Federal Reserve Bank of Chicago

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REVISED
March 2015

WP 2013-05

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March 23, 2015

Abstract

Holst (1985) introduced a discrete spacings model that is related to the Bose-Einstein distribution and obtained the distribution of the number of vacant positions in an associated circle covering problem. We correct his expression for its probability mass function, obtain the first two moments, and describe their limiting properties. We then examine the properties of the vacancy statistic when the number of covering arcs in the associated circle covering problem is random. We also discuss applications of our results to a study of contagion in networks.

Key Words: *occupancy problems; spacings; Bose-Einstein distribution; sampling without replacement; sampling with replacement.*

2010 Mathematics Subject Classification: Primary: 60C05; Secondary: 62E15.

1 Introduction

This paper examines the discrete version of the circle covering problem first introduced by Stevens (1939) in which m arcs of length a are randomly placed on the unit circle, and the question of interest is the fraction of the circle left uncovered by any arc. The discrete version of this problem can be described as follows. Consider r (≥ 2) boxes arranged in a ring numbered $0, 1, ..., r-1$. Draw $m-1$ boxes by simple random sampling without replacement from the boxes numbered $1, 2, ..., r-1$, where $2 \leq m \leq r$. Let $1 \leq R_1 < \cdots < R_{m-1} \leq r-1$ be the drawn numbers, and set $R_0 = 0$ and $R_m = r$. Define

$$S_k = R_k - R_{k-1}$$

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for $k = 1, 2, ..., m$, i.e. S_k are spacings. Next, for an integer b where $1 \leq b \leq r$, define

$$V = \sum_{k=1}^{m} (S_k - b)_+ \tag{1}$$

where $(x)_+ = \max\{x, 0\}$. This setup can be interpreted as follows. We can think of $\{R_0, \ldots, R_{m-1}\}$ as m distinct starting points whose location among the r boxes is chosen at random. From each starting point, we designate the next b boxes including the starting point as covered. Any remaining uncovered boxes are designated as vacant. The random variable (rv) V represents the total number of vacant boxes. For reference, it will be convenient to also define

$$N = \sum_{k=1}^{m} I(S_k > b) \tag{2}$$

where $I(C) = 1$ if condition C is true and 0 else. Thus, N represents the number of distinct blocks of vacant boxes.

Our interest is in characteristics of V, specifically its distribution, some of its moments, and its behavior when m is random. Holst (1985) derived the marginal and joint distributions of $\{S_k\}_{k=1}^{m}$ and showed they are exchangeable. He also explored the connection between these random variables (rv's) and the Bose-Einstein distribution. Feller (1968, Sec II.5(a)) provides a nice introduction to the Bose-Einstein urn model.

As anticipated by our comments above, in the limit as $r \to \infty$ while $b = ar$ for some constant $a < 1$ this problem converges to the circle covering problem in which m points are chosen uniformly from the circumference of a circle, and each of the m points forms the end point of an arc of length a. The latter problem has been extensively analyzed; see for example, Siegel (1978). The limit of V/r in our discrete setup corresponds to the fraction of the circumference that is uncovered. However, the finite version of the problem has been less studied, even though as we discuss later this version arises in certain applications.

We derive in Section 2 an explicit expression for the probability mass function (pmf) of V including an exploration of the range of its values; in this process we correct an error in the expression for the pmf given in Holst (1985). We give explicit expressions for the first two moments of V in Section 3 using several properties of the joint distribution of S_k derived by Holst (1985). We discuss an extension of the model in Section 4 in which the number of starting points m is random, allowing us among other things to discuss an alternative version of our model in which the boxes are chosen with replacement. We establish limiting properties of V in Section 5 and link our results to those of Siegel (1978). In Section 6, we discuss an application concerning financial contagion that coincides with the discrete circle covering problem under certain conditions. This application suggests generalizations of the circle covering problem that have not been explored in previous work.

2 Exact Distribution of V

2.1 The Range

The value of V must be non-negative, and the lowest value it can assume is $r - mb$. Further, the largest possible value of V occurs when the chosen boxes are consecutive (implying $N \leq 1$) and V takes on the value $r - m - b + 1$. Thus, the support of V is the set $\{(r - mb)_+, \ldots, (r - m - b + 1)_+\}$, and so V is degenerate at 0 whenever $r < m + b$. Further when $r - mb \geq 0$, the total number of points in the support of V is $(m-1)(b-1)+1$ independently of r. Hence when $b = 1$, we have a single support point at $r - m > 0$. We now examine the form of the pmf $P(V = x)$ for various x values when $r \geq m + b$ and $b > 1$.

2.2 Probability Mass Function of V

Holst (1985; Theorem 2.2) argues that $P(V = x)$ is given by

$$\sum_{y=1}^{m} \binom{m}{y} \sum_{t=0}^{m-y} (-1)^t \binom{m-y}{t} \binom{x-1}{y-1} \binom{r-(y+t)\,b-x-1}{m-y-1} \Big/ \binom{r-1}{m-1}. \quad (3)$$

Note that expression (3) may include improper binomial coefficients $\binom{n}{k}$ where either $n < 0$ or $k \notin \{0, \ldots, n\}$. Such terms are traditionally set to 0. We now argue that this convention may yield an incorrect expression for $P(V = x)$ for $x = 0$ and for $x = r - mb$, and offer correct expressions for $P(V = x)$ for these cases.

Observe first that for $x = 0$, the right-hand side of (3) would under the usual convention equal to 0 due to the presence of the $\binom{x-1}{y-1}$ term. But this is at odds with the fact that $P(V = 0) = 1$ whenever $r < m + b$.

To obtain a proper expression for $P(V = 0)$, we use the observation noted by Holst that

$$P(V = 0) = P\left(\sum_{j=1}^{m} I(S_j > b) = 0\right) = P(N = 0). \quad (4)$$

Holst (1985) derives an expression for the right hand side of (4) in part (a) of his Theorem 2.2. Using his result, we can deduce that for $x = 0$, (3) must be replaced by

$$P(V = 0) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{r-jb-1}{m-1} \Big/ \binom{r-1}{m-1}. \quad (5)$$

We next turn to the case where $r \geq m + b$ and $x > 0$, and examine the range of values for y and t for which the associated terms on the right side of (3) are all positive, i.e. when $0 \leq t \leq \min\{m - y, (r - m - x - (b-1)y)/b\}$, and

$1 \leq y \leq \min\{m-1, x, (r-m-x)/(b-1)\}$ for $b > 1$ and $1 \leq y \leq \min\{m-1, x\}$ for $b = 1$.

We shall now argue that for this range, (3) holds except when $x = r - mb \geq m$. To see this, we begin with the observation by Holst in proving his Theorem 2.2 that if $I_k = I(S_k > b)$, then $P(V = x)$ must equal

$$\sum_{y=1}^{m} \binom{m}{y} \sum_{t=0}^{m-y} (-1)^t \binom{m-y}{t} P\left(\sum_{k=1}^{y} (S_k - b)_+ = x, I_1 = \cdots = I_{y+t} = 1\right). \quad (6)$$

Expression (6) can in turn be rewritten as

$$\sum_{y=1}^{m-1} \binom{m}{y} \sum_{t=0}^{m-y-1} (-1)^t \binom{m-y}{t} P\left(\sum_{k=1}^{y} (S_k - b)_+ = x, I_1 = \cdots = I_{y+t} = 1\right)$$
$$+ \sum_{y=1}^{m} \binom{m}{y} (-1)^{m-y} P\left(\sum_{k=1}^{y} (S_k - b)_+ = x, I_1 = \cdots = I_m = 1\right). \quad (7)$$

Holst then computes the following probabilities based respectively on parts (E) and (D) of his Theorem 2.1:

$$P\left(I_1 = \cdots = I_{y+t} = 1\right) = \frac{\binom{r-(y+t)b-1}{m-1}}{\binom{r-1}{m-1}}$$

and

$$P\left(\sum_{k=1}^{y} (S_k - b)_+ = x \,\middle|\, I_1 = \cdots = I_{y+t} = 1\right) = \frac{\binom{x-1}{y-1}\binom{r-(y+t)b-x-1}{m-y-1}}{\binom{r-(y+t)b-1}{m-1}} \quad (8)$$

and thus concludes that

$$P\left(\sum_{k=1}^{y} (S_k - b)_+ = x, I_1 = \cdots = I_{y+t} = 1\right) = \frac{\binom{x-1}{y-1}\binom{r-(y+t)b-x-1}{m-y-1}}{\binom{r-1}{m-1}}. \quad (9)$$

The expression for the conditional probability in (8) is valid and nonzero whenever $y \leq m-1$, and $y + t < m$.

Next we consider the last sum on the right in (7). Since the event $\{I_1 = \cdots = I_m = 1\}$ implies $S_i > b$ for $i = 1, \ldots, m$, the sum $\sum_{k=1}^{y} (S_k - b)_+$ is strictly increasing in y for $y \leq m$ and $\sum_{k=1}^{y} (S_k - b)_+ < \sum_{k=1}^{m} (S_k - b)_+ \equiv r - mb$. This means

$$\sum_{y=1}^{m} \binom{m}{y} (-1)^{m-y} P\left(\sum_{k=1}^{y} (S_k - b)_+ = x, I_1 = \cdots = I_m = 1\right)$$

is equal to 0 when $x > r - mb$ and is equal to the last term in the sum, $P\left(\sum_{k=1}^{m} (S_k - b)_+ = r - mb, I_1 = \cdots = I_m = 1\right)$, when $x = r - mb$. Since each term in $\sum_{k=1}^{m}(S_k-b)_+$ is at least one, the sum should be at least m. In other words, the only nonzero term in the last sum on the right side of (7) is

$$\begin{aligned} &P\left(\sum_{k=1}^{m}(S_k-b)_+=r-mb, I_1=\cdots=I_m=1\right)\\ &\quad=P\left(I_1=\cdots=I_m=1\right)=P\left(S_1>b,\cdots,S_m>b\right)=P(S_1>mb)\\ &\qquad\qquad\qquad\qquad=\tbinom{r-mb-1}{m-1}\Big/\tbinom{r-1}{m-1} \end{aligned} \tag{10}$$

provided $r-mb\geq m$. Upon collecting all of our findings (in (4), (6), (8), and (9)), we have the following modification of Theorem 2.2 of Holst (1985).

Theorem 1. *The support of the rv V representing the length of the vacant region is given by $\{(r-mb)_+,\ldots,(r-m-b+1)_+\}$. When $r<m+b$, V is degenerate at 0. When $r>m$ and $b=1$, V is degenerate at $(r-m)$. When $r\geq m+b$ and $r-mb\leq 0$, $P(V=0)$ is given by (5). In all other cases, $P(V=x)$ is given by*

$$\begin{aligned} &\left\{\sum_{y=1}^{m-1}\binom{m}{y}\sum_{t=0}^{m-y-1}(-1)^t\binom{m-y}{t}\binom{x-1}{y-1}\binom{r-(y+t)\,b-x-1}{m-y-1}\right.\\ &\left.+I(x=r-mb\geq m)\binom{r-mb-1}{m-1}\right\}\Big/\binom{r-1}{m-1}. \end{aligned} \tag{11}$$

Remark 1: The actual range of values for y and t for which the associated terms are positive is more restricted than given by the limits in the double sum in (11) in a way that depends on x. For example, when $x=r-mb$, the lowest value V can assume, the terms are positive for all $1\leq y\leq m-1$ and $0\leq t\leq\min\{m-y-1, m-y-(m-y)/b\}$. In contrast, when $x=r-m-b+1$, the highest value V can assume, $(y,t)=(1,0)$ is the only combination that produces a positive term. In that case, (11) yields

$$P(V=r-m-b+1)=m\Big/\binom{r-1}{m-1},$$

a quantity free of b.

Table 1 below provides the pmf of V for $r=10$ and $m=5$. It shows how the probability mass shifts towards values close to 0 as b increases.

3 Moments of V

Instead of using the pmf for V to compute the first two moments of V, we take advantage of an exchangeability argument to derive them from those of S_k. We will need the following expressions for the first two moments of nonnegative integer valued rv's X and Y.

$$E(X)=\sum_{i=0}^{\infty}P(X>i); \tag{12}$$

Table 1: **The pmf $P(V = x)$ for $r = 10, m = 5$ for various values of b**

	x					
b	0	1	2	3	4	5
1	0	0	0	0	0	1
2	0.008	0.159	0.476	0.317	0.040	
3	0.405	0.397	0.159	0.040		
4	0.802	0.159	0.040			
5	0.960	0.040				
6	1					

$$E(X^2) = 2\sum_{i=0}^{\infty} iP(X > i) + E(X); \tag{13}$$

$$E(XY) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P(X > i, Y > j). \tag{14}$$

The first two are well-known. Equations (12) and (13) are given in, for example, David and Nagaraja (2003), p. 43, and go back to Feller's classical work. Expression (14) is similar to known results for the continuous case; see, for example, the expression for the covariance in Barlow and Proschan (1981, p. 31), and the idea goes back to Hoeffding (1940); see Wellner (1994).

Here we give a short proof of (14) when X and Y are nonnegative integer valued rv's.

$$\begin{aligned} E(XY) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} ijP(X = i, Y = j) \\ &= \sum_{i=0}^{\infty} i \sum_{j=0}^{\infty} P(X = i, Y > j) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P(X > i, Y > j) \end{aligned}$$

upon using the idea of the form on the right side of (12) twice.

The moment expressions simplify further by the use of the following well-known identity: For positive integers $c \le a$,

$$\sum_{k=c}^{a} \binom{k-1}{c-1} = \binom{a}{c}. \tag{15}$$

Harris, Hirst, and Mossinghoff (2008) derive this identity using an induction argument (see their equation (2.10)). We give a simple probabilistic proof.

Proof. Multiply both sides by $(1/2)^a$. Then the right side, $\binom{a}{c}(1/2)^a$ represents the probability that in a tosses of a fair coin there are exactly c heads. Now if we have c heads, this event can be written as the union of disjoint events $E_c, \ldots, E_a$ where E_k is the event that we have exactly c heads and the cth head appears at the kth toss. By the negative binomial type argument we know that this probability is

$$\binom{k-1}{c-1}(1/2)^c(1/2)^{a-c} = \binom{k-1}{c-1}(1/2)^a.$$

Now sum this over k from c to a. □

Theorem 2. *Let $W_i = (S_i - b)_+$, for $i = 1, 2$. The W_i are exchangeable and for $r \geq m + b$*

$$E(W_1) = \binom{r-b}{m} \Big/ \binom{r-1}{m-1}, \tag{16}$$

and

$$E(W_1^2) = (2(r-b)+1)E(W_1) - 2m\frac{\binom{r-b+1}{m+1}}{\binom{r-1}{m-1}}. \tag{17}$$

For $r \geq m + 2b$,

$$E(W_1W_2) = \binom{r-2b+1}{m+1} \Big/ \binom{r-1}{m-1}, \tag{18}$$

and $E(W_1W_2) = 0$ if $r < m + 2b$.

Proof. Exchangeability follows from Theorem 2.1 of Holst (1985). Now

$$\begin{aligned} E(W_1) &= \sum_{i=0}^{r-m-b} P(W_1 > i) \quad \text{[from (12)]} \\ &= \sum_{j=b}^{r-m} P(S_1 > j) \\ &= \sum_{j=b}^{r-m} \binom{r-j-1}{m-1} \Big/ \binom{r-1}{m-1} \text{ [from Th 2.1(B), Holst].} \end{aligned} \tag{19}$$

From (15), the numerator on the right side of (19) reduces to $\binom{r-b}{m}$.

To establish (17), we use the expression for the second moment in (13).

Consider

$$
\begin{aligned}
\sum_{i=0}^{\infty} iP(W_1 > i) &= \sum_{j=b}^{r-m} (j-b)P(S_1 > j) \\
&= \sum_{j=b}^{r-m} \{r-(r-j)\}P(S_1 > j) - bE(W_1) \\
&= r\sum_{j=b}^{r-m} P(S_1 > j) - \sum_{j=b}^{r-m} (r-j)P(S_1 > j) - bE(W_1) \\
&= (r-b)E(W_1) - \sum_{j=b}^{r-m} (r-j)\binom{r-j-1}{m-1} \Big/ \binom{r-1}{m-1} \qquad (20)
\end{aligned}
$$

from the expression for $P(S_1 > i)$ in Theorem 2.1, Part (B) of Holst (1986). The numerator in the second term in (20) above can be expressed as

$$
m\sum_{i=b}^{r-m} \binom{r-i}{m} = m\sum_{j=m+1}^{r-b+1} \binom{j-1}{m} = m\binom{r-b+1}{m+1}, \qquad (21)
$$

where the last equality follows from (15). Upon using (13) with $W_1 = X$ and applying (20) and (21) we obtain (17).

Using (14) with $W_1 = X$ and $W_2 = Y$, and applying Theorem 2.1 Part (E), and Part (B) of Holst (1985) in succession, we obtain

$$
\begin{aligned}
E(W_1W_2) &= \sum_{i\geq b}\sum_{j\geq b} P(S_1 > i, S_2 > j) \\
&= \sum_{i=b}^{r-m-b}\sum_{j=b}^{r-m-i} P(S_1 > i, S_2 > j) \\
&= \sum_{i=b}^{r-m-b}\sum_{j=b}^{r-m-i} P(S_1 > i+j) \\
&= \sum_{i=b}^{r-m-b}\sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} \Big/ \binom{r-1}{m-1}.
\end{aligned}
$$

Now, with $k = r-i-j$,

$$
\sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} = \sum_{k=m}^{r-b-i} \binom{k-1}{m-1} = \binom{r-b-i}{m}
$$

from (15). Hence

$$
\sum_{i=b}^{r-m-b}\sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} = \sum_{i=b}^{r-m-b} \binom{r-b-i}{m}.
$$

With $k = r - b - i + 1$, the above sum can be expressed as

$$\sum_{k=m+1}^{r-2b+1} \binom{k-1}{m} = \binom{r-2b+1}{m+1}.$$

Hence the claim in (18) holds. Clearly, when $r < m + 2b$, we cannot have both W_1, W_2 be positive simultaneously and hence $E(W_1 W_2) = 0$. □

From (1) and the exchangeability of the W_i we see that

$$\begin{aligned} E(V) &= mE(W_1) \\ \mathrm{Var}(V) &= m\mathrm{Var}(W_1) + m(m-1)\mathrm{Cov}(W_1, W_2) \\ &= m[E(W_1^2) - \{E(W_1)\}^2] + m(m-1)[E(W_1 W_2) - \{E(W_1)\}^2] \\ &= mE(W_1^2) + m(m-1)E(W_1 W_2) - m^2\{E(W_1)\}^2 \qquad (22) \end{aligned}$$

where the expectations on the right side of (22) are given by Theorem 2. Thus we have the following result.

Theorem 3. *If $r \geq m + b$, the first two moments of the rv V representing the number of vacant boxes are given by*

$$E(V) = m\frac{\binom{r-b}{m}}{\binom{r-1}{m-1}}, \qquad (23)$$

$$\begin{aligned} Var(V) &= \frac{m(2(r-b)+1)\binom{r-b}{m} - 2m^2\binom{r-b+1}{m+1} + m(m-1)\binom{r-2b+1}{m+1}}{\binom{r-1}{m-1}} \\ &\quad -m^2\left\{\frac{\binom{r-b}{m}}{\binom{r-1}{m-1}}\right\}^2 \qquad (24) \end{aligned}$$

where the coefficient of $m(m-1)$ in (24) is taken to be 0 whenever $r < m + 2b$.

Notes:

1. After deriving the expression for $E(V)$, we discovered it was previously reported in Ivchenko (1994, p. 108). However, he does not derive a formula for the variance of V.

2. Ivchenko (1994, p. 108) also derives an expression for $E(N)$. Using similar exchangeability arguments, we can derive the same expression as well as an expression for the variance of N. In particular, from Theorem 2.1 of Holst (1985), we see that

$$E(I_1) = E(I_1^2) = P(S_1 > b) = \binom{r-b-1}{m-1} \Big/ \binom{r-1}{m-1} \equiv p_1\ ,$$

and

$$E(I_1 I_2) = P(S_1 > b, S_2 > b) = \binom{r-2b-1}{m-1} \Big/ \binom{r-1}{m-1} \equiv p_2\ .$$

Thus from (2) we obtain

$$E(N) = mE(I_1) = m\binom{r-b-1}{m-1} \Big/ \binom{r-1}{m-1}$$

and

$$\begin{aligned} \mathrm{Var}(N) &= m\mathrm{Var}(I_1) + m(m-1)\mathrm{Cov}(I_1, I_2) \\ &= mp_1(1-p_1) + m(m-1)(p_2 - p_1^2) \\ &= m(m-1)p_2 + mp_1 - (mp_1)^2 \end{aligned} \tag{25}$$

where the p_i's are given above.

3. For $m \leq r-b$,

$$E(V) = m\binom{r-b}{m} \Big/ \binom{r-1}{m-1} = r\binom{r-b}{m} \Big/ \binom{r}{m} \tag{26}$$

$$= \frac{(r-b)!\,(r-m)!}{(r-b-m)!\,(r-1)!}. \tag{27}$$

   As seen from the expression in (27), $E(V)$ is symmetric in b and m, even though the pmf for V is not symmetric in these parameters. The symmetry also does not hold for the second moment.

4. As mentioned in Theorem 1, if $r < m+b$, $P(V=0)=1$. Thus if $b \geq r-m+1$ or $m+b > r$, all the S_i are b or less. Thus, $E(V) = \mathrm{Var}(V) = 0$ whenever $r < m+b$. When $b=1$ and $r > m$, V is degenerate at $r-m$ and in that case $E(V) = r-m$ and $\mathrm{Var}(V) = 0$.

4 Distribution and Moments with Random m

Up to now we assumed the number of starting points m is fixed and restricted to values in $\{2, \ldots, r\}$. We now consider an extension in which the number of starting points is a rv M with support $\{0, ..., r\}$ and pmf $P(M=m) = p_M(m)$. We let V_M denote the number of vacant boxes in $\{0, \ldots, r-1\}$ in this case to highlight that the number of starting points in this case is allowed to be random.

Formally, we construct V_M as follows. We first draw a value for M. If $M=0$, we designate all boxes as vacant and set $V_M = r$. If $M = m > 0$, we draw m starting points without replacement from the boxes labelled $0, \ldots, r-1$. Let $\{R_0, \ldots, R_{m-1}\}$ denote the identities of these starting points. For each $k \in \{0, \ldots, m-1\}$ we designate the boxes

$$\{R_k, (R_k+1) \mod r, \ldots, (R_k+b-1) \mod r\}$$

as covered. Any box that is not covered is labelled vacant. Define J_i for each $i = 0, \ldots, r-1$, as equal to 1 if box i is vacant and 0 if covered. Then the number of vacant boxes is

$$V_M = \sum_{i=0}^{r-1} J_i. \tag{28}$$

The discrete circle covering problem we started with is thus a special case of this formulation in which M is degenerate with full mass at a single value m. We now describe two different approaches for deriving the mean and variance of V_M in the general case where M can have a nondegenerate distribution. We also obtain an expression for the pmf of V_M.

4.1 A Conditioning Approach

When $2 \leq m \leq r$, the rv V_M given $M = m$ has the same distribution as V in the circle covering problem with m starting points. When $M = 0$ and $M = 1$, the rv V_M has a degenerate distribution will full mass at r and $r - b$, respectively. We can therefore use a simple conditioning argument to obtain the following:

Theorem 4. *The pmf, mean, and variance of V_M are respectively given by*

$$P(V_M = x) = \sum_{m=0}^{r} P(V_m = x) p_M(m), \tag{29}$$

$$\begin{aligned} E(V_M) &= \sum_{m=0}^{r} E(V_m) p_M(m) \\ &= r \sum_{m=0}^{r-b} \frac{\binom{r-b}{m}}{\binom{r}{m}} p_M(m), \end{aligned} \tag{30}$$

and

$$Var(V_M) = Var(E(V_M|M)) + E(Var(V_M|M)), \tag{31}$$

where, for $2 \leq m \leq r$, $P(V_m = x)$ is given in Theorem 1 and expressions for $E(V_m)$ and $Var(V_m)$ are given in Theorem 3. Further, V_0 is degenerate at r and V_1 is degenerate at $(r - b)_+$.

In (30) we have used the second form of $E(V_m)$ in (26) and the fact that $E(V_1) = (r - b)_+ = r\binom{r-b}{1}/\binom{r}{1}$.

We now describe three specific mechanisms that are covered by the above result. The first two models have attracted attention in the application to financial contagion discussed later in Section 6, although the first one turns out to be of more general interest as we discuss below.

Binomial M. Let M assume a Binomial(r, p) distribution, i.e. $p_M(m) = \binom{r}{m} p^m (1 - p)^{r-m}, 0 \leq m \leq r$. This is equivalent to assuming that each box $i \in \{0, \ldots, r - 1\}$ is chosen as a starting point with probability p independently of whether any other box is chosen as a starting point. The expression for the mean in (30) simplifies substantially;

$$E(V_M) = r \sum_{m=0}^{r-b} \binom{r-b}{m} p^m (1-p)^{r-m} = r(1-p)^b. \tag{32}$$

While an expression for $Var(V_M)$ can be obtained using (31), the algebra is quite involved; we compute it using another approach in Section 4.2.

Mixture of Binomials. As above let M be Binomial(r,p) and assume p is also a rv with support $(0, 1)$, i.e. M is a mixture of binomials. In this case, we can compute the conditional mean of V_M given p as $rE(1-p)^b$. If p has a Beta distribution, this expectation can be easily calculated and expressed in terms of a beta function.

Sampling With Replacement. Suppose we choose starting points by taking n (≥ 1) draws from $\{0, \ldots, r-1\}$ with replacement. The resulting number of starting points M will be random with support $\{1, \ldots, n\}$, and its pmf is given by (see, e.g., Feller, 1968, p. 102)

$$\begin{aligned} p_M(m) &= \binom{r}{m} \frac{1}{r^n} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n \\ &= \frac{r(r-1)\cdots(r-m+1)}{r^n} S(n, m) \end{aligned} \tag{33}$$

where $S(n, m)$ is the Stirling's number of second kind (see, e.g., Johnson and Kotz, 1977, pages 110, 8 for the Stirling number connection).

When the values of all the parameters are known, the expressions for pmf, mean, and variance (29)–(31) given in Theorem 4 can be numerically evaluated in a straightforward manner. However, we now introduce another approach that greatly simplifies the evaluation of these moments.

4.2 An Alternative Approach for Moments

An alternative approach to computing moments for V_M makes use of (28) which defines V_M as the sum of the indicator variables J_i where $J_i = 1$ if box i is vacant. As long as the sampling mechanism is completely symmetric with respect to the r available boxes, these J_i are identically distributed (but not exchangeable) and

$$\begin{aligned} E(V_M) &= rP(J_0 = 1), \\ Var(V_M) &= rP(J_0 = 1)P(J_0 = 0) + r \sum_{k=1}^{r-1} Cov(J_0, J_k); \end{aligned} \tag{34}$$

and $Cov(J_0, J_k) = P(J_0 = 1, J_k = 1) - [P(J_0 = 1)]^2$. We will now illustrate the utility of this approach with the examples considered above.

Binomial M (Contd.) Note that box 0 is vacant if and only if none of the boxes labeled $0, r-1, \ldots, r-b+1$ are starting points, i.e. $\{R_0, \ldots, R_{M-1}\} \cap \{0, r-1, \ldots, r-b+1\} = \varnothing$. Recall that when M is binomial, each box i is chosen as a starting point with probability p independently of whether any other box is a starting point. Hence, $P(J_0 = 1) = q^b$, where $q = 1-p$, and $E(V_M) = rq^b$ as seen earlier in (32). When $2b \leq r$, we have

$$P(J_0 = 1, J_k = 1) = \begin{cases} q^{b+k}, & \text{if } 1 \leq k \leq b-1 \\ q^{b+r-k}, & \text{if } r-b+1 \leq k \leq r-1 \\ q^{2b}, & \text{if } b \leq k \leq r-b; \end{cases}$$

$$Cov(J_0, J_k) = \begin{cases} q^{b+k} - q^{2b}, & \text{if } 1 \leq k \leq b-1 \\ q^{b+r-k} - q^{2b}, & \text{if } r-b+1 \leq k \leq r-1 \\ 0, & \text{if } b \leq k \leq r-b. \end{cases}$$

and consequently, we obtain

$$\begin{aligned} \sum_{k=1}^{r-1} Cov(J_0, J_k) &= \sum_{k=1}^{b-1} q^{b+k} + \sum_{k=r-b+1}^{r-1} q^{b+r-k} - 2(b-1)q^{2b} \\ &= 2q^b \frac{q-q^b}{p} - 2(b-1)q^{2b}. \end{aligned} \tag{35}$$

It follows that when $2b \leq r+1$,

$$Var(V_M) = rq^b \left\{ 1 + \frac{2q}{p} - q^b\left(\frac{2}{p} + 2b - 1\right) \right\}. \tag{36}$$

For larger b values, the associated geometric series can be summed similarly. The rv V_M is never degenerate whatever be b. For $b \geq r$, V_M has a two point distribution with $E(V_M) = rq^r$, and $Var(V_M) = r^2 q^r (1-q^r)$.

Sampling With Replacement (Contd.) We will now use (34) and obtain the first two moments of V_M when M is determined by drawing n times with replacement from all boxes. Note that $P(J_0 = 1) = \{(r-b)/r\}^n$; thus

$$E(V_M) = r\left(1 - \frac{b}{r}\right)^n, 1 \leq b \leq r. \tag{37}$$

Since

$$P(J_0 = 1, J_k = 1) = \begin{cases} \left(\frac{r-(b+k)}{r}\right)^n, & \text{if } 1 \leq k \leq b-1 \\ \left(\frac{k-b}{r}\right)^n, & \text{if } r-b+1 \leq k \leq r-1 \\ \left(\frac{r-2b}{r}\right)^n, & \text{if } b \leq k \leq r-b, \end{cases} \tag{38}$$

we obtain

$$Cov(J_0, J_k) = \begin{cases} \left(\frac{r-(b+k)}{r}\right)^n - \left(\frac{r-b}{r}\right)^{2n}, & \text{if } 1 \le k \le b-1 \\ \left(\frac{k-b}{r}\right)^n - \left(\frac{r-b}{r}\right)^{2n}, & \text{if } r-b+1 \le k \le r-1 \\ \left(\frac{r-2b}{r}\right)^n - \left(\frac{r-b}{r}\right)^{2n}, & \text{if } b \le k \le r-b. \end{cases}$$

Thus,

$$\sum_{k=1}^{r-1} Cov(J_0, J_k)$$
$$= 2\sum_{k=1}^{b-1} \left(\frac{r-b-k}{r}\right)^n + (r-2b+1)\left(\frac{r-2b}{r}\right)^n - (r-1)\left(\frac{r-b}{r}\right)^{2n},$$

and for $2b \le r+1$, (34) yields the following expression for $Var(V_M)/r$:

$$\left(1-\frac{b}{r}\right)^n + 2\sum_{k=1}^{b-1}\left(1-\frac{b+k}{r}\right)^n + (r-2b+1)\left(1-\frac{2b}{r}\right)^n - r\left(1-\frac{b}{r}\right)^{2n}. \quad (39)$$

For $b \ge r$, V_M is degenerate at 0; for other b values, the third term above vanishes and k is summed from 1 to $(r-b)$ in the second term.

Without-Replacement Sample. In this setup, we have

$$P(J_0 = 1) = \binom{r-b}{m} \Big/ \binom{r}{m}$$

leading to the second form for $E(V)$ given in (26). Using a representation for $P(J_0 = 1, J_k = 1)$ that parallels (38), an expression for the variance can be written using (34). We will not pursue it as we already have a compact expression available in (24).

5 Limiting Properties of V

5.1 Limiting Distributions

We now return to the case where the number of starting points m is fixed. Holst (1985; Theorem 3.2) argues that as $r, b \to \infty$ with $b/r \to a$ for some $0 < a < 1$, $V/r \xrightarrow{d} V_a$ where V_a has the same distribution as the length of non-covered segments when m arcs of length a are dropped at random on a circle with unit circumference. Siegel (1978; Theorem 3) has shown that the distribution of V_a can be expressed as the mixture of a degenerate and a continuous rv. Specifically, he shows that $P(V_a(m) = (1-ma)_+) = p_a(m)$ where

$$p_a(m) = \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} (1-ia)_+^{m-1}, ma > 1 \quad (40)$$
$$= (1-ma)^{m-1}, ma \le 1, \quad (41)$$

and with probability $1 - p_a(m)$, $V_a(m)$ behaves like a continuous rv $W_a(m)$ having the pdf $f(w; a, m)$ given by

$$\begin{aligned} f(w; a, m) &= \\ \frac{m}{1 - p_a(m)} \sum_{i=1}^{m} \sum_{j=1}^{m-1} &(-1)^{i+j} \binom{m-1}{i-1} \binom{m-1}{j} \binom{i-1}{j-1} w^{j-1} (1 - ia - w)_+^{m-j-1}, \\ &(1 - ma)_+ < w < 1 - a. \end{aligned} \tag{42}$$

with the convention that $(1 - ia - w)_+^0$ is interpreted as 1 if $1 - ia - w \geq 0$, and as 0, otherwise. We now show the following.

Lemma 1. *If $r, b \to \infty$ such that $b/r \to a, 0 < a < 1$, then*

$$P\{V = (r - mb)_+\} \to p_a(m) \equiv P\{V_a = (1 - ma)_+\},$$

given by (40) when $ma > 1$, and by (41) when $ma < 1$. When $ma = 1$, both (40) and (41) reduce to 0.

Proof. When $ma > 1$, $r - mb$ is eventually negative, our interest then is in the limiting form of $P(V = 0)$ given in (5). Consider the jth term there, excluding the factor $(-1)^m \binom{m}{j}$:

$$\frac{\binom{r-jb-1}{m-1}}{\binom{r-1}{m-1}} = \frac{(r - jb - 1) \cdots (r - jb - m + 1)}{(r - 1) \cdots (r - m + 1)},$$

if $r - jb = r(1 - j(b/r)) > m - 1$; and it is 0 if $r(1 - j(b/r)) \leq m - 1$. So if $b/r \to a$ with $1 - ma < 0$, the above ratio converges to $(1 - ja)_+^{m-1}$. Thus the limit is given by (40).

Whenever $ma < 1$, since $r - mb = r(1 - m(b/r))$, $r - mb$ eventually exceeds any fixed m. In that case the term (10) converges to $(1 - ma)^{m-1}$. The remaining finite number of terms in the numerator on the right in (11) are finite and each is of $o(r^{m-1})$ whereas the denominator is $O(r^{m-1})$. Thus, the only nonzero term in the limit is that of (10) and it coincides with (41).

If $ma = 1$, (41) is obviously 0 and now we show that (40) also converges to 0 as $a \to (1/m)^+$. For this we consider the continuous uniform spacing problem where one chooses at random $m - 1$ points $U_1, \ldots, U_{m-1}$ from the interval $(0, 1)$. With spacings defined as $Y_i = U_{i:m-1} - U_{i-1:m-1}, i = 1, \ldots, m$, where $U_{0:m-1} = 0$ and $U_{m:m-1} = 1$, it is known that the distribution function of the continuous rv $Y_{(m)}$ representing the maximal spacing can be expressed as (see, e.g., David and Nagaraja, 2003, p. 135)

$$P(Y_{(m)} \leq a) = 1 - P(Y_{(m)} > a) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (1 - ia)_+^{m-1}, \tag{43}$$

for all a in $(0, 1)$. Since by construction the maximal spacing $Y_{(m)}$ exceeds $1/m$ with probability 1, the right side sum in (43) is 0 whenever $a \leq 1/m$ or $ma \leq 1$

and thus approaches 0 as $a \to (1/m)^+$. The difference between this sum and the sum in (40), $(-1)^m(1-am)_+^{m-1}$, is 0 whenever $a \geq (1/m)$. Thus we conclude that as $a \to (1/m)^+$ the expression in (40) converges to 0. $\square$

Notes:

5. When $b = ar$, with $ma < 1$, we have seen that the expression in (10) converges to $(1-ma)^{m-1}$, while the other terms contributing to $P(V = r-mb)$ converge to 0, indicating the dominant nature of this term missing in Holst's Theorem 2.2 (1985). That is, the term missing from Holst's expression is precisely what converges to the degenerate component of the rv in the continuous case.

6. Holst's Theorem 3.2 gives expressions for $P(V_a = 0)$ and the pdf of the continuous part. Lemma 1 reveals that his expressions are imprecise and fail to properly account for the range of V.

7. Siegel's (1978) version of (40) [his expression (3.23)] has the summation that includes an additional term with $i = m$. In view of the assumption that $ma > 1$, the corresponding term is 0, and hence they coincide. Further, in view of the above lemma, we can conclude that when $ma = 1$ both (40) and (41) hold.

8. Consider the case where M is random, specifically where M is generated by taking n draws with replacement from $\{1, \ldots, r\}$. As $r \to \infty$ while n is held fixed, the first factor in the expression for the pmf of M in (33) converges to 0 whenever $m < n$ and to 1 when $m = n$. Since $S(n,n) = 1$, it follows that $M \xrightarrow{P} n$, the sample size, and the limiting distribution of V_M is the same as the limiting distribution of V when m is replaced by n. More generally, given any process M that converges in probability to a degenerate distribution as $r \to \infty$, the distribution for V_M will converge to the same limiting distribution as V with m corresponding to the value that M collapses to.

For $m = 5$ and selected r values, Figure 1 and Figure 2 respectively provide the normalized conditional pmf of V given the event $\{V > (r-mb)_+\}$, for $a = 0.1$ and 0.25. The case of $r = \infty$ corresponds to the conditional pdf $f(w; a, m)$ of the continuous case, given in (42). Both these figures suggest that by the time r reaches 500, we are close to the limiting result, indicating that when the sampling fraction is under 1%, $f(w; a, m)$ provides a close approximation to the conditional pmf of V.

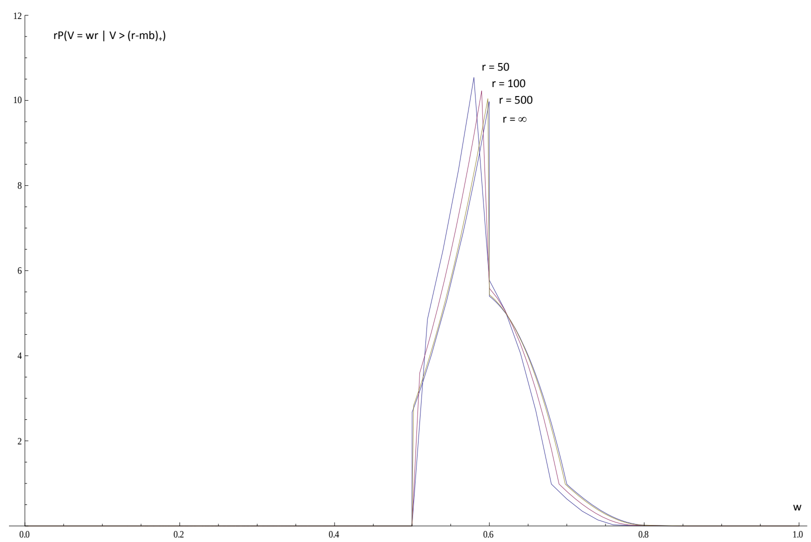

Figure 1: $rP\{V = rw | V > (r - bm)_+\}$ for $m = 5$, $a = 0.1$, $b = ar$, for selected r; $f(w; a, m)$, given in (42), corresponds to $r = \infty$.

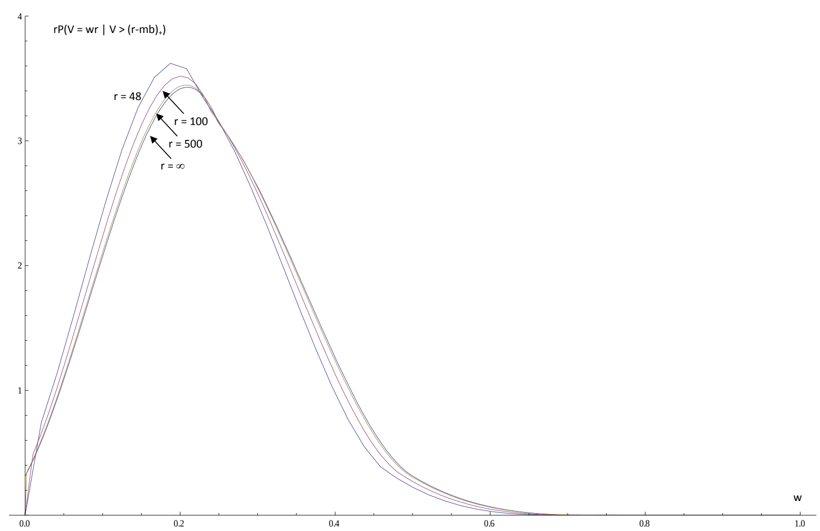

Figure 2: $rP\{V = rw|V > (r - bm)_+\}$ for $m = 5$, $a = 0.25$, $b = ar$, for selected r; $f(w; a, m)$, given in (42), corresponds to $r = \infty$.

5.2 Limiting Moments

Limits for Large r and b

Since V/r is uniformly bounded, convergence in distribution implies that $E(V/r)^k \to E(V_a^k)$ when $r \to \infty$ and $b/r \to a$ and $ma \neq 1$. Siegel has shown in his Theorem 2 that,

$$E(V_a^k) = \binom{k+m-1}{m}^{-1} \sum_{i=1}^{k} \binom{k}{i} \binom{m-1}{i-1} (1-ia)_+^{m+k-1}, k \geq 1. \tag{44}$$

Hence we can obtain approximations to any moment of V when m is small and r and b are large using the moments of V_a.

Table 2: **Properties of V and V_a for $m = 5$, $r = 20, 50$, and selected b values; $a = b/r$.**

b	$P(V = (r - mb)_+)$	$p_a(m)$	$E(V)$	$rE(V_a)$	$SD(V)$	$rSD(V_a)$
			$r = 20$			
1	1	0.3164	15	15.48	0	0.49
2	0.008	0.0625	11.05	11.14	0.80	1.14
3	0.405	0.0040	7.98	8.87	1.42	1.71
4	0.802	0	5.63	6.55	1.84	2.11
5	0.960	0.0040	3.87	4.75	2.04	2.31
			$r = 50$			
1	1	0.6561	45	45.196	0	0.341
2	0.6407	0.4096	40.408	40.769	0.586	0.899
3	0.3882	0.2401	36.199	36.695	1.213	1.536
4	0.2189	0.1296	32.348	32.954	1.875	2.196
5	0.1121	0.0625	28.832	29.525	2.532	2.842
6	0.0502	0.0256	25.628	26.387	3.156	3.452
7	0.0183	0.0081	22.716	23.521	3.726	4.009
8	0.0047	0.0016	20.075	20.911	4.229	4.503
9	0.0006	0.0001	17.685	18.537	4.658	4.925
10	0.0000	0	15.528	16.384	5.007	5.270
11	0.0006	0.0001	13.587	14.436	5.273	5.538
12	0.0047	0.0016	11.845	12.678	5.457	5.728
13	0.0179	0.0081	10.287	11.095	5.560	5.842
14	0.0452	0.0246	8.897	9.675	5.586	5.881
15	0.0885	0.0545	7.661	8.404	5.541	5.852
16	0.1467	0.0989	6.566	7.27	5.430	5.759
17	0.2158	0.1561	5.601	6.262	5.261	5.609
18	0.2912	0.2226	4.752	5.369	5.041	5.408
19	0.3689	0.2944	4.010	4.581	4.780	5.164
20	0.4455	0.3680	3.363	3.888	4.486	4.886

Table 2 provides some key facts about the features of the distributions of V and the limiting rv V_a for $m = 5$, $r = 20, 50$ and b values up to 20 corresponding to a good range of a values. It shows that as a increases $p_a(m)$ decreases for $a \leq 1/m$, it is 0 when $ma = 1$ and then $p_a(m)$ increases. Note that whenever a reaches $1/m$ from below, the lower limit of the support of V_a moves towards 0 and whenever $ma > 1$, the lower limit remains at 0. This limiting pattern is closely followed by V when $r = 50$, but not that closely when $r = 20$. The moments converge fairly quickly to the limiting values. The mean is better approximated by the limit for small b $(= ar)$ whereas for the standard deviation, large b values tend to be slightly more efficient.

Limits for Large r and m

When b is held fixed and $r, m \to \infty$ such that $m/r \to p = 1 - q, 0 < p < 1$, Holst (1985, Theorem 4.2) has shown that V is asymptotically normal and has given the first two moments of the limit distribution. We now derive asymptotic approximations for the expressions for $E(V)$ and $Var(V)$ in (23) and (24) to study their limiting properties. When b is held fixed, we have

$$C(r, m; b) \equiv \frac{(r-b)!(r-m)!}{r!(r-m-b)!} = \frac{(r-m)(r-m-1)\cdots(r-m-b+1)}{r(r-1)\cdots(r-b+1)} \approx q^b.$$

Hence

$$\frac{\binom{r-b}{m}}{\binom{r-1}{m-1}} = \frac{r}{m} C(r, m; b) \approx \frac{r}{m} q^b;$$

$$\frac{\binom{r-b+1}{m+1}}{\binom{r-1}{m-1}} = \frac{r(r-b+1)}{m(m+1)} C(r, m; b) \approx \frac{r(r-b+1)}{m(m+1)} q^b;$$

$$\frac{\binom{r-2b+1}{m+1}}{\binom{r-1}{m-1}} = \frac{r(r-2b+1)}{m(m+1)} C(r, m; 2b) \approx \frac{r(r-2b+1)}{m(m+1)} q^{2b}.$$

Upon plugging in these approximations in the expressions given in Theorem 3, we obtain

$$E(V) \approx rq^b, \tag{45}$$

and $Var(V)/r$ is

$$\begin{aligned}
&\approx q^b \left\{ 2(r-b) + 1 - 2\frac{m}{m+1}(r-b+1) \right\} + q^{2b} \left\{ \frac{m-1}{m+1}(r-2b+1) - r \right\} \\
&= \frac{q^b}{m+1} \left\{ 2(r-b) + 1 - m \right\} - \frac{q^{2b}}{m+1} \left\{ 2r + (2b-1)(m-1) \right\} \\
&\approx q^b \frac{1+q}{p} - q^{2b} \left\{ \frac{2}{p} + 2b - 1 \right\}. \qquad (46)
\end{aligned}$$

We note that the approximations of the mean and variance of V above where m grows deterministically with r match with the exact moments of V_M in the case where M is Binomial(r, p) with p equal to the limiting value of m/r. In particular, (45) matches with (32) and (46) matches the expression in (36). The expressions we obtain for the variances differ from the variance of the limiting normal distribution reported in Theorem 4.2 of Holst (1985). While the convergence in distribution and convergence of moments are not directly related, it appears his expression is incorrect.

Remark 2: To appreciate why the asymptotic approximations for the mean and variance of V coincide with the exact moments of V_M when M has a binomial distribution, observe that when $m/r \to p$, if we take any pair of boxes i and j, the probability that each is a drawn as a starting point converges to p while the probability that both are drawn converges to p^2, i.e. the two events are asymptotically independent. But recall that this independence is what distinguishes the case where M is binomial when r is finite. Consistent with this, J_i converges in probability to a Bernoulli rv with success probability $(1-p)^b$, which is the exact distribution of J_i when M is binomial. In other words, when the number of starting points m is deterministic but grows proportionately with r, the number of vacant boxes in a block of boxes of fixed size behaves asymptotically in the same way as the number of vacant boxes in that block for the same r where M is distributed Binomial(r, p).

Sampling with Replacement: Limits for Large r and n

Finally, consider the case where M is random and generated by drawing n times with replacement from all boxes. As we discussed above, this case converges to the discrete circle covering problem with n boxes. Consistent with this, suppose $r, n \to \infty$ such that $n/r \to \theta, 0 < \theta < 1$. Using (37) and (39) we obtain

$$
\begin{aligned}
E(V_M) &\approx re^{-b\theta} = rq^b && (47)\\
\frac{\mathrm{Var}(V_M)}{r} &\approx e^{-b\theta} + 2\sum_{k=1}^{b-1} e^{-(b+k)\theta} + (r-2b+1)e^{-2b\theta} - re^{-2b\theta} \\
&= q^b\frac{1+q}{p} - q^{2b}\left\{\frac{2}{p} + 2b - 1\right\} && (48)
\end{aligned}
$$

upon simplification with $-\log(q) = \theta$, and $p = 1-q$. This matches (32) and (36). Under this scheme, the probability that a particular box is never among the n boxes drawn is $(1-r^{-1})^n$, and thus the expected number of distinct boxes chosen is

$$E(M) = r\left\{1 - (1-\frac{1}{r})^n\right\} \approx r(1-e^{-\theta}) = r(1-q) = rp.$$

In other words $E(M) \approx rp$.

6 Application to Financial Contagion

We conclude by showing how the discrete circle covering problem we analyzed is related to the literature on financial contagion. One of the workhorse model of financial contagion is due to Eisenberg and Noe (2001). Their model posits that banks are connected via a directed network based on the obligations banks owe one another. If some banks incur losses, they will be unable to meet their required payments to other banks, inflicting losses on other banks. Thus, shocks that affect certain nodes in a network can propagate to other nodes. Eisenberg and Noe derive the vector of clearing payments given the network structure and the identity of the nodes that incur direct losses. One can use this vector to deduce which banks will be adversely affected when certain banks are hit.

Subsequent work has extended the Eisenberg and Noe model by assuming that the number of banks that experience losses as well as their identity is random. For tractability, this literature has focused on simple network structures. For example, Caballero and Simsek (2013), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014) and Alvarez and Barlevy (2015) all consider contagion in circular networks in which the network of obligations across banks is isomorphic to a circular graph. While these networks bear little resemblance to the pattern of obligations across banks in practice, these settings still provide useful intuition about what determines contagion and how banks might behave when they are uncertain about the extent of contagion. We now show how contagion in circular networks is related to the discrete circle covering problem. We further argue that the connection between the two suggests generalizations of the circle covering problem that to our knowledge have not been noted previously.

Suppose there are r banks, that each bank owes an amount λ to one other bank, and that each bank is in turn owed λ by another bank. Banks can be viewed as connected to one another via a directed network in which a bank points to another bank if the former owes something to the latter. Formally, index banks in the network by $i \in \{0, \dots r-1\}$. A circular network is one where bank i owes λ to bank $i+1 \bmod r$. We drop the reference to mod r in what follows. Following Alvarez and Barlevy (2015), we impose the following assumptions: (1) Each bank owns μ worth of assets that it can sell to repay its outstanding obligations if it needs to, (2) Among the r banks, a random number M with pmf $P(M = m) = p_M(m)$ will be "bad", meaning they incur a loss of size ϕ that must be subtracted from their initial asset holdings μ, (3) Given $M = m$, each of the $\binom{r}{m}$ groups of size m is equally like to be those which are bad, and (4) $\mu < \phi < \frac{r}{m}\mu$. The last assumption implies bad banks incur losses that exceed what they can afford to pay by liquidating their assets, but total losses across all m bad banks are still less than the combined value of all assets held among all r banks in the network. Banks are required to pay their full obligation λ if possible, and must sell their asset holdings if they fall short. Although the distribution of the number of bad banks M is unrestricted, Alvarez and Barlevy (2015) draw particular attention to the cases where M has a degenerate distribution (i.e. $p_M(m) = 1$ for some m), a binomial distribution, and a mixture of binomials as instructive special cases.[1]

Let x_i denote the amount bank i pays bank $i+1$ and assume that the bad banks are labeled R_k, $k = 0, \ldots, m-1$, and these are the ones who have incurred a direct external loss of ϕ and the others have no external losses. Given that banks must pay their obligations if they can, the payments $\{x_i\}_{i=0}^{r-1}$ satisfy the following system of equations

$$\begin{aligned} x_i &= \min\{(x_{i-1} + \mu - \phi)_+, \lambda\}, \qquad i \in \{R_0, ..., R_{m-1}\} \\ &= \min\{x_{i-1} + \mu, \lambda\}, \qquad i \notin \{R_0, ..., R_{m-1}\}, \end{aligned}$$

with $x_{-1} \equiv x_{r-1}$. For $\phi < \frac{r}{m}\mu$, there exists a unique solution $\{x_i\}_{i=0}^{r-1}$ to this system. Bank i is said to be insolvent if $x_i < \lambda$, i.e. if it cannot meet its obligation, and solvent otherwise. Each of the m bad banks are insolvent, since even if they received the full amount λ from the bank that is obligated to them, the fact that $\mu < \phi$ implies $x_{i-1} + \mu - \phi < \lambda$ and so they would be unable to pay in full even after liquidating their assets. Beyond these m bad banks, banks that do not directly suffer losses may still end up insolvent because they are exposed to bad banks either directly – meaning the bank that owes them λ is bad – or indirectly – meaning the bank that owes them λ is good but is exposed to a bad bank. A central question in this framework is to determine the number of banks that are insolvent, i.e. to gauge the extent of contagion when only M banks suffer direct losses.

The results turn out to depend on the parameter λ. Suppose $\lambda \leq \phi - \mu$, and $\lambda = b\mu$, for some integer b. Then $(b+1)\mu \leq \phi$, and $x_{R_k} = 0$, and $x_{R_k+j} = j\mu$, $j = 1, \ldots, \min\{b-1, S_{k+1} - 1\}$, for $k = 0, \ldots, m-1$. Hence, the number of insolvent banks starting from each bad bank is a fixed number b, unless one of those b banks is itself bad. It should be clear that the number of bad banks corresponds to the number of starting points M, while the number of solvent banks corresponds to the number of vacant boxes V with b equal to $\frac{\lambda}{\mu}$. Thus our results provide the small as well as large sample properties of the number of solvent banks in a circular network with a random number of bad banks M when $\lambda \leq \phi - \mu$.

The situation where $\lambda > \phi - \mu$ provides a new generalization of the discrete circle covering problem. In this case, b_k, the number of insolvent banks induced by the kth bad bank, becomes a rv that depends on the location of the other bad banks. However, unlike in Siegel and Holst (1982) who discuss the continuous case of the circle covering problem assuming the length of the arc b starting at any point is an i.i.d. rv, here the number of insolvent banks starting at each R_k will depend on the distribution of the spacings between the bad banks. To elaborate, $x_{R_k} = (x_{R_k - 1} - (\phi - \mu))_+$ is no longer identically 0, and this will affect the number of banks immediately following bank R_k that are insolvent. That is, the number of banks that must be covered starting from the kth bad bank is a rv

[1]Specifically, the unconditional probability that a bank is bad is higher than, equal to, and lower than the probability that a bank is bad conditional on news that another bank is good when M is degenerate, binomial, and a mixture of binomials, respectively. These distinctions turn out to matter when there is some possibility of news about some banks might be revealed.

that depends on the entire collection of spacings $\{S_k\}_{k=1}^{M}$. Alvarez and Barlevy (2015) show that when M has a degenerate distribution so that $p_M(m) = 1$ for some m, if $\lambda > m(\phi - \mu)$ then the number of solvent banks V (vacant boxes) is degenerate and equals $r - mb$ where $b = \lambda/\mu$. In the intermediate case where $\phi - \mu < \lambda < m(\phi - \mu)$ the distribution of V is non-degenerate. We leave the investigation of the closed-form expression for this distribution where b_k is a function of $\{S_k\}_{k=1}^{m}$ for future work. More generally, results for Bose-Einstein statistics may prove useful for analyzing contagion in networks that are not circular but still symmetric.

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