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Can Standard Preferences Explain the Prices of Out-of-the-Money S&P 500 Put Options?

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Abstract

The 1987 stock market crash occurred with minimal impact on observable economic variables (e.g., consumption), yet dramatically and permanently changed the shape of the implied volatility curve for equity index options. Here, we propose a general equilibrium model that captures many salient features of the U.S. equity and options markets before, during, and after the crash. The representative agent is endowed with Epstein-Zin preferences and the aggregate dividend and consumption processes are driven by a persistent stochastic growth variable that can jump. In reaction to a market crash, the agent updates her beliefs about the distribution of the jump component. We identify a realistic calibration of the model that matches the prices of short-maturity at-the-money and deep out-of-the-money S&P 500 put options, as well as the prices of individual stock options. Further, the model generates a steep shift in the implied volatility 'smirk' for S&P 500 options after the 1987 crash. This 'regime shift' occurs in spite of a minimal impact on observable macroeconomic fundamentals. Finally, the model's implications are consistent with the empirical properties of dividends, the equity premium, as well as the level and standard deviation of the risk-free rate. Overall, our findings show that it is possible to reconcile the stylized properties of the equity and option markets in the framework of rational expectations, consistent with the notion that these two markets are integrated.

Key words. Volatility Smile; Volatility Smirk; Implied Volatility; Option Pricing; Portfolio Insurance; Market Risk.

JEL Classification Numbers. G12, G13.

1 Introduction

The 1987 stock market crash has generated many puzzles for financial economists. Although there was little change in observable macroeconomic fundamentals, market prices dropped 20-25% and interest rates dropped about 2%. Moreover, the crash generated a permanent regime shift in the prices of index options. Indeed, prior to the market crash, implied 'volatility smiles' for index options were relatively flat. However, since the crash the Black-Scholes (B/S) formula has been significantly underpricing short-maturity, deep out-of-the-money S&P 500 put options. This property, termed the 'volatility smirk,' has been documented by, e.g., Rubinstein (1994) and Bates (2000) using S&P 500 option data up to the early 1990s. Here we show that this regime shift has persisted up to the present date. Indeed, Figure 1 reports the spread of in-the-money (ITM) and out-of-the-money (OTM) implied volatilities relative to at-the-money implied volatilities from 1985-2006. Prior to the crash, 10% OTM puts had an average implied volatility spread of 1.83% with standard deviation of 1.18%. Similarly, the spread for 2.5% ITM put options averaged -0.13% prior to the crash with a standard deviation of 0.34%. On some dates the implied volatility function had the shape of a mild 'smile' and on others it was shaped like a mild 'smirk'. Overall, the Black Scholes formula priced all options relatively well prior to the crash, underpricing deep OTM options only slightly. This all changed on October 19, 1987, when the spread for OTM puts spiked up to a level above 10%. Since then, implied volatilities for deep OTM puts have averaged 8.21% higher than ATM implied volatilities, with standard deviation of 1.66%. Moreover, since the crash, implied volatilities for ITM options have been systematically lower than ATM implied volatilities, with an average spread of -1.34%.

Another puzzle associated with option prices is that the implied volatility functions for individual stock options are much flatter and more symmetric compared to the steep 'volatility smirk' associated with S&P 500 options (see, e.g., Bollen and Whaley (2004), Bakshi, Kapadia, and Madan (2003), and Dennis and Mayhew (2002)). Indeed, from their analysis, Bollen and Whaley (2004) conclude that the relative difference in the implied volatility functions for options on individual firms and the S&P 500 cannot be explained solely by the underlying asset return distribution.

In this paper, we attempt to capture these empirical features within a rational-expectations general equilibrium setting. In particular, we propose a framework that can simultaneously explain:

- prices of deep OTM put options for both individual stocks and the S&P 500 index;
- why the slope of the implied volatility curve changed so dramatically after the crash;
- why the regime shift in the 'volatility smirk' has persisted for the past twenty years;
- how the market can crash with little change in observable macroeconomic variables.

Motivated by the empirical failures of the B/S model in post-crash S&P 500 option data, prior studies have examined more general option pricing models (see, e.g., Bates (1996), Duffie et al.

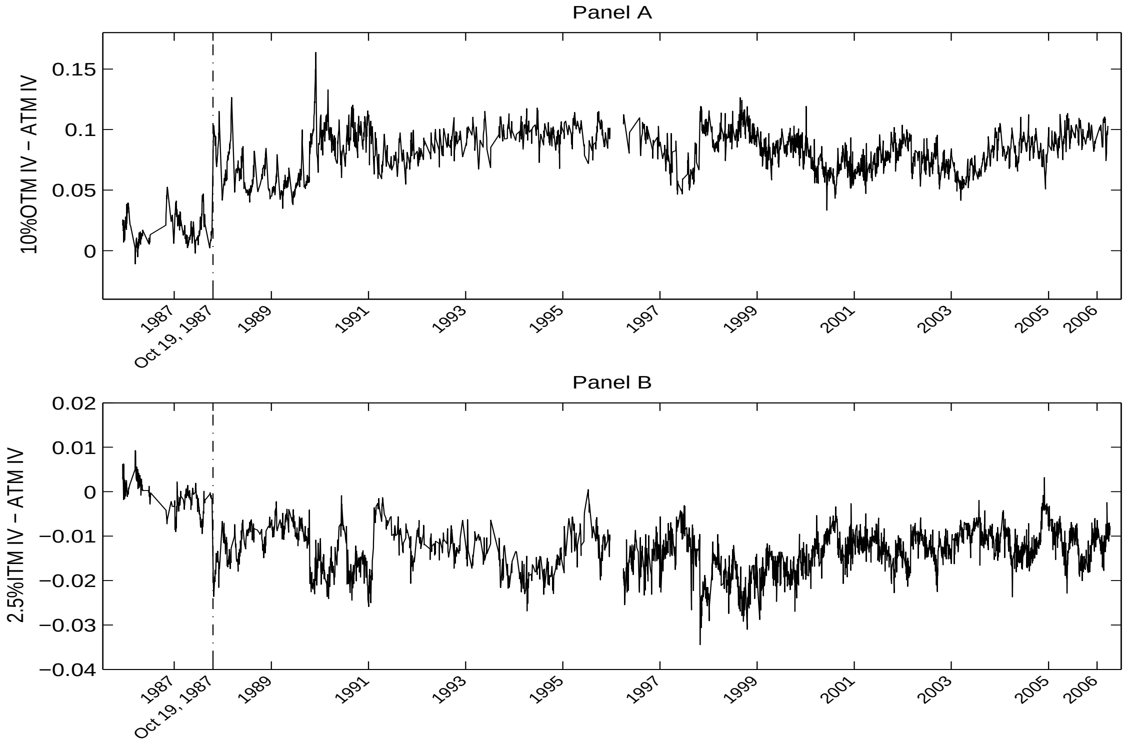

Figure 1: Pre- and Post-Crash Implied Volatility Smirk for S&P 500 Options with One Month to Maturity. The plot in Panel A depicts the spread between implied volatilities for S&P 500 options with a strike-to-price ratio $X = K/S - 1 = -10\%$ and at-the-money implied volatilities. The plot in Panel B depicts the spread between implied volatilities for options with a strike-to-price ratio $X = K/S - 1 = 2.5\%$ and at-the-money implied volatilities. Additional details on how the series are constructed are given in Appendix A.

(2000), Heston (1993)). Several authors have tested these extensions empirically.[1] Overall, this literature concurs that a model with stochastic volatility and jumps significantly reduces the pricing and hedging errors of the B/S model, both in- and out-of-sample.[2] These previous studies, however, focus on post-1987 S&P 500 option data. Further, they follow a partial equilibrium approach and let statistical evidence guide the *exogenous* specification of the underlying return dynamics.

Reconciling the findings of this literature in a rational expectations general equilibrium setting

[1] Among recent contributions, Bakshi et al. (1997, 2000), Bates (2000), and Huang and Wu (2004) extract information about the model parameters of the underlying returns process from derivatives prices alone. Pan (2002), Broadie et al. (2004), Chernov and Ghysels (2000), Jones (2003), Eraker (2004), and Benzoni (2002) use data on both underlying and derivatives prices to fit the model.

[2] A related literature investigates the profits of option trading strategies (e.g., Coval and Shumway (2001) and Santa-Clara and Saretto (2004)) and the economic benefits of giving investors access to derivatives when they solve the portfolio choice problem (e.g., Constantinides et al. (2004), Driessen and Maenhout (2004) and Liu and Pan (2003)). Overall, these papers suggest that derivatives are non-redundant securities and, in particular, that volatility risk is priced. These findings are consistent with the evidence in Bakshi and Kapadia (2003) and Buraschi and Jackwerth (2001), as well as with the results of the studies that use data on both underlying and derivatives prices to fit parametric stochastic volatility models.

has proven difficult. For instance, Pan (2002) notes that the compensation demanded for the 'diffusive' return risk is very different from that for jump risk. Consistent with Pan's finding, Jackwerth (2000) shows that the risk aversion function implied by S&P 500 index options and returns post-1987 crash is partially negative and increasing in wealth (similar results are presented in Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2000)). This evidence eludes the standard general equilibrium model with constant relative risk aversion utility and suggests that there may be a lack of integration between the option market and the market for the underlying stocks.

Here, we examine a representative-agent general equilibrium endowment economy that simultaneously captures the stylized properties of the S&P 500 options, the options on individual stocks, and the underlying stock returns. To this end, we expand on the insights of Bansal and Yaron (2004, BY) by considering Epstein and Zin (1989) preferences and specifying the expected growth rate of dividends to be driven by a persistent stochastic variable that follows a jump-diffusion process. As noted by BY and Shephard and Harvey (1990), it is very difficult to distinguish between a purely i.i.d. process and one which incorporates a small persistent component. As such, the dividend process implied by the model fits the properties of actual dividends well. Nevertheless, the presence of a small persistent component can have important asset pricing implications.

We solve the model using standard results in recursive utility (e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003)). We show that the price-dividend ratio satisfies an integro-differential equation that is non-linear when the EIS is different from the inverse of the coefficient of risk aversion. To solve such equations, we use the approximation method of Collin-Dufresne and Goldstein (2005), which is itself an extension of the Campbell-Shiller approximation (see Campbell and Shiller (1988)).

We illustrate the properties of the model through a realistic calibration of its coefficients. Consistent with the findings of BY, the model matches the empirical properties of dividends and consumption, and generates a realistic 1% real risk-free rate, a 6% equity premium, and a price-dividend ratio of 20. Furthermore, the model also captures certain features of the stock market that elude the BY specification.[3] Specifically, an unexpected jump in the predictable component of consumption and dividend *growth rate* can generate a market crash without a jump in the consumption process itself. In addition, the model is consistent with a large drop in the risk-free rate on crash dates, consistent with the 1987 market crash.

In our baseline case, a put option with maturity of one month and a strike price that is 10% out-of-the-money has an implied volatility of approximately 24%. In contrast, a one-month, at-the-money option has an implied volatility of approximately 14%. That is, consistent with empirical evidence, we find a 10% volatility smirk. Sensitivity analysis shows that the main qualitative results are robust to a wide range of parameter calibrations.

[3]Most existing models that capture the equity premium (e.g., BY and Campbell and Cochrane (1999)) specify the dividend growth rate process as continuous. As such, these models cannot account for the high premium on near-term out-of-the-money put options, nor the possibility of a market crash.

The intuition for these results is similar to that discussed in BY. Epstein and Zin preferences allow for a separation of the elasticity of intertemporal substitution (EIS) and risk aversion. When the EIS is larger than one, the intertemporal substitution effect dominates the wealth effect. Thus, in response to higher expected growth the demand function for assets of the representative agent increases, and consequently prices rise. The opposite occurs when there is a decrease in expected growth, e.g., because of an unexpected downward jump in the predictable component of dividends that triggers a market crash. In this framework, the risky asset exhibits positive returns when the state is good, while it performs poorly in the bad state. As such, investors demand a high equity premium and are willing to pay a high price for a security that delivers insurance in the bad state, like, e.g., a put option on the S&P 500 index.

The model also reproduces the stylized properties of the implied volatility functions for individual stock option prices. We specify individual firm stock dynamics by first taking our model for S&P 500 index dynamics, and then adding on idiosyncratic shocks, both of the diffusive-type and the jump-type. We then calibrate the coefficients of the idiosyncratic components to match the distribution of returns for the 'typical' stock. In particular, we match the cross-sectional average of the high-order moments (variance, skewness, and kurtosis) for the stocks in the Bollen and Whaley (2004) sample. We simulate option prices from this model and compute B/S implied volatilities across different moneyness. Consistent with the evidence in Bollen and Whaley (2004), Bakshi, Kapadia, and Madan (2003), and Dennis and Mayhew (2002), we find an implied volatility function that is considerably flatter than that for S&P 500 options. Bakshi, Kapadia, and Madan (2003) conclude that the differential pricing of individual stock options is driven by the degree of skewness/kurtosis in the underlying return distribution in combination with the agent's high level of risk aversion. Here, we propose a plausible endowment economy that in combination with recursive utility yields predictions consistent with their empirical findings.

Further, the model captures the stark change in the S&P 500 options implied volatility pattern that has been observed since the 1987 market crash. We note that an extreme event such as the 1987 crash is likely to dramatically change the investor's perception about the nature of possible future market fluctuations. To formalize this intuition, we consider a Bayesian setting in which the agent formulates a prior on the average value of the jump size, and then updates her prior when she observes an extreme event such as the 1987 crash. Note that the updating of beliefs only occurs at crash dates. As such, her posterior beliefs on the average value of the jump size are potentially very long lived, and hence can explain why the volatility smirk has remained high even twenty years after the crash.

We find that the model can capture the implied volatility pattern of option prices both before and after the 1987 crash. Specifically, we present simulation results in which the steepness of the volatility smirk (i.e., the difference between implied volatilities of 10%-out-of-the-money and at-the-money puts) is lower than 3%, a number that is consistent with the pre-crash evidence. At the same time, the occurrence of a jump triggers the updating of the agent's beliefs about the expected value of the jump size. As such, after the crash, out-of-the-money put options are perceived to be more valuable, and the volatility smirk becomes as steep as 10%. Furthermore, consistent with observation the model predicts a downward jump in the risk-free rate during crash events.

Consistent with the 1987 crash, the model produces these results in spite of minimal change in observable macroeconomic variables. At the time of the crash, dividends remain smooth. Indeed, jumps occur only for the estimate of the magnitude of future crashes and the expected consumption growth rate. It is the updating of the agent's beliefs about the likelihood of future jumps of this magnitude that generates a regime shift. As such, the jump risk premium increases and this effect pushes stock prices further down and makes out-of-the-money puts more valuable.

The rest of the paper is organized as follows. We first discuss related literature. Then, in Section 3, we present an option pricing model that explains the post-1987 volatility smirk in S&P 500 prices as well as the pricing of individual stock options. In Section 4, we extend our setting to incorporate Bayesian updating of the agent's believes. We use this setting to show that an event such as the 1987 market crash can generate a change in the S&P 500 price that is qualitatively consistent with what we observe in the data. In Section 5 we conclude and discuss possible extensions of our analysis.

2 Related Literature

Several papers have investigated the ability of equilibrium models to explain post-1987 S&P 500 option prices. Liu, Pan, and Wang (2005, LPW) consider an economy in which the endowment is an i.i.d. process that is subject to jumps. They show that in this setting neither constant relative risk aversion, nor Epstein and Zin (1989) preferences can generate a volatility smirk consistent with post-1987 evidence on S&P 500 options. They argue that in order to reconcile the prices of options and the underlying index, agents must exhibit 'uncertainty aversion' towards rare events that is different from the standard 'risk-aversion' they exhibit towards diffusive risk. As such, they provide a decision-theoretic basis to the idea of crash aversion advocated by Bates (2001), who proposed an extension of the standard power utility that allows for a special risk-adjustment parameter for jump risk distinct from that for diffusive risk. These prior studies assume that the dividend *level* is subject to jumps, while the *expected* dividend growth rate is constant. That is, in these models a crash like that observed in 1987 is due to a 20-25% downward jump in the dividend level.[4] Further, their model predicts no change in the risk free rate during the crash event. In contrast, in our setting, it is not the consumption or dividend process itself, but rather the expected growth rate that is subject to jumps (and the Bayesian-updated distribution of future jumps). In combination with recursive utility, our model delivers different option pricing implications, while allowing for a

[4]Barro (2006) makes a similar assumption about the output dynamics in his model. His model captures the contractions associated with the Great Depression and the two World Wars, but it does not match the evidence around the 1987 crash, when the output level remained smooth.

smooth dividend process. Further, it predicts a large downward jump in the riskfree rate around crash events, consistent with empirical evidence.

David and Veronesi (2002) consider an economy in which the expected growth rate in dividends switches between two unobservable states, interpreted as 'booms' and 'recessions.' Investors use past realizations of dividends to infer the current drift rate of the economy. They fit the model on S&P 500 real earnings growth rates for the 1960-1998 period. In their model, the shape of the volatility smirk depends on the agent's belief about the state of the economy. In economic expansions, the model generates a downward-sloping volatility smirk across exercise prices, i.e., out-of-the-money S&P 500 puts are expensive. However, it also implies an upward sloping smirk during recessions (i.e., out-of-the-money S&P 500 puts are cheap in the early 1990s). In contrast our model predicts a permanent change in the shape of the volatility smirk after 1987, consistent with the evidence observed in S&P 500 option data since the crash and, in particular, during the early 1990s recession (see Figure 1).

Other studies have explored the option pricing implications of a model with state dependence in preferences and/or fundamentals. For instance, Chabi-Yo, Garcia, and Renault (2004) show that the regime switching model of Garcia, Luger, and Renault (2003) for the endowment process, in combination with state-dependent preferences, can reproduce the features identified by Jackwerth (2000) and Aït-Sahalia and Lo (2000).[5] We note, however, that their regime switching model for the endowment process, as well as the one in Garcia et al. (2001), will share the same properties of the setting considered by David and Veronesi (2002). That is, the shape of the volatility smirk will depend on the state of the economy and, in particular, the smirk will be upward sloping (i.e., out-of-the-money S&P 500 put options are cheap) during economic recessions, which is inconsistent with the evidence in Figure 1.

In focusing on the pricing of both out-of-the-money put options and the equity index, our paper is related to the recent literature that searches for a pricing kernel derived within a general equilibrium setting that can simultaneously capture the salient features of equity returns, risk-free rates, and the prices of derivative securities. For example, Chen et al. (2004) investigate the ability of the BY and Campbell and Cochrane (1999) models to jointly price equity and risky (defaultable) corporate debt. Bansal et al. (2007) examine the implications of the BY and Campbell and Cochrane (1999) models for the pricing of at-the-money options on a stock market index as well as on consumption and wealth claims.

[5]Related, Garcia et al. (2001) consider a model with regime shifts in the conditional mean and the volatility of the dividend and consumption growth rates. They show that such a model can produce various shapes of the implied volatility function. Brown and Jackwerth (2004) consider a representative agent model in which the marginal utility of the representative agent is driven by a second state variable that is a function of a 'momentum' state variable. Bondarenko (2003) argues that in order to explain S&P 500 put prices a candidate equilibrium model must produce a path-dependent projected pricing kernel. Finally, Buraschi and Jiltsov (2006) consider a model in which heterogeneity in beliefs over the dividend growth rate generates state dependent utility. They focus on the volume of trading in the option market.

Also related is a growing literature that investigates the effect of changes in investors' sentiment (e.g., Han (2005)), market structure, and net buying pressure (e.g., Bollen and Whaley (2004), Dennis and Mayhew (2002), and Gârleanu et al. (2005)) on the shape of the implied volatility smile.[6] These papers, however, do not address why end users buy these options at high prices relative to the B/S value or why the 1987 crash changed the shape of the volatility smile so dramatically and permanently. Our paper offers one possible explanation.

3 A General Equilibrium Model of Equity and Option Prices

We specify the consumption and dividend dynamics as

$$\frac{dC}{C} = (\mu_C + x)\, dt + \sqrt{\Omega}\, dz_C \tag{1}$$

$$\frac{dD}{D} = (\mu_D + \phi\, x)\, dt + \sigma_D \sqrt{\Omega}\, \left(\rho_{C,D} dz_C + \sqrt{1-\rho_{C,D}^2} dz_D\right) \tag{2}$$

$$dx = -\kappa_x x\, dt + \sigma_x \sqrt{\Omega}\, dz_x + \widetilde{\nu}\, dN\, . \tag{3}$$

Here, $\{dz_C,\, dz_D,\, dz_x\}$ are uncorrelated Brownian motions, the Poisson jump process dN has a jump intensity equal to λ and the jump size $\widetilde{\nu}$ is normally distributed:

$$\mathrm{E}\,[dN] = \lambda\, dt \tag{4}$$

$$\widetilde{\nu} \rightsquigarrow N(\mu_\nu, \sigma_\nu). \tag{5}$$

It is convenient to define $c \equiv \log C$ and $\delta \equiv \log D$. Itô's formula then yields

$$dc = \left(\mu_C + x - \frac{1}{2}\Omega\right) dt + \sqrt{\Omega}\, dz_C \tag{6}$$

$$d\delta = \left(\mu_D + \phi\, x - \frac{1}{2}\sigma_D^2 \Omega\right) dt + \sigma_D \sqrt{\Omega}\, \left(\rho_{C,D} dz_C + \sqrt{1-\rho_{C,D}^2} dz_D\right)\, . \tag{7}$$

We note that our specification is similar to the so-called one-channel BY model, in which the expected growth rates in dividend and consumption are stochastic. There is however one important difference—in our setting, the state variable driving the expected growth rate in consumption and dividend (i.e., the x process) is subject to jumps. Consistent with BY, we calibrate the mean reversion parameter κ_x to be relatively low, implying that the effect of a downward jump may be very long-lived. As we demonstrate below, this persistence causes the agent in our model to be

[6] This literature argues that due to the existence of limits to arbitrage, market makers cannot always fully hedge their positions (see, e.g., Green and Figlewski (1999), Figlewski (1989), Hugonnier et al. (2005), Liu and Longstaff (2004), Longstaff (1995), and Shleifer and Vishny (1997)). As such, they are likely to charge higher prices when asked to absorb large positions in certain option contracts. Consistent with this view, Han (2005) finds that the S&P 500 option volatility smile tends to be steeper when survey evidence suggests that investors are more bearish, when large speculators hold more negative net positions in the S&P 500 index futures, and when the index level drops relative to its fundamentals. Related, Bollen and Whaley (2004) and Gârleanu et al. (2005) identify an excess of buyer-motivated trades in out-of-the-money SPX puts and find a positive link between demand pressure and the steepness of the volatility smirk.

willing to pay a high premium to buy out-of-the-money S&P 500 put options in order to hedge downside risk.

Here, we intentionally focus on a rather minimal version of the model. In particular, we present results only for the so-called 'one-channel' BY case, in which the dividend and consumption dynamics have constant volatility. In unreported results, we have also considered different extensions to our analysis. It is straightforward to solve a model in which the growth rate in dividends and consumption exhibit stochastic volatility (the 'two-channel' BY case), extended for the possible presence of jumps in volatility. The results, available upon request, are qualitatively similar to those discussed below. As in BY, stochastic volatility adds additional flexibility to match the moments of the underlying returns—in combination with jumps (in the predictable component of dividends and possibly in volatility), the model generates a steep volatility smirk. Further, as shown by BY the presence of stochastic volatility yields higher time-variation in risk premia, i.e., the two-channel model generates higher return predictability.[7]

Two aspects of the volatility smirk are evident from Figure 1. First, as mentioned previously, there has been a permanent shift in the shape of the implied volatility function due to the crash. Second, there are daily fluctuations in the shape of the smirk. This second feature has been studied extensively in the literature. Prior contributions have shown that these fluctuations can be understood in both a general equilibrium framework (e.g., David and Veronesi (2002)) and a partial equilibrium setting (e.g., Bakshi et al. (1997 and 2000), Bates (2003), Pan (2002), and Eraker (2004)). Such daily fluctuations can be captured within the context of our model by introducing additional state variables that drive high-frequency changes in expected dividend growth and/or volatility. However, since these daily fluctuations have already been explained, we do not investigate such variables in order to maintain parsimony. Instead, the focus of the paper is explaining the permanent shift in the implied volatility curve, and how stock crashes can occur with minimal changes in observable macroeconomic variables. To our knowledge, this paper is the first to investigate this issue.

3.1 Recursive Utility

Following Epstein and Zin (1989), we assume that the representative agent's preferences over a consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the following recursive equation:

$$U(t) = \left\{ (1 - e^{-\beta dt}) C_t^{1-\rho} + e^{-\beta dt} \mathrm{E}_t \left(U(t+dt)^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} . \tag{8}$$

With $dt = 1$, this is the discrete time formulation of Kreps-Porteus/Epstein-Zin (KPEZ), in which $\Psi \equiv 1/\rho$ is the EIS and γ is the risk-aversion coefficient.

[7] We note that the one-channel BY model generates constant risk-premia when considering the first order Campbell-Shiller approximation to the model. Higher order approximation (or 'exact' numerical solutions) of the model generate some, albeit small, time-variation in expected returns.

The properties of the stochastic differential utility in (8) and the related implications for asset pricing have been previously studied by, e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003). In Appendix B, we extend their results to the case in which the aggregate output has jump-diffusion dynamics.[8] The solution to our model is a special case of such general results and follows immediately from Propositions 1 and 2 in Appendix B. Specifically, when $\rho, \gamma \neq 1$ Proposition 1 shows that the agent's value function, which is defined via $J \equiv \frac{U^{1-\gamma}}{1-\gamma}$, takes the form:

$$J = \frac{e^{c(1-\gamma)}}{1-\gamma}\, \beta^{\theta}\, I(x)^{\theta}\,, \tag{9}$$

where I denotes the price-consumption ratio and satisfies the following equation

$$\begin{aligned} 0 \;=\;& I\left[(1-\gamma)\mu_C + (1-\gamma)x - \frac{\gamma}{2}(1-\gamma)\Omega - \beta\theta\right] - \kappa_x x\theta I_x \\ &+\frac{1}{2}\sigma_x^2\Omega\theta\left[(\theta-1)\left(\frac{I_x}{I}\right)^2 I + I_{xx}\right] + \lambda I\,\mathcal{J} I^{\theta} + \theta\,, \end{aligned} \tag{10}$$

and where we have defined the operator

$$\mathcal{J} h(x) = \mathrm{E}\left[\frac{h(x+\nu)}{h(x)}\right] - 1. \tag{11}$$

To obtain an approximate solution for $I(x)$, we use the method of Collin-Dufresne and Goldstein (2005), which itself is in the spirit of the Campbell-Shiller approximation. In particular, we note that $I(x)$ would possess an exponential affine solution *if* the last term on the right-hand-side (RHS) of equation (10) (the θ term) were absent. As such, we move θ to the left-hand-side (LHS) and then add to both sides of the equation the term $h(x) \equiv (n_0 + n_1 x)\, e^{A+Bx}$. Hence, we re-write equation (10) as

$$\begin{aligned} (n_0 + n_1 x)\, e^{A+Bx} - \theta \;=\;& (n_0 + n_1 x)\, e^{A+Bx} \\ &+ I\left[(1-\gamma)\mu_C + (1-\gamma)x - \frac{\gamma}{2}(1-\gamma)\Omega - \beta\theta\right] - \kappa_x x\theta I_x \\ &+\frac{1}{2}\sigma_x^2\Omega\theta\left[(\theta-1)\left(\frac{I_x}{I}\right)^2 I + I_{xx}\right] + \lambda I\mathcal{J} I^{\theta}\,. \end{aligned} \tag{12}$$

We then approximate the RHS to be identically zero and look for a solution of the form

$$I(x) = e^{A+Bx}\,. \tag{13}$$

We find this form to be self-consistent in that the only terms that show up are either linear in or independent of x. This approach provides us with two equations, which we interpret as identifying

[8] A related literature studies the general equilibrium properties of a jump-diffusion economy in which the agent has non-recursive utility; see, e.g., Ahn and Thompson (1988) and Naik and Lee (1990). Also related, Cvitanić et al. (2005) and Liu, Longstaff, and Pan (2003) examine the optimal portfolio choice problem when asset returns (or their volatility) are subject to jumps.

the $\{n_0, n_1\}$ coefficients in terms of B

$$-n_0 = (1-\gamma)\mu_C - \frac{\gamma}{2}(1-\gamma)\Omega - \beta\theta + \frac{1}{2}\sigma_x^2\Omega(\theta B)^2 + \lambda(\chi^P_{\theta B} - 1) \tag{14}$$

$$-n_1 = (1-\gamma) - \kappa_x \theta B\,, \tag{15}$$

where we have defined

$$\chi^P_a \equiv \mathrm{E}\left[e^{a\widetilde{\nu}}\right] = e^{a\mu_\nu + \frac{1}{2}a^2\sigma^2_\nu}\,. \tag{16}$$

The model solution is specified by the four parameters $\{A,\, B,\, n_0,\, n_1\}$. To this end, the system (14)-(15) provides two identifying conditions. The last two equations necessary to identify the remaining parameters are obtained from minimizing the following unconditional expectation:

$$\min_{\{A,\,B\}} \mathrm{E}_{-\infty}\left\{(LHS)^2\right\} = \min_{\{A,\,B\}} \mathrm{E}_{-\infty}\left\{\left((n_0 + n_1 x)\, e^{A+Bx} - \theta\right)^2\right\}. \tag{17}$$

The logic of this condition is as follows. Recall that we have set the RHS to zero above. Here, we are choosing the parameters so that the LHS is as close to zero as possible (in a least-squares error metric). Collin-Dufresne and Goldstein (2005) show that this approach provides an accurate approximation to the problem solution.

We note that the Campbell-Shiller approximation is similar in that their first two equations are as in (14)-(15) above. However, their last two equations satisfy[9]

$$\begin{aligned} 0 &= \left[LHS(x)\right]\Big|_{x=\mathrm{E}_{-\infty}[x]} \\ 0 &= \frac{\partial}{\partial x}\left[LHS(x)\right]\Big|_{x=\mathrm{E}_{-\infty}[x]}\,. \end{aligned}$$

3.2 Risk-Free Rate and Risk-Neutral Dynamics

When $\rho, \gamma \neq 1$, Proposition 1 in Appendix B gives the pricing kernel as

$$\Pi(t) = e^{\int_0^t ds\left((\theta-1)I(x_s)^{-1} - \beta\theta\right)}\, \beta^\theta e^{-\gamma c_t} I(x_t)^{\theta-1}\,, \tag{18}$$

which has dynamics

$$\frac{d\Pi}{\Pi} = -rdt - \gamma\sqrt{\Omega}dz_C + (\theta-1)B\sigma_x\sqrt{\Omega}\,dz_x + \left[\frac{I^{\theta-1}(x+\widetilde{\nu})}{I^{\theta-1}(x)} - 1\right] dN - \lambda\,\mathcal{J}I(x)^{\theta-1}dt\,, \tag{19}$$

where the risk-free rate r is given by Proposition 2 in Appendix B ($\rho, \gamma \neq 1$):

$$r = r_0 + \rho x \tag{20}$$

$$r_0 \equiv \beta + \rho\mu_C - \frac{\gamma}{2}\Omega(1+\rho) - \sigma_x^2\Omega(1-\theta)\frac{B^2}{2} - \lambda(\chi^P_{(\theta-1)B} - 1) + \frac{\theta-1}{\theta}\lambda(\chi^P_{\theta B} - 1)\,. \tag{21}$$

[9]Note that $\mathrm{E}_{-\infty}[x] \neq 0$ since we have written the state vector dynamics without compensator terms on the jumps.

Given the pricing kernel dynamics, it is straightforward to determine the risk-neutral dynamics. We find

$$
\begin{aligned}
dc &= \left(\mu_C + x - \Omega\left(\frac{1}{2} + \gamma\right)\right) dt + \sqrt{\Omega}\, dz_C^Q && (22) \\
d\delta &= \left(\mu_D + \phi x - \sigma_D \Omega\left(\frac{1}{2}\sigma_D + \rho_{C,D}\gamma\right)\right) dt + \sigma_D\sqrt{\Omega}\left(\rho_{C,D}\, dz_C^Q + \sqrt{1-\rho_{C,D}^2}\, dz_D^Q\right) && (23) \\
dx &= \left(-\kappa_x x - (1-\theta)B\sigma_x^2\Omega\right) dt + \sigma_x\sqrt{\Omega}\, dz_x^Q + \widetilde{\nu}\, dN\,, && (24)
\end{aligned}
$$

where the three Brownian motions $\{dz_C^Q,\, dz_x^Q,\, dz_\Omega^Q\}$ are uncorrelated, and the Q-intensity of the Poisson jump process N is

$$
\lambda^Q = \lambda\, \chi^P_{(\theta-1)B}\,. \tag{25}
$$

Furthermore, the Q-probability density of the jump amplitudes is

$$
\begin{aligned}
\pi^Q(\widetilde{\nu} = \nu) &= \pi(\widetilde{\nu} = \nu)\, \frac{I^{\theta-1}(x+\nu)}{\mathrm{E}\left[I^{\theta-1}(x+\widetilde{\nu})\right]} \\
&= \frac{1}{\sqrt{2\pi\sigma_\nu^2}} \exp\left\{\left(-\frac{1}{2\sigma_\nu^2}\right)\left[\nu - \mu_\nu - (\theta-1)B\sigma_\nu^2\right]^2\right\}. && (26)
\end{aligned}
$$

That is,

$$
\begin{aligned}
\widetilde{\nu}^Q &\rightsquigarrow N(\mu_\nu^Q, \sigma_\nu) \\
\mu_\nu^Q &= \mu_\nu + (\theta-1)B\sigma_\nu^2\,. && (27)
\end{aligned}
$$

3.3 Dividend Claim

Define $V(D,\, x)$ as the claim to dividend. By construction, the expected return under the risk neutral measure is the risk-free rate:

$$
\mathrm{E}_t^Q\left[\frac{dV + D\,dt}{V}\right] = r\, dt. \tag{28}
$$

It is convenient to define the price-divided ratio $I^D \equiv \frac{V}{D}$. Equation (28) can thus be written as

$$
r - \frac{1}{I^D} = \frac{1}{dt}\mathrm{E}^Q\left[\frac{dV}{V}\right] = \frac{1}{dt}\mathrm{E}^Q\left[\frac{dI^D}{I^D} + \frac{dD}{D} + \frac{dD}{D}\frac{dI^D}{I^D}\right]. \tag{29}
$$

We look for a solution of the form

$$
I^D(x) = e^{F+Gx}\,. \tag{30}
$$

We use the risk-neutral dividend and x-dynamics (23)-(24) to re-write equation (29) as

$$
r - \frac{1}{I^D} = \mu_D + \phi\, x - \gamma\rho_{C,D}\sigma_D\Omega - \kappa_x x G - (1-\theta)BG\sigma_x^2\Omega + \frac{1}{2}G^2\sigma_x^2\Omega + \lambda^Q(\chi_G^Q - 1)\,, \tag{31}
$$

where we have defined

$$
\chi_a^Q \equiv \mathrm{E}^Q\left[e^{a\widetilde{\nu}}\right] = e^{a\mu_\nu^Q + \frac{1}{2}a^2\sigma_\nu^2}. \tag{32}
$$

As above, we find an approximate solution for I^D by moving r to the RHS, multiplying both sides by I^D, and adding $(m_0 + m_1 x)\, I^D$ to both sides. These calculations give

$$\begin{aligned} LHS &= (m_0 + m_1 x)\, e^{F+Gx} - 1 \qquad (33) \\ \left(\frac{1}{I^D}\right) RHS &= (m_0 + m_1 x) - r + \mu_D + \phi\, x - \gamma \rho_{C,D} \sigma_D \Omega - \kappa_x x G \\ &\quad -(1-\theta) B G \sigma_x^2 \Omega + \frac{1}{2} G^2 \sigma_x^2 \Omega + \lambda^Q (\chi_G^Q - 1)\,. \qquad (34) \end{aligned}$$

From equation (20), $r = r_0 + \rho x$. Hence, if we approximate the RHS to be identically zero, and then collect terms linear in and independent of x, respectively, we obtain the system:

$$\begin{aligned} -m_0 &= -r_0 + \mu_D - \gamma \rho_{C,D} \sigma_D \Omega - (1-\theta) B G \sigma_x^2 \Omega + \frac{1}{2} G^2 \sigma_x^2 \Omega + \lambda^Q (\chi_G^Q - 1) \qquad (35) \\ -m_1 &= -\rho - \kappa_x G + \phi\,. \qquad (36) \end{aligned}$$

Equations (35)-(36) specify $\{m_0, m_1\}$ in terms of G. In turn, we identify $\{F, G\}$ by minimizing the unconditional squared error:

$$\min_{\{F,G\}} \mathrm{E}_{-\infty} \left[(LHS)^2 \right] = \min_{\{F,G\}} \mathrm{E}_{-\infty} \left[\left((m_0 + m_1 x)\, e^{F+Gx} - 1 \right)^2 \right]. \qquad (37)$$

Finally, note that $V = I^D\, D$. Thus, Itô's Lemma yields an expression for the V-dynamics under the Q measure,

$$\begin{aligned} \frac{dV}{V} &= \left(r - \frac{1}{I^D} \right) dt + G \sigma_x \sqrt{\Omega}\, dz_x^Q \\ &\quad + \sigma_D \sqrt{\Omega} \left(\rho_{C,D} dz_C^Q + \sqrt{1-\rho_{C,D}^2}\, dz_D^Q \right) + dN(e^{G\widetilde{\nu}} - 1) - \lambda^Q (\chi_G^Q - 1)\, dt\,, \qquad (38) \end{aligned}$$

where the drift term $(r - \frac{1}{I^D})$ is given in equation (31), and under the physical probability measure,

$$\begin{aligned} \frac{dV}{V} &= \left(\mu_D + \phi x - \kappa_x x G + \frac{1}{2} G^2 \sigma_x^2 \Omega \right) dt \\ &\quad + G \sigma_x \sqrt{\Omega}\, dz_x + \sigma_D \sqrt{\Omega} \left(\rho_{C,D} dz_C + \sqrt{1-\rho_{C,D}^2}\, dz_D \right) + dN(e^{G\widetilde{\nu}} - 1)\,. \qquad (39) \end{aligned}$$

3.4 The Equity Premium

The general form of the risk premium on the risky asset is given in equation (105) of Proposition 2 in Appendix B. Here, such expression simplifies to

$$\text{Equity Premium} = \gamma\, \sigma_D \rho_{C,D}\, \Omega + (1-\theta) B G \sigma_x^2 \Omega - \lambda \left[\, \chi_{G+(\theta-1)B}^P - \chi_G^P - \chi_{(\theta-1)B}^P + 1 \,\right], \qquad (40)$$

where the transform $\chi_\bullet^P$ was previously defined in equation (16).

In equation (40), the second and third terms represent the risk premia on the diffusive and jump components of expected growth risk. We note that in the constant relative risk aversion (CRRA) case, γ equals $1/\Psi$, and therefore $\theta = 1$. As such, the last two terms in equation (40) vanish and the

CRRA equity premium reduces to $(\gamma\,\sigma_D \rho_{C,D}\,\Omega)$. Thus, as in BY, with CRRA utility a persistent endowment process cannot generate a realistic equity premium, let alone explain out-of-the-money put prices.

On the other hand, in the KPEZ case with $\Psi > 1$, the risk premium on expected growth risk is positive. As in BY, the mechanism for this results is as follows. When $\Psi > 1$, the intertemporal substitution effect dominates the wealth effect. Thus, in response to higher expected growth, the demand function for assets of the representative agent increases, and consequently the wealth-to-consumption ratio increases. That is, in this scenario the coefficient B in the wealth-to-consumption-ratio function (13) is positive. In addition, due to the effect of leverage the coefficient G in the price-to-dividend ratio function (30) is larger than B. Hence, the last two terms in equation (40) are positive. Intuitively, with KPEZ utility and $\Psi > 1$, the stock exhibits positive returns when the state is good, while it performs poorly in the bad state. As such, investors demand a higher risk premium.

3.5 Valuing Options on the S&P 500 Index

The date-t value of an European call option on the dividend claim $V_t = D_t e^{F+Gx_t}$, with maturity T and strike price K, is given by

$$C(V_t, x_t, K, T) = \mathrm{E}_t^Q \left[e^{-\int_t^T r(x_s)\,ds} \; (V_T - K)^+ \right] . \tag{41}$$

We note that our model is affine. As such, the option pricing problem can be solved using standard inverse Fourier transform techniques (see, e.g., Bates (1996), Duffie et al. (2000), and Heston (1993)). In Appendix C, we report a semi-closed form formula for the price of an option given in equation (41).

3.6 Valuing Options on Individual Stocks

As in Bakshi, Kapadia, and Madan (2003), we specify return dynamics on an individual stock, $\frac{dV_i}{V_i}$, as a sum of a systematic component and an idiosyncratic component. In particular, we assume individual firm dynamics follow

$$\frac{dV_i}{V_i} = \frac{dV}{V} + \sigma_i\,dz_i + \left[\left(e^{\widetilde{\nu}_i} - 1\right) dN_i - \mathrm{E}\left[e^{\widetilde{\nu}_i} - 1\right] \lambda_i\,dt \right] , \tag{42}$$

where the market return dynamics $\frac{dV}{V}$ are in equation (39). Here, σ_i captures the volatility of the idiosyncratic diffusive shock, while the diversifiable jump component has Poisson arrival rate N_i with constant intensity λ_i and normally-distributed jump size $\widetilde{\nu}_i \rightsquigarrow N(\mu_{\nu_i}, \sigma_{\nu_i})$. The free parameters $(\sigma_i,\, \lambda_i,\, \mu_{\nu_i},\, \sigma_{\nu_i})$ are chosen to match historical moments of the return distribution on individual firms. By definition, the diversifiable shocks do not command a risk premium, while the risk adjustments on the systematic component are identical to those that we have applied to price the options on the S&P 500 index. As such, the price of an option on an individual stock is given by a formula similar to equation (41).

We acknowledge that there is a *potential* concern that the dynamics (42) for the individual firms and the dynamics (39) for the aggregate index are not self-consistent. Indeed, it is not obvious a priori that the terminal value of a strategy that invests an amount $V(0) = \sum_{i=1}^{N} V_i(0)$ in the index will have the same terminal value of a strategy that invests an amount $V_i(0)$ in each of the individual stocks, $i = 1, \ldots, N$. However, as we demonstrate in Appendix D, the discrepancy is negligible, i.e., $V(T) \approx \sum_{i=1}^{N} V_i(T)$. Intuitively, the idiosyncratic shocks that we specify are in fact diversifiable when the portfolio is composed of a sufficiently large number of firms.

3.7 Model Calibration

To illustrate the implications of the model, we consider a realistic calibration of its coefficients. In the next section, we will show that our main result is robust to a wide range of parameter calibrations.

1. *Consumption and Dividend Dynamics:*

   To calibrate the consumption process in equations (6), we rely on the model coefficients reported in BY. BY use the convention to express their parameters in decimal form with *monthly* scaling. Here, instead, we express them in decimal form with *yearly* scaling. After adjusting for differences in scaling, we fix $\mu_C = 0.018$ and $\Omega = 0.00073$.

   We note that corporate leverage justifies a higher expected growth rate in dividends than in consumption (see, e.g., Abel (1999)). This can be modeled by setting $\mu_D > \mu_C$ and $\phi > 1$ in equation (7). As such, we fix $\mu_D = 0.025$ and $\phi = 1.5$. We note the difference with BY, who assume $\mu_D = \mu_C$ and model leverage entirely through the ϕ coefficient, which they choose to be in the 3-3.5 range. We use $\sigma_D = 4.5$, the same value of BY. Finally, we allow for a 60% correlation between consumption and dividend, i.e., $\rho_{C,D} = 0.6$.

   In the x-dynamics (3), we use $\kappa_x = 0.3$. This is in line with the value used by BY (if we adjust for differences in scaling and we map the BY AR(1) ρ coefficient into the κ_x of our continuous-time specification, we find $\kappa_x = 0.2547$). We fix $\sigma_x = 0.4472$, a value similar to, but slightly lower than that of BY (i.e., 0.5280, after adjusting for differences in scaling). A slightly lower value of σ_x is justified by the fact that part of the variance of the x process is driven by the jump component, which is absent in the BY model.

   Finally, we calibrate the Poisson jump intensity process to yield, on average, one jump every fifty years, i.e., $\lambda = 0.02$. This is consistent with the intuition that our jump process captures extreme and very rare price fluctuations such as the 1987 market crash. Further, we fix $\mu_\nu = -0.094$. This approach implies that one jump of average size produces a fall in market prices of approximately 23%, which is in line with the 24.5% drop in the S&P 500 index observed in between the close of Thursday, October 15, and Monday, October 19, 1987. Finally, we fix the standard deviation of the jump size to $\sigma_{\nu} = 0.015$.

2. *Individual Stock Returns:*

   For each of the 20 stocks in the Bollen and Whaley (2004) study, we compute standard deviation, skewness, and kurtosis by using daily return series for the sample period from January 1995 to December 2000 (the same period considered by Bollen and Whaley).[10] For each of these statistics, we evaluate cross sectional averages. We find an average standard deviation of 37.6% per year, and average skewness and kurtosis of 0.12 and 7.12, respectively. Four coefficients characterize the distribution of the idiosyncratic shocks in equation (42): the standard deviation of the diffusive firm-specific shock, σ_i, the intensity of the diversifiable jump component, λ_i, and the mean and standard deviation of the jump size, μ_{ν_i} and σ_{ν_i}. After some experimentation, we fix the jump intensity to $\lambda_i = 5$, which corresponds to an expected arrival rate of 5 jumps per year. We choose the remaining coefficients to match the average standard deviation, skewness, and kurtosis reported above. This approach yields $\sigma_i = 0.3205$, $\mu_{\nu_i} = 0.0038$, and $\sigma_{\nu_i} = 0.0658$. We have confirmed that the results reported below are robust to the choice of the λ_i coefficient. To this end, we have solved for σ_i, μ_{ν_i}, and σ_{ν_i} when λ_i takes different values in the 1-10 range. The results were similar to those discussed below.

3. *Preferences:*

   We use a time discount factor coefficient $\beta = 0.023$.

   Mehra and Prescott (1985) argue that reasonable values of the relative risk aversion coefficient γ are smaller than 10. BY consider $\gamma = 7.5$ and 10. Bansal et al. (2007) report $\gamma = 7.1421$. As such, we fix $\gamma = 7.5$ in our baseline case.

   The magnitude of the coefficient ρ is more controversial. Hall (1988) argues that the EIS is below 1. However, Attanasio and Weber (1989), Bansal et al. (2007), Guvenen (2001), Hansen and Singleton (1982), among others, estimate the EIS to be in excess of 1. In particular, Attanasio and Weber (1989) find estimates that are close to 2. Bansal et al. (2006) construct a proxy for total wealth that comprises corporate equity and debt, durable goods (houses), and human capital. They use such measure of wealth to estimate the EIS, and they find it to be well in excess of 1. Bansal et al. (2007) estimate the EIS to be in the 1.5-2.5 region, and fix it at 2 in their application. Here we follow Bansal et al. (2007) and use $\Psi = 1/\rho = 2$ for our baseline case.

   In the next section, we document the sensitivity of our results to different values of γ and Ψ.

[10] CRSP data, Center for Research in Security Prices, Graduate School of Business, University of Chicago, used with permission.

4. *Initial Conditions:*

   In the plots below, we fix the state variable x at its steady-state mean value $x_0 = \mu_\nu \lambda / \kappa_x$. We note that x_0 is nearly zero in our calibration, i.e., it is very close to the steady-state mean value of x in the BY model. Further, we emphasize that our results are robust to the value assigned to the state x. Specifically, when option prices are computed at values of x that are within $\pm$ 3 standard deviations from the steady-state mean x_0, we obtain implied volatility plots that are very similar to those reported below.

3.8 Simulation Results

Our calibration yields realistic values of the risk-free rate, the equity premium, and the price-dividend ratio. Specifically, in the baseline case we find that the steady-state real risk-free rate is 0.93%, and its standard deviation is 1.2%. The equity premium predicted by the model is 5.76%, while the standard deviation of the stock market return is 13.1%. Further, we find that the steady-state price-dividend ratio is 20.

Most importantly, the model produces a volatility smirk that is consistent with post-1987 market crash observation. Figure 2 reports implied volatilities for options on the S&P 500 index with one month to maturity for the baseline case. The main result is that put options that are 10% out-of-the-money have a 23.8% implied volatility. At-the-money options have a 13.8% implied volatility. As such, consistent with the evidence in Figure 1 the model predicts a realistic 10% volatility smirk, as measured by the difference in 10%-out-of-the-money and at-the-money implied volatilities.

3.9 Sensitivity Analysis

Here we investigate the sensitivity of our findings to changes in the underlying parameters:

Jump Coefficients

Figure 3 illustrates the sensitivity of our results to the jump coefficients λ and μ_ν. In the left panel we lower the jump intensity coefficient λ to 0.01, which corresponds to an expected arrival rate of one jump every 100 years. Interestingly, we find that most of the volatility smirk remains intact. As intuition would suggest, increasing the jump intensity to 0.03, i.e., one jump every 33 years, makes our results much stronger.

In the right panel, we illustrate the effect of a one-standard-deviation perturbation of the average jump size coefficient. We note that in the model a value of $\mu_\nu = (-0.094 + \sigma_\nu) = -0.079$ implies that a jump of average size determines a 20.6% fall in stock prices, which is smaller than the 24.5% drop in the S&P 500 index observed in between the close of Thursday, October 15, and Monday, October 19, 1987.[11] Still, the model predicts a steep volatility smirk.

[11] Note, however, that the drop in prices between the close of Friday October 16 and Monday October 19 was 20.46%. Furthermore, the S&P 500 closing prices over that week are as follows. 1987-10-13: \$314.52; 1987-10-14: \$305.23; 1987-10-15: \$298.08; 1987-10-16: \$282.94; 1987-10-19: \$225.06; 1987-10-20: \$236.84.

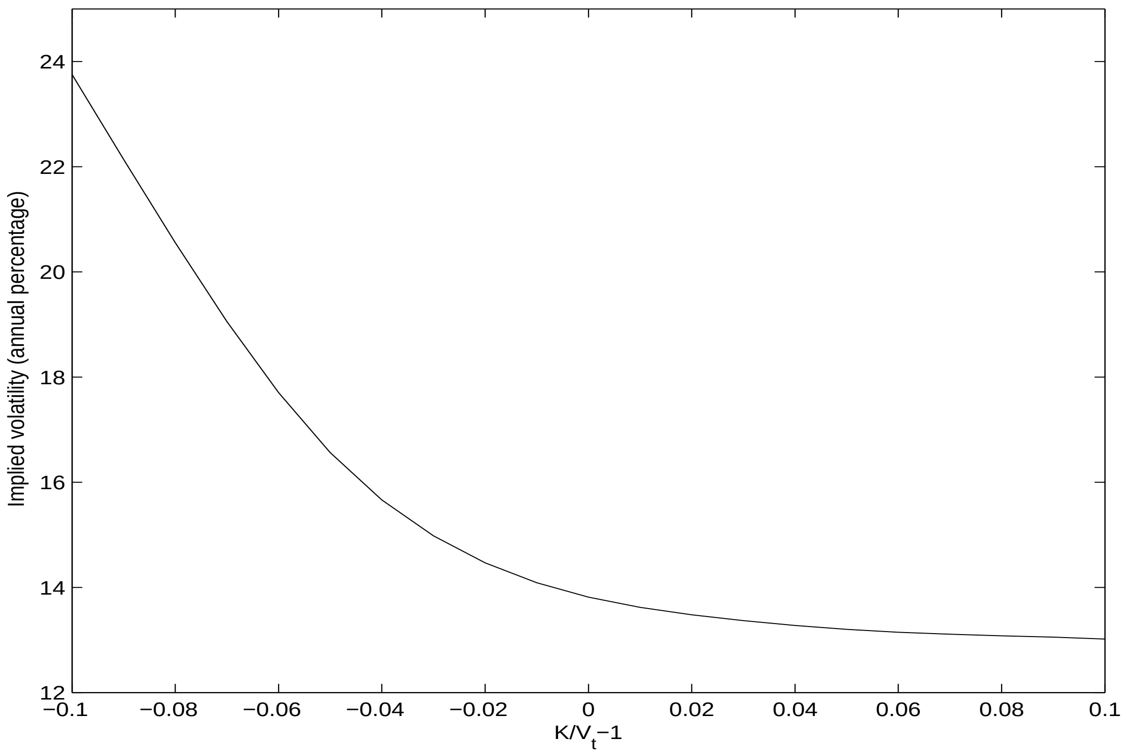

Figure 2: The plot depicts the implied volatility smirk for S&P 500 options with one month to maturity. The model coefficients are set equal to the baseline values.

Preferences Coefficients

Figure 4 illustrates the sensitivity of our results to the preferences coefficients γ and $\Psi \equiv 1/\rho$. The left panel shows that when the coefficient of risk aversion is lowered to 5, most of the volatility smirk remains intact. Further, we note that when $\gamma = 10$ (the upper bound of the range that Mehra and Prescott (1985) consider reasonable) the volatility smirk becomes considerably steeper.

The right panel illustrates the sensitivity of the volatility smirk to the EIS coefficient. As noted previously, researchers have obtained a wide array of estimates for this parameter. Our base case estimate of $\rho = 2$ is consistent with that of Bansal et al. (2007). Here we demonstrate that even lower estimates for ρ, such as 1.25 and 1.5, still produce steep volatility smirks.

3.10 The Pricing of Options on Individual Stocks

In this section we illustrate the model implications for the pricing of individual stock options. We simulate option prices for a typical stock and extract B/S implied volatilities. Figure 5 contrasts such implied volatility function to the volatility smirk for S&P 500 options. Consistent with the evidence in Bollen and Whaley (2004), Bakshi, Kapadia, and Madan (2003), Dennis and Mayhew (2002), our model predicts that the volatility smile for individual stock options is considerably flatter than that for S&P 500 options.

Bakshi, Kapadia, and Madan (2003) conclude that the differential pricing of individual stock

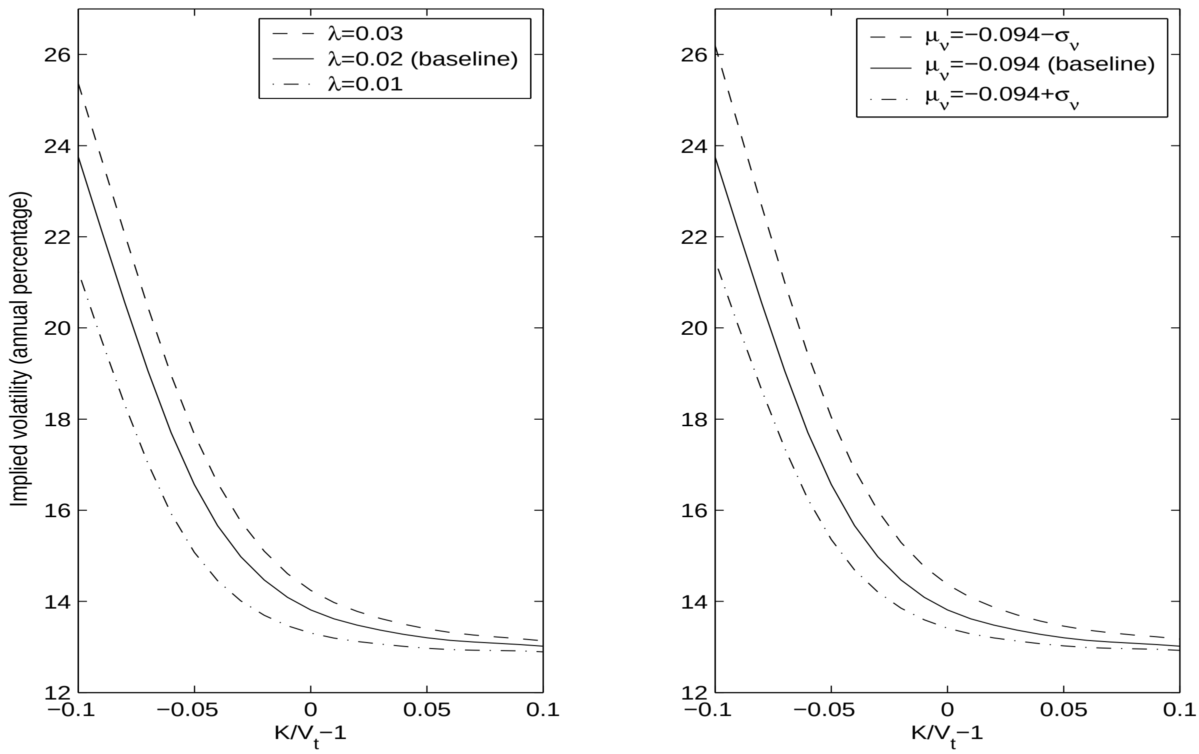

Figure 3: The plot illustrates the sensitivity of the implied volatility smirk to the agent's preferences coefficients, i.e., the jump intensity coefficient λ and the average jump size coefficient μ_ν. Implied volatilities are from S&P 500 options with one month to maturity.

options is driven by the degree of skewness/kurtosis in the underlying return distribution in combination with the agent's high level of risk aversion. Here, we propose a plausible endowment economy that in combination with recursive utility yields predictions consistent with their empirical findings. Combined with our results discussed above, this evidence suggests that the market of S&P 500 and individual stock options, as well as the market for the underlying stocks, are well integrated.

4 Bayesian Updating of Jump Beliefs

In this section, we examine whether our model can also explain the stark change in the implied volatility pattern that has maintained since the 1987 market crash. In the previous section, we assumed that the specified parameters of the model are known to the agent. In what follows, we will assume that, because stock market crashes are so rare, the agent does not know the exact distribution of the jump size. As such, she will update her prior beliefs about the distribution of jump size after observing a crash. Note that this Bayesian updating only occurs at crash dates. As such, the effect on the implied volatility pattern can be extremely long-lived.

We specify the model so that, prior to the first crash, given the agent's information set, the distribution of the jump size $\widetilde{\nu}_1$ is a normal random variable whose mean value $\bar{\mu}_\nu$ is itself an

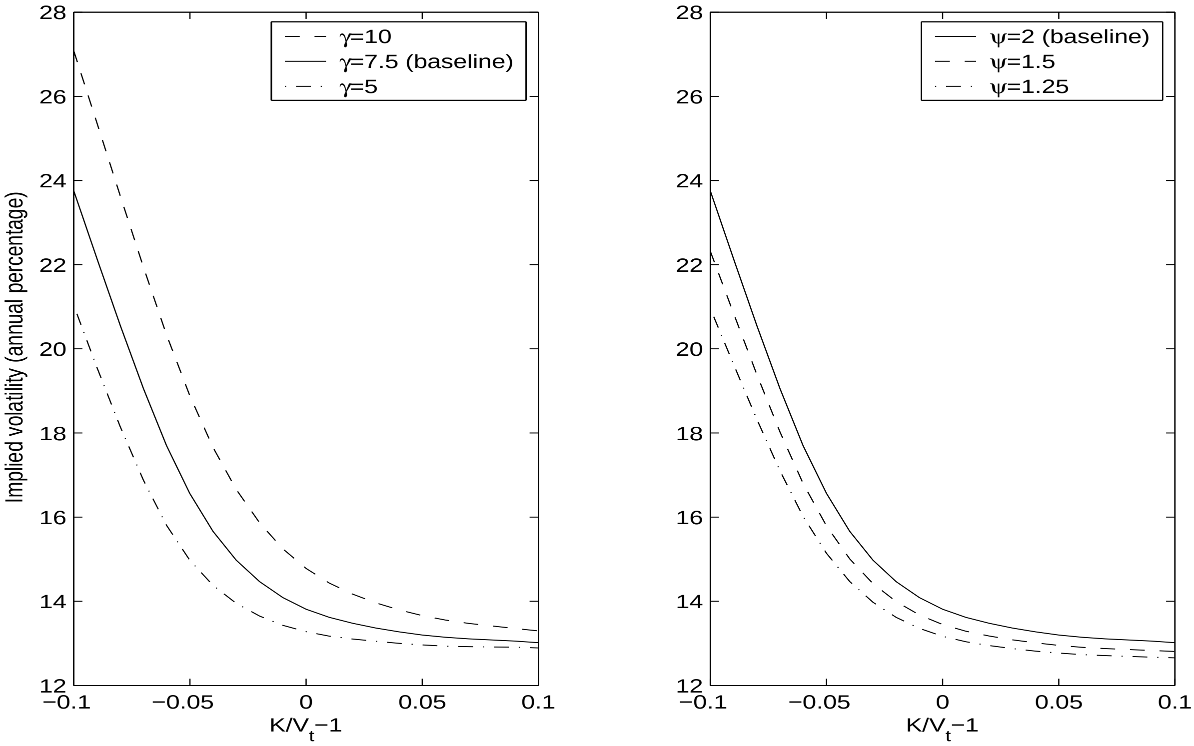

Figure 4: The plot illustrates the sensitivity of the implied volatility smirk to the agent's preferences coefficients, i.e., the coefficient of relative risk aversion γ and the EIS $\Psi = \frac{1}{\rho}$. Implied volatilities are from S&P 500 options with one month to maturity.

unknown quantity, and is selected from a normal distribution:

$$\begin{aligned} \widetilde{\nu}_1|\bar{\mu}_\nu &\rightsquigarrow N(\bar{\mu}_\nu, \bar{\sigma}_\nu^2) &(43)\\ \bar{\mu}_\nu &\rightsquigarrow N(\bar{\bar{\mu}}_\nu, \bar{\bar{\sigma}}_\nu^2)\,. &(44)\end{aligned}$$

That is, before the first crash occurs, the agent's prior is

$$\widetilde{\nu}_1 \rightsquigarrow N(\bar{\bar{\mu}}_\nu, \bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2)\,. \tag{45}$$

After the first crash occurs and the agent observes the realization of $\widetilde{\nu}_1$, she updates her beliefs about the distribution of $\bar{\mu}_\nu$ via the projection theorem:

$$\begin{aligned} \mathrm{E}[\bar{\mu}_\nu|\widetilde{\nu}_1] &= \mathrm{E}[\bar{\mu}_\nu] + \frac{\mathrm{Cov}(\bar{\mu}_\nu, \widetilde{\nu}_1)}{\mathrm{Var}(\widetilde{\nu}_1)}\left(\widetilde{\nu}_1 - \mathrm{E}[\widetilde{\nu}_1]\right) \\ &= \bar{\bar{\mu}}_\nu \left(\frac{\bar{\sigma}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2}\right) + \widetilde{\nu}_1 \left(\frac{\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2}\right) &(46)\\ \mathrm{Var}[\bar{\mu}_\nu|\widetilde{\nu}_1] &= \mathrm{Var}(\bar{\mu}_\nu) - \frac{\mathrm{Cov}(\bar{\mu}_\nu, \widetilde{\nu}_1)^2}{\mathrm{Var}(\widetilde{\nu}_1)} \\ &= \frac{\bar{\sigma}_\nu^2 \bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2}\,. &(47)\end{aligned}$$

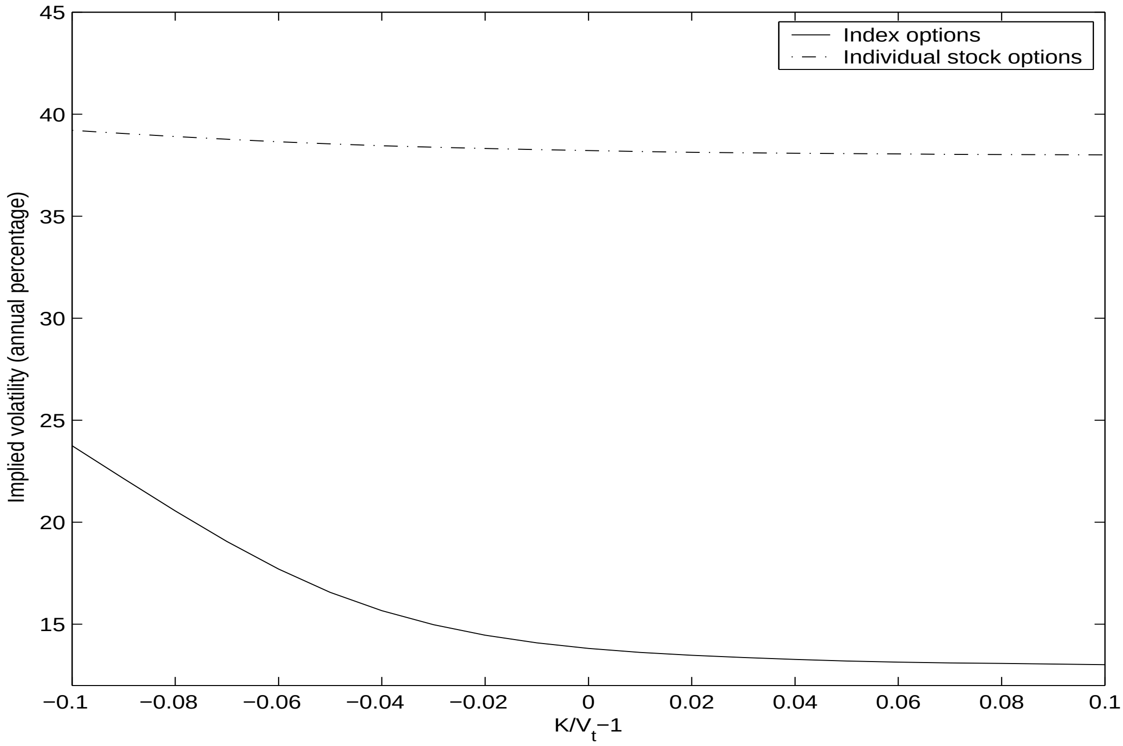

Figure 5: The plot contrasts the implied volatility function for individual stock options to the volatility smirk for S&P 500 options with one month to maturity. The model coefficients are set equal to the baseline values.

Hence, the agent sees the second crash size as distributed normally:

$$\widetilde{\nu}_2 \quad \rightsquigarrow \quad N\left(\mathrm{E}[\bar{\mu}_\nu|\widetilde{\nu}_1],\, \bar{\sigma}_\nu^2 + \mathrm{Var}[\bar{\mu}_\nu|\widetilde{\nu}_1]\right). \tag{48}$$

We see from equation (46) that if the realization of $\widetilde{\nu}_1$ is substantially worse than the pre-crash estimate $\bar{\mu}_\nu$, then, after the first crash, the expected size of the next crash is considerably worse. Further, we emphasize that the random variable $\bar{\mu}_\nu$ is chosen only once at date-0, and hence uncertainty about its value is reduced at the crash date, as noted in equation (47). Indeed, prior to the crash the uncertainty about the value of $\bar{\mu}_\nu$ is $\bar{\bar{\sigma}}_\nu^2$, as can be seen from equation (44). However, after the crash, this uncertainty reduces to $\frac{\bar{\sigma}_\nu^2\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2+\bar{\bar{\sigma}}_\nu^2} = \frac{\bar{\sigma}_\nu^2}{1+\frac{\bar{\sigma}_\nu^2}{\bar{\bar{\sigma}}_\nu^2}}$. Below, we will parameterize the model so that $\bar{\bar{\sigma}}_\nu^2 \ll \bar{\sigma}_\nu^2$. As such, most of the uncertainty regarding the value of $\bar{\mu}_\nu$ is determined from the first crash. While the agent would typically continue to update her beliefs about the distribution of $\bar{\mu}_\nu$ when subsequent crashes occur, given the parametrization of the model we choose below, there would be little change in the subsequent posterior beliefs. Therefore, and because it considerably simplifies the analysis, we make the assumption that the updating of jump beliefs occurs only once, when the agent observes a jump for the first time. Effectively this approach implies that the pre- and post-crash jump distributions are given by, respectively:

$$\widetilde{\nu}_1 \quad \rightsquigarrow \quad N(\bar{\bar{\mu}}_\nu, \bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2) \tag{49}$$

$$\widetilde{\nu}_j \rightsquigarrow N\left\{\left[\bar{\bar{\mu}}_\nu\left(\frac{\bar{\sigma}_\nu^2}{\bar{\sigma}_\nu^2+\bar{\bar{\sigma}}_\nu^2}\right)+\widetilde{\nu}_1\left(\frac{\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2+\bar{\bar{\sigma}}_\nu^2}\right)\right], \bar{\sigma}_\nu^2+\frac{\bar{\sigma}_\nu^2\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2+\bar{\bar{\sigma}}_\nu^2}\right\} \qquad j=2,3,,....\infty. \quad (50)$$

4.1 Model Solution with Bayesian Updating

We have assumed that the agent updates her beliefs only once, when she observes the first jump. As such, we only need to consider two cases when solving our problem. First, the case in which the agent is aware that stock market prices can jump, but she has not yet seen a jump occur. Second, the case in which the agent has witnessed a jump in market prices and therefore has updated her beliefs on the jump distribution. Intuitively, we can think of the first case as a description of the pre-1987 crash economy, while the second one depicts the post-1987 regime.

Once the agent has updated her beliefs in reaction to the occurrence of the first jump, the post-crash problem reduces to the setting without Bayesian updating that we have already considered in Sections 3.1-3.3. As such, the solution to the problem is unchanged, except that the mean μ_ν and variance σ_ν^2 in the jump distribution (5) are replaced by those of the post-crash jump distribution (50).

When solving the pre-crash problem, instead, we need to account for the fact that the agent rationally anticipates that the occurrence of a crash will determine an updating of the prior on the jump coefficients. To this end, we proceed as follows: As before, we exogenously specify the aggregate consumption and dividends dynamics as in equations (3)-(7). However, we now assume that the pre-crash jump size distribution is given by equation (49). Further, we consider a representative agent's whose preferences over the consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the recursive equation (8).

Proposition 1 in Appendix B still applies. As such, when $\rho, \gamma \neq 1$ the pre-crash value function J_{pre} has the form:

$$J_{pre} = \frac{e^{c(1-\gamma)}}{1-\gamma}\beta^\theta I_{pre}(x)^\theta, \quad (51)$$

where the price-consumption ratio I_{pre} satisfies the following equation

$$\begin{aligned} 0 &= I_{pre}\Big((1-\gamma)\mu_C+(1-\gamma)x-\frac{\gamma}{2}(1-\gamma)\Omega-\beta\theta\Big)-\kappa_x x\theta I_{pre,x} \\ &+\frac{1}{2}\sigma_x^2\Omega\theta\left[(\theta-1)\left(\frac{I_{pre,x}}{I_{pre}}\right)^2 I_{pre}+I_{pre,xx}\right]+\lambda I_{pre}\,\mathrm{E}_{\widetilde{\nu}_1}\left[\frac{I^\theta_{post}(x+\widetilde{\nu}_1)}{I^\theta_{pre}(x)}-1\right]+\theta. \quad (52)\end{aligned}$$

We note the effect of Bayesian updating on the pre-crash price-consumption ratio I_{pre}. The agent anticipates that if a crash occurs, the price-consumption ratio will take the post-crash form

$$I_{post} = e^{\tilde{A}+\tilde{B}x}, \quad (53)$$

where, for each different possible realization of $\widetilde{\nu}_1$, the coefficients $\tilde{A} \equiv A(\widetilde{\nu}_1)$ and $\tilde{B} \equiv B(\widetilde{\nu}_1)$ minimize the squared error in equation (17).

An approach similar to that followed in Section 3.1 delivers an approximate solution of the form

$$I_{pre}(x) = e^{A_{pre}+B_{pre}x}\,. \tag{54}$$

Specifically, we re-write equation (52) as

$$\begin{aligned}(p_0+p_1x)\,e^{A_{pre}+B_{pre}x}-\theta-\lambda e^{(1-\theta)(A_{pre}+B_{pre}x)}\,\mathrm{E}_{\widetilde{\nu}_1}\left[e^{\theta(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right] &= (p_0+p_1x)\,e^{A_{pre}+B_{pre}x}\\ +e^{A_{pre}+B_{pre}x}\left[(1-\gamma)\mu_C+(1-\gamma)x-\frac{\gamma}{2}(1-\gamma)\Omega-\beta\theta-\kappa_x x\theta B_{pre}+\frac{1}{2}\sigma_x^2\Omega(\theta B_{pre})^2-\lambda\right].\end{aligned} \tag{55}$$

We set the RHS of (55) to zero and obtain a system of two equations, which identify the $\{p_0, p_1\}$ coefficients in terms of B_{pre}:

$$-p_0 = (1-\gamma)\mu_C-\frac{\gamma}{2}(1-\gamma)\Omega-\beta\theta+\frac{1}{2}\sigma_x^2\Omega(\theta B_{pre})^2-\lambda \tag{56}$$

$$-p_1 = (1-\gamma)-\kappa_x B_{pre}\theta\,. \tag{57}$$

We then choose $\{A_{pre},\, B_{pre}\}$ by minimizing the unconditional squared error:

$$\min_{\{A_{pre},\,B_{pre}\}}\ \mathrm{E}_{-\infty}\left\{\left((p_0+p_1x)\,e^{A_{pre}+B_{pre}x}-\theta-\lambda e^{(1-\theta)(A_{pre}+B_{pre}x))}\,\mathrm{E}_{\widetilde{\nu}_1}\left[e^{\theta(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right]\right)^2\right\}. \tag{58}$$

Next, we derive the dynamics of the pre-crash pricing kernel:

$$\begin{aligned}\frac{d\Pi_1}{\Pi_1} &= -r_{pre}dt-\gamma\sqrt{\Omega}dz_C+(\theta-1)B_{pre}\sigma_x\sqrt{\Omega}\,dz_x\\ &+\left[\frac{e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}}{e^{(\theta-1)(A_{pre}+B_{pre}x)}}-1\right]dN-\lambda\mathrm{E}_{\widetilde{\nu}_1}\left[\frac{e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}}{e^{(\theta-1)(A_{pre}+B_{pre}x)}}-1\right]dt\,,\end{aligned} \tag{59}$$

where the pre-crash risk-free rate r_{pre} is no longer an affine function of x:

$$\begin{aligned}r_{pre} &= r_{pre,0}+\rho x-\lambda\frac{(1-\theta)}{\theta}\,e^{-\theta(A_{pre}+B_{pre}x)}\,\mathrm{E}_{\widetilde{\nu}_1}\left[e^{\theta(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right]\\ &\quad-\lambda\,e^{(1-\theta)(A_{pre}+B_{pre}x)}\,\mathrm{E}_{\widetilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right]\\ r_{pre,0} &= \beta+\rho\mu_C-\frac{\gamma}{2}\Omega(1+\rho)-\frac{1}{2}\sigma_x^2\Omega(1-\theta)B_{pre}^2+\frac{\lambda}{\theta}\,.\end{aligned} \tag{60}$$

Further, we obtain pre-crash risk-neutral dynamics:

$$dc = \left(\mu_C+x-\Omega\left(\frac{1}{2}+\gamma\right)\right)dt+\sqrt{\Omega}\,dz_C^Q \tag{61}$$

$$d\delta = \left(\mu_D+\phi x-\sigma_D\Omega\left(\frac{1}{2}\sigma_D+\rho_{C,D}\gamma\right)\right)dt+\sigma_D\sqrt{\Omega}\left(\rho_{C,D}dz_C^Q+\sqrt{1-\rho_{C,D}^2}dz_D^Q\right) \tag{62}$$

$$dx = \left(-\kappa_x x-(1-\theta)B_{pre}\sigma_x^2\Omega\right)dt+\sigma_x\sqrt{\Omega}\,dz_x^Q+\widetilde{\nu}_1\,dN\,, \tag{63}$$

where the three Brownian motions $\{dz_C^Q,\, dz_x^Q,\, dz_\Omega^Q\}$ are uncorrelated, and the Q-intensity of the Poisson jump process N is

$$\lambda^Q = \lambda\,e^{(1-\theta)(A_{pre}+B_{pre}x)}\,\mathrm{E}_{\widetilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right]. \tag{64}$$

Furthermore, the Q-probability density of the jump amplitudes is

$$\pi^Q(\widetilde{\nu}_1=\nu_1) = \pi(\widetilde{\nu}_1=\nu_1)\,\frac{e^{(\theta-1)(A(\nu_1)+B(\nu_1)(x+\nu_1)}}{\mathrm{E}_{\widetilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\widetilde{\nu}_1))}\right]}\,. \tag{65}$$

4.2 Pre-Crash Dividend Claim

We denote the pre-crash claim to dividend by $V_{pre}(D,\, x)$. By construction, its expected return under the risk neutral measure is the risk-free rate:

$$\mathrm{E}_t^Q\left[\frac{dV_{pre} + D\,dt}{V_{pre}}\right] = r_{pre}\,dt. \tag{66}$$

We proceed as in Section 3.3. That is, we define the pre-crash price-dividend ratio $I^D_{pre} \equiv \frac{V_{pre}}{D}$ and then look for a solution of the form

$$I^D_{pre}(x) = e^{F_{pre} + G_{pre}x}\,. \tag{67}$$

We combine equations (66)-(67) with the risk-neutral dynamics (62)-(63) to obtain:

$$\begin{aligned} r_{pre} - \frac{1}{I^D_{pre}} &= \mu_D + \phi\, x - \gamma \rho_{C,D}\sigma_D \Omega - \kappa_x x G_{pre} - (1-\theta) B_{pre} G_{pre} \sigma_x^2 \Omega + \frac{1}{2} G^2_{pre}\sigma_x^2 \Omega \\ &\quad + \lambda^Q e^{-(F_{pre}+G_{pre}x)}\, \mathrm{E}^Q_{\tilde{\nu}_1}\left[e^{\tilde{F}+\tilde{G}(x+\tilde{\nu}_1)}\right] - \lambda^Q\,, \end{aligned} \tag{68}$$

As above, we find an approximate solution for I^D_{pre} by moving r_{pre} to the RHS, arranging the non-affine terms to the LHS, multiplying both sides by I^D_{pre}, and adding $(q_0 + q_1 x)\, I^D_{pre}$ to both sides. These calculations give

$$\begin{aligned} LHS &= (q_0 + q_1 x)\, e^{F_{pre}+G_{pre}x} - 1 - \lambda \frac{(1-\theta)}{\theta}\, e^{(F_{pre}+G_{pre}x) - \theta(A_{pre}+B_{pre}x)}\, \mathrm{E}_{\tilde{\nu}_1}\left[e^{\theta(\tilde{A}+\tilde{B}(x+\tilde{\nu}_1))}\right] \\ &\quad - \lambda\, e^{(F_{pre}+G_{pre}x)+(1-\theta)(A_{pre}+B_{pre}x)}\, \mathrm{E}_{\tilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\tilde{\nu}_1))}\right] - \lambda^Q\, \mathrm{E}^Q_{\tilde{\nu}_1}\left[e^{\tilde{F}+\tilde{G}(x+\tilde{\nu}_1)}\right] \end{aligned} \tag{69}$$

$$\begin{aligned} \left(\frac{1}{I^D_{pre}}\right) RHS &= (q_0 + q_1 x) - r_{pre,0} + \mu_D - \sigma_D \rho_{C,D} \gamma\, \Omega - (1-\theta) B_{pre} G_{pre} \sigma_x^2 \Omega + \frac{1}{2} G^2_{pre} \sigma_x^2 \Omega - \lambda^Q \\ &\quad + \phi\, x - \kappa_x x G_{pre} - \rho x\,, \end{aligned} \tag{70}$$

where the constant $r_{pre,0}$ is defined in equation (60). We note the effect of Bayesian updating on the pre-crash price-dividend ratio I^D_{pre}. The agent anticipates that if a crash occurs the price-dividend ratio will take the post-crash form

$$I^D_{post} = e^{\tilde{F}+\tilde{G}x}\,, \tag{71}$$

where, for each possible realization of $\widetilde{\nu}_1$, the coefficients $\tilde{F} \equiv F(\widetilde{\nu}_1)$ and $\tilde{G} \equiv G(\widetilde{\nu}_1)$ minimize the squared error in equation (37).

We approximate the RHS to be identically zero, and then collect terms linear in and independent of x, respectively. We obtain a system of two equations that identify $\{q_0,\, q_1\}$ in terms of G_{pre}:

$$-q_0 = -r_{pre,0} + \mu_D - \sigma_D \rho_{C,D} \gamma\, \Omega - (1-\theta) B_{pre} G_{pre} \sigma_x^2 \Omega + \frac{1}{2} G^2_{pre} \sigma_x^2 \Omega - \lambda^Q \tag{72}$$

$$-q_1 = \phi - \kappa_x G_{pre} - \rho\,. \tag{73}$$

In turn, we identify $\{F_{pre}, G_{pre}\}$ by minimizing the unconditional squared error:

$$\min_{\{F_{pre},G_{pre}\}} \mathrm{E}_{-\infty}\left[(LHS)^2\right]\,. \tag{74}$$

4.3 The Pre-Crash Equity Premium

In the pre-crash economy, the expression for the risk premium on the risky asset simplifies to:

$$\begin{aligned}\text{Equity Premium}_{pre} \;&=\; \gamma\,\sigma_D\rho_{C,D}\,\Omega + (1-\theta)B_{pre}G_{pre}\sigma_x^2\Omega \\ &-\; \lambda\, I_{pre}^{(1-\theta)}\,(I_{pre}^{D})^{-1}\,\mathrm{E}_{\tilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\tilde{\nu}_1))+\tilde{F}+\tilde{G}\,(x+\tilde{\nu}_1)}\right] \\ &+\; \lambda\, I_{pre}^{(1-\theta)}\,\mathrm{E}_{\tilde{\nu}_1}\left[e^{(\theta-1)(\tilde{A}+\tilde{B}(x+\tilde{\nu}_1))}\right] + \lambda\,(I_{pre}^{D})^{-1}\,\mathrm{E}_{\tilde{\nu}_1}\left[e^{\tilde{F}+\tilde{G}(x+\tilde{\nu}_1)}\right] - \lambda\,.\end{aligned} \tag{75}$$

where I_{pre} and I_{pre}^{D} were previously defined in equations (54) and (67), respectively.

The intuition for this formula is similar to that discussed previously in Section 3.4. That is, the first term in equation (75) is identical to the risk premium in a model with CRRA. The following terms are the risk premia on diffusive and jump components of expected growth risk. Again, in the KPEZ with $\Psi > 1$ case, the agent demands a positive premium on expected growth risk, which increases the risk premium on the risky asset.

4.4 Valuing Options on the Dividend Claim

The option pricing problem for the pre-crash economy is outside of the affine class. Thus, we lack an analytical formula for the option price. However, the problem is easily handled via Monte Carlo simulation. Specifically, we simulate two antithetic samples of 50,000 paths of the dividend δ and the process x from the Q-dynamics (23) and (24). For each simulated case, we use the x-path from time t to maturity T to approximate the discount factor $e^{-\int_t^T r(x_s)ds}$. Further, we use the simulated value of x_T to obtain the price-dividend ratio $I_{pre}^{D}(T) = e^{F_{pre}+G_{pre}x_T}$. Next, we compute the simulated value of the contingent claim $V_{pre}(T) = D_T I_{pre}^{D}(T)$, where $D = \exp\delta$. Finally, we average across the simulated discounted realizations of $|V_{pre}(T) - K|^{+}$ to approximate the expectation in (41).

4.5 Model Calibration

We note that the requirements imposed on this model is considerably higher than in the previous section in that here we want to explain not only the post-1987 volatility smirk, but also the regime shift in option prices that was observed immediately after the 1987 crash. As such, we consider a slightly different baseline calibration. We argue that the coefficient values that we use below are still consistent with observation and similar to those used in, e.g., BY and Bansal et al. (2007).

1. *Consumption and Dividend Dynamics:*

   In the consumption dynamics (6), we fix $\mu_C = 0.018$ and $\Omega = 0.00078$.

   For the dividend process (7), we use $\mu_D = 0.018$, $\phi = 2.1$, and $\sigma_D = 3.5$. We fix the correlation between shocks to dividend and consumption at 25%, i.e., $\rho_{C,D} = 0.25$.

In the x-dynamics (3), we use $\kappa_x = 0.34$ and $\sigma_x = 0.6325$. We fix the Poisson jump intensity process at $\lambda = 0.007$, which on average corresponds to less than one jump every hundred years.

In equations (49)-(50), we fix $\bar{\bar{\mu}}_\nu = -0.011$ and we assume that at the time of the crash $\widetilde{\nu}_1$ takes value -0.094. Further, we set $\bar{\sigma}_\nu = 0.0023$ and $\bar{\bar{\sigma}}_\nu = 0.022$.

The intuition for this calibration is as follows. Before a crash occurs, the agent does not fully appreciate the extent to which prices can fall. As such, her prior is that the jump size $\widetilde{\nu}_1$ has nearly zero mean, $\bar{\bar{\mu}}_\nu = -0.011$. The agent realizes however that there is considerable uncertainty about the magnitude of a possible jump, as reflected by the large standard deviation of $\widetilde{\nu}_1$, which equals $\sqrt{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2} = 0.0221$.

Suddenly, she unexpectedly observes a crash of the proportion of the 1987 event. When that happens, she updates her beliefs about the post-crash jump distribution according to (50). As such, the mean and standard deviation of the post-crash jump size $\widetilde{\nu}_2$ become, respectively,

$$\bar{\bar{\mu}}_\nu \left(\frac{\bar{\sigma}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2} \right) + \widetilde{\nu}_1 \left(\frac{\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2} \right) = -0.0931 \tag{76}$$

$$\sqrt{\bar{\sigma}_\nu^2 + \frac{\bar{\sigma}_\nu^2 \bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2}} = 0.0032\,. \tag{77}$$

That is, immediately after the crash the agent updates her prior on the average jump size in a way that reflects the possibility of a large, although very rare, stock price fall.

Further, we note that the occurrence of a crash determines a stark increase in the precision of the agent's belief about the jump size. Specifically, the standard deviation of the post-crash jump size is over seven times smaller than its pre-crash value. As discussed above, this observation is consistent with the intuition that a single event of the proportion of the 1987 market crash can generate most of the updating of the agent's beliefs.

2. *Preferences:*

   We use a time discount factor coefficient $\beta = 0.017$. We fix the coefficient of relative risk aversion at $\gamma = 10$. Finally, we follow Bansal et al. (2007) and we use $\Psi = 1/\rho = 2$ for our baseline case.

3. *Initial Conditions:*

   In the plots below, we fix the state variable x at the its steady-state mean value. In the pre-crash economy, such value is $x_{pre,0} = \bar{\bar{\mu}}_\nu \lambda/\kappa_x$, while in the post-crash economy it is $x_{post,0} = \left[\bar{\bar{\mu}}_\nu \left(\frac{\bar{\sigma}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2} \right) + \widetilde{\nu}_1 \left(\frac{\bar{\bar{\sigma}}_\nu^2}{\bar{\sigma}_\nu^2 + \bar{\bar{\sigma}}_\nu^2} \right) \right] \lambda/\kappa_x$. We also confirmed, however, that our results are robust to the choice of a wide range of values for the state x.

4.6 Simulation Results

We note that our calibration yields realistic values of the risk-free rate and the equity premium, both pre- and post-crash. Specifically, we find that the pre-crash steady-state real risk-free rate is 1.33%, while the equity premium predicted by the model is 4.48%. Post-crash, the steady-state value of the risk-free rate drops to 0.7%, while the equity premium becomes 6.4%. Further, the calibration matches other aspects of the economy. For instance, we find that the steady-state value of the price-dividend ratio is around 27, a value that drops to approximately 19 in the post-crash economy.

Before the crash, our calibration produces a mild smirk that is qualitatively consistent with the evidence in Figure 1. Figure 6 shows that the difference between the implied volatilities from 10%-out-of-the-money and at-the-money puts is approximately 2.8%. Immediately after the crash, however, the agent updates her beliefs about the expected value of the jump size. As such, the volatility smirk steepens dramatically. In Figure 6, we show that post-1987 the difference between the implied volatilities from 10%-out-of-the-money and at-the-money puts becomes nearly 10%.

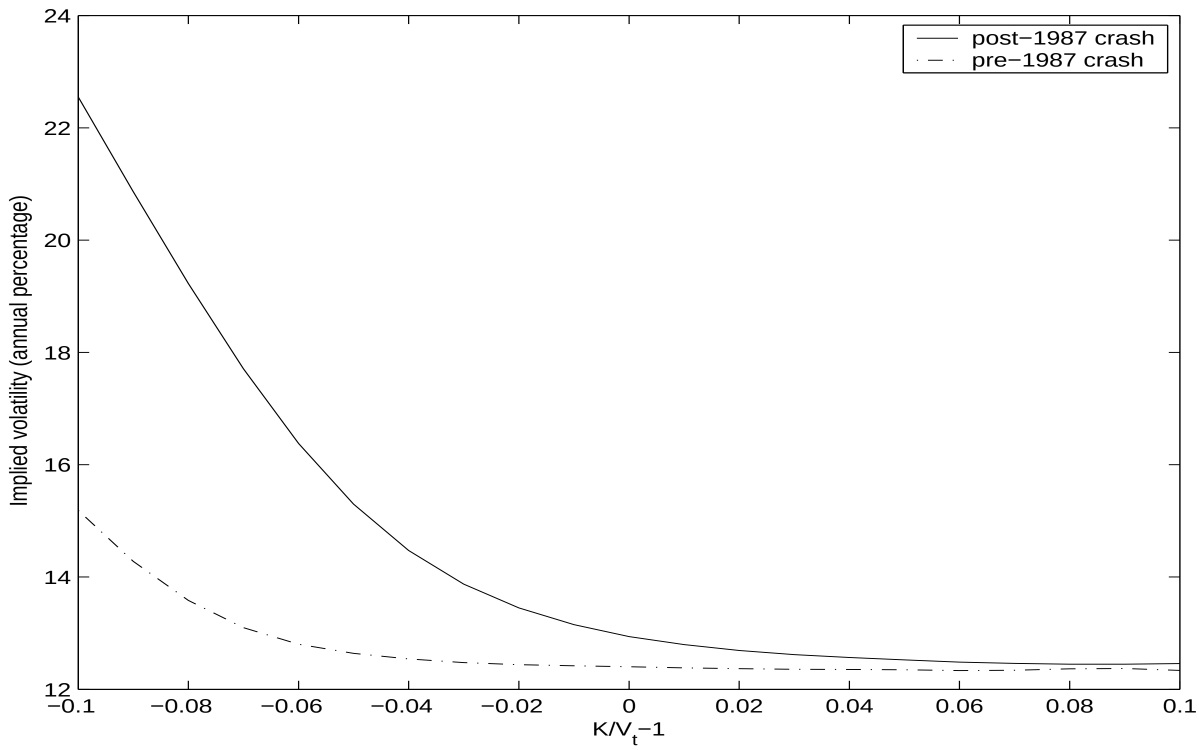

Figure 6: The plot depicts the implied volatility smirk pre- and post-1987 market crash. Implied volatilities are from S&P 500 options with one month to maturity. The model coefficients are set equal to the baseline values.

Finally, we note a drawback of our calibration. During the two weeks after the 'Black Monday' in October 1987, the 3-month Treasury bill rate was on average 1.5% lower than the same rate during the two weeks preceeding the crash.[12] Consistent with observation, our model predicts a fall in the risk-free rate at the time of a market crash. However, the magnitude of the drop is larger than what was observed in October 1987. To study this model implication, we use data from the COMPUSTAT database and compute the price-dividend ratio for the S&P 500 index as of the end of September 1987. We find it to be 36.24. Next, we infer the pre-crash value x_t of the latent process x (where t is the end of September 1987) by matching the pre-crash price-dividend ratio predicted by the model with the value observed in the data:

$$I^D_{pre}(x_t) = e^{F_{pre} + G_{pre} x_t} \equiv 36.24 \,. \tag{78}$$

Then, we use equation (60) to compute the change in the risk-free rate determined by a jump in x from the pre-crash value x_t to the post-crash level $x_t + \widetilde{\nu}_1$. We find the jump in the risk-free rate to be -5.2%.

Related, we can use our model to predict the drop in stock prices at the time of the 1987 crash. We do so by following an approach similar to that we used above to determine the jump in the interest rate. That is, assuming that the level of the dividend is unaffected by the crash, the jump in price around the crash event is given by

$$\frac{I^D(x_t + \widetilde{\nu}_1)}{I^D_{pre}(x_t)} - 1 = \frac{e^{F + G(x_t + \widetilde{\nu}_1)}}{e^{F_{pre} + G_{pre} x_t}} - 1 \,, \tag{79}$$

where x_t is determined by equation (78) and $\widetilde{\nu}_1$ is the jump in x at the time of the crash. The model predicts a nearly fifty percent fall in the stock price, a drop twice as large as that observed in 1987. We note however that there are institutional features that may have attenuated the fluctuation in interest rates and market prices during the crash day.[13]

5 Conclusions

We examine a representative-agent general equilibrium model that can explain the salient features of the U.S. equity options markets both before and after the 1987 crash, and investigate their linkage with the underlying stock market. The agent is endowed with Epstein-Zin preferences and

[12] The bank discount rates on the 3M T-bill were as follows. 1987-10-05: 6.68; 1987-10-06: 6.55; 1987-10-07: 6.56; 1987-10-08: 6.69; 1987-10-09: 6.75; 1987-10-12: N.A.; 1987-10-13: 6.74; 1987-10-14: 7.19; 1987-10-15: 7.07; 1987-10-16: 6.93; 1987-10-19: 6.39; 1987-10-20: 5.86; 1987-10-21: 5.60; 1987-10-22: 5.36; 1987-10-23: 5.29; 1987-10-26: 5.22; 1987-10-27: 5.23; 1987-10-28: 5.10; 1987-10-29: 5.03; 1987-10-30: 5.27.

[13] On October 19 and 20, 1987, the S&P 500 Futures price was considerably lower than the index price, which suggests that the drop in the index level does not fully represent the magnitude of the market adjustment in prices. This evidence can be explained by the existence of significant delays in the submission and execution of limit orders during the crash events, magnified by the standard problem of 'stale' prices (see, e.g., Kleidon (1992)). Moreover, interventions of the exchange might have further contained the fluctuations in stock prices during the crash. Finally, the Fed assured that it would provide adequate liquidity to the U.S. financial system necessary to calm the equity and other markets (see, e.g., p. 3 of the November 3, 1987, 'Notes for FOMC Meeting' document available from the Federal Reserve web site http://www.federalreserve.gov/fomc/transcripts/1987/871103StaffState.pdf).

the aggregate dividend and consumption processes are driven by a persistent stochastic growth variable that can jump. In reaction to a market crash, the agent updates her beliefs about the distribution of the jump component.

We identify a realistic calibration of the model that matches four stylized properties of the equity option market and the underlying stocks. First, the model implies a deep volatility smirk for S&P 500 options. In the baseline case, the implied volatility of 10% out-of-the-money put options with one month to maturity is close to 24%. At-the-money options, instead, have an implied volatility of approximately 14%. That is, consistent with empirical evidence we find a 10% volatility smirk. Second, the model implies a mild volatility smile for individual stock options, as illustrated in Bollen and Whaley (2004).

Third, the model explains the stark regime shift in S&P 500 option prices observed around the time of the 1987 market crash. Before the crash, the difference between implied volatilities from 10%-out-of-the-money and at-the-money puts is approximately 2.8%. However, the occurrence of a jump triggers the updating of the agent's beliefs about the distribution of future jumps. As such, after the crash out-of-the-money put options are perceived to be more valuable, and the volatility smirk becomes as steep as 10%, consistent with the post-1987 evidence. Fourth, such paradigm change occurs in spite of a minimal change in observed macroeconomic variables (in particular, the level of consumption or dividends). Finally, the model's implications are consistent with the empirical properties of dividends, equity returns, and the riskfree rate. In the baseline calibration, the equity premium is approximately 6%, the price-dividend ratio is 20, the riskfree rate is 1% and its standard deviation is 1%.

Overall, our findings show that it is possible to reconcile the stylized properties of the equity and option markets in the framework of rational expectations, consistent with the notion that these two markets are integrated.

In the current version of the paper we intentionally focus on a rather minimal version of the model. In particular, we present results only for the so-called 'one-channel' BY case, in which the dividend and consumption dynamics have constant volatility. We find that a single channel (a rare jump in consumption growth) suffices to reconcile option and index prices. In unreported results, we have also considered different extensions to our analysis. It is straightforward to solve a model in which the growth rate in dividends and consumption exhibit stochastic volatility (the 'two-channel' BY case), extended for the possible presence of jumps in volatility. The results, available upon request, are qualitatively similar to those discussed here. As in BY, stochastic volatility adds additional flexibility to match the moments of the underlying returns—in combination with jumps (in the predictable component of dividends and possibly in volatility), the model generates a steep volatility smirk. Further, as shown by BY the presence of stochastic volatility yields a time-varying risk premium, i.e., the two-channel model generates return predictability.

More interestingly, stochastic volatility has the potential to improve the model predictions around the time of the crash. In the baseline calibration, we find that on the day of the crash the interest rate and the stock market return jump more than what they had in reality. In the two-channel model, stochastic volatility contributes to generate a deeper smirk. As such, a smaller jump in the predictable component in dividends, x, suffices to explain the volatility smirk. The loadings on volatility for underlying stock returns and changes in interest rates are smaller than those on the variable x. Thus, we conjecture that such a model would produce a smaller jump in prices and interest rates while still capturing the volatility smirk. Another extension is to examine the case in which after a crash the agent updates her assessment of the likelihood of future crashes rather than the expected magnitude of a future crash. As in Collin Dufresne et al. (2003), we can allow for Bayesian updating on the intensity of a jump, i.e., on the probability that a jump will occur.

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A Appendix A: Pre- and Post-Crash Implied Volatility Patterns

Figure 1 shows the permanent regime shift in pre- and post-1987 market crash implied volatilities for S&P 500 options. The plot in Panel A depicts the spread between implied volatilities for S&P 500 options that have a strike-to-price ratio $X = K/S - 1 = -10\%$ and at-the-money implied volatilities. The plot in panel B depicts the spread between implied volatilities for options that have a strike-to-price ratio $X = K/S - 1 = 2.5\%$ and at-the-money implied volatilities.

A.1 American Options on the S&P 500 Futures

We construct implied volatility functions from 1985 to 1995 by using transaction data on American options on S&P 500 futures. As in Bakshi et al. (1997), prior to analysis we eliminate observations that have price lower than \$(3/8) to mitigate the impact of price discreteness on option valuation. Since near-maturity options are typically illiquid we also discard observations with time-to-maturity lower than 10 calendar days. For the same reason we do not use call and put contracts that are more than 3% in-the-money. Finally, we disregard observations on options that allow for arbitrage opportunities, e.g., calls with a premium lower than the early exercise value.

We consider call and put transaction prices with the three closest available maturities. For each contract we select the transaction price nearest to the time of the market close and we pair it with the nearest transaction price on the underlying S&P 500 futures. This approach typically results in finding a futures price that is time stamped within 6 seconds from the time of the option trade. We approximate the risk-free rate with the three-month Treasury yield and we compute implied volatilities using the Barone-Adesi and Whaley (1987) approximate option pricing formula.

At each date and for each of the three closest maturities we interpolate the cross section of implied volatilities with a parabola. This approach is similar to the one used in Shimko (1993). In doing so we require that we have at least three implied volatility observations, one of which with a strike-to-price ratio $X = K/S - 1$ no higher than -9%, one with X no lower than 1.5%, and one in between these two extremes. We record the interpolated implied volatility at $X = 0$ and the implied volatility computed at the available X-values closest to -10% and 2.5%.

Then at each date and for each of the three X choices we interpolate the implied volatility values across the three closest maturities using a parabola. We use the fitted parabola to obtain the value of implied volatility at 30 days to maturity. If only two maturities are available, we replace the parabola with a linear interpolation. If only one maturity is available we retain the value of implied volatility observed at that maturity provided that such maturity is within 20 to 40 days.

Trading in American options on the S&P 500 futures contract began on January 28, 1983. Prior to 1987, only quarterly options maturing in March, June, September, and October were available. Additional serial options written on the next quarterly futures contracts and maturing in the nearest two months were introduced in 1987 (e.g., Bates (2000)). This data limitation, combined with the relatively scarce size and liquidity of the option market in early years, renders it difficult to obtain smirk observations at the 30-day maturity with -10% moneyness. As such, we start the plot in December 1985. After this date we find implied volatility values with the desired parameters for most trading days. Relaxing the time-to-maturity and moneyness requirements results in longer implied volatility series going back to January 1983. Qualitatively, the plot during the period from January 1983 to December 1985 remains similar to that for the period from December 1983 to October 1987 (see, e.g., Bates (2000)).

A.2 European Options on the S&P 500 Index

After April 1996, we use data on S&P 500 index European options. We obtain daily SPX implied volatilities from April 1996 to April 2006 from the Optionmetrics database. Similar to what discussed in Section A.1, we exclude options with price lower than \$(3/8), time-to-maturity lower than 10 calendar days, and contracts that are more than 3% in-the-money.

At each date and for each of the three closest maturities we interpolate the cross-section of implied volatilities using a parabola. We have also considered a spline interpolation, which has produced similar results. We use the fitted parabola to compute the value of implied volatilities for strike-to-index-price ratios $X = K/S - 1 = -10\%$, zero, and 2.5%. Finally, we interpolate implied volatilities at each of these three levels of moneyness across the three closest maturities. We use the fitted parabola to compute the value of implied volatility at the 30-day maturity.

B Appendix B: Equilibrium Prices in a Jump-Diffusion Exchange Economy with Recursive Utility

There are several formal treatments of stochastic differential utility and its implications for asset pricing (see, e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003)). For completeness, in this Appendix we offer a very simple informal derivation of the pricing kernel that obtains in an exchange economy where the representative agent has a KPEZ recursive utility. Our contribution is to characterize equilibrium prices in an exchange economy where aggregate output has particular jump-diffusion dynamics (Propositions 1 and 2).

B.1 Representation of Preferences and Pricing Kernel

We assume the existence of a standard filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ on which there exists a vector $z(t)$ of d independent Brownian motions and one counting process $N(t) = \sum_i \mathbf{1}_{\{\tau_i \leq t\}}$ for a sequence of inaccessible stopping times τ_i, $i = 1, 2, \ldots$.[14]

[14]We note that $N(t)$ is a pure jump process by construction and hence is independent of $z(t)$ by construction (in the sense that their quadratic co-variation is zero).

Aggregate consumption in the economy is assumed to follow a continuous process, with stochastic growth rate and volatility, which both may experience jumps:

$$d\log C_t \;=\; \mu_{_C}(X_t)dt + \sigma_{_C}(X_t)dz(t) \tag{80}$$

$$dX_t \;=\; \mu_{_x}(X_t)dt + \sigma_{_x}(X_t)dz(t) + \widetilde{\nu}dN(t)\,. \tag{81}$$

We note that X_t is a n-dimensional Markov process (we assume sufficient regularity on the coefficient of the stochastic differential equation (SDE) for it to be well-defined, e.g., Duffie (2001) Appendix B). In particular $\mu_{_x}$ is an $(n,1)$ vector, $\sigma_{_x}$ is an (n,d) matrix and $\widetilde{\nu}$ is a $(n,1)$ vector of i.i.d. random variable with joint density (conditional on a jump $dN(t)=1$) of $g(\nu)$. We further assume that the counting process has a (positive integrable) intensity $\lambda(X_t)$ in the sense that $N(t)-\int_0^t\lambda(X_s)ds$ is a $(P,\mathcal{F}_t)$ martingale.

Following Epstein and Zin (1989), we assume that the representative agent's preferences over a consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the following recursive equation:

$$U(t)=\left\{(1-e^{-\beta dt})C_t^{1-\rho}+e^{-\beta dt}\mathrm{E}_t\left(U(t+dt)^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}}\,. \tag{82}$$

With $dt=1$, this is the discrete time formulation of KPEZ, in which $\Psi\equiv 1/\rho$ is the EIS and γ is the risk-aversion coefficient.

To simplify the derivation let us define the function

$$u_\alpha(x)=\begin{cases}\frac{x^{1-\alpha}}{(1-\alpha)} & 0<\alpha\neq 1\\ \log(x) & \alpha=1\,.\end{cases}$$

Further, let us define

$$g(x)=u_\rho(u_\gamma^{-1}(x))\equiv\begin{cases}\frac{((1-\gamma)x)^{1/\theta}}{(1-\rho)} & \gamma,\rho\neq 1\\ u_\rho(e^x) & \gamma=1,\rho\neq 1\\ \frac{\log((1-\gamma)x)}{(1-\gamma)} & \rho=1,\gamma\neq 1\,,\end{cases}$$

where

$$\theta=\frac{1-\gamma}{1-\rho}\,.$$

Then defining the 'normalized' utility index J as the increasing transformation of the initial utility index $J(t)=u_\gamma(U(t))$ equation (82) becomes simply:

$$g(J(t))=(1-e^{-\beta dt})u_\rho(C_t)+e^{-\beta dt}\,g\left(\mathrm{E}_t\left[J(t+dt)\right]\right)\,. \tag{83}$$

Using the identity $J(t+dt)=J(t)+dJ(t)$ and performing a simple Taylor expansion we obtain:

$$0=\beta u_\rho(C_t)dt-\beta g(J(t))+g'(J(t))\,\mathrm{E}_t\left[dJ(t)\right]\,. \tag{84}$$

Slightly rearranging the above equation, we obtain a backward recursive stochastic differential equation which could be the basis for a formal definition of stochastic differential utility (see Duffie Epstein (1992), Skiadas (2003)):

$$\mathrm{E}_t[dJ(t)] = -\frac{\beta u_\rho(C_t) - \beta g(J(t))}{g'(J(t))}dt\,. \tag{85}$$

Indeed, let us define the so-called 'normalized' aggregator function:

$$f(C,J) = \frac{\beta u_\rho(C) - \beta g(J)}{g'(J)} \equiv \begin{cases} \frac{\beta u_\rho(C)}{((1-\gamma)J)^{1/\theta-1}} - \beta\theta J & \gamma,\rho \neq 1 \\ (1-\gamma)\beta J\log(C) - \beta J\log((1-\gamma)J) & \gamma\neq 1, \rho = 1 \\ \frac{\beta u_\rho(C)}{e^{(1-\rho)J}} - \frac{\beta}{1-\rho} & \gamma = 1, \rho\neq 1\,. \end{cases} \tag{86}$$

We obtain the following representation for the normalized utility index:

$$J(t) = \mathrm{E}_t\left(\int_t^T f(C_s, J(s)) + J(T)\right)\,. \tag{87}$$

Further, if the transversality condition $\lim_{T\to\infty}\mathrm{E}_t(J(T)) = 0$ holds, letting T tend to infinity, we obtain the simple representation:

$$J(t) = \mathrm{E}_t\left(\int_t^\infty f(C_s, J(s))ds\right)\,. \tag{88}$$

Fisher and Gilles (1999) discuss many alternative representations and choices of the utility index and associated aggregator as well as their interpretations. Here we note only the well-known fact that when $\rho = \gamma$ (i.e., $\theta = 1$) then $f(C,J) = \beta u_\rho(C) - \beta J$ and a simple application of Itô's lemma shows that

$$J(t) = \mathrm{E}_t\left(\int_t^\infty e^{-\beta(s-t)}\beta u_\rho(C_s)ds\right)\,.$$

To obtain an expression for the pricing kernel note that under the assumption (which we maintain throughout) that an 'interior' solution to the optimal consumption-portfolio choice of the agent exists, a necessary condition for optimality is that the gradient of the Utility index is zero for any small deviation of the optimal consumption process in a direction that is budget feasible. More precisely, let us define the utility index corresponding to such a small deviation by:

$$J^\delta(t) = \mathrm{E}_t\left(\int_t^\infty f\left(C_s^* + \delta\tilde{C}(s), J^\delta(s)\right)ds\right)\,.$$

Then we may define the gradient of the utility index evaluated at the optimal consumption process $C^*(t)$ in the direction $\tilde{C}(t)$:

$$\begin{aligned} \nabla J(C_t^*;\tilde{C}_t) &= \lim_{\delta\to 0}\frac{J^\delta(t) - J(t)}{\delta} \\ &= \lim_{\delta\to 0}\mathrm{E}_t\left[\int_t^\infty \frac{f(C_s^* + \delta\tilde{C}(s), J^\delta(s)) - f(C_s, J^\delta(s))}{\delta}ds\right] \\ &= \mathrm{E}_t\left[\int_t^\infty f_C(C_s^*, J(s))\tilde{C}_s + f_J(C_s, J(s))\nabla J(C_s^*;\tilde{C}_s)ds\right]\,. \end{aligned} \tag{89}$$

Assuming sufficient regularity (essentially the gradient has to be a semi-martingale and the transversality condition has to hold: $\lim_{T\to\infty} \mathrm{E}_t[e^{\int_t^T f_J(C_s,J_s)ds}\nabla J(C_T^*;\tilde{C}_T) = 0)$, a simple application of the generalized Itô-Doeblin formula gives the following representation:

$$\nabla^\delta J(C_t^*;\tilde{C}_t) = \mathrm{E}_t\left(\int_t^\infty e^{\int_t^s f_J(C_u,J_u)du} f_C(C_s,J_s)\tilde{C}_s ds\right). \tag{90}$$

This shows that

$$\Pi(t) = e^{\int_0^t f_J(C_s,J_s)ds} f_C(C_t,J_t) \tag{91}$$

is the Riesz representation of the gradient of the normalized utility index at the optimal consumption. Since a necessary condition for optimality is that $\nabla J(C_t^*;\tilde{C}_t) = 0$ for any feasible deviation $\tilde{C}_t$ from the optimal consumption stream C_t^*, we conclude that $\Pi(t)$ is a pricing kernel for this economy.[15]

B.2 Equilibrium Prices

Assuming equilibrium consumption process given in (80)-(81) above we obtain an explicit characterization of the felicity index J and corresponding pricing kernel Π.

For this we define the operator for any $h(\cdot):\mathbb{R}^n - \mathbb{R}$:

$$\mathcal{J}h(x) = \int\dots\int \frac{h(x+\nu)}{h(x)} g(\nu)d\nu_1\dots d\nu_n - 1$$

and the standard Dynkin operator:

$$\mathcal{D}h(x) = h_x(x)\mu_x(x) + \frac{1}{2}\mathrm{trace}(h_{xx}\sigma_x(x)\sigma_x(x)^\top),$$

where h_x is the $(n,1)$ Jacobian vector of first derivatives and h_{xx} denotes the (n,n) Hessian matrix of second derivatives. with these notations, we find:

Proposition 1 *Suppose $I(x):\mathbb{R}^n\to\mathbb{R}$ solves the following equation:*

$$\begin{cases} 0 = I(x)\left((1-\gamma)\mu_C(x) + (1-\gamma)^2\frac{||\sigma_C(x)||^2}{2} - \beta\theta\right) + & \\ \quad \frac{\mathcal{D}I(x)^\theta}{I(x)^{(\theta-1)}} + (1-\gamma)\theta\sigma_C(x)\sigma_x(x)^\top I_x(x) + \theta + I(x)\lambda(x)\mathcal{J}I(x)^\theta & \text{for} \quad \rho,\gamma\neq 1 \\ 0 = I(x)\left((1-\rho)\mu_C(x) - \beta\right) + I(x)\mathcal{D}\log I(x) + 1 + I(x)\lambda(x)\log\left(1+\mathcal{J}I(x)\right) & \text{for} \quad \gamma = 1, \rho\neq 1 \\ 0 = I(x)\left((1-\gamma)\mu_C(x) + (1-\gamma)^2\frac{||\sigma_C(x)||^2}{2}\right) + \mathcal{D}I(x) + & \\ \quad (1-\gamma)\sigma_C(x)\sigma_x(x)^\top I_x(x) - \beta I(x)\log I(x) + I(x)\lambda(x)\mathcal{J}I(x) & \text{for} \quad \rho = 1,\gamma\neq 1 \end{cases} \tag{92}$$

and satisfies the transversality condition ($\lim_{T\to\infty}\mathrm{E}[J(T)] = 0$ for $J(t)$ defined below) then the value function is given by:

$$\begin{cases} J(t) = u_\gamma(C_t)(\beta I(x_t))^\theta & \text{for} \quad \rho,\gamma\neq 1 \\ J(t) = \log(C_t) + \frac{\log(\beta I(x_t))}{1-\rho} & \text{for} \quad \gamma = 1,\rho\neq 1 \\ J(t) = u_\gamma(C_t)I(x_t) & \text{for} \quad \rho = 1,\gamma\neq 1\,. \end{cases} \tag{93}$$

[15]Further discussion is provided in Chapter 10 of Duffie (2001).

The corresponding pricing kernel is:

$$\begin{cases} \Pi(t) = e^{-\int_0^t (\beta\theta + \frac{(1-\theta)}{I(x_s)})ds} (C_t)^{-\gamma} (I(x_t))^{(\theta-1)} & \textit{for} \quad \rho, \gamma \neq 1 \\ \Pi(t) = e^{-\int_0^t \frac{\beta}{I(x_s)})ds} \frac{1}{(C_t I(x_t))} & \textit{for} \quad \gamma = 1, \rho \neq 1 \\ \Pi(t) = e^{-\int_0^t \beta(1+\log I(x_s))ds} (C_t)^{-\gamma} I(x_t) & \textit{for} \quad \rho = 1, \gamma \neq 1 \,. \end{cases} \tag{94}$$

Proof 1 *We provide the proof for the case $\gamma, \rho \neq 1$. The special cases are treated similarly.*

From its definition

$$J(t) = \mathrm{E}_t \left(\int_t^\infty f(c_s, J(s)) \right) . \tag{95}$$

Thus, $J(x_t, c_t) + \int_0^t f(c_s, J(x_s, c_s))ds$ is a martingale. This observation implies that:

$$\mathrm{E}[dJ(x_t, c_s) + f(c_t, J(x_t, c_t))dt] = 0 \,. \tag{96}$$

Using our guess ($J(t) = u_\gamma(c_t)\beta^\theta I(x)^\theta$) and applying the Itô-Doeblin formula we obtain:

$$(1-\gamma)\mu_c(x_t) + (1-\gamma)^2 \frac{||\sigma_c(x_t)||^2}{2} + \frac{\mathcal{D}I^\theta(x_t)}{I(x_t)^\theta} + \lambda(x)\mathcal{J}I(x)^\theta + \frac{\theta}{I(x_t)} - \beta\theta = 0 \,, \tag{97}$$

where we have used the fact that

$$\frac{f(c,J)}{J} = \frac{u_\rho(c_t)}{((1-\gamma)J)^{1/\theta-1}J} - \beta\theta = \frac{\theta}{I(x)} - \beta\theta$$

and the definition of the Dynkin operator $\mathcal{D}I(x) = I_x(x)^\top \mu_x(x) + \frac{1}{2}Trace(I_{xx}(x)\sigma_x(x)\sigma_x(x)^\top)$. Rearranging we obtain the equation of the proposition.

Now suppose that $I(\cdot)$ solves this equation. Then, applying the Itô Doeblin formula to our candidate $J(t)$ we obtain

$$\begin{aligned} J(T) &= J(t) + \int_t^T \mathcal{D}J(s)ds + \int_t^T J_c\sigma_c dz_c(s) + \int_t^T J_x\sigma_x dz_x(s) + \int_t^T J(s^-)(\frac{I(X_{s^-}+\nu)^\theta}{I(X_{s^-})^\theta} - 1)dN(s) \\ &= J(t) - \int_t^T f(c_s, J_s)ds + \int_t^T J(s)(1-\gamma)\sigma_c(x_s)dz_c(s) + \int_t^T \theta J(s)\sigma_I(x_s)dz_x(s) + \int_t^T dM(s), \end{aligned} \tag{98}$$

where we have defined $\sigma_I(x) = \frac{I'(x)}{I(x)}\sigma_x(x)$ and the pure jump martingale

$$M(t) = \int_0^t J(s^-)(\frac{I(X_{s^-}+\nu)^\theta}{I(X_{s^-})^\theta} - 1)dN(s) - \int_0^t \lambda(X_{s^-})J(s^-)\mathcal{J}I(X_s)^\theta ds \,.$$

If the stochastic integral is a martingale,[16] *and if the transversality condition is satisfied, then we obtain the desired result by taking expectations and letting T tend to infinity:*

$$J(t) = \mathrm{E}\left[\int_t^\infty f(c_s, J_s)\, ds\right] , \tag{99}$$

which shows that our candidate $J(t)$ solves the recursive stochastic differential equation. Uniqueness follows (under some additional technical conditions) from the appendix in Duffie, Epstein, Skiadas (1992).

[16] Sufficient conditions are:

$$\mathrm{E}\left[\int_0^T J(s)^2 \left(\sigma_c(x_s)^2 + \sigma_I(x_s)^2\right) ds\right] < \infty \ \forall T > 0 \,.$$

The next result investigates the property of equilibrium prices.

Proposition 2 *The risk-free interest rate is given by:*

$$\begin{cases} r(x_t) = \beta + \rho(\mu_C(x_t) + \frac{||\sigma_C(x_t)||^2}{2}) - \gamma(1+\rho)\frac{||\sigma_C(x)||^2}{2} - \\ \qquad (1-\theta)\sigma_I(x)^\top(\sigma_C(x_t) + \frac{1}{2}\sigma_I(x_t)) + \lambda(x_t)\left(\frac{\theta-1}{\theta}\mathcal{J}I^\theta - \mathcal{J}I^{(\theta-1)}\right) & \text{for} \quad \rho \neq 1 \\ r(x_t) = \beta + \mu_C(x_t) + \frac{||\sigma_C(x_t)||^2}{2} - \gamma||\sigma_C(x_t)||^2 & \text{for} \quad \rho = 1\,. \end{cases} \tag{100}$$

Further, the value of the claim to aggregate consumption is given by:

$$\begin{cases} S(t) = C(t)I(x_t) & \text{for} \quad \rho \neq 1 \\ S(t) = \frac{C(t)}{\beta} & \text{for} \quad \rho = 1\,. \end{cases} \tag{101}$$

Thus

$$\frac{dS_t}{S_t} = \mu_S(x_t)dt + (\sigma_C(x_t) + \sigma_I(x_t))\,dz(t) + \nu_I(x_t)dN(t)\,, \tag{102}$$

where we have defined:

$$\sigma_I(x) = \frac{1}{I(x)}I_x(x)^\top\sigma_x(x)\mathbf{1}_{\{\rho\neq 1\}} \tag{103}$$

$$\nu_I(x) = \left(\frac{I(x+\widetilde{\nu})}{I(x)} - 1\right)\,. \tag{104}$$

The risk premium on the stock is given by

$$\mu_S(x) + \frac{1}{I(x_t)} - r(x_t) = (\gamma\sigma_C(x_t) + (1-\theta)\sigma_I(x))^\top(\sigma_C(x) + \sigma_I(x)) + \lambda(x_t)\left(\mathcal{J}I(x)^{\theta-1} - \mathcal{J}I(x)^\theta\right)\,. \tag{105}$$

Proof 2 *To prove the result for the interest rate, apply Itô-Doeblin to the pricing kernel and it follows from* $r(t) = -\mathrm{E}[\frac{d\Pi(t)}{\Pi(t)}]/dt$ *that we obtain:*

$$r(x_t) = \beta\theta + \frac{(1-\theta)}{I(x_t)} + \gamma\mu_c(x_t) - \frac{1}{2}\gamma^2||\sigma_c(x_t)||^2 - \frac{\mathcal{D}I(x_t)^{(\theta-1)}}{I(x_t)^{(\theta-1)}} - \lambda(X_t)\mathcal{J}I(x)^{\theta-1}\,. \tag{106}$$

Now substitute the expression for $\frac{1}{I(x)}$ *from the equation in (92) to obtain the result.*

To prove the result for the consumption claim, define $S(t) = c_tI(x_t)$*. Then using the definition of*

$$\Pi(t) = e^{-\beta\theta t - \int_0^t \frac{(1-\theta)}{I(x_s)}ds}c_t^{-\gamma}I(x_t)^{\theta-1}$$

we obtain:

$$d\left(\Pi(t)\,S(t)\right) = e^{-\beta\theta t - \int_0^t \frac{(\theta-1)}{I(x_s)}ds}\left(dJ(t) - J(t)\left(\beta\theta + \frac{(1-\theta)}{I(x_t)}\right)dt\right)\,. \tag{107}$$

Now, note that by definition we have:

$$\begin{aligned} dJ(t) &= -f(c_t, J)dt + dM_t \\ &= -J(t)\left(\frac{\theta}{I(x_t)} - \theta\beta\right)dt + dM_t \end{aligned} \tag{108}$$

for some P-martingale M. Combining this observation with (107), we get:

$$
\begin{aligned}
d\left(\Pi(t)\,S(t)\right) &= e^{-\beta\theta t-\int_0^t \frac{(1-\theta)}{I(x_s)}ds}\frac{(-J(t))}{I(x_t)}\,dt + e^{-\beta\theta t-\int_0^t \frac{(1-\theta)}{I(x_s)}ds}dM_t \\
&= -\Pi(t)\,c(t)\,dt + e^{-\beta\theta t-\int_0^t \frac{(1-\theta)}{I(x_s)}ds}dM_t\,. \qquad (109)
\end{aligned}
$$

Thus integrating we obtain

$$
\Pi(T)S(T)+\int_t^T \Pi(s)c_s ds = \Pi(t)S(t)+\int_t^T e^{-\beta\theta(u-t)-\int_t^u \frac{(1-\theta)}{I(x_s)}ds}\,dM_u\,. \qquad (110)
$$

Taking expectations and letting $T\to\infty$ and assuming the transversality condition holds (i.e., $\lim_{T\to\infty}\mathrm{E}[\Pi(T)S(T)]=0$), we obtain the desired result:

$$
\Pi(t)S(t)=\mathrm{E}_t\left[\int_t^\infty \Pi(s)\,c_s\,ds\right]. \qquad (111)
$$

C Appendix C: The Price of an Option in the Post-Crash Affine Model

We note that the model in Section 3 is affine. In particular, the value of the dividend claim $V_t = D_t e^{F+Gx_t}$ has the following risk-neutral dynamics:

$$
\frac{dV_t}{V_t}=\left(\mu_0^Q+\mu_1^Q x_t\right)dt+\sigma_D\sqrt{\Omega}\left(\rho_{C,D}dz_c^Q(t)+\sqrt{1-\rho_{C,D}^2}\,dz_D^Q(t)\right)+G\sigma_x\sqrt{\Omega}dz_x^Q(t)+(e^{G\tilde{\nu}}-1)dN(t),
$$

where

$$
\begin{aligned}
\mu_0^Q &= \mu_D-\gamma\rho_{C,D}\sigma_D\Omega-(1-\theta)B\sigma_x^2\Omega G+\frac{1}{2}\sigma_x^2\Omega G^2 \qquad (112)\\
\mu_1^Q &= \phi-\kappa_x G. \qquad (113)
\end{aligned}
$$

As such, the option pricing problem can be solved using standard inverse Fourier transform techniques. In particular, the date-t value of a European call option on the dividend claim V_t, with maturity T and strike price K, is given by

$$
\begin{aligned}
C(V_t,x_t,K,T) &= \mathrm{E}_t^Q\left[e^{-\int_t^T ds\,r_s}\,(V_T-K)\,\mathbf{1}_{\{V_T>K\}}\right] \\
&= \Psi_{t,1}(\log K)-K\,\Psi_{t,0}(\log K)\,, \qquad (114)
\end{aligned}
$$

where we have defined:

$$
\Psi_{t,a}(k)\equiv \mathrm{E}_t^Q\left[e^{-\int_t^T r_s\,ds}\;e^{a\,\log V_T}\;\mathbf{1}_{\{\log V_T>k\}}\right]. \qquad (115)
$$

Following Bates (1996), Heston (1993), Duffie et al. (2000), and others, we use the Fourier inversion theorem for the random variable ($\log V_t$) to obtain:

$$
\Psi_{t,a}(k)=\frac{\psi_t(a)}{2}+\frac{1}{\pi}\int_0^\infty \frac{\mathcal{IM}\left[\psi_t(a+\mathcal{I}v)\,e^{-\mathcal{I}vk}\right]}{v}\,dv\,. \qquad (116)
$$

In equation (116), the transform $\psi_t(\alpha) \equiv \mathrm{E}_t^Q \left[e^{-\int_t^T r_s ds} \, e^{\alpha \log V_T} \right]$ admits the following closed form solution:

$$\psi_t(\alpha) = \exp\left(M(T-t) + N(T-t)x_t + \alpha \log V_t \right) \, , \tag{117}$$

where the functions $M(\tau)$, $N(\tau)$ satisfy

$$N(\tau) = \left(\alpha \mu_1^Q - \rho \right) \left(\frac{1 - e^{-\kappa_x \tau}}{\kappa_x} \right) \tag{118}$$

$$M(\tau) = \int_0^\tau d\tau \left\{ \frac{N^2 \sigma_x^2 \Omega}{2} - N(1-\theta) B \sigma_x^2 \Omega + \alpha \mu_0^Q - r_0 + \alpha(\alpha - 1) \frac{\Omega(\sigma_D^2 + G^2 \sigma_x^2)}{2} + \lambda^Q \left(\chi_{N+\alpha G}^Q - 1 \right) \right\} . \tag{119}$$

Proof 3 *The proof simply consists in showing that* $\left[e^{-\int_0^t r_s ds} \exp\left(M(T-t) + N(T-t)x_t + \alpha \log V(t) \right) \right]$ *is a Q-martingale. Indeed, in that case*

$$\begin{aligned} e^{-\int_0^t r_s ds} \exp\left(M(T-t) + N(T-t)x_t + \alpha \log V(t) \right) &= \mathrm{E}^Q \left[e^{-\int_0^T r_s ds} \exp\left(M(0) + N(0)x_t + \alpha \log V(T) \right) \right] \\ &= \mathrm{E}^Q \left[e^{-\int_0^T r_s ds} \exp\left(\alpha \log S(T) \right) \right] \\ &= e^{-\int_0^t r_s ds} \psi_t(\alpha) \, , \end{aligned} \tag{120}$$

which is the desired result. To verify the martingale condition we apply Itô-Doeblin formula to

$$Y_t \equiv e^{-\int_0^t r_s ds} \exp\left(M(T-t) + N(T-t)x_t + \alpha \log V(t) \right)$$

and obtain that

$$\mathrm{E}_t[dY_t] = 0$$

holds when N and M satisfy equations (118)-(119) above. A standard argument then shows that Y_t is a Q-martingale.

D Appendix D: Pricing the Market Portfolio and Individual Stocks in General Equilibrium

In general equilibrium the valuation of the market portfolio must equal the valuation of the portfolio that invests in the individual stocks. Here we show that this condition approximately holds when the individual stock returns have dynamics (42).

We simulate 2,000 stock prices from an exponential distribution with mean coefficient $\lambda = \$(100/2,000)$ and compute the price of the market portfolio by summing the values of the individual stocks. We generate one-year return paths from the market return dynamics $\frac{dV}{V}$ and we compute the terminal value of a strategy that holds the market portfolio. Then we simulate 2,000 return paths from the individual stock dynamics $\frac{dV_i}{V_i}$ and compute the terminal value for the 2,000 positions in each of the individual stocks. We sum the terminal values of these 2,000 positions to obtain the total value of the strategy that invests in the individual stocks.

We repeat this analysis 10,000 times. In each case we record the value of the market portfolio computed using simulated aggregate returns and the value obtained by summing the terminal price of the 2,000 stocks computed using simulated individual-stock returns. We find the correlation coefficient between the two series to be 99.59%. Further, Table 1 shows that the sample moments of the two distributions are nearly identical. This evidence suggests that to a good approximation the dynamics of individual stock returns are consistent with the aggregate index return dynamics in general equilibrium.

Table 1: Summary Statistics for the Market Value Variable. We report mean, standard deviation, skewness, and kurtosis for two measures of the market value variable with a one-year holding period. The first measure is constructed using market returns simulated from aggregate market return dynamics. The second measure is formed by summing the terminal value of 2,000 individual stock positions computed using individual-stock returns.

	Mean	Standard deviation	Skewness	Kurtosis
Market value computed from aggregate return dynamics	106.46	13.96	0.31	3.38
Market value computed from individual stock return dynamics	106.43	14.03	0.31	3.36

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