so the optimal contract reduces x_2^n. From constraint (i), this implies r_2^n will be as small as possible. Since constraint (3) implies $r_2^n \geq 0$, it follows that the optimal contract involves $r_2^n = 0$. From constraint (1), this implies

$$x_2^n = \frac{D - (1 + r_2^e)\, p_2}{D - p_2}$$

The condition on r_2^e follows from the fact that lenders earn zero expected profits in equilibrium. ■

Proof of Proposition 4: Consider first an agent who intends to hold on to his asset. In this case, the contract is given by

$$\max_{x_1^n, r_1^{n,\text{late}}} \; x_1^n \left[\epsilon \left(1 + r_1^{n,\text{late}}\right) - 1\right]$$

subject to

$$\begin{array}{ll} \text{i.} & \epsilon \left(\dfrac{D}{p_1} - \left(1 + r_1^{n,\text{late}}\right)\right) x_1^n = \epsilon \left(\dfrac{D}{p_1} - (1 + r_1^e)\right) \\ \text{ii.} & r_1^{n,\text{late}} \geq 0 \end{array}$$

where we know from the proof of Proposition 5 that we can omit the constraint that the entrepreneur must prefer to produce than buy assets. Since this problem is identical to the period 2 problem, the same argument implies that when $p > \epsilon D$, in equilibrium $r_1^{n,\text{late}} = 0$ and

$$x_2^n = \frac{D - (1 + r_2^e)\, p_2}{D - p_2}$$

When $p_1 = \epsilon D$, profits are independent of x_1, and so any pair (x_1^n, r_1^n) that yields expected utility $V_1(n, e)$ to the agent is an equilibrium.

Next, consider an agent who intends to sell his asset if d remains uncertain. In this case, the equilibrium contract solves

$$\max_{x_1^n, r_1^{n,\text{early}}} \; x_1^n \left[(q\epsilon + 1 - q)\left(1 + r_1^{n,\text{early}}\right) - 1\right]$$

subject to

$$\begin{array}{ll} \text{i.} & \left[q\epsilon \left(\dfrac{D}{p_1} - 1 - r_1^{n,\text{early}}\right) + (1 - q)\left(\dfrac{p_2}{p_1} - 1 - r_1^{n,\text{early}}\right)\right] x_1^n = V_1(n, e) \\ \text{ii.} & r_1^{n,\text{early}} \geq 0 \end{array}$$

Rearranging constraint (i) yields

$$\left[\frac{q\epsilon D + (1 - q)\, p_2}{p_1}\right] x_1^n - V_1(n, e) = (q\epsilon + 1 - q)\left(1 + r_1^{n,\text{early}}\right) x_1^n$$

So the objective function can be written as

$$\left[\frac{q\epsilon D + (1 - q)\, p_2}{p_1} - 1\right] x_1 - V_1(n, e)$$

When $p_1 > q\epsilon D + (1 - q)\, p_2$, this expression is decreasing in x_1^n, so the optimal contract reduces x_1^n and sets $r_1^{n,\text{early}} = 0$. Since $p_1 \geq q\epsilon D + (1 - q)\, p_2$ or else the asset market would not clear at date 1, in the only other case profits are independent of x_1, and so any pair (x_1^n, r_1^n) that yields expected utility $V_1(n, e)$ to the agent is an equilibrium. ■

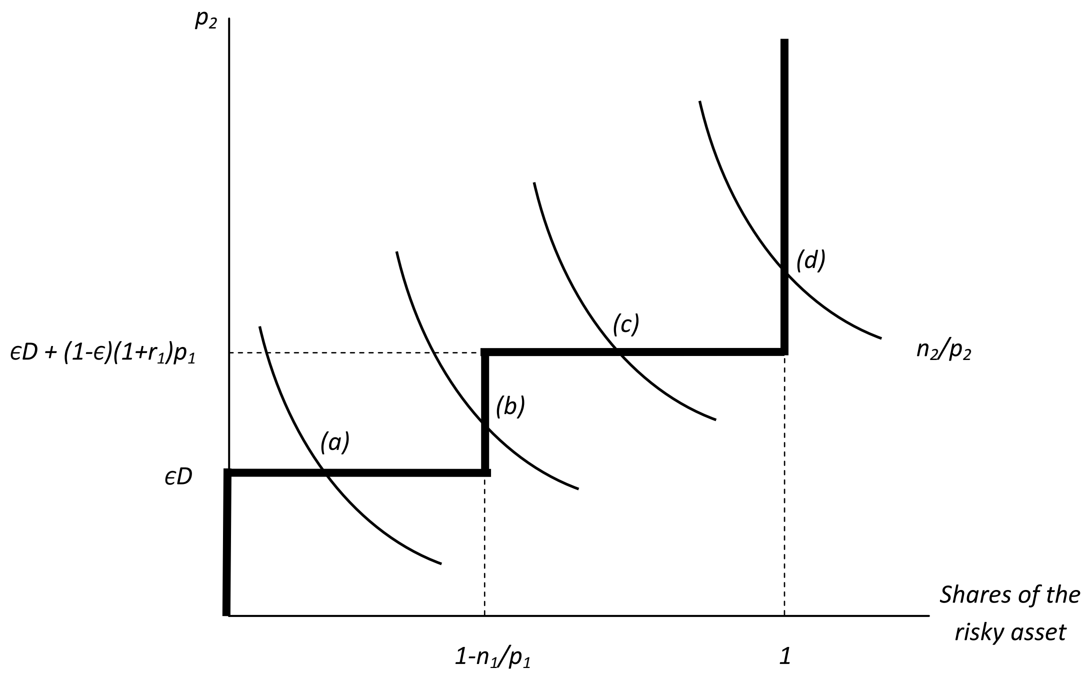

Figure 1: Supply and demand in the asset market at $t = 2$ for $\mathcal{I}_2 = \emptyset$,

assuming a single interest rate is charged to all borrowers at $t = 1$

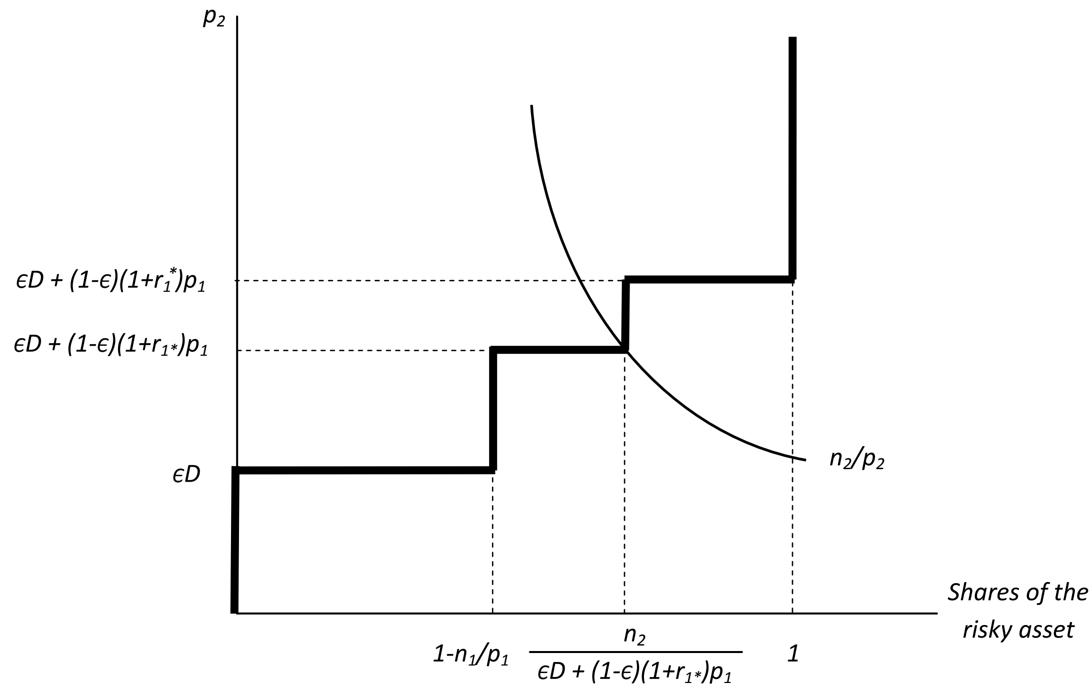

Figure 2: Supply and demand in the asset market at $t = 2$ for $\mathcal{I}_2 = \emptyset$,

when some non-entrepreneurs who buy at $t = 1$ sell at $t = 2$.

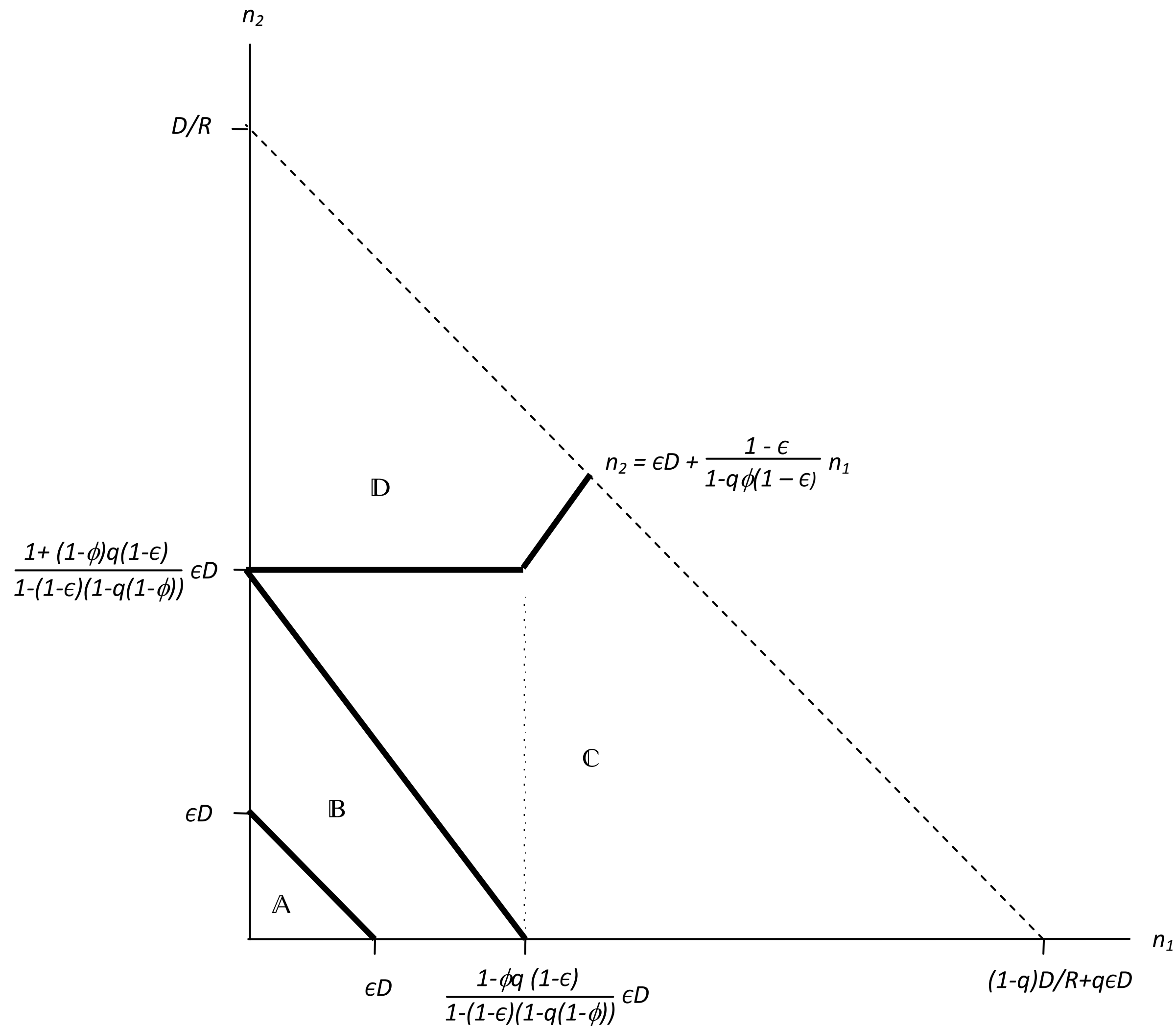

Figure 3: Regions corresponding to different types of equilibrium at $t = 2$ for $\mathcal{I}_2 = \emptyset$

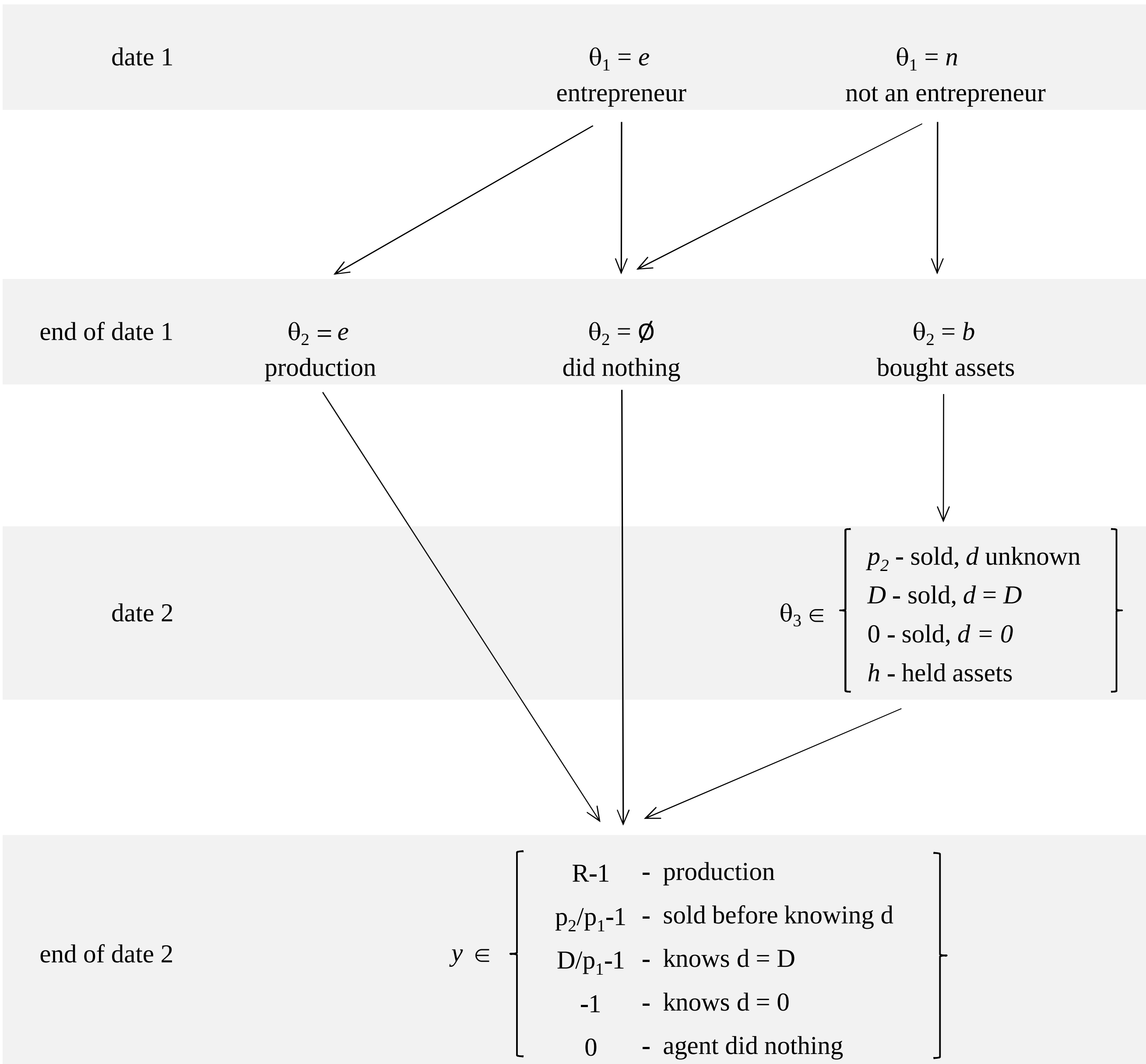

Figure 4: Reports and Transfers Mandated by Contract

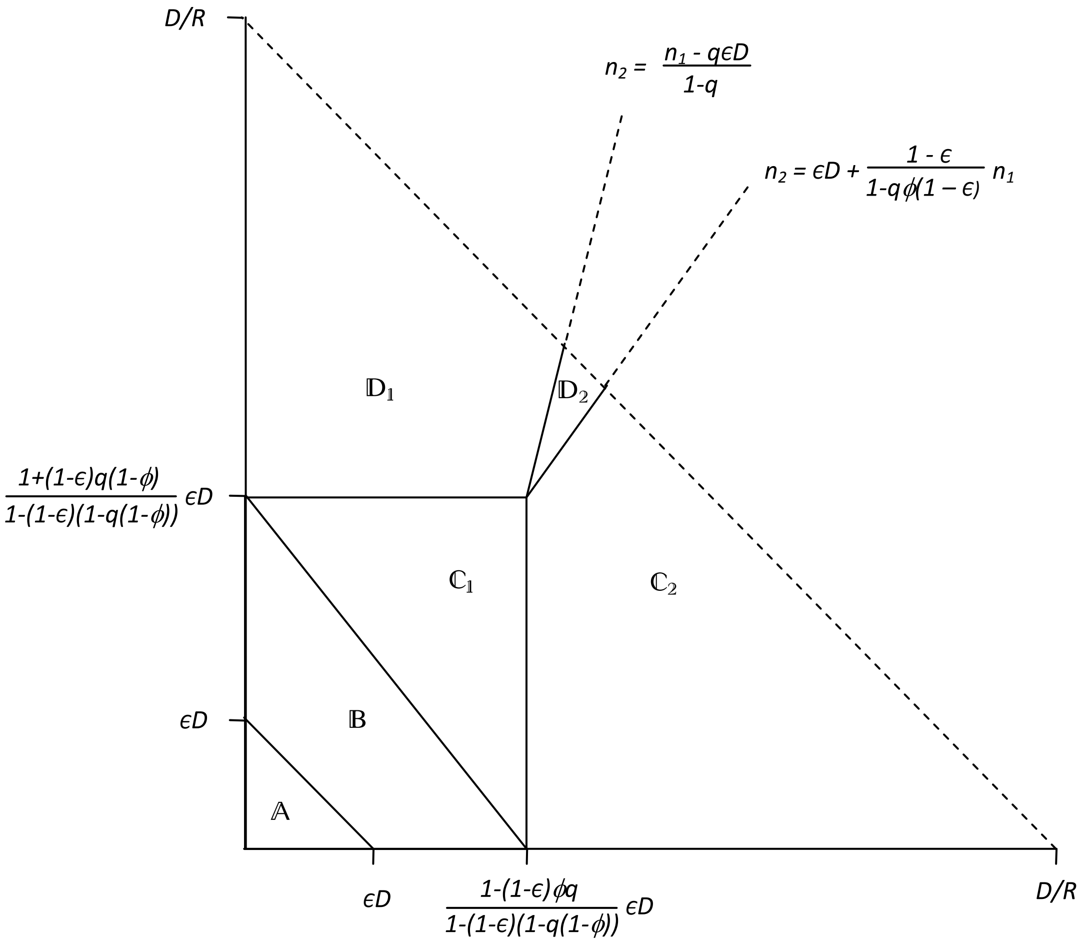

Figure A1: Equilibrium types for different pairs (n_1,n_2)

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